

The essentially free spectrum of a variety

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ABSTRACT: We partially prove a conjecture from [MeSh] which says that the spectrum of almost free, essentially free, non-free algebras in a variety is either empty or consists of the class of all successor cardinals.

Introduction and notation

Suppose that T is a **variety** in a countable vocabulary τ . This means that τ is a countable set of function symbols and T is a set of equations, i.e. sentences of the form $\forall x_1, \dots, x_n (\sigma_1(x_1, \dots, x_n) = \sigma_2(x_1, \dots, x_n))$ where σ_i are τ -terms. The class of all

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models of T will be denoted by $\text{Mod}(T)$, and a member of $\text{Mod}(T)$ is called an **algebra** in the variety T . Let $M \in \text{Mod}(T)$. For $A \subseteq M$, $\langle A \rangle$ denotes the submodel of M generated by A . Such A is called a **free basis** (of $\langle A \rangle$) if no distinct $a_1, \dots, a_n \in A$ satisfy an equation which is not provable from T . Moreover, M is called **free** if there exists a free basis of M , i.e. one which generates M . By F_λ we denote the free algebra with free basis of size λ , where λ is a cardinal. For $M_1, M_2 \in \text{Mod}(T)$, the **free product** of M_1 and M_2 is denoted by $M_1 * M_2$. Formally it is obtained by building all formal terms in the language τ with constants belonging to the disjoint union of M_1 and M_2 , and then identifying them according to the laws in T . For $M, \langle M_\nu : \nu < \alpha \rangle$ such that $M, M_\nu \in \text{Mod}(T)$ and M is a submodel of M_ν for all $\nu < \alpha$, the **free product of the M'_ν 's over M** is defined similarly, and it is denoted by $*_M \{M_\nu : \nu < \alpha\}$; the intention being that distinct $M_\nu, M_{\nu'}$ are disjoint outside M except for those equalities which follow from the laws in T and the equations in $\text{Diag}(M_\nu) \cup \text{Diag}(M_{\nu'})$. Here Diag denotes the **diagram** of a model. For $M, N \in \text{Mod}(T)$ we say “ N/M is free” if M is a submodel of N and there exists a free basis A of N over M , i.e. A is a free basis, $N = \langle M \cup A \rangle$ and between members of $\langle A \rangle$ and M only those equations hold which follow from T and $\text{Diag}(M)$.

Suppose $|M| = \lambda$. Then M is called **almost free** if there exists an increasing continuous family $\langle M_\nu : \nu < \text{cf}(\lambda) \rangle$ of free submodels of size $< \lambda$ with union M . Moreover, M is called **essentially free** if there exists a free $M' \in \text{Mod}(T)$ such that $M * M'$ is free, **essentially non-free** otherwise. The essentially free spectrum of the variety T which is denoted by $\text{EINC}(T)$, is the class of cardinals λ such that there exists $M \in \text{Mod}(T)$ of size λ which is almost free and essentially free, but not free.

In [MeSh] the **essentially non-free spectrum**, i.e. the spectrum of cardinalities of almost free and essentially non-free algebras in a variety T , has been investigated, and it is shown that this spectrum has no simple description in ZFC, in general. Here we will show that the situation is different for $\text{EINC}(T)$. Firstly, by a general compactness theorem due to the second author (see [Sh]), $\text{EINC}(T)$ contains only regular cardinals. Secondly, we will show that $\text{EINC}(T)$ is contained in the class of successor cardinals. Our conjecture is that $\text{EINC}(T)$ is either empty or equals the class of all successor cardinals (depending on T). Motivating examples for this conjecture are among others $\mathbb{Z}/4\mathbb{Z}$ -modules (where EINC is empty) and $\mathbb{Z}/6\mathbb{Z}$ -modules (where EINC consists of all successor cardinals) (see [EkMe, p.90]). We succeed to prove the conjecture to a certain extent. Namely, we prove

the following theorem.

Theorem. *If for some cardinal μ , $(\mu^{\aleph_0})^+ \in \text{EINC}(T)$, then every successor cardinal belongs to $\text{EINC}(T)$.*

For the proof we will isolate a property of T , denoted $\text{Pr}_1(T)$, which says that a countable model of T with certain properties exists, and then show that, on the one hand, the existence of $M \in \text{Mod}(T)$ in any cardinality of the form $(\mu^{\aleph_0})^+$ implies that $\text{Pr}_1(T)$ holds, and on the other hand, from $\text{Pr}_1(T)$ an algebra $M \in \text{Mod}(T)$ can be constructed in every successor cardinality.

1. $\text{EINC}(T)$ is contained in the class of successor cardinals

Theorem. *For every variety T , $\text{EINC}(T)$ is contained in the class of successor cardinals.*

PROOF: Suppose $\lambda \in \text{EINC}(T)$. By the main result of [Sh], λ must be regular. So suppose λ is a regular limit cardinal. Let $\mathfrak{M} \in \text{Mod}(T)$ be generated by $\{a_\alpha : \alpha < \lambda\}$ and suppose that \mathfrak{M} is almost free and essentially free. We will show that then \mathfrak{M} must be free, and hence does not exemplify $\lambda \in \text{EINC}(T)$.

By assumption and a Löwenheim-Skolem argument, $\mathfrak{M} * F_\lambda$ is free. Let $\{c_\nu : \nu < \lambda\}$, $\{b_\nu : \nu < \lambda\}$ be a free basis of $\mathfrak{M} * F_\lambda$, F_λ , respectively.

Let χ be a large enough regular cardinal, and let $C \subseteq \lambda$ be the club consisting of all α such that for some substructure $\mathcal{A} \prec \langle H(\chi), \in, \prec_\chi \rangle$ of size $< \lambda$ which contains $\mathfrak{M}, F_\lambda, \{a_\nu : \nu < \lambda\}, \{b_\nu : \nu < \lambda\}$ and $\{c_\nu : \nu < \lambda\}$, we have $\mathcal{A} \cap \lambda = \alpha$. Here $H(\chi)$ is the set of all sets which are hereditarily of cardinality $< \chi$, and \prec_χ is a fixed well-ordering of $H(\chi)$. Note that the information about \mathfrak{M} reflects to each $\alpha \in C$, especially $\langle \{c_\nu : \nu < \alpha\} \rangle = \langle \{a_\nu : \nu < \alpha\} \rangle * \langle \{b_\nu : \nu < \alpha\} \rangle$.

Since \mathfrak{M} is supposed to be almost free, the set

$$C_0 = \{\alpha \in C : \langle \{a_\nu : \nu < \alpha\} \rangle \text{ is free}\}$$

is still a club. Let $\alpha, \beta \in C_0$ be cardinals with $\alpha < \beta$. We will show that $\langle \{a_\nu : \nu < \beta\} \rangle / \langle \{a_\nu : \nu < \alpha\} \rangle$ is free. This will suffice to conclude that \mathfrak{M} is free since the cardinals below λ are a club and hence $C_1 = \{\alpha \in C_0 : \alpha \text{ is a cardinal}\}$ is a club such that for every $\alpha, \beta \in C_1$ with $\alpha < \beta$, $\langle \{a_\nu : \nu < \beta\} \rangle / \langle \{a_\nu : \nu < \alpha\} \rangle$ is free.

For the proof, let $\{d_\nu : \nu < \beta\}$ be a free basis of $\langle \{a_\nu : \nu < \beta\} \rangle$. As $\alpha = |\alpha| < |\beta| = \beta$ we may assume $\langle \{a_\nu : \nu < \alpha\} \rangle \subseteq \langle \{d_\nu : \nu < \alpha\} \rangle$. Hence easily

$$\langle \{a_\nu : \nu < \beta\} \rangle \cong_{\langle \{a_\nu : \nu < \alpha\} \rangle} \langle \{a_\nu : \nu < \beta\} \rangle * F_\beta,$$

i.e. there exists an isomorphism which leaves $\langle \{a_\nu : \nu < \alpha\} \rangle$ fixed. But $\langle \{a_\nu : \nu < \beta\} \rangle * F_\beta \cong \langle \{c_\nu : \nu < \beta\} \rangle$ and $\langle \{c_\nu : \nu < \beta\} \rangle / \langle \{c_\nu : \nu < \alpha\} \rangle$ is free. Moreover $\langle \{c_\nu : \nu < \alpha\} \rangle = \langle \{a_\nu : \nu < \alpha\} \rangle * \langle \{b_\nu : \nu < \alpha\} \rangle$ and hence $\langle \{c_\nu : \nu < \alpha\} \rangle / \langle \{a_\nu : \nu < \alpha\} \rangle$ is free. Consequently $\langle \{a_\nu : \nu < \beta\} \rangle / \langle \{a_\nu : \nu < \alpha\} \rangle$ is free. \square

2. EINC(T) is either empty or contains almost all successor cardinals

Definition 2.1. The property $\text{Pr}_1(T)$ says: There exist $N, M \in \text{Mod}(T)$ such that N is countably generated, M is a subalgebra of N and the following clauses hold:

- (i) M has a free basis;
- (ii) $N * F_{\aleph_0} / M$ is free;
- (iii) $*_M\{N : n \in \omega\} * F_{\aleph_0} / M * F_{\aleph_0}$ is not free.

Theorem 2.2. *Suppose that $\text{Pr}_1(T)$ holds and λ is a successor cardinal. Then $\lambda \in \text{EINC}(T)$.*

PROOF: Let $\lambda = \mu^+$. Let M, N witness $\text{Pr}_1(T)$. Let $\mathfrak{N} = *_M\{N : \alpha < \lambda\}$. We claim that $\mathfrak{M} = \mathfrak{N} * F_\mu$ exemplifies that $\lambda \in \text{EINC}(T)$. Let $\{c_\alpha : \alpha < \mu\}$ be a free basis of F_μ .

Firstly, \mathfrak{M} is almost free: For $\alpha < \lambda$ let $\mathfrak{N}_\alpha = *_M\{N : \nu < \alpha\}$. Then clearly $\langle \mathfrak{N}_\alpha * F_\mu : \alpha < \lambda \rangle$ is a λ -filtration of \mathfrak{M} . Moreover $\mathfrak{N}_\alpha * F_\mu$ is free for every $\alpha < \lambda$, since easily $\mathfrak{N}_\alpha * F_\mu \cong *_M\{N * F_{\aleph_0} : \nu < \alpha\}$ and by $\text{Pr}_1(T)$, M is free and $N * F_{\aleph_0} / M$ is free.

Secondly, $\mathfrak{M} * F_\lambda \cong \mathfrak{N} * F_\lambda$ is free, since $\mathfrak{N} * F_\lambda \cong *_M \{N * F_{\aleph_0} : \alpha < \lambda\}$ is free as in the proof of almost freeness.

Thirdly, \mathfrak{M} is not free. By contradiction, suppose that $I = \{d_\nu : \nu < \lambda\}$ were a free basis of \mathfrak{M} .

Let χ be a large enough regular cardinal, and let $\mathcal{A} \prec \langle H(\chi), \in, \prec_\chi \rangle$ such that $|\mathcal{A}| = \mu$, $\mu + 1 \subseteq \mathcal{A}$, and $\lambda^+, N, M, \mathfrak{N}, \mathfrak{M}, F_\mu, I \in \mathcal{A}$. Next choose $\mathcal{B} \prec \langle H(\chi), \in, \prec_\chi \rangle$ such that $|\mathcal{B}| = \aleph_0$, and $\mathcal{A}, \lambda^+, N, M, \mathfrak{N}, \mathfrak{M}, F_\mu, I \in \mathcal{B}$.

Let $u = \mathcal{B} \cap \lambda \setminus (\mathcal{A} \cap \lambda)$, $v = \mathcal{A} \cap \mathcal{B} \cap \lambda$, $w = \mathcal{A} \cap \mathcal{B} \cap \mu$. Notice that $w = \mathcal{B} \cap \mu$. Define $M_1 = \mathcal{A} \cap \mathcal{B} \cap \mathfrak{M}$. Now easily M_1 is countably generated and it has the form

$$M_1 = *_M \{N : \alpha \in v\} * \langle \{c_\alpha : \alpha \in w\} \rangle.$$

Hence $M_1 \cong_M *_M \{N : n \in \omega\} * F_{\aleph_0} \cong_M *_M \{N * F_{\aleph_0} : n \in \omega\} \cong_M M * F_{\aleph_0}$, where for the last isomorphism we applied (ii) from $\text{Pr}_1(T)$. Next define $M_2 = \mathcal{B} \cap \mathfrak{M}$. Then easily

$$M_2 = *_M \{N : \alpha \in u\} *_M M_1.$$

Hence by the isomorphism above we have

$$M_2 \cong *_M \{N : n \in \omega\} * F_{\aleph_0}.$$

By (iii) from $\text{Pr}_1(T)$ we conclude that M_2/M_1 is not free. On the other hand, $\{d_\nu : \nu \in u\}$ witnesses that M_2/M_1 is free, a contradiction. \square

Theorem 2.3. *Suppose λ, μ are cardinals such that $\lambda = \mu^+$, $\mu^{\aleph_0} = \mu$ and $\lambda \in \text{EINC}(T)$. Then $\text{Pr}_1(T)$ holds.*

PROOF: Let \mathfrak{M} exemplify $\lambda \in \text{EINC}(T)$. Let $\{a_\nu : \nu < \lambda\}$ generate \mathfrak{M} . Let F be free such that $\mathfrak{M} * F$ is free. Without loss of generality we may assume that $F = F_\lambda$; in fact, if $|F| < \lambda$ then we may replace F by $F * F_\lambda$ which is isomorphic to F_λ , and if $|F| > \lambda$ use a Löwenheim–Skolem argument. So let $\{b_\nu : \nu < \lambda\}$ be a free basis of F , and let $\{c_\nu : \nu < \lambda\}$ be a free basis of $\mathfrak{N} = \mathfrak{M} * F$.

Let χ be a large enough regular cardinal, and let N_α , for every $\alpha < \lambda$, be a countable elementary substructure of $\langle H(\chi), \in, \prec_\chi \rangle$ such that $\alpha, \mathfrak{M}, F, \mathfrak{N}, \{a_\nu : \nu < \lambda\}, \{b_\nu : \nu < \lambda\}, \{c_\nu : \nu < \lambda\}$ belong to N_α . Let $u_\alpha = N_\alpha \cap \lambda$.

By assumption on \mathfrak{M} (\mathfrak{M} is almost free), the set

$$\{\alpha < \lambda : \langle \{a_\nu : \nu < \alpha\} \rangle \text{ is free} \wedge \langle \{c_\nu : \nu < \alpha\} \rangle = \langle \{a_\nu : \nu < \alpha\} \rangle * \langle \{b_\nu : \nu < \alpha\} \rangle\}$$

contains a club; let C be the \prec_χ -least one. Hence $C \in N_\alpha$ for every $\alpha < \lambda$.

Using elementarity, it is easy to see that for every $\alpha \in C$ the following three clauses hold ((1) holds for every $\alpha < \lambda$):

- (1) $\langle \{c_\nu : \nu \in u_\alpha\} \rangle = \langle \{a_\nu : \nu \in u_\alpha\} \rangle * \langle \{b_\nu : \nu \in u_\alpha\} \rangle$;
- (2) $\langle \{c_\nu : \nu \in u_\alpha \cap \alpha\} \rangle = \langle \{a_\nu : \nu \in u_\alpha \cap \alpha\} \rangle * \langle \{b_\nu : \nu \in u_\alpha \cap \alpha\} \rangle$;
- (3) $\langle \{a_\nu : \nu \in u_\alpha \cap \alpha\} \rangle$ is free, and $\langle \{a_\nu : \nu \in \alpha\} \rangle / \langle \{a_\nu : \nu \in u_\alpha \cap \alpha\} \rangle$ is free.

To prove (3), let $\langle d_\nu : \nu \in I \rangle$ be the \prec_χ -least free basis of $\langle \{a_\nu : \nu \in \alpha\} \rangle$. So by elementarity $\langle d_\nu : \nu \in I \rangle \in N_\alpha$ and $\langle \{d_\nu : \nu \in I \cap N_\alpha\} \rangle = \langle \{a_\nu : \nu \in u_\alpha \cap \alpha\} \rangle$. Hence $\langle d_\nu : \nu \in I \cap N_\alpha \rangle$ and $\langle d_\nu : \nu \in I \setminus N_\alpha \rangle$ witness that (3) holds.

Moreover it is not difficult to see that $C_0 = \{\alpha \in C : \alpha = \bigcup \{u_\nu : \nu < \alpha\}\}$ is still a club. Hence $S_0 = \{\alpha \in C_0 : \text{cf}(\alpha) > \omega\}$ is stationary. By Fodor's Lemma, for some $\alpha^* < \lambda$, $S_1 = \{\alpha \in S_0 : u_\alpha \cap \alpha \subseteq \alpha^*\}$ is stationary. By assumption, $|\alpha^*|^{\aleph_0} \leq \mu^{\aleph_0} < \lambda$. So by thinning out S_1 further (using this assumption and the λ -completeness of the nonstationary ideal on λ), we may find a stationary $S_2 \subseteq S_1$ and $u^* \subseteq \alpha^*$ such that for every $\delta_1, \delta_2 \in S_2$ the following hold:

- (4) $u_{\delta_1} \cap \delta_1 = u^*$;
- (5) $\text{o.t.}(u_{\delta_1}) = \text{o.t.}(u_{\delta_2})$, and the unique order-preserving map $h = h_{\delta_1 \delta_2} : u_{\delta_1} \rightarrow u_{\delta_2}$ induces (by $c_\nu \rightarrow c_{h(\nu)}$) an isomorphism from $\langle \{c_\nu : \nu \in u_{\delta_1}\} \rangle$ onto $\langle \{c_\nu : \nu \in u_{\delta_2}\} \rangle$ which maps a_ν to $a_{h(\nu)}$ and b_ν to $b_{h(\nu)}$.

Let $\delta^* = \min(S_2 \setminus \mu)$, $M = \langle \{a_\nu : \nu \in u^*\} \rangle$ and $N = \langle \{a_\nu : \nu \in u_{\delta^*}\} \rangle$.

As $\delta^* \in C$, by elementarity we know that M is free.

As $\{c_\nu : \nu \in \lambda\}$ is a free basis, clearly $\langle \{c_\nu : \nu \in u_{\delta^*}\} \rangle / \langle \{c_\nu : \nu \in u^*\} \rangle$ is free, and by (2) and as $\delta^* \in S_2 \subseteq C$, also $\langle \{c_\nu : \nu \in u^*\} \rangle / M$ is free. Finally, $\langle \{c_\nu : \nu \in u_{\delta^*}\} \rangle \cong N * F_{\aleph_0}$ by (1). Hence we conclude that $N * F_{\aleph_0} / M$ is free.

Hence, if the pair M, N does not exemplify $\text{Pr}_1(T)$, then (iii) in its definition fails. We will use this to show that then \mathfrak{M} is free, which contradicts our assumption. Then we conclude that $\text{Pr}_1(T)$ holds.

By induction on $\zeta < \lambda$ we choose $w_\zeta \subseteq \lambda$ such that the following requirements are satisfied:

- (6) $w_0 = \delta^*$;
(7) $|w_\zeta| < \lambda$;
(8) for ζ limit, $w_\zeta = \bigcup\{w_\nu : \nu < \zeta\}$;
(9) if $\gamma(\zeta) = \min(\lambda \setminus w_\zeta)$, then $w_{\zeta+1} = w_\zeta \cup \{\gamma(\zeta)\} \cup \{\beta(\zeta, n) : n \in \omega\}$, where the $\beta(\zeta, n)$ belong to S_2 , and for any $m, n \in \omega$ with $m < n$, $\bigcup\{u_{\gamma(\zeta)}, u_\nu : \nu \in w_\zeta\} < \min(u_{\beta(\zeta, n)} \setminus u^*)$ and $\sup(u_{\beta(\zeta, m)}) < \min(u_{\beta(\zeta, n)} \setminus u^*)$ hold.
By (6) and $\delta^* \in C_0 \subseteq C$ we conclude that $\bigcup\{u_\nu : \nu \in w_0\} = \delta^*$ and $\langle\{a_\alpha : \alpha \in \delta^*\}\rangle$ is free. By (8) and (9) it is clear that the sequence

$$\langle\langle\{a_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in w_\zeta\}\}\rangle : \zeta < \lambda\rangle$$

is increasing and continuous with limit \mathfrak{M} . Hence the following claim gives the desired contradiction:

Claim. For every $\zeta < \lambda$, $\langle\{a_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in w_{\zeta+1}\}\}\rangle / \langle\{a_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in w_\zeta\}\}\rangle$ is free.

PROOF: Let us introduce the following notation. For $x \in \{a, b, c\}$ and $I \subseteq \lambda$ set:

$$\begin{aligned} Z_I^x &= \langle\{x_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in I\}\}\rangle \\ W_\zeta^x &= Z_{w_\zeta}^x \\ K^x &= \langle\{x_\alpha : \alpha \in u^*\}\rangle, \text{ so } K^a = M. \end{aligned}$$

The Claim will follow from the following three facts:

- (10) $Z_I^c = Z_I^a * Z_I^b$;
(11) $\langle W_\zeta^a \cup Z_{\{\gamma(\zeta)\}}^a \rangle * F_{\aleph_0} / W_\zeta^a$ is free;
(12) $W_{\zeta+1}^a = *_{K^a} \{Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a, Z_{\{\beta(\zeta, n)\}}^a : n \in \omega\}$.

For (10), to prove $Z_I^c = \langle Z_I^a \cup Z_I^b \rangle$ is rather straightforward by using (1). Moreover there exists a homomorphism $h : \mathfrak{M} * F \rightarrow Z_I^c$ which is the identity on Z_I^c and maps \mathfrak{M} onto Z_I^a ; h can be defined by letting

$$h(c_\alpha) = \begin{cases} c_\alpha & \text{if } \alpha \in \bigcup\{u_\nu : \nu \in I\}, \\ a_0 & \text{otherwise.} \end{cases}$$

Now suppose that $\langle Z_I^a \cup Z_I^b \rangle \models \phi(\bar{a}, \bar{b})$, where ϕ is an equation and $\bar{a} \subseteq \{a_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in I\}\}$, $\bar{b} \subseteq \{b_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in I\}\}$ are finite. Then this equation holds in

$\mathfrak{M} * F$, of course. As $\{b_\nu : \nu < \lambda\}$ is a free basis of F , we conclude that $\phi(\bar{a}, \bar{b})$ is provable from finitely many equations in $\text{Diag}(\mathfrak{M})$ and the laws of the variety. But h maps this proof to a proof from $\text{Diag}(Z_I^a)$ and leaves \bar{a}, \bar{b} fixed. Consequently $Z_I^c = Z_I^a * Z_I^b$ holds.

To prove (11), first clearly $Z_{w_\zeta \cup \{\gamma(\zeta)\}}^c / W_\zeta^c$ is free. Hence by (10), $Z_{w_\zeta \cup \{\gamma(\zeta)\}}^c / W_\zeta^a$ is free. We may assume that $|w_\zeta| = \mu$; hence $Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_\mu / W_\zeta^a$ is free. How to get μ down to \aleph_0 ? Choose $\bar{N} \prec (H(\chi), \in, \prec_\chi)$ such that \bar{N} is countable and contains everything relevant, especially $W_\zeta^a, Z_{\gamma(\zeta)}^a, F_\mu$. Let $X \in \bar{N}$ be a free basis of $Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_\mu$ over W_ζ^a . Then $X \cap \bar{N}$ is a free basis of $\bar{N} \cap (Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_\mu)$ over $\bar{N} \cap W_\zeta^a$. Moreover $\bar{N} \cap (Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_\mu) = \langle (\bar{N} \cap W_\zeta^a) \cup Z_{\gamma(\zeta)}^a \rangle * (\bar{N} \cap F_\mu) \cong \langle (\bar{N} \cap W_\zeta^a) \cup Z_{\gamma(\zeta)}^a \rangle * F_{\aleph_0}$. We claim that $X \cap \bar{N}$ is a free basis over W_ζ^a of what it generates over W_ζ^a , namely $Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_{\aleph_0}$. Otherwise there were finite sets $X_0 \subseteq X$ and $Y_0 \subseteq W_\zeta^a$ such that $X_0 \cup Y_0$ satisfies an equation which does not follow from the laws of the variety and the equalities in $\text{Diag}(W_\zeta^a)$. By elementarity we can find $Y_1 \subseteq W_\zeta^a \cap \bar{N}$ such that $X_0 \cup Y_1$ satisfies the same equation, a contradiction.

To prove (12), if a is replaced by c , then (12) is easily verified by using the free basis $\{c_\nu : \nu < \lambda\}$. But then using (10) and $K^c = K^a * K^b$ we easily finish.

Finally, as $\delta^* \in C$ and δ^* has uncountable cofinality, by (3) we may choose $F_{\aleph_0} \subseteq \langle \{a_\nu : \nu \in \delta^*\} \rangle$ (i.e. an algebra isomorphic to F_{\aleph_0}) such that $M \cap F_{\aleph_0} = \emptyset$ and even $\langle M \cup F_{\aleph_0} \rangle = F_{\aleph_0} * M$. By (12) we conclude

$$W_{\zeta+1}^a = Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a *_{F_{\aleph_0} * M} (F_{\aleph_0} * *_{M} \{Z_{\beta(\zeta, n)}^a : n \in \omega\}).$$

Moreover by construction (as $\beta(\zeta, n) \in S_2$), $Z_{\beta(\zeta, n)}^a \cong_M N$ for every $n \in \omega$. Hence $F_{\aleph_0} * *_{M} \{Z_{\beta(\zeta, n)}^a : n \in \omega\} \cong_M F_{\aleph_0} * *_{M} \{N : n \in \omega\}$. By assumption $F_{\aleph_0} * *_{M} \{N : n \in \omega\} / F_{\aleph_0} * M$ is free, and so clearly of rank \aleph_0 . We conclude that $W_{\zeta+1}^a \cong_{W_\zeta^a} Z_{w_\zeta \cup \{\gamma(\zeta)\}}^a * F_{\aleph_0}$, and so $W_{\zeta+1}^a / W_\zeta^a$ is free by (11). $\square \square$

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