ADVANCES IN CARDINAL ARITHMETIC
SH420

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Annotated Content

§1 $I[\lambda]$ is quite large

[If $\mathrm{cf}\kappa = \kappa, \kappa^+ < \mathrm{cf}\lambda = \lambda$ then there is a stationary subset S of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ in $I[\lambda]$. Moreover, we can find $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of λ, $\mathrm{otp}(C_\delta) = \kappa$, guessing clubs and for each $\alpha < \lambda$ we have: $\{C_\delta \cap \alpha : \alpha \in \text{ nacc } C_\delta\}$ has cardinality $< \lambda$.]

§2 Measuring $\mathscr{S}_{<\kappa}(\lambda)$

[We prove that e.g. there is a stationary subset of $\mathscr{S}_{<\aleph_1}(\lambda)$ of cardinality $\mathrm{cf}(\mathscr{S}_{<\aleph_1}(\lambda), \subseteq)$.]

§3 Nice filters revisited

[We prove the existence of nice filters when instead being normal filters on ω_1 they are normal filters with larger domains, which can increase during a play. They can help us transfer situation on $\aleph_1$-complete filters to normal ones].

§4 Ranks

[We reconsider ranks and niceness of normal filters, such that we can pass say from $pp_{\Gamma(\aleph_1)}(\mu)$ (where $\mathrm{cf}\mu = \aleph_1$) to $\mathrm{pp}_{\mathrm{normal}}(\mu)$.]

§5 More on ranks and higher objects

§6 Hypotheses

[We consider some weakenings of G.C.H. and their consequences. Most have not been proved independent of ZFC.]

§1 $I[\lambda]$ is Quite Large and Guessing Clubs

On $I[\lambda]$ see [Sh 108], [Sh 88a], [Sh 351, §4] (but this section is self-contained; see Definition 1.1 and Claim 1.3 below). We shall prove that for regular κ, λ, such that $\kappa^+ < \lambda$, there is a stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ in $I[\lambda]$. We then investigate "guessing clubs" in (ZFC).

1.1 Definition. For a regular uncountable cardinal $\lambda, I[\lambda]$ is the family of $A \subseteq \lambda$ such that $\{\delta \in A : \delta = \text{cf}(\delta)\}$ is not stationary and for some $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ we have:

- (a) $\mathscr{P}_\alpha$ is a family of $< \lambda$ subsets of α
- (b) for every limit $\alpha \in A$ of cofinality $< \alpha$ there is $x \subseteq \alpha$, $\text{otp}(x) < \alpha = \sup(x)$ such that $\zeta < \alpha \Rightarrow x \cap \zeta \in \{\mathscr{P}_\gamma : \gamma < \alpha\}$.

1.2 Observation. In Definition 1.1 we can weaken (b) to:

for some club E of x for every limit $\alpha \in A \cap E$ of cofinality $< \alpha \ldots$.

Proof. Just replace $\mathscr{P}_\alpha$ by $\{x \cap \alpha : x \in \cup\{\mathscr{P}_\beta : \beta \leq \text{ Min}(E \backslash \alpha + 1)\}\}$.

We know (see [Sh 108], [Sh 88a] or below)

1.3 Claim. *Let $\lambda > \aleph_0$ be regular.*
1) $A \in I[\lambda]$ iff (note: by (c) below the set of inaccessibles in A is not stationary and) there is $\langle C_\alpha : \alpha < \lambda \rangle$ such that:

- (a) *C_α is a closed subset of α*
- (b) *if $\alpha^* \in \text{nacc}(C_\alpha)$ then $C_{\alpha^*} = C_\alpha \cap \alpha$ (nacc stands for "non-accumulation")*
- (c) *for some club E of λ, for every $\delta \in A \cap E$, we have: $\text{cf}(\delta) < \delta$ and $\delta = \sup(C_\delta)$, and $\text{cf}(\delta) = \text{otp}(C_\delta)$*
- (d) *$\text{nacc}(C_\alpha)$ is a set of successor ordinals.*

2) $I[\lambda]$ is a normal ideal.

Proof. 1) The "if" part:
Assume $\langle C_\beta : \beta < \lambda \rangle$ satisfy $(a), (b), (c)$ with a club E for (c). For each limit $\alpha < \lambda$ choose a club e_α of order type $\text{cf}(\alpha)$. We define, for $\alpha < \lambda$:

$$\mathscr{P}_\alpha =: \{C_\beta : \beta \le \alpha\} \cup \{e_\beta : \beta \le \alpha\} \cup \{e_\gamma \cap \alpha : \gamma \le \text{ Min}(E \backslash (\alpha+1)\}.$$

It is easy to check that $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ exemplify "$A \in I[\lambda]$".

<u>The "only if" part</u>:

Let $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ exemplify "$A \in I[\lambda]$" (by Definition 1.1). Without loss of generality

(∗) if $C \in \mathscr{P}_\alpha$, and $\zeta \in C$ then $C \backslash \zeta \in \mathscr{P}_\alpha$ and $C \cap \zeta \in \mathscr{P}_\alpha$

For each limit $\beta < \lambda$ let e_β be a club of β satisfying $\text{otp}(e_\beta) = \text{cf}(\beta)$ and $\text{cf}(\beta) < \beta \Rightarrow \text{cf}(\beta) < \min(e_\beta)$. Let $\langle \gamma_i : i < \lambda \rangle$ be strictly increasing continuous, each γ_i a non-successor ordinal $< \lambda$, $\gamma_0 = 0$, and $\gamma_{i+1} - \gamma_i \ge \aleph_0 + |\bigcup\limits_{\alpha \le \gamma_i} \mathscr{P}_\alpha| + |\gamma_i|$ and $\gamma_i \in A \Rightarrow \text{cf}(\gamma_i) < \gamma_i$.
(Why? Let E' be a club of λ such that $\gamma \in E \cap A \Rightarrow \text{cf}(\gamma) < \gamma$, and then choose $\gamma_i \in E$ by induction on $i < \lambda$.)

Let F_i be a one to one function from $(\bigcup\limits_{\alpha \le \gamma_i} \mathscr{P}_\alpha) \times \gamma_i$ into $\{\zeta+1 : \gamma_i < \zeta+1 < \gamma_{i+1}\}$.
Now we choose $C_\alpha \subseteq \alpha$ as follows. First, for $\aleph = 0$ let $C_\alpha = \emptyset$. Second, assume α is a successor ordinal, let $i(\alpha)$ be such that $\gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1}$. If $\alpha \notin \text{ Rang}(F_{i(\alpha)})$, let $C_\alpha = \emptyset$. If $\alpha = F_{i(\alpha)}(x, \beta)$ hence necessarily $x \in \bigcup\limits_{\epsilon \le \gamma_{i(\alpha)}} \mathscr{P}_\epsilon, \beta < \gamma_{i(\alpha)})$ and x, β are unique. Let C_α be the closure (in the order topology) of C^-_α, which is defined as:

$$\big\{F_j(x \cap \zeta, \beta) : \text{ the sequence } (j, \zeta, \beta) \text{ satisfies } (*)^{x,\beta}_{j,\zeta} \text{ below}\big\} \text{ where}$$

$\boxtimes^{x,\beta}_{j,\zeta}$ (i) $\zeta \in x$

(ii) $\text{otp}(x \cap \zeta) \in e_\beta$,

(iii) $j < i(\alpha)$ is minimal such that $x \cap \zeta \in \bigcup\limits_{\epsilon \le \gamma_j} \mathscr{P}_\epsilon$

(iv) if $\xi \in x \cap \zeta$, $\text{otp}(x \cap \xi) \in e_\beta$ then $(\exists j(1) < j)[x \cap \xi \in \bigcup\limits_{\epsilon \le \gamma_{j(1)}} \mathscr{P}_\epsilon]$

(v) $\beta < \text{ Min}(x)$.

Third, for $\alpha < \lambda$ limit, choose C_α: if possible, $\mathrm{nacc}(C_\alpha)$ is a set of successor ordinals, C_α is a club of α, $[\beta \in \mathrm{nacc}(C_\alpha) \Rightarrow C_\beta = \beta \cap C_\alpha]$; if this is impossible, let $C_\delta = \emptyset$. Lastly, let $C_0 = \emptyset$ and let $E =: \{\gamma_i : i \text{ is a limit ordinal } < \lambda\}$.
Now we can check the condition in 1.3(1).

Note that for α successor $C^-_\alpha = \mathrm{nacc}(C_\alpha)$.

<u>Clause (a)</u>: C_α a closed subset of α.

If $\alpha = 0$ trivial as $C_\alpha = \emptyset$ and if α is a limit ordinal, this is immediate by the definition. So let α be a successor ordinal, hence, by the choice of $\langle \gamma_i : i < \lambda \rangle$ as an increasing continuous sequence of nonsuccessor ordinals with $\gamma_0 = 0$, clearly $i(\alpha)$ is well defined, $\gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1}$. Now if $\alpha \notin \mathrm{Rang}(F_{i(\alpha)})$ then $C_\alpha = \emptyset$ and we are done so for some x, β we have $\alpha = F_{i(\alpha)}(x, \beta)$ hence necessarily $x \in \bigcup\limits_{\epsilon \le \gamma_{i(\alpha)}} \mathscr{P}_\epsilon$ and $\beta < \gamma_{i(\alpha)}$. By the definition of C_α (the closure in the order topology on α, of the set of C^-_α i.e. the set of $F_j(x \cap \zeta, \beta)$ for the pair (j, ζ) satisfying $\boxtimes^{x,\beta}_{j,\zeta}$ it suffices to show $C^-_\alpha \subseteq \alpha$, i.e.

(∗) if the pair (j, ζ) satisfies $\boxtimes^{x,beta}_{j,\zeta}$ then $F_j(x \cap \zeta, \beta) < \alpha$.

So assume (j, ζ) satisfies $\boxtimes^{x,\beta}_{j,\zeta}$ but by clause (iii) we know that $j < i(\alpha)$ and so $\mathrm{Rang}(F_j) \subseteq \gamma_{j+1} \subseteq \gamma_{i(\alpha)} < \alpha$ as required.

<u>Clause (*b*)</u>: If $\alpha^* \in \mathrm{nacc}(C_\alpha)$ then $C_{\alpha^*} = C_\alpha \cap \alpha^*$.

If it is enough to show $C^-_{\alpha^*} = \alpha^* \cap C^-_\alpha$ and as $C^-_\alpha = \mathrm{nacc}(C_\alpha)$, we have $\alpha^* \in C^-_\alpha$. As $\alpha^* \in C^-_\alpha$ necessarily for some ζ, j satisfying $\boxtimes^{x,\beta}_{j,\zeta}$ we have $\alpha^* = F_j(x \cap \zeta, \beta)$. By the choice of F_j necessarily α^* is a successor ordinal and $\gamma_j < \alpha^* < \gamma_{j+1}$.

Now any member $\alpha(1)$ of $\alpha^* \cap C^-_\alpha$ has the form $F_{j(1)}(x \cap \zeta(1), \beta)$ with $j(1), \zeta(1)$ satisfying $\boxtimes^{x,\beta}_{j,\zeta}$; clearly $\gamma_{j(1)} < \alpha(1) = F_{j(*)}(x \cap \zeta(1), \beta) < \gamma_{j(1)+1}$ and $\gamma_j < \alpha^* = F_j(x \cap \zeta, \beta) < \gamma_{j+1}$. But $\alpha(1) < \alpha^*$ (being in $\alpha^* \cap C^-_\alpha$) so necessarily $j(1) + 1 \le j$. So $j(1)$, $\zeta(1)$ satisfy $(i) - (v)$ with x replaced by $x \cap \zeta$, i.e., satisfy $\boxtimes^{x,\beta}_{j,\zeta}$; recall by $\alpha^* = F_j(x \cap \zeta, \beta)$, so $F_{j(x)}(x \cap \zeta(1), \beta) \in C^-_{\alpha^*}$. So $\alpha^* \cap C^-_\alpha \subseteq C^-_{\alpha^*}$; similarly $C^-_{\alpha^*} \subseteq \alpha^* \cap C^-_\alpha$, so we get the desired equality.

<u>Clause (*c*)</u>: We shall show that $E = \{\gamma_i : i \text{ is a limit ordinal } < \lambda\}$ is as required in closed (c).

Clearly E is a club of λ. So assume that $\delta \in A \cap E$ we should prove: $\mathrm{cf}(\delta) < \delta, \delta = \sup(C_\delta)$, $\mathrm{cf}(\delta) = \mathrm{otp}(C_\delta)$.
Now $\delta \in E \cap A \Rightarrow \delta > \mathrm{cf}(\delta)$ holds as we assume $\gamma_i \in A \Rightarrow \mathrm{cf}(\gamma_i) < \gamma_i$. As $\delta \in E$, by E's definition for some limit ordinal $i(*)$ we have $\delta = \gamma_{i(*)}$. By the choice of C_δ it is enough to find a set C closed unbounded in δ of order type $\mathrm{cf}(\delta)$ such that $\alpha \in \mathrm{nacc}(C) \Rightarrow \alpha$ successor & $C_\alpha = C \cap \alpha$.

By the choice of $\bar{\mathscr{P}}$, for some $x \subseteq \delta$, $\mathrm{otp}(x) < \delta = \sup(x)$ and $\bigwedge\limits_{\zeta<\delta} x \cap \zeta \in \bigcup\limits_{\gamma<\delta} \mathscr{P}_\gamma$.
By $(*)$ above also $\xi \in x \ \ \& \ \ \bar{S} \in x \backslash \xi \Rightarrow x \cap \zeta \backslash \xi \in \bigcup\limits_{\gamma<\delta} \mathscr{P}_\gamma$ so without loss of generality $\mathrm{otp}(x) <$ Min(x). Let $\beta =$ otp(x), so we know that β is a limit ordinal, moreover cf$(\beta) =$ cf(δ). Remember e_β is a club of β of order type cf(β) which is cf(δ). Let

$$y =: \{\zeta \in x : \ \mathrm{otp}(x \cap \zeta) \in e_\beta\}.$$

Clearly y is a subset of x of order type otp$(e_\beta) =$ cf(δ). Define $h : y \rightarrow i(*)$ by $h(\zeta) =$ Min$\{j : x \cap \zeta \in \bigcup\limits_{\epsilon \le \gamma_j} \mathscr{P}_\epsilon\}$, so by $(*)$ we know that h is non-decreasing, and by the choice of x, $\bigwedge\limits_{\zeta \in y} \gamma_{h(\zeta)} < \delta$, equivalently $\bigwedge\limits_{\zeta \in y} h(\zeta) < i(*)$.
Let $z = \{\zeta \in y : \mathit{for}$ every $\xi \in y \cap \zeta$ we have $h(\xi) < h(\zeta)\}$. Let $C^- = \big\{F_{h(\zeta)}(x \cap \zeta, \beta) : \zeta \in z\big\}$; it satisfies: $C^- \subseteq \delta = \sup{}^\alpha \delta_\alpha$ and it is easy to check, as in the proof of clause (c) that $[\alpha \in C^- \Rightarrow C^-_\alpha = C^- \cap \alpha]$. So by the choice of C^- its closure in δ is as required.

<u>Clause (*d*)</u>: nacc(C_α) is a set of successor ordinals.
Check.

Remark. 1) We could also strengthen $(*)$ to make $z \cap \zeta \in \mathscr{P}_{h(\zeta)}$.
2) By Definition 1.1 we know that $I[\lambda]$ is an ideal; by 1.3(1) we know that $I[\lambda]$ includes the ideal of non-stationary subsets of λ. By the last phrase and Definition 1.1, clearly $I[\lambda]$ is normal. $\square_{1.3}$

1.4 Claim. *If κ, λ are regular, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}, S \in I[\lambda], S$ stationary, $\kappa^+ < \lambda$ <u>then</u> we can find $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ such that for $\delta(*) =: \kappa$ we have:*

$\oplus^{\lambda,\delta(*)}_{\mathscr{P}_S}$ (*i*) *$\mathscr{P}_\alpha$ is a family of closed subsets of $\alpha, |\mathscr{P}_\alpha| < \lambda$*

(*ii*) $\mathrm{otp}(C) \le \delta(*)$ *for* $C \in \bigcup\limits_\alpha \mathscr{P}_\alpha$

(*iii*) *for some club E of λ, we have:*
$[\alpha \notin E \Rightarrow \mathscr{P}_\alpha = \emptyset]$ *and*
$[\alpha \in E \Rightarrow (\forall C \in \mathscr{P}_\alpha)(\mathrm{otp}(C) \le \delta(*))]$
$[\alpha \in E \backslash (S \cap \mathrm{acc}(E)) \Rightarrow (\forall C \in \mathscr{P}_\alpha)[\mathrm{otp}(C) < \delta(*)]$
$[\alpha \in S \cap \mathrm{acc}(E) \Rightarrow (\exists! C \in \mathscr{P}_\alpha)(\mathrm{otp}(C) = \delta(*))]$
$[\alpha \in S \cap \mathrm{acc}(E) \ \ \& \ \ C \in \mathscr{P}_\alpha \ \ \& \ \ \mathrm{otp}(C) = \delta(*) \Rightarrow \alpha = \sup(C))]$

(iv) $C \in \mathscr{P}_\alpha$ & $\beta \in \mathrm{nacc}(C) \Rightarrow \beta \cap C \in \mathscr{P}_\beta$

(v) *for any club E' of λ for some $\delta \in S \cap E'$ and $C \in \mathscr{P}_\delta$ we have $C \subseteq E'$ & $\mathrm{otp}(C) = \delta(*)$.*

Proof. Let $\langle C_\alpha : \alpha < \lambda \rangle$ witness "$S \in I[\lambda]$" be as in 1.3(1); without loss of generality $\mathrm{otp}(C_\alpha) \leq \delta(*)$. For any club E, consisting of limit ordinals for simplicity, let us define $\mathscr{P}_E^\alpha$ by induction on $\alpha < \lambda$:

$$\begin{aligned}\mathscr{P}_E^\alpha =: &\{\alpha \cap g\ell(C_\beta, E) : \alpha \in E \text{ and } \alpha \leq \beta < \ \mathrm{Min}[E \backslash (\alpha+1)]\} \\ &\cup \{C \cup \{\beta\} : \beta \in E \cap \alpha, C \in \mathscr{P}_E^\beta \text{ and } \mathrm{otp}(C) < \delta(*)\}\end{aligned}$$

where

$$g\ell(C_\beta, E) =: \{\sup(E \cap (\gamma+1)) : \gamma \in C_\beta \text{ and } \gamma > \ \mathrm{Min}(E)\}.$$

Note that $|\mathscr{P}_E^\alpha| \leq |\mathrm{Min}(E \backslash (\alpha+1)| < \lambda$.
We can prove that for some club E of λ the sequence $\langle \mathscr{P}_E^\alpha : \alpha < \lambda \rangle$ is as required except possibly clause (v) which can be corrected gotten by a right of E (just by trying successively κ^+ clubs E_ζ (for $\zeta < \kappa^+$) decreasing with ζ, see [Sh 365]). Note that clause (iv) guaranteed by demanding E to consist of limit ordinals only and the second set in the union defining $\mathscr{P}_E^\alpha$. $\square_{1.4}$

The following lemma gives sufficient condition for the existence of "quite large" stationary sets in $I[\lambda]$ of almost any fixed cofinality.

1.5 Lemma. *Suppose*

(i) *$\lambda > \kappa > \aleph_0, \lambda$ and κ are regular*

(ii) *$\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \kappa \rangle$, $\mathscr{P}_\alpha$ a family of $< \lambda$ closed subsets of α*

(iii) *$I_{\bar{\mathscr{P}}} =: \{S \subseteq \kappa :$ for some club E of κ for no $\delta \in S \cap E$ is there a club C of δ, such that $C \subseteq E$ and $[\alpha \in \mathrm{nacc}(C) \Rightarrow C \cap \alpha \in \bigcup_{\beta < \alpha} \mathscr{P}_\beta]\}$ is a proper ideal on κ.*

<u>Then</u> there is $S^* \in I[\lambda]$ such that for stationarily many $\delta < \lambda$ of cofinality $\kappa, S^* \cap \delta$ is stationary in δ, moreover for some club E of δ of order type κ

$$\{\mathrm{otp}(\alpha \cap E) : \alpha \in E \backslash S^*\} \in I_{\mathscr{P}}.$$

1.6 Remark. 1) The "for stationarily many" in the conclusion can be strengthened to: a set whose complement is in the ideal defined in [Sh 371, §2].
2) So if $\kappa^\sigma < \lambda$ then we can have $\{i < \kappa : \mathrm{cf}(i) = \sigma\} \in I_{\bar{\mathscr{P}}}$.

Proof. Let χ be regular large enough, N^* be an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality λ such that $(\lambda + 1) \subseteq N^*$, $\bar{\mathscr{P}} \in N$. Let $\bar{C} = \langle C_i : i < \lambda \rangle$ list $N^* \cap \{A \subseteq \lambda : |A| < \kappa\}$ and let

$$S^* = \{\delta < \lambda : \mathrm{cf}(\delta) < \kappa \text{ and for some } A \subseteq \delta \text{ satisfying } \delta = \sup(A), \text{ we have } \mathrm{otp}(A) < \kappa \text{ and } (\forall \alpha < \delta)[A \cap \alpha \in \{C_i : i < \delta\}]\}.$$

Clearly $S^* \in I[\lambda]$; so we should only find enough $\delta < \lambda$ of cofinality κ as required in the conclusion of 1.5. So let E^* be a club of λ and we shall prove that such $\delta \in E^*$ exists. We can choose M_ζ by induction on $\zeta \leq \kappa$ such that:

- (a) $M_\zeta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
- (b) $\|M_\zeta\| < \lambda, M_\zeta \cap \lambda$ an ordinal
- (c) M_ζ is increasing continuous
- (d) $N, \kappa, \bar{\mathscr{P}}, \bar{C}, E^*$ belongs to M_0
- (e) $\langle M_\epsilon : \epsilon \leq \zeta \rangle \in M_{\zeta+1}$.

Let $\delta_\zeta = \sup(M_\zeta \cap \lambda)$, clearly $\delta_\zeta \in E^*$ for every $\zeta \leq \kappa$ and $\langle \delta_\zeta : \zeta \leq \kappa \rangle$ is a (strictly) increasing continuous, so $\delta =: \delta_\kappa$ has cofinality κ. Hence there is a (strictly) increasing continuous sequence $\langle \alpha_\zeta : \zeta < \kappa \rangle \in N^*$ with limit δ, and clearly $E = \{\zeta < \kappa : \alpha_\zeta = \delta_\zeta$ and ζ is a limit ordinal$\}$ is a club of κ. We know that

$$T =: \{\zeta < \kappa : \zeta \in E \text{ and for some club } C \text{ of } \zeta, C \subseteq E \text{ and } \bigwedge_{\epsilon < \zeta} [C \cap \epsilon \in \bigcup_{\xi < \zeta} \mathscr{P}_\xi]\}.$$

is stationary; moreover, $\kappa \backslash T \in I_{\bar{\mathscr{P}}}$ (see assumption (iii)) and clearly $T \subseteq E$.
Clearly it suffices to show

- $(*)$ $\zeta \in T \Rightarrow \delta_\zeta \in S^*$.

Suppose $\zeta \in T$, so there is C, a club of ζ such that $C \subseteq E$ and $\bigwedge_{\epsilon<\zeta}[C \cap \epsilon \in \bigcup_{\xi<\zeta} \mathscr{P}_\xi]$. Let $C^* = \{\delta_\epsilon : \epsilon \in C\}$, so C^* is a club of δ_ζ of order type $\leq \zeta < \kappa$ (which *is* $< \delta_0 \leq \delta_\zeta$). It suffices to show for $\xi \in C$ that $\{\delta_\epsilon : \epsilon \in \xi \cap C\} \in \{C_i : i < \delta_\zeta\}$. For this end we shall show

- (α) $\{\delta_\epsilon : \epsilon \in C \cap \xi\} \in \{C_i : i < \lambda\}$
- (β) $\{\delta_\epsilon : \epsilon \in C \cap \xi\} \in M_{\xi+1}$.

This suffices as $\langle C_i : i < \lambda\rangle \in M_0 \prec M_{\xi+1}$ and $M_{\xi+1} \cap \{C_i : i < \lambda\} = \{C_i : i \in \lambda \cap M_{\xi+1}\} = \{C_i : i < \delta_{\xi+1}\}$.

Proof of (α). Remember $\langle \alpha_\epsilon : \epsilon < \kappa\rangle \in N^*$. Also $\bar{\mathscr{P}} = \langle \mathscr{P}_\epsilon : \epsilon < \kappa\rangle \in N^*$ hence $\bigcup_{\epsilon<\kappa} \mathscr{P}_\epsilon \subseteq N^*$ (as $\kappa < \lambda, |\mathscr{P}_\epsilon| < \lambda, \lambda+1 \subseteq N, \bar{\mathscr{P}} \in N^*$ so now for $\xi \in C$ we have $C \cap \xi \in \bigcup_{\epsilon<\kappa} \mathscr{P}_\epsilon$; hence $C \cap \xi \in N^*$. Together $\{\alpha_\epsilon : \epsilon \in \xi \cap C\} \in N^*$; as $\epsilon \in C \Rightarrow \epsilon \in E \Rightarrow \alpha_\epsilon = \delta_\epsilon$ (as $C \subseteq E$ and the definition of E), and the definition of $\langle C_i : i < \lambda\rangle$, we are done.

Proof of (β). We know $\bar{\mathscr{P}} \in M_0$; as $|\mathscr{P}_\epsilon| < \lambda, \kappa < \lambda$ clearly $|\bigcup_{\epsilon<\kappa} \mathscr{P}_\epsilon| < \lambda$ so as $M_\epsilon \cap \lambda$ is an ordinal, clearly $\bigcup_{\epsilon<\kappa} \mathscr{P}_\epsilon \subseteq M_0$. So for $\epsilon < \zeta$ we have $C \cap \epsilon \in \bigcup_{\gamma<\zeta} \mathscr{P}_\gamma \subseteq M_0 \subseteq M_{\xi+1}$. As $\langle M_i : i \leq \xi\rangle \in M_{\xi+1}$ clearly $\langle \delta_i : i \leq \xi\rangle \in M_{\xi+1}$ hence by the previous sentence also $\langle \delta_i : i \in C \cap \xi\rangle \in M_{\xi+1}$, as required. $\square_{1.5}$

1.7 Conclusion. If κ, λ are regular, $\kappa^+ < \lambda$ then there is a stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ in $I[\lambda]$.

Proof. If $\lambda = \kappa^{++}$ - use [Sh 351, 4.1]. So assume $\lambda > \kappa^{++}$. By [Sh 351, 4.1] the pair (κ, κ^{++}) satisfies the assumption of 1.4 for $S = \{\delta < \kappa^{++} : \mathrm{cf}(\delta) = \kappa\}$; (i.e. κ, λ there stands for κ, κ^{++} here). Hence the conclusion of 1.4 holds for some $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \kappa^{++}\rangle$, $|\mathscr{P}_\alpha| < \kappa^{++}$. Now apply 1.5 with (κ^{++}, λ) here standing for (κ, λ) there (we have just proved $I_{\bar{\mathscr{P}}}$ is a proper ideal, so assumption (ii) holds). Note:

- $(*)$ $\{\delta < \kappa^{++} : \mathrm{cf}(\delta) = \kappa\} \notin I_{\bar{\mathscr{P}}}$.

Now the conclusion of 1.5 (see the moreover and choice of $\bar{\mathscr{P}}$ i.e. $(*)$) gives the desired conclusion. $\square_{1.7}$

1.8 Conclusion. If $\lambda > \kappa$ are uncountable regular, $\kappa^+ < \lambda$, <u>then</u> for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and some $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ we have: $\oplus^{\lambda,\kappa}_{\bar{\mathscr{P}},S}$ from the conclusion of 1.4 holds.

Proof. As κ is regular apply 1.7 and then 1.4. $\square_{1.8}$

Now 1.8 was a statement I have long wanted to know, still sometimes we want to have "$C_\delta \subseteq E$, $\mathrm{otp}(C) = \delta(*)$", $\delta(*)$ not a regular cardinal. We shall deal with such problems.

1.9 Claim. *Suppose*

(*i*) $\lambda > \kappa > \aleph_0, \lambda$ *and* κ *are regular cardinals*

(*ii*) $\bar{\mathscr{P}}_\ell = \langle \mathscr{P}_{\ell,\alpha} : \alpha < \kappa\rangle$ *for* $\ell = 1,2$, *where* $\mathscr{P}_{1,\alpha}$ *is a family of* $< \lambda$ *closed subsets of* α, $\mathscr{P}_{2,\alpha}$ *is a family of* $\leq \lambda$ *clubs of* α *and* $[C \in \mathscr{P}_{2,\alpha}\ \ \&\ \ \beta \in C \Rightarrow C \cap \beta \in \bigcup\limits_{\gamma<\alpha} \mathscr{P}_{1,\gamma}]$

(*iii*) $I_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_2} =: \{S \subseteq \kappa :$ *for some club* E *of* κ *for no* $\delta \in S \cap E$ *is there* $C \in \mathscr{P}_{2,\alpha}$, $C \subseteq E\}$ *is a proper ideal on* κ.

<u>*Then*</u> *we can find* $\bar{\mathscr{P}}^*_\ell = \langle \mathscr{P}^*_{\ell,\alpha} : \alpha < \lambda\rangle$ *for* $\ell = 1,2$ *such that:*

(*A*) $\mathscr{P}^*_{1,\alpha}$ *is a family of* $< \lambda$ *closed subsets of* α

(*B*) $\beta \in \mathrm{nacc}(C)\ \ \&\ \ C \in \mathscr{P}^*_{1,\alpha} \Rightarrow C \cap \beta \in \mathscr{P}^*_{1,\beta}$

(*C*) $\mathscr{P}^*_{2,\delta}$ *is a family of* $\leq \lambda$ *clubs of* δ *(for* δ *limit* $< \lambda$ *such that)* $[\beta \in \mathrm{nacc}(C)\ \ \&\ \ C \in \mathscr{P}^*_{2,\delta} \Rightarrow C \cap \beta \in \mathscr{P}^*_{1,\beta}]$

(*D*) *for every club* E *of* λ *for some strictly increasing continuous sequence* $\langle \delta_\zeta : \zeta \leq \kappa\rangle$ *of ordinals* $< \lambda$ *we have* $\{\zeta < \kappa : \zeta$ *limit, and for some* $C \in \mathscr{P}_{2,\zeta}$ *we have:*
$\{\delta_\epsilon : \epsilon \in C\} \in \mathscr{P}^*_{2,\delta_\zeta}$ *(hence* $[\xi \in \mathrm{nacc}(C) \Rightarrow \{\delta_\epsilon : \epsilon \in C \cap \xi\} \in \mathscr{P}^*_{1,\delta_\xi}]\} \equiv \kappa$ *mod* $I_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_2}$

(*E*) *we have* e_δ *a club of* δ *of order type* $\mathrm{cf}(\delta)$ *for any limit* $\delta < \lambda$*; such that for any* $C \in \bigcup\limits_{\alpha<\lambda} \mathscr{P}^*_{2,\alpha}$ *for some* $\delta < \lambda, \mathrm{cf}(\delta) = \kappa$ *and* $C' \in \bigcup\limits_{\beta<\kappa} \mathscr{P}_{2,\beta}$ *we have* $C = \{\gamma \in e_\delta : \mathrm{otp}(e_\delta \cap \gamma) \in C'\}$.

Proof. Same proof as 1.5. (Note that without loss of generality $[C \in \mathscr{P}_{1,\alpha}\ \&\ \beta < \alpha < \kappa \Rightarrow C \cap \beta \in \mathscr{P}_{1,\beta}]$).

1.10 Conclusion. If $\delta(*)$ is a limit ordinal and $\lambda = \mathrm{cf}(\lambda) > |\delta(*)|^+$ <u>then</u> we can find $\bar{\mathscr{P}}^*_\ell = \langle \mathscr{P}^*_{\ell,\alpha} : \alpha < \lambda \rangle$ for $\ell = 1, 2$ and stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))\}$ such that:

$\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}^*_1,\bar{\mathscr{P}}^*_2}$ (A) $\mathscr{P}^*_{1,\alpha}$ is a family of $< \lambda$ closed subsets of α each of order type $< \delta(*)$

(B) $\beta \in \mathrm{nacc}(C)\ \&\ C \in \mathscr{P}^*_{1,\alpha} \Rightarrow C \cap \beta \in \mathscr{P}^*_{1,\beta}$

(C) $\mathscr{P}^*_{2,\delta}$ is a family of $\leq \lambda$ clubs of δ (yes, maybe $= \lambda$) of order type $\delta(*)$, and $[\beta \in \mathrm{nacc}(C)\ \&\ C \in \mathscr{P}^*_{2,\delta} \Rightarrow C \cap \beta \in \mathscr{P}^*_{1,\beta}]$

(D) for every club E of λ for some $\delta \in E \cap S$, $\mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$ and there is $C \in \mathscr{P}^*_{2,\beta}$ such that $C \subseteq E$.

Proof. If $\lambda = |\delta(*)|^{++}$ (or any successor of regulars) use [Sh:e, ChIII,6.4](2) or [Sh 365, 2.14](2)((c)+(d)).

If $\lambda > |\delta(*)|^{++}$ let $\kappa = |\delta(*)|^{++}$ and let $S_1 = \{\delta < \kappa^{++} : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))\}$; applying the previous sentence we get $\bar{\mathscr{P}}^*_1$, $\bar{\mathscr{P}}^*_2$ satisfying $\oplus^{\kappa^{++},\delta(*)}_{\bar{\mathscr{P}}^*_1,\bar{\mathscr{P}}^*_2,S_1}$, hence satisfying the assumption of 1.9 so we can apply 1.9. $\square_{1.10}$

1.11 Definition. $^+\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_{2,S}}$ is defined as in 1.10 except that we replace (C) by

$(C)^+$ $\mathscr{P}^*_{2,\delta}$ is a family of $< \lambda$ clubs of δ of order type $\delta(*)$.

1.12 Remark. Note that if $\mathscr{P}_\alpha = \mathscr{P}_{1,\alpha} \cup \mathscr{P}_{2,\alpha}$, $|\mathscr{P}_{2,\alpha}| \leq 1$, $\mathscr{P}_{1,\alpha} = \{C \in \mathscr{P}_\alpha : \mathrm{otp}(C) < \delta(*)\}$, $\mathscr{P}_{2,\alpha} = \{C \in \mathscr{P}_\alpha : \mathrm{otp}(C) = \delta(*)\}$ then $^+\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_{2,S}} \Leftrightarrow \oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_S}$ mod.

1.13 Claim. *Suppose $\lambda = \mathrm{cf}(\lambda) > |\delta(*)|^+$, $\delta(*)$ a limit ordinal, additively indecomposable (i.e. $\alpha < \delta(*) \Rightarrow \alpha + \alpha < \delta(*)$), $\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_{2,S}}$ from 1.10 and*

$(*)$ $\alpha \in S \Rightarrow |\mathscr{P}_{2,\alpha}| \leq |\alpha|$.

(Note: a non-stationary subset of S does not count; e.g. for λ successor cardinal the α with $|\alpha|^+ < \lambda$. Note: ${}^+\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_{2,S}}$ holds by $(*)$ and if λ is successor then ${}^+\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,\bar{\mathscr{P}}_{2,S}}$ suffice).
Then for some stationary $S_1 \subseteq S$ and $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ we have: $\mathscr{P}_\alpha \subseteq \mathscr{P}_{1,\alpha} \cup \mathscr{P}_{2,\alpha}$ and:

${}^*\otimes^{\lambda,\delta(*)}_{\bar{\mathscr{P}},S_1}$ *(i)* $\mathscr{P}_\alpha$ *is a family of closed subsets of* α, $|\mathscr{P}_\alpha| < \lambda$
(ii) $\mathrm{otp} C < \delta(*)$ *if* $C \in \mathscr{P}_\alpha, \alpha \notin S_1$
(iii) if $\alpha \in S_1$ *then:* $\mathscr{P}_\alpha = \{C_\alpha\}, \mathrm{otp}(C_\alpha) = \delta(*)$,
C_α *a club of* α *disjoint to* S_1
(iv) $C \in \mathscr{P}_\alpha$ & $\beta \in \mathrm{nacc}(C) \Rightarrow \beta \cap C \in \mathscr{P}_\beta$
(v) for any club E *of* λ *for some* $\delta \in S_1$ *we have* $C_\delta \subseteq E$.

1.14 Remark. Note there are two points we gain: for $\alpha \in S_1$, $\mathscr{P}_\alpha$ is a singleton (similarly to 1.4 where we have $(\exists^{\le 1} C \in \mathscr{P}_\delta)[\mathrm{otp}(C) = \delta(*)])$, and an ordinal α cannot have a double role $-C_\alpha$ a guess (i.e. $\alpha \in S_1$) and C_α is a proper initial segment of such C_δ. When $\delta(*)$ is a regular cardinal this is easier.

Proof. Let $\mathscr{P}_{2,\alpha} = \{C_{\alpha,i} : i < \alpha\}$ (such a list exists as we have assumed $|\mathscr{P}_{2,\alpha}| \le |\alpha|$, we ignore the case $\mathscr{P}_{2,\alpha} = \emptyset$). Now

$(*)_0$ for some $i < \lambda$ for every club E of λ for some $\delta \in S \cap E$ we have $C_{\delta,i} \backslash E$ is bounded in α
[Why? If not, for every $i < \lambda$ there is a club E_i of λ such that for no $\delta \in S \cap E$ is $C_{\delta,i} \backslash E$ bounded in α. Let $E^* = \{j < \lambda : j$ a limit ordinal, $j \in \bigcap_{i<j} E_i\}$, it is a club of λ, hence for some $\delta \in S \cap E^*$ and $C \in \mathscr{P}_{2,\delta}$ we have $C \subseteq E^*$. So for some $i < \alpha, C = C_{\delta,i}$, so $C \subseteq E^* \subseteq E_i \cup i$ hence $C_{\delta,i} \backslash i \subseteq E_i$, contradicting the choice of E_i.].

$(*)_1$ for some $i < \lambda$ and $\gamma < \delta(*)$, letting $C_\delta =: C_{\delta,i} \backslash \{\zeta \in C_{\delta,i} : \mathrm{otp}(\zeta \cap C_{\delta,i}) < \gamma\}$ we have: for every club E of λ for some $\delta \in S \cap E$ we have: $C_\delta \subseteq E$
[Why? Let $i(*)$ be as in $(*)_0$, and for each $\gamma < \delta(*)$ suppose E_γ exemplify the failure of $(*)_1$ for $i(*)$ and γ, now $\bigcap_{\gamma < \delta(*)} E_\gamma$ is a club of λ exemplifying the failure of $(*)_0$ for $i(*)$ contradiction. So for some $\gamma < \delta(*)$ we succeed.]

$(*)_2$ Without loss of generality $|\mathscr{P}_{2,\alpha}| \le 1$, so let $\mathscr{P}_{2,\alpha} = \{C_\alpha\}$
[Why? Let i, γ and C_δ (for $\delta \in S$) be as in $(*)_1$ and use $\mathscr{P}'_{1,\alpha} = \{C \backslash \{\zeta \in C : \mathrm{otp}(\zeta \cap C) < \gamma\} : C \in \mathscr{P}_{1,\alpha}\}, \mathscr{P}'_{2,i} = \{C_\delta\}$.]

$(*)_3$ for some $h : \lambda \to |\delta(*)|^+$, for every $\alpha \in S$ we have $h(\alpha) \notin \{h(\beta) : \beta \in C_\alpha\}$
[Why? Choose $h(\alpha)$ by induction on α.]

$(*)_4$ for some $\beta < |\delta(*)|^+$ for every club E of λ, for some $\delta \in S \cap h^{-1}(\{\beta\}), C_\delta \subseteq E$
[Why? If for each β there is a counterexample E_β then $\cap\{E_\beta : \beta < |\delta(*)|^+\}$ is a counterexample for $(*)_2$.]

Now we have gotten the desired conclusion. $\square_{1.13}$

1.15 Claim. *If $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}, S \in I[\lambda], \kappa^+ < \lambda = \mathrm{cf}(\lambda)$, then for some stationary $S_1 \subseteq S$ and $\bar{\mathscr{P}}_1$ we have ${}^*\oplus^{\lambda,\delta(*)}_{\bar{\mathscr{P}}_1,S_1}$.*

Proof. Same proof as 1.4 (plus $(*)_3, (*)_4$ in the proof of 1.10). $\square_{1.15}$

1.16 Claim. *Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$ and $\mathrm{cf}(\delta(*)) \neq \mathrm{cf}(\mu)$.*
Then we can find stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\delta)(*)\}$ and $\bar{\mathscr{P}}$ such that ${}^*\otimes^{\lambda,\delta(*)}_{\bar{\mathscr{P}},S}$.

Remark. This strengthens 1.10.

Proof. Case (α).μ regular.
By [Sh:e, Ch.III,6.4](2), [Sh 365, 2.14](2)((c)+(d)).

Case β. μ singular.
Let $\theta =: \mathrm{cf}(\mu), \sigma =: |\delta(*)|^+ + \theta^+$ and $\mu = \sum\limits_{\zeta<\theta} \mu_\zeta$, $\langle \mu_\zeta : \zeta < \theta\rangle$ strictly increasing, $\mu_0 > \sigma$ and for each $\alpha < \lambda$ let $\alpha = \bigcup\limits_{\zeta<\theta} A_{\alpha,\zeta}$, $\langle A_{\alpha,\zeta} : \zeta < \theta\rangle$ increasing, $|A_{\alpha,\zeta}| \leq \mu_\zeta$.

By 1.8 there is a sequence $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ and stationary $S_1 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\}$ such that $\oplus^{\lambda,\sigma}_{\bar{\mathscr{P}},S_1}$ of 1.4 holds. *Let* $\cup\{\mathscr{P}_\alpha : \alpha < \lambda\} \cup \{\emptyset\}$ be $\{C_\alpha : \alpha < \lambda\}$ such that $C_\alpha \subseteq \alpha$, $[\alpha \in S_1 \Rightarrow C_\alpha \in \mathscr{P}_\alpha$ & $\mathrm{otp}(C_\alpha) = \sigma]$ and $[\alpha \notin S_1 \Rightarrow \mathrm{otp}(C_\alpha) < \sigma]$. For some club E^*_1 of $\lambda, [\alpha \in E^*_1 \Rightarrow \bigcup\limits_{\beta<\alpha} \mathscr{P}_\beta = \{C_\beta : \beta < \alpha\}]$.

Looking again at $\oplus^{\lambda,\sigma}_{\bar{\mathscr{P}},S_1}$, we can assume $S_1 \subseteq E^*_1$ & $(\forall\delta)[\delta \in S_1 \Rightarrow C_\delta \subseteq E^*_1]\}$, hence

$(*)$ $\delta \in S_1$ & $\alpha \in \mathrm{nacc}\, C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \mathrm{Min}(C_\delta \backslash (\alpha + 1))\}$.

So as we can replace every C_α by $\{\beta \in C_\alpha : \text{otp}(C_\alpha \cap \beta)\}$ is even, without loss of generality [because we can replace every C_α by $\{\beta \in C_\alpha : \text{otp}(\beta \cap C_\alpha)$ is even$\}$, without loss of generality (check)]

$(*)^+$ $\delta \in S_1$ & $\alpha \in$ nacc $C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \alpha\}$.

Without loss of generality $[\beta \in A_{\alpha,\zeta} \Rightarrow C_\beta \subseteq A_{\alpha,\zeta}]$ (just note $|C_\beta| \le \sigma < \mu_\zeta$) and $\alpha \in A_{\beta,\zeta} \Rightarrow A_{\alpha,\zeta} \subseteq A_{\beta,\zeta}$. For $\alpha \in S_1$ let $C_\alpha = \{\beta_{\alpha,\epsilon} : \epsilon < \sigma\}$($\beta_{\alpha,\epsilon}$ increasing in ϵ) and let $\beta^*_{\alpha,\epsilon} \in [\beta_{\alpha,\epsilon}, \beta_{\alpha,\epsilon+1})$ be mimimal such that $C_\alpha \cap \beta_{\alpha,\epsilon+1} = C_{\beta^*_{\alpha,\epsilon}}$ (exists as $\delta \in S_1 \Rightarrow C_\delta \subseteq E^*_1$). Without loss of generality every C_α is an initial segment of some C_β, $\beta \in S_1$ (if not, we redefine it as $\emptyset$).

$(*)_1$ there are $\gamma = \gamma(*) < \theta$ and stationary $S_2 \subseteq S_1$ such that for every club E of λ, for some $\delta \in S_2$ we have: $C_\delta \subseteq E$, and for arbitrarily large $\epsilon < \sigma$, $\beta^*_{\delta,\epsilon} \in A_{\beta_{\delta,\epsilon+1},\gamma}$.
[Why? If not, for every $\gamma < \theta$ (by trying $\gamma(*) = \gamma$) there is a club E_γ of λ exemplifying the failure of $(*)_1$ for γ. Let $E = \bigcap\limits_{\gamma<\theta} E_\gamma \cap E^*_1$, so E is a club of λ, hence

$$S' =: \{\delta : \delta < \lambda, \delta \in S_1 (\text{so cf}(\delta) = \sigma) \text{ and } C_\delta \subseteq E\}$$

is a stationary subset of λ. For each $\delta \in S'$ and $\epsilon < \sigma$ for some $\gamma = \gamma(\delta,\epsilon) < \theta$ we have $\beta^*_{\delta,\epsilon} \in A_{\beta_{\delta,\epsilon+1},\gamma}$, but as $\sigma =$ cf$(\sigma) \ne$ cf$(\theta) = \theta$ for some $\gamma(\delta)$, $\{\epsilon < \sigma : \epsilon\gamma(\delta,\epsilon) = \gamma(\delta)\}$ is unbounded in σ. But $\delta \in E_{\gamma(\delta)}$, contradiction.]

$(*)_2$ Without loss of generality: if $\beta \in$ nacc$(C_\alpha), \alpha < \lambda$ then $(\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi > \sup(\beta \cap C_\alpha)$ & $\beta \cap C_\alpha = C_\xi]$.
[Why? Define C'_α for $\alpha < \lambda$:
$C^0_\alpha = \{\beta : \beta \in$ nacc(C_α) and $(\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi \ge \sup(\beta \cap C_\alpha)$ & $\beta \cap C_\alpha = C_\xi]\}$.
C'_α is: $\emptyset$ if $\alpha \in S_2$, $\alpha > \sup(C^0_\alpha)$
$\alpha \cap$ closure of C^0_α otherwise.] Now $\langle C_\alpha : \alpha < \lambda\rangle$ can be replaced by $\langle C'_\alpha : \alpha < \lambda\rangle$.]

$(*)_3$ For some $\gamma_1 = \gamma_1(*) < \theta$ for every club E of λ for some $\delta \in E$: cf$(\delta) =$ cf$(\delta(*))$, and there is a club e of δ satisfying: $e \subseteq E$, otp(e) is $\delta(*)$, and for arbitrarily large $\beta \in$ nacc(e) we have $e \cap \beta \in \{C_\zeta : \zeta \in A_{\delta,\gamma_1}\}$.
[Why? If not, for each $\gamma_1 < \theta$ there is a club E_{γ_1} of λ for which there is no δ as required. Let $E =: \bigcap\limits_{\gamma_1<\theta} E_{\gamma_1}$, so E is a club of λ hence for some $\alpha \in$ acc$(E) \cap S_2$, $C_\alpha \subseteq E$. Letting again $C_\alpha = \{\beta_{\alpha,\epsilon} : \epsilon < \sigma\}$ (increasing), $C_\alpha \cap \beta_{\alpha,\epsilon} = C_{\delta,\beta^*_{\delta,\epsilon}}$ where $\beta^*_{\delta,\epsilon} \in A_{\beta_{\delta,\epsilon+1},\gamma(*)}$ clearly $\delta =: \beta_{\alpha,\delta(*)}$, $e = \{\beta_{\delta,\epsilon} :$

$\epsilon < \delta(*)\}$ satisfies the requirements except the last. As $\mathrm{cf}(\delta(*)) \neq \mathrm{cf}(\mu)$, for some $\gamma_1(*) < \theta$, $\gamma_1(*) \geq \gamma(*)$ and $\{\epsilon < \delta(*) : \beta^*_{\delta,\epsilon} \in A_{\beta_{\delta,\delta(*)},\gamma_1(*)}\}$ is unbounded in $\delta(*)$. Clearly $\delta =: \beta_{\alpha,\delta(*)}$, $e =: C_\alpha \cap \delta$ satisfies the requirement. Now this contradicts the choice of $E_{\gamma_1(*)}$.]

$(*)_4$ For some club E^a of λ, for every club $E^b \subseteq E^a$ of λ, for some $\delta \in E^b$ we have:

- (a) $\mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$
- (b) for some club e of $\delta : e \subseteq E^b$, $\mathrm{otp}(e) = \delta(*)$, and for arbitrarily large $\beta \in \mathrm{nacc}(e)$ we have $e \cap \beta \in \{C_\xi : \epsilon \in A_{\delta,\gamma_1(*)}\}$
- (c) for every $\beta \in A_{\delta,\gamma_1(*)}$ we have: $C_\beta \subseteq E^a \Rightarrow C_\beta \subseteq E^b$ (we could have demanded $C_\beta \cap E^a = C_\beta \cap E^b$).
  [Why? If not we choose E_i for $i < \mu^+_{\gamma_1(*)}$ by induction on i, $[j < i \Rightarrow E_i \subseteq E_j]$, E_i a club of λ, and E_{i+1} exemplify the failure of E_i as a candidate for E^a. So $\bigcap\limits_i E_i$ is a club of λ hence by $(*)_3$ there are δ and e as there. Now $\langle\{\beta \in A_{\delta,\gamma_1(*)} : C_\beta \subseteq E_i\} : i < \mu^+_{\gamma_1(*)}\rangle$ is a decreasing sequence of subsets of $A_{\delta,\gamma_1(*)}$ of length $\mu^+_{\gamma_1(*)}$, and $|A_{\delta,\gamma_1(*)}| \leq \mu_{\gamma_1(*)}$, hence it is eventually constant. So for every i large enough, δ contradicts the choice of E_{i+1}.]

* * *

Let $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$, and there is a club $e = e_\delta$ of δ satisfying: $e \subseteq E^a$, $\mathrm{otp}(e) = \delta(*)$, $\alpha \in \mathrm{nacc}(e) \Rightarrow e \cap \alpha \in A_{\alpha,\gamma(*)}$ and for arbitrarily large $\beta \in \mathrm{nacc}(e)$ we have $e \cap \beta \in \{C_\xi : \xi \in A_{\delta,\gamma(*)}\}\}$.

So S is stationary, let for $\delta \in S$, C^*_δ be an e as above. For $\alpha < \lambda$ let $\mathscr{P}_{1,\alpha} = \{C_\beta : \beta \leq \alpha, \beta \in A_{\alpha,\gamma_2(*)}\}$

$(*)_5(a)$ for every club E of λ, for some $\delta \in S$, $C^*_\delta \subseteq E$

- (b) C^*_δ is a club of δ, $\mathrm{otp}(C^*_\delta) = \delta(*)$
- (c) if $\beta \in \mathrm{nacc}\ C^*_\delta(\delta \in S)$ then $C^*_\delta \cap \beta \in \mathscr{P}_{1,\beta}$
- (d) $|\mathscr{P}_{1,\beta}| \leq \mu_{\gamma(*)}$, $\mathscr{P}_{1,\beta}$ is a family of closed subsets of β of order type $< \delta(*)$,
  [Why? This is what we have proved in $(*)_4$; noting that in $(*)_4$ in (b), (e) is not uniquely determined, but by (c) every "reasonable" candidate is O.K.]

Now repeating $(*)_3$, $(*)_4$ of the proof of 1.13, and we finish. $\square_{1.16}$

1.17 Claim. *1) Assume $\lambda = \mu^+$, $|\delta(*)| < \mu, \aleph_0 < \mathrm{cf}(\delta(*)) = \mathrm{cf}(\mu)(< \mu)$; $\underline{\text{then}}$ we can find stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))\}$ and $\bar{\mathscr{P}}$ such that ${}^*\otimes^{\lambda,\delta(*)}_{\bar{\mathscr{P}},S}$, except when:*

- $\oplus$ *for every regular $\sigma < \mu$, we can find $h : \sigma \to \mathrm{cf}(\mu)$ such that for no δ, ϵ do we have: if $\delta < \sigma, \mathrm{cf}(\delta) = \mathrm{cf}(\mu), \epsilon < \mathrm{cf}(\mu)$ $\underline{\text{then}}$ $\{\alpha < \delta : h(\alpha) < \epsilon\}$ is not a stationary subset of δ.*

2) In 1.16 and 1.17(1) we can have $\mu > \sup\{|\mathscr{P}_\alpha| : \alpha < \lambda\}$.
3) If 1.17(2) if μ is strong limit we can have $|\mathscr{P}_\alpha| \leq 1$ for each α.

Remark. Compare with [Sh 186, §3].

Proof. Left to the reader (reread the proof of 1.16 and [Sh 186, §3].

1.18 Claim. *1) Let κ be regular uncountable and we have global choice (or restrict ourselves to $\lambda < \lambda^*$). We can choose for each regular $\lambda > \kappa^+$, $\bar{\mathscr{P}}^\lambda = \langle \mathscr{P}^\lambda_\alpha : \alpha < \lambda\rangle$ (assuming global choice) such that:*

- (a) *for each λ, $\mathscr{P}^\lambda_\alpha$ is a family of $\leq \lambda$ of closed subsets of α of order type $< \kappa$.*
- (b) *if χ is regular, F is the function $\lambda \mapsto \bar{\mathscr{P}}^\lambda$ (for λ regular $< \chi$), $\aleph_0 < \kappa = \mathrm{cf}(\kappa), \kappa^{++} < \chi, x \in \mathscr{H}(\chi)$ then we can find $\bar{N} = \langle N_i : i \leq \kappa\rangle$, an increasing continuous chain of elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi, F), \langle N_j : j \leq i\rangle \in N_{i+1}$, $\|N_i\| = \aleph_0 + |i|$, $x \in N_0$ such that:*
  - $(*)$ *if $\kappa^+ < \theta = \mathrm{cf}(\theta) \in N_i$, then for some club C of $\sup(N_\kappa \cap \theta)$ of order type κ; for any $j^i_1 < j < \kappa$ we have:*
    $C \cap \sup(N_j \cap \theta) \in N_{j+1}, \mathrm{otp}(C \cap \sup(N_j \cap \theta)) = j$.

2) We can above have $|\mathscr{P}^\lambda_\alpha| < \lambda$.

Proof. 1) Let $\langle C_\alpha : \alpha \in S\rangle$ be such that $S \subseteq \{\alpha \leq \kappa^{++} :\ \mathrm{cf}(\alpha) \leq \kappa\}$ is stationary, $\mathrm{otp}(C_\alpha) \leq \kappa$, $[\beta \in C_\alpha \Rightarrow C_\beta = \beta \cap C_\alpha], C_\alpha$ a closed subset of α, [α limit $\Rightarrow \alpha = \sup(C_\alpha)$], $\{\alpha \in S : \mathrm{cf}(\alpha) = \kappa\}$ stationary, and for every club E of κ^{++} there is $\delta \in S$, $\mathrm{cf}(\delta) = \kappa$, $C_\delta \subseteq E$. For $i \in \kappa^{++} \backslash S$ let $C_i = \emptyset$. Now for every regular $\lambda > \kappa^+$ and $\alpha \leq \lambda$, let $e^\lambda_\alpha \subseteq \alpha$ be a club of α of order type $\mathrm{cf}(\alpha)$. For λ as above and for $\alpha \leq \lambda$ limit let $\bar{\mathscr{P}}^\lambda_\alpha = \{\{i \in e_\delta : i < \alpha, \mathrm{otp}(e_\delta \cap i) \in C_\beta\} : \delta < \lambda$ has cofinality κ^{++}, and $\beta \in S\}$. Given $x \in H(\chi)$, we choose by induction on $i < \kappa^{++}$, M_i, N_i such that:

$N_i \prec M_i \prec (\mathscr{H}(\chi), \in, <^*_\chi, F)$
$\|M_i\| = |i| + \aleph_0$
$\|N_i\| = |C_i| + \aleph_0$
$M_i (i < \kappa^{++})$ is increasing continuous
$x \in M_0$,
$\langle M_j : j \leq i \rangle \in M_{i+1}$
N_i is the Skolem Hull of $\{\langle N_j : j \in C_\zeta \rangle : \zeta \in C_i\}$.

We leave the checking to the reader.
2) We imitate the proof of 1.5. $\square_{1.18}$

§2 Measuring $[\lambda]^{<\kappa}$

We prove here that two natural ways to measure $\mathscr{S}_{<\kappa}(\lambda)$ for κ regular uncountable, give the same cardinal: the minimal cardinality of a cofinal subset; i.e. its cofinality (i.e. $\text{cov}(\lambda,\kappa,\kappa,2)$) and the minimal cardinality of a stationary subset. The theorem is really somewhat stronger: for appropriate normal ideal on $\mathscr{S}_{<\kappa}(\lambda)$, some member of the dual filter has the right cardinality.

The problem is natural and I did not trace its origin, but until recent years it seems (at least to me) it surely is independent, and find it gratifying we get a clean answer. I thank P. Matet and M. Gitik of reminding me of the problem.

We then find applications to Δ-systems and largeness of $\check{I}[\lambda]$.

2.1 Definition. 1) Let $(\bar{C},\bar{\mathscr{P}},Z) \in \mathscr{T}^*[\theta,\kappa]$ <u>when</u>:

- (i) $\aleph_0 < \kappa = \text{cf}(\kappa) < \theta = \text{cf}(\theta)$,
- (ii) $S \subseteq \theta, S$ is stationary
- (iii) $\bar{C} = \langle C_\delta : \delta \in S\rangle$ (and we shall write $S = S(\bar{C})$), $\bar{\mathscr{P}} = \langle \mathscr{P}_\delta : \delta \in S\rangle, Z = \langle <_{\mathscr{P}_\delta} : \delta \in S\rangle$
- (iv) C_δ is an unbounded subset of δ, (not necessarily closed)
- (v) $\text{id}^a(\bar{C})$ is a proper ideal (i.e. for every club E of θ for some $\delta \in S$, $C_\delta \subseteq E$)
- (vi) $\bigwedge\limits_{\delta \in S} \text{otp}(C_\delta) < \kappa$, (hence $[\delta \in S \Rightarrow \text{cf}(\delta) < \kappa]$)
- (vii) (α) $\mathscr{P}_\delta$ is a family of bounded subsets of C_δ, directed by the partial order $<_{\mathscr{P}_\delta}$ which is a partial order on $\mathscr{P}^* = \{x \cap \alpha : x \in \mathscr{P}_\delta$ for some $\delta \in S$ and $\alpha < \theta\}$ satisfying $y <_{\mathscr{P}_\delta} z \Rightarrow y \subseteq z$, (but see parts (1A),(1B))
  - (β) $\bigcup\limits_{x \in \mathscr{P}_\delta} x = C_\delta$, and $|\mathscr{P}_\delta| < \kappa$
- $(viii)$ for some[1] list $\langle b^*_i : i < \theta\rangle$ of $\bigcup\limits_{\alpha \in S} \mathscr{P}_\alpha \cup \{\emptyset\}$ satisfying $b^*_i \subseteq i$ we have: for every $\alpha \in S$ we have $\mathscr{P}_\alpha \subseteq \{b^*_j : j < \alpha\}$
- (ix) for $x \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta$ we have the set $\mathscr{P}_x := \{y \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta : y <_{\mathscr{P}_\delta} x\}$ has cardinality $< \kappa$.

[1] a sufficient condition is:

$(viii)^+$ for every $\alpha < \theta$ the set $\mathscr{P}^*_\alpha =: \{a \cap \alpha :$ for some $\delta \in S$ we have $\alpha < \delta \in S$, $a \in \mathscr{P}_\delta$ and $\alpha \in C_\delta\}$ has cardinality $< \theta$ or at least

1A) If each $<_{\mathscr{P}_\delta}$ is inclusion we may omit it.
1B) If $<_*$ is a partial order of $\bigcup\limits_{\delta \in S} \mathscr{P}_\delta$ and $\delta \in S \Rightarrow <_{\mathscr{P}_\delta} = <_* \restriction \mathscr{P}_\delta$ then we may write $<_*$ instead of Z.
2) $\bar{C} \in \mathscr{T}^0[\theta,\kappa]$, if $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^*[\theta,\kappa]$ where $\delta \in S(\bar{C}) \Rightarrow \mathscr{P}_\delta = \{C_\delta \cap \alpha : \alpha \in C_\delta\}$.
3) $\bar{C} \in \mathscr{T}^1[\theta,\kappa]$ if $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^*[\theta,\kappa]$ where $\delta \in S(\bar{C}) \Rightarrow \mathscr{P}_\delta = [C_\delta]^{<\aleph_0}$.

Note that:

2.2 Claim. *1) If $\theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \sigma = \mathrm{cf}(\sigma)$, then there is $\bar{C} \in \mathscr{T}^1[\theta,\kappa]$ such that:*

$$\{\delta \in S(\bar{C}) : \mathrm{cf}(\delta) = \sigma\} \neq \emptyset \bmod \mathrm{id}^a(\bar{C}).$$

2) If $S \subseteq \{\delta < \theta : \mathrm{cf}(\delta) < \kappa\}$ is stationary, $\bar{C}$ an S-club system, $|C_\delta| < \kappa$, and $\mathrm{id}^a(\bar{C})$ a proper ideal, then $\bar{C} \in \mathscr{T}^1[\theta,\kappa]$.
3) In (2) if in addition for each $\alpha < \theta$ we have $|\{C_\delta \cap \alpha : \alpha \in C_\delta, \delta \in S\}| < \theta$ then $\bar{C} \in \mathscr{T}^0[\theta,\kappa]$.
4) If θ is a successor of regular then in part (2) we can demand $\bar{C} \in \mathscr{T}^0[\theta,\kappa]$ and each C_δ closed.
5) If $\theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \sigma = \mathrm{cf}(\sigma)$, then there is $\bar{C} \in \mathscr{T}^0[\theta,\kappa]$ such that: $\{\delta \in S(\bar{C}) : \mathrm{cf}(\delta) = \sigma\} \neq \emptyset \bmod \mathrm{id}^a(\bar{C})$.
6) If $\theta = \mathrm{cf}(\theta) > \kappa =$ cf$(\kappa) > \sigma = \mathrm{cf}(\sigma)$ and $S \in \check{I}[\theta]$ is stationary then there is $\bar{C} \in \mathscr{T}^0[\theta,\kappa]$ such that $S(\bar{C}) = S$.

Proof. 1) Let $S_0 \subseteq \{\delta < \theta : \mathrm{cf}(\delta) = \sigma\}$ be stationary, C^0_δ a club of δ of order type σ for every $\delta \in S_0$. By [Sh 365, §2], for some club E of θ letting $S = S_0 \cap \mathrm{acc}(E)$ and letting, for $\delta \in S, C_\delta = g\ell(C^0_\delta, E) = \{\sup(\alpha \cap E) : \alpha \in C^0_\delta\}$ we have $S \notin \mathrm{id}^a(\langle C_\delta : \delta \in S_0\rangle)$, now use part (2).
2) Check.
3) Check.
4) By [Sh 351, §4], [Sh:e, Ch.IV,3.4](2) or [Sh 365, 2.14](2)((c)+(d)) but see [Sh:E12].

5) By 1.7 and 1.15 (so we use the non-accumulation points).
6) Similarly. $\square_{2.2}$

Remember (see [Sh 52, §3]).

2.3 Definition. 1) $\mathscr{D}_\kappa$ is the filter generated by the family of clubs of κ.
2) $\mathscr{D}^\kappa_{<\kappa}(\lambda)$ is the filter on $[\lambda]^{<\kappa}$ defined by:
$\mathscr{D}^\kappa_{<\kappa}(\lambda)$ is the filter on $[\lambda]^{<\kappa}$ defined by:
for $X \subseteq [\lambda]^{<\kappa}$:

> $X \in \mathscr{D}^\kappa_{<\kappa}(\lambda)$ <u>iff</u> there is a function F with domain the set of sequences of length $< \kappa$ with elements from $[\lambda]^{<\kappa}$ and F is into $[\lambda]^{<\kappa}$ such that: if $a_\zeta \in [\lambda]^{<\kappa}$ for $\zeta < \kappa$, is $\subseteq$-increasing continuous and for each $\zeta < \kappa$ we have $F(\langle \ldots, a_\xi, \ldots \rangle)_{\xi \le \zeta} \subseteq a_{\zeta+1}$ then $\{\zeta < \kappa : a_\zeta \in X\} \in \mathscr{D}_\kappa$.

Similarly

2.4 Definition. For $\lambda \ge \theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \aleph_0$, $(\bar C, \bar{\mathscr{P}}) \in \mathscr{T}^*[\theta, \kappa]$ we define a filter $\mathscr{D}_{(\bar C, \bar{\mathscr{P}})}(\lambda)$ on $[\lambda]^{<\kappa}$; (letting, e.g. $\chi = \beth_{\omega+1}(\lambda)$):
$Y \in \mathscr{D}_{(\bar C, \bar{\mathscr{P}})}(\lambda)$ <u>iff</u> $Y \subseteq [\lambda]^{<\kappa}$ and for some $\mathbf{x} \in \mathscr{H}(\chi)$, for every $\langle N_\alpha, N^*_a : \alpha < \theta, a \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta \rangle$ satisfying $\otimes$ below, also there is $A \in \mathrm{id}^a(\bar C)$ such that: $\delta \in S(\bar C) \backslash A \Rightarrow \bigcup\limits_{a \in \mathscr{P}_\delta} N^*_a \cap \lambda \in Y$ where, letting $\mathscr{P} = \cup\{\mathscr{P}_\delta : \delta \in S\}$,

- $\otimes(i)$ $N_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
- (ii) $\|N_\alpha\| < \theta$,
- (iii) $\langle N_\beta : \beta \le \alpha \rangle \in N_{\alpha+1}$
- (iv) $\langle N_\alpha : \alpha < \theta \rangle$ is increasing continuous
- (v) $N^*_a \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ for $a \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta$
- (vi) $\|N^*_a\| < \kappa$, $N^*_a \cap \kappa$ an initial segment of κ
- (vii) $b \subseteq a$ (both in $\bigcup\limits_{\delta \in S} \mathscr{P}_\delta$) implies $N^*_b \prec N^*_a$
- $(viii)$ if $\alpha \in a \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta$ then $\langle N_\beta, N^*_b : \beta \le \alpha, b \subseteq a, b \in \{b^*_i : i \le \alpha\} \subseteq \mathscr{P} \rangle$ belongs to N^*_a
- (ix) $\langle N_\beta, N^*_b : \beta \le \alpha, b \subseteq \alpha + 1, b \in \{b^*_i : i \le \alpha + 1\} \subseteq \mathscr{P} \rangle$ belongs to $N_{\alpha+1}$
- (x) $a \subseteq N^*_a$ and $\alpha \in a \Rightarrow \alpha \cap a \in N^*_a$
- (xi) $a \subseteq \alpha, a \in \mathscr{P}$ implies $N^*_a \in N_{\alpha+1}$ (follows from (ix) by clause (viii) of Definition 2.1(1))
- (xii) $a \in \mathscr{P}_\delta$ & $\delta \in S$ & $\alpha < \theta \Rightarrow \mathbf{x} \in N^*_a$ & $\mathbf{x} \in N_\alpha$.

Clearly

2.5 Claim. *1) If $\chi > \lambda^{<\kappa}$ then $\mathscr{H}(\chi)$ can serve, and $\mathbf{x} = (Y, \lambda, \bar{C}, \bar{\mathscr{P}})$ is enough.*
2) $\mathscr{D}_{(\bar{C},\bar{\mathscr{P}})}(\lambda)$ is a (non-trivial) fine $(< \kappa)$-complete filter on $[\lambda]^{<\kappa}$ when $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^*[\theta, \kappa]$, $\lambda \geq \theta$, hence it extends $\mathscr{D}_{<\kappa}(\lambda)$. (Remember $\mathrm{id}^a(\bar{C})$ is a proper ideal).

Proof. Should be clear. $\square_{2.5}$

2.6 Theorem. *Suppose $\lambda > \theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $\theta = \kappa^+$. Then the following four cardinals are equal for any $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^*[\theta, \kappa]$, recalling there are such $(\bar{C}, \bar{\mathscr{P}})$ by 2.2:*

$\mu(0) = \mathrm{cf}([\lambda]^{<\kappa}, \subseteq)$

$\mu(1) = \mathrm{cov}(\lambda, \kappa, \kappa, 2) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{<\kappa}$, *and for every* $a \subseteq \lambda, |a| < \kappa$ *there is* $b \in \mathscr{P}$ *satisfying* $a \subseteq b\}$

$\mu(2) = \mathrm{Min}\{|S| : S \subseteq [\lambda]^{<\kappa}$ *is stationary*$\}$

$\mu(3) = \mu_{(\bar{C},\bar{\mathscr{P}})} = \mathrm{Min}\{|Y| : Y \in \mathscr{D}_{(\bar{C},\bar{P})}(\lambda)\}$.

2.7 Remark. 0) We thank M. Shioya for asking for a correction of an inaccuracy in the proof in a meeting in the summer of 1999 in which we answer him; this and other minor changes are done here. I thank P. Komjath for helpful comments and S. Garti for help in proofreading.
1) It is well known that if $\lambda > 2^{<\kappa}$ then the equality holds as they are all equal to $\lambda^{<\kappa}$.
2) This is close to "strong covering".
3) Note that only $\mu(3)$ has $(\bar{C}, \bar{\mathscr{P}})$ in its definition, so actually $\mu(3)$ does not depend on $(\bar{C}, \bar{\mathscr{P}})$, recalling that by Claim 2.2 we know that $\mathscr{T}^*[\theta, \kappa]$ is not empty.
4) $\mu(0), \mu(1)$ are equal trivially.

2.8 Remark. 0) We can concentrate on the case $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^1[\theta, \kappa]$ or $\mathscr{T}^0[\theta, \kappa]$. This somewhat simplifies and is enough.
1) We can weaken in Definition 2.1(1) demand (ix) as follows:

$(ix)'$ there is a sequence $\langle a_i, \mathscr{P}^*_i : i < \lambda \rangle$ such that

- (a) $|a_i| < \kappa, \mathscr{P}^*_i$ is a family of $< \kappa$ subsets of a_i
- (b) for every $\delta \in S$ and $x \in \mathscr{P}_\delta$ for some $i < \delta, a_i = x$ and $(\forall b)[b \in \mathscr{P}_\delta \ \& \ b \subseteq a \Rightarrow b \in \mathscr{P}^*_i]$.

In this case 2.6, 2.7(4) (and 2.5) remain true and we can strengthen 2.2.
2) We can even use $\mathscr{P}_\delta$ with another order (not $\subseteq$).

Proof. Clearly $\lambda \le \mu(0) = \mu(1) \le \mu(2) \le \mu(3)$ (the last — by 2.5(2)). So we shall finish by proving $\mu(3) \le \mu(1)$, and let $\mathscr{Q}$ exemplify $\mu(1) = \operatorname{cov}(\lambda, \kappa, \kappa, 2)$. Let $S = S(\bar{C})$, etc.

Let χ be e.g. $\beth_3(\lambda)^+$ and let M^*_λ be the model with universe $\lambda + 1$ and all functions definable in $(\mathscr{H}(\chi), \in, <^*_\chi, \lambda, \kappa, \mu(1))$. Let M^* be an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality $\mu(1)$ such that $\mathscr{Q} \in M^*, M^*_\lambda \in M^*$, $(\bar{C}, \bar{\mathscr{P}}) \in M^*$ and $\mu(1) + 1 \subseteq M^*$ hence $\mathscr{Q} \subseteq M^*$. It is enough to prove that $M^* \cap [\lambda]^{<\kappa}$ belongs to $\mathscr{D}_{(\bar{C},\bar{P})}(\lambda)$.

So let N_i (for $i < \theta$), N^*_x (for $x \in \bigcup\limits_{\delta \in S} \mathscr{P}_\delta$) be such that: they satisfy $\otimes$ of Definition 2.4 for $\mathbf{x} := \langle M^*_\lambda, M^*, \mathscr{P}, \mathscr{Q}, \lambda, \kappa, (\bar{C}, \bar{\mathscr{P}})\rangle$ so it belongs to every N_α, N^*_x. It is enough to prove that $\{\delta \in S : [\lambda]^{<\kappa} \cap \bigcup\limits_{x \in \mathscr{P}_\delta} N^*_x \in M^*\} = \theta \mod \mathrm{id}^a(\bar{C})$. For $i \in S$ clearly $x \subseteq y$ (or $x <_{\mathscr{P}_i} y) \Rightarrow N^*_x \prec N^*_y$ and $\mathscr{P}_i$ is directed (by the partial order $\subseteq$ or $<_{\mathscr{P}_i}$ recalling clause (vii) of $\otimes$ of Definition 2.4) hence $N'_i := \cup\{N^*_x : x \in \mathscr{P}_i\}$ is $\prec (\mathscr{H}(\chi), \in, <^*_\chi)$ and even $\prec N_{i+1}$ and N'_i has cardinality $< \kappa$ (as $|\mathscr{P}_i| < \kappa$ and each N^*_x has cardinality $< \kappa$ and κ is regular) and we have to show that $\{i \in S : [\lambda]^{<\kappa} \cap N'_i \in M^*\} = \theta \mod \mathrm{id}^a(\bar{C})$.

For each $i \in S$ by the choice of $\mathscr{Q}$, there is a set a_i such that $N'_i \cap \lambda = (\bigcup\limits_{y \in \mathscr{P}_i} N^*_y) \cap \lambda \subseteq a_i \in \mathscr{Q}$; so as $\mathscr{Q}$ and $\langle N^*_y : y \in \mathscr{P}_i\rangle$ belong to N_{i+1}, see clause (ix) of Definition 2.4 without loss of generality $a_i \in N_{i+1}$. Let $\mathfrak{a}_i =:$ Reg $\cap\, a_i \cap \lambda^+ \backslash \theta^+$, so $\mathfrak{a}_i$ is a set of $< \kappa$ regular cardinals $\ge \theta^+$ and $\mathfrak{a}_i \in N_{i+1}$ too, so there is a generating sequence $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \operatorname{pcf}(\mathfrak{a}_i)\rangle$ as in [Sh:g, VII,2.6] = [Sh 371, 2.6], without loss of generality it is definable from $\mathfrak{a}_i$ (in $(\mathscr{H}(\chi), \in, <^*_\chi)$ say the $<^*_\chi$-first such object). Also $a_i \in \mathscr{P} \subseteq M^*$ and Reg, $\lambda^+, \theta^+ \in M^*$ so $\mathfrak{a}_i \in M^*$. As $\mathfrak{a}_i \in N_{i+1}$ we have $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \operatorname{pcf}(\mathfrak{a}_i)\rangle \in N_{i+1} \cap M^*$, and also there is $\langle f^{\mathfrak{a}_i}_{\partial,\alpha} : \alpha < \partial, \partial \in \operatorname{pcf}(\mathfrak{a}_i)\rangle$ as in [Sh:g, VIII,1.2] = [Sh 371, 1.2], and again without loss of generality it belongs to $N_{i+1} \cap M^*$. As max pcf$(\mathfrak{a}_i) \le \operatorname{cov}(\lambda, \kappa, \kappa, 2) = \mu(1)$, (first inequality by [Sh:g, II,5.4] = [Sh 355, 5.4]) clearly each $f^{\mathfrak{a}_i}_{\partial,\alpha} \in M^*$.

Let

$\odot_1$ h be the function with domain $\mathfrak{a} := \bigcup\limits_{i \in S} \mathfrak{a}_i$ defined by $h(\sigma) = \sup(\sigma \cap \bigcup\limits_{i<\theta} N_i)$.

So by [Sh:g, VIII,2.3](1) = [Sh 371, 2.3](1)

$\odot_2$ if $i \in S$ then $h \restriction \mathfrak{a}_i$ has the form $\text{Max}\{f^{\mathfrak{a}_i}_{\partial_\ell,\alpha_\ell} : \ell < n\}$ for some $n < \omega, \partial_\ell \in \text{pcf}(\mathfrak{a}_\ell)$ and $\alpha_\ell < \partial_\ell$ for $\ell < n$
hence

$\odot_3$ if $i \in S$ then $h \restriction \mathfrak{a}_i$ belongs to M^*
and obviously (as $\sigma \in \mathfrak{a}_i \wedge i < j_1 < j_2 \Rightarrow \sup(\sigma \cap N_{j_1}) < \sup(\sigma \cap N_{j_2})$)

$\odot_4$ $\sigma \in \text{Dom}(h) \Rightarrow \text{cf}(h(\sigma)) = \theta$.

Let e be a definable function in $(\mathscr{H}(\chi), \in, <^*_\chi, \lambda, \kappa)$ with $\text{Dom}(e) = \lambda + 1$ such that $e(\alpha) = e_\alpha$ is a club of α of order type $\text{cf}(\alpha)$, enumerated as $\langle e_\alpha(\zeta) : \zeta < \text{cf}(\alpha)\rangle$. Now for each $\sigma \in \bigcup\limits_{i<\theta} \mathfrak{a}_i$ let

$\odot_5$ $E_\sigma =: \{i < \theta : (\forall \zeta < \theta)[e_{h(\sigma)}(\zeta) \in N_i \Leftrightarrow \zeta < i], i$ is a limit ordinal and $\sup(N_i \cap \sigma) = \sup\{e_{h(\sigma)}(\zeta) : \zeta < i\}\}$.

Clearly E_σ is a club of θ, hence (on $\langle b^*_j : j < \theta\rangle$, see clause (viii) of Definition 2.1)

$$\begin{aligned} E = \{\delta < \theta :&\delta \text{ is a limit ordinal and } \sigma \in \cup\{\mathfrak{a}_i : i < \delta\} \subseteq \\ &\text{Reg } \cap \lambda^+ \backslash \theta^+ \Rightarrow \delta \in \text{ acc}(E_\sigma) \text{ and } N_\delta \cap \theta = \delta\} \end{aligned}$$

is a club of θ. For each $\delta \in E \cap S$ such that $C_\delta \subseteq E$, let $\delta^* := \sup(\kappa \cap N'_\delta) = \sup(\kappa \cap \bigcup\limits_{y \in \mathscr{P}_\delta} N^*_y)$ so $\delta^* < \kappa$, and we define by induction on $n \in \omega$ models $M_{y,\delta,n}$ for every $y \in \mathscr{P}_\delta$.

First, $M_{y,\delta,0}$ is the Skolem Hull in M^*_λ of $\{i : i \in y\} \cup (N'_\delta \cap \kappa)$.
Second, $M_{y,\delta,n+1}$ is the Skolem Hull in M^*_λ of $M_{y,\delta,n} \cup \{e_{h(\sigma)}(\zeta) : \sigma \in (\text{Reg} \cap \lambda^+ \backslash \theta^+) \cap M_{y,\delta,n}$ and $\zeta \in y\}$. Now we note

$(*)_0$ if $y \in \{b^*_i : i < \zeta\}, \zeta \in C_\delta$ and $\delta \in E$ then $N^*_y \in N_\zeta$ hence $N^*_y \prec N_\zeta$.

[Why? By clause (ix) of $\otimes$ of Definition 2.4 we have $N^*_y \in N_\zeta$ so $\|N^*_y\| \in N_j$; as $\|N^*_y\| < \kappa < \theta$ and $N_\zeta \cap \theta \in \theta$ as $\zeta \in C_\delta \subseteq E$ we have $N^*_y \subseteq N_\zeta$ hence $N^*_y \prec N_\zeta$.]

$(*)_1$ if $\zeta \in E (\subseteq \theta)$ and $\sigma \in \text{ Reg } \cap N_\zeta \cap \lambda^+ \backslash \theta^+$ then $e_{h(\sigma)}(\zeta) = \sup(N_\zeta \cap \sigma)$.

[Why? By the choice of E.]

$(*)_2$ assume $\delta \in S$ satisfies $\delta \in E$, moreover $C_\delta \subseteq E$; if $y \in \mathscr{P}_\delta$ and $\sigma \in N^*_y \cap \text{ Reg } \lambda^+ \backslash \theta^+$ then ($h(\sigma)$ has cofinality θ, the sequence $\langle e_{h(\sigma)}(\zeta) : \zeta < \theta\rangle$ is increasing continuous with limit $h(\sigma)$ and):

  (i) if $y \in \{b^*_i : i < \zeta\}$ and $\zeta \in C_\delta$ then $\sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$

(*ii*) if $y \in \{b^*_i : i < \zeta\}, \zeta \in z \in \mathscr{P}_\delta$ and $y <_{\mathscr{P}_\delta} z$ <u>then</u> $y \in N^*_z, N^*_y \in N^*_z, N^*_y \prec N^*_z$ and $e_{h(\sigma)}(\zeta) \in N^*_z$

(*iii*) $\{e_{h(\sigma)}(\zeta) : \zeta \in C_\delta\}$ is a subset of $N'_\delta = \bigcup\limits_{z \in \mathscr{P}_\delta} N^*_z$

(*iv*) the set above is an unbounded subset of $N'_\delta \cap \sigma$.

[Why? <u>Clause (*i*)</u>: So we assume $\zeta \in C_\delta$ and $y \in \{b^*_i : i < \zeta\}$.

By $(*)_0$ (and recall that $\delta \in E$) we have $N^*_y \prec N_\zeta$. By the definition of E_σ as $\sigma \in N^*_y \prec N_\zeta \wedge \zeta \in E$ clearly $\zeta \in E_\sigma$ hence $\sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$ by $(*)_1$.

<u>Clause (*ii*)</u>: So assume $y \in \{b^*_i : i < \zeta\}, \zeta \in z$ and $y <_{\mathscr{P}_\delta} z$ (so $y, z \in \mathscr{P}_\delta$) hence $\mathscr{P}_{z,\zeta} = \{x \in \bigcup\limits_{\alpha \in S} \mathscr{P}_\alpha : x \subseteq z \cap \zeta\}$ has cardinality $< \kappa$ and $z \cap \zeta \in N^*_z$ by clause (x) of 2.4, so $\mathscr{P}_{z,\zeta} = \{x \in \cup\{\mathscr{P}_\alpha : \alpha \in S\} : x \subseteq z \cap \zeta\} \in N^*_z$, so (as $N^*_z \cap \kappa \in \kappa$, $|\mathscr{P}_{z,\zeta}| < \kappa$) clearly $\mathscr{P}_{z,\zeta} \subseteq N^*_z$ hence $y \in N^*_z$. By clause (viii) of $\otimes$ of Definition 2.4 it follows that $N^*_y \in N^*_z$. But $\|N^*_y\| < \kappa \wedge N^*_z \cap \kappa \in \kappa$ hence $N^*_y \subseteq N^*_z$ so $N^*_y \prec N^*_z$. But $\sigma \in N^*_y$ hence $\sigma \in N^*_z$. Also $N_\zeta \in N^*_z$ as $\zeta \in z \subseteq N^*_z$ recalling (*viii*) of 2.4 hence $e_{h(\sigma)}(\zeta) = \sup(N_\zeta \cap \sigma) \in N^*_z$ recalling $(*)_1$ so we have shown all clauses of (*ii*).

<u>Clause (*iii*)</u>: So let $\zeta \in C_\delta$; by clause (vii)(β) of Definition 2.1 we know that $C_\delta = \cup\{y : y \in \mathscr{P}_\delta\}$ hence for some $y_1 \in \mathscr{P}_\delta$ we have $\zeta \in y_1$. By clause (x) of $\otimes$ from Definition 2.4 we have $y_1 \subseteq N^*_{y_1}$ hence $\zeta \in N^*_{y_1}$. Also we are assuming in $(*)_2$ that $\sigma \in N^*_y, y \in \mathscr{P}_\delta$, so recalling $\mathscr{P}_\delta$ is directed, we can find $y_2 \in \mathscr{P}_\delta$ which is a common $\subseteq$-upper bound of y, y_1 hence $N^*_y \prec N^*_{y_2}, N^*_{y_1} \prec N^*_{y_2}$ hence $\sigma, \zeta \in N^*_{y_2}$.

By the choice of the function e and the model M^*_λ clearly $e(-,-)$ is a function of M^*_λ, but the object $\mathbf{x}$ belongs to $N^*_{y_2}$ and by its choice this implies that $e \in N^*_{y_2}$. By clause (viii) of 2.4 recalling $\zeta \in N^*_{y_2}$ we know that $N_\zeta \in N^*_{y_2}$ but $\sigma \in N^*_{y_2}$ hence $\sup(N_\zeta \cap \sigma) \in N^*_{y_2}$. But we are assuming in $(*)_2$ that $C_\delta \subseteq E$ and, see above, $\zeta \in C_\delta$ so $\zeta \in E$ and $\zeta \in C_\delta \subseteq N_\zeta, \sigma \in N^*_{y_2} \subseteq N'_\delta \subseteq N_\zeta$ so $\sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$ so by the previous sentence $e_{h(\sigma)}(\zeta) \in N^*_{y_2}$, hence $e_{h(\sigma)}(\zeta) \in \cup\{N^*_x : x \in \mathscr{P}_\delta\} = N'_\delta$ as required.

<u>Clause (*iv*)</u>: By clause (*iii*) it is $\subseteq N'_\delta$, and by the choice of the function e it is $\subseteq \sigma$ hence it is $\subseteq N'_\delta \cap \sigma$. Now $N'_\delta = \cup\{N^*_z : z \in \mathscr{P}_\delta\}$ and $z \in \mathscr{P}_\delta \Rightarrow N^*_z \prec N_\delta$ by $(*)_0$ hence $N'_\delta \subseteq N_\delta$. Now we know that $\langle e_{h(\sigma)}(\zeta) : \zeta < \delta\rangle$ is increasing with limit $e_{h(\sigma)}(\delta) = \sup(N_\delta \cap \sigma)$ hence is unbounded in it and even $\langle e_{h(\sigma)}(\zeta) : \zeta \in C_\delta\rangle$ is an unbounded subset of $e_{h(\sigma)}(\delta)$ and it is included in N'_δ as required.

So $(*)_2$ indeed holds.

Now (A), (B), (C), (D), (E) below clearly suffice to finish.

(A) (a) for $\delta \in S, y \in \mathscr{P}_\delta$ and $n < \omega$ we have $M_{y,\delta,n} \subseteq N'_\delta = \bigcup\limits_{z \in \mathscr{P}_\delta} N^*_z$.

[Why? We prove this by induction on n. First assume $n = 0, M_{y,\delta,n}$ is the Skolem hull of $y \cup (N'_\delta \cap \kappa)$ in the model M^*_λ, well defined as $y \subseteq \lambda$ hence $y \subseteq M^*_\lambda$ and $N' \cap \kappa \subseteq \kappa \subseteq \lambda$. As $y \subseteq N^*_y \subseteq N'_\delta$ and $M^*_\lambda \in N^*_y \subseteq N'_\delta$ clearly $M_{y,\delta,n} \subseteq N'_\delta$. Second, assume $n = m+1$ and $M_{y,\delta,m} \subseteq N'_\delta$. Now $M_{y,\delta,n}$ in the Skolem hull of $M_{y,\delta,m} \cup \{e_{h(\sigma)}(\zeta) : \sigma \in M_{y,\delta,m} \cap \text{ Reg} \cap (\lambda^+ \backslash \theta^+)$ and $\zeta \in y\}$, so it is enough to show that: if $\sigma \in M_{y,\delta,m}$ (hence $\sigma \in N'_\delta$) and $\sigma \in \text{ Reg } \cap \lambda^+ \backslash \theta^+$ and $\zeta \in y$ then $e_{h(\sigma)}(\zeta) \in N'_\delta$. But by $(*)_2(iii)$ this holds.

(b) for $z \subseteq y$ in $\mathscr{P}_\delta$ we have $M_{z,\delta,n} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on $n < \omega$.]

(c) for $y \in \mathscr{P}_\delta$ and $m \leq n$ we have $M_{y,\delta,m} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on n.]

(d) $M'_\delta := \cup\{M_{y,\delta,n} : y \in \mathscr{P}_\delta$ and $n < \omega\}$ is $\prec N'_\delta$.

[Why? By the above.]

(e) if $\zeta \in z$ (hence $\zeta \in C_\delta \subseteq E$), $\{y,z\} \subseteq \mathscr{P}_\delta$, $\sup(y) < \zeta, y <_{\mathscr{P}_\delta} z$ and $\sigma \in \text{ Reg } \cap \lambda^+ \backslash \theta^+$ then: $\sigma \in N^*_y \prec N_\zeta \Rightarrow e_{h(\sigma)}(\zeta)$ $= \sup(\sigma \cap N_\zeta) \in N^*_z$.

[Why? By $(*)_2(i) + (ii)$ this holds.]

(B) We can also prove that $\langle M_{y,\delta,n} : n < \omega, y \in \mathscr{P}_\delta \rangle$ is definable in $(\mathscr{H}(\chi), \in, <^*_\chi)$ from the parameters $\delta, M^*_\lambda, (\bar{C}, \bar{\mathscr{P}})$ and $h \restriction \mathfrak{a}_i$, all of them belong to M^*_λ, hence the sequence, and $M'_\delta = \cup\{M_{y,\delta,n} : n < \omega, y \in \mathscr{P}_\delta\}$, belong to M^*_λ

(C) $M'_\delta \cap \text{ Reg } \cap (\theta, \lambda^+)$ is a subset of $\mathfrak{a}_\delta$.

[Why? Use (A)(a) and definition of $a_i, \mathfrak{a}_i$).]

(D) if $\sigma \in M'_\delta$ and $\sigma \in \text{ Reg } \cap \lambda^+ \backslash \kappa$ then $\sigma \cap M'_\delta$ is unbounded in $\sigma \cap N'_\delta$.

[Why? When $\sigma > \theta$ use $(*)_2(iii), (iv)$. For $\sigma = \theta$ we have $N'_\delta \cap \theta \subseteq N_\delta \cap \theta = \delta$ as $\delta \in E$ and $C_\delta \subseteq \delta = \sup(C_\delta)$ so it is enough to show $C_\delta \subseteq N'_\delta$, but C_δ is equal to $\bigcup\limits_{y \in \mathscr{P}_\delta} y$. For $\sigma = \kappa$ see the choice of $M_{y,\delta,0}$. So as $\theta = \kappa^+$ we are done.]

(E) $M'_\delta \cap \lambda = N'_\delta \cap \lambda$.

[Why? By (A)(a) we have one inclusion, the $\subseteq$. By the choice of M^*_λ and clause (D) the result follows by [Sh 400, 3.3A,5.1A] recalling $N'_\delta \cap \kappa \in \kappa$.] $\square_{2.6}$

But to get normality of the filter we better define

2.9 Definition. Assume $\theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0, (\bar C, \bar{\mathscr P}) \in \mathscr T^*[\theta,\kappa]$ and X is a set, of cardinality $\geq \theta$ for simplicity and let χ be large enough. We define a filter $\mathscr D_{(\bar C,\bar{\mathscr P})}[X]$ on $[X]^{<\kappa}$ as the set of $Y \subseteq [X]^{<\kappa}$ such that for some $\mathbf x \in \mathscr H(\chi)$, for every sequence $\langle N_\alpha, N^*_a : \alpha < \theta, a \in \bigcup\limits_{\delta \in S} \mathscr P_\delta\rangle$ satisfying $\otimes$ below, there is $A \in \text{id}^a(\bar C)$ such that $\mathbf x \in \bigcup\limits_{a \in \mathscr P_\delta} N^*_a$ & $\delta \in S(\bar C)\backslash A \Rightarrow \bigcup\limits_{a \in \mathscr P_\delta} N^*_a \cap [X]^{<\kappa} \in Y$ where

$\otimes$ as in Definition 2.4 omitting $\mathbf x \in N_\alpha$.

2.10 Claim. *Let $(\bar C, \bar{\mathscr P}) \in \mathscr T^*[\theta,\kappa]$.*
1) Any χ such that $\mathscr P(X) \subseteq \mathscr H(\chi)$ can serve in Definition 2.9, and $\mathbf x = Y$ can serve.
2) If X_1, X_2 are sets of cardinality $\lambda \geq \chi$ and f is a one-to-one function from X_1 onto X_2, <u>then</u> f maps $\mathscr D_{(\bar C,\bar{\mathscr P})}(X_1)$ onto $\mathscr D_{(\bar C,\bar{\mathscr P})}(X_2)$.
3) If $X_1 \subseteq X_2$ has cardinality $\geq \theta$ then $Y \in \mathscr D_{(\bar C,\bar{\mathscr P})}[X_1] \Rightarrow \{u \in [X_2]^{<\kappa} : u \cap X_1 \in Y\} \in \mathscr D_{(\bar C,\bar{\mathscr P})}[X_2]$ and $Y \in \mathscr D_{(\bar C,\bar{\mathscr P})}(X_2) \Rightarrow \{u \cap X_1 : u \in Y\} \in \mathscr D_{(\bar C,\bar{\mathscr P})}(X_1)$.
4) For any set X of cardinality $\geq \kappa$, really $\mathscr D_{(\bar C,\bar{\mathscr P})}(X)$ is a fine normal filter on X, i.e.:

- (a) *fine: $t \in X \Rightarrow \{u \in [X]^{<\kappa} : t \in u\} \in \mathscr D_{(\bar C,\bar{\mathscr P})}(X)$*
- (b) *normal: if $Y_t \in \mathscr D_{(\bar C,\bar{\mathscr P})}(X)$ for $t \in X$ then $Y \in \mathscr D_{(\bar C,\bar{\mathscr P})}(X)$, when $Y := \Delta\{Y_t : t \in X\} = \{u \in [X]^{<\kappa} : u \neq \emptyset$ and $t \in u \Rightarrow u \in Y_t\}$.*

Proof. 1),2) Easy.
3) The "fine" is trivial and for normal let $\mathbf x_t$ be a witness for $Y_t \in \mathscr D_{(\bar C,\bar{\mathscr P})}[X]$ now $\mathbf x = \langle \mathbf x_t : t \in X\rangle$ witness that $Y \in \mathscr D_{(\bar C,\bar{\mathscr P})}[X]$.

2.11 Claim. *Let $(\bar C, \bar{\mathscr P}) \in \mathscr T^*[\theta,\kappa]$.*
1) $\mathscr D_{(\bar C,\bar{\mathscr P})}(\lambda) \supseteq \mathscr D_{(\bar C,\bar{\mathscr P})}[\lambda]$.
2) In 2.6 we can replace $\mathscr D_{(\bar C,\bar{\mathscr P})}(\lambda)$ by $\mathscr D_{(\bar C,\bar{\mathscr P})}[\lambda]$.
3) Assume that $\text{cf}(\lambda) \geq \kappa$ and $\beta < \alpha \Rightarrow \lambda > \text{cov}(|\beta|,\kappa,\kappa,2)$. <u>Then</u> there is $S \in \mathscr D_{(\bar C,\bar{\mathscr P})}(\lambda)$ such that $\alpha < S \Rightarrow \lambda > |\{u \in S : u \subseteq \alpha\}|$.

Proof. 1) Trivial.
2) Repeat the proof, the change is minor.
3) We can find $\mathscr{Q} = \{u_i : i < \lambda\} \subseteq [\lambda]^{<\kappa}$ which is cofinal such that $(\forall \alpha < \lambda)(\exists \beta)[\alpha \leq \beta < \lambda \wedge [\{u_i : i < \beta, u_i \subseteq \alpha\}]$ is cofinal in $[\alpha]^{<\kappa}$.

2.12 Remark. In 2.6 we can replace $\theta = \kappa^+$ by $\theta > \kappa_\sigma > \sigma = \mathrm{cf}(\sigma)$ and $\alpha < \theta \Rightarrow |\alpha|^{<\sigma>_{\mathrm{tr}}} < \theta$ and $\delta \in S(\bar{C}) \Rightarrow \mathrm{cf}(\delta) = \sigma$.

Proof. Fill.

2.13 Conclusion. Suppose $\lambda > \kappa > \aleph_0$ are regular cardinals and $(\forall \mu < \lambda)[\mathrm{cov}(\mu, \kappa, \kappa, 2) < \lambda]$.
1) If for $\alpha < \lambda$, a_α is a subset of λ of cardinality $< \kappa$ and $S \in \mathscr{D}_{<\kappa}(\lambda)$ and $T_1 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \geq \kappa\}$ is stationary, <u>then</u> we can find a stationary $T_2 \subseteq T_1, c \subseteq \lambda$ and $\langle b_\delta : \delta \in T_2\rangle$ such that:

$$a_\delta \subseteq b_\delta \in S \text{ for } \delta \in T_2$$

$$b_\delta \cap \delta = c \text{ for } \delta \in T_2.$$

2) If in addition $(\bar{C}, \bar{\mathscr{P}}) \in \mathscr{T}^*[\kappa^+, \kappa]$ and $S \in (\mathscr{D}_{(\bar{C},\bar{\mathscr{P}})}(\lambda))^+$ <u>then</u> part (1) holds for this S.

Remark. See on this and on 2.15 Rubin Shelah [RuSh 117, 4.12,pg.76] and [Sh 371, §6]. There we do not know that $(\forall \mu < \lambda)[\mathrm{cov}(\mu, \kappa, \kappa, 2) < \lambda]$ implies (as proved here) that

$\boxtimes_{\lambda,\kappa}$ for each $\alpha < \lambda$ we can find S_α a stationary $S_\alpha \subseteq [\alpha]^{<\lambda}$ of cardinality $< \lambda$; moreover such that $\{\{\alpha\} \cup u : u \in S_\alpha, \alpha < \lambda\} \subseteq [\lambda]^{<\kappa}$ is stationary, (if λ is a successor cardinal, the moreover follows. So the assumption there seems just what was used now. So we could just quote.

Proof. 1) By part (2).
2) For each $\alpha < \lambda$ let $S_\alpha \in \mathscr{D}_{(\bar{C},\bar{\mathscr{P}})}[\alpha]$ be of cardinality $\mathrm{cov}(|\alpha|, \kappa, \kappa, 2)$.

Let $S = \{u \in [\lambda]^{<\kappa}$: if $\alpha \in u \backslash \kappa^+$ then $u \cap \alpha \in S_\alpha\}$, so by 2.10 we know that $S \in \mathscr{D}_{(\bar{C},\bar{\mathscr{P}})}[\lambda]$; and by 2.11(3) without loss of generality

$(*)$ $\alpha < \lambda \Rightarrow \{u \in S : u \subseteq \alpha\}$ has cardinality $< \lambda$.

Now for each $\alpha < \lambda$ let $b_\alpha \in S$ be such that $a_\alpha \subseteq b_\alpha$, clearly exist and let $h : T_1 \to \lambda$ be defined by $h(\delta) = \sup(b_\delta \cap \delta)$ so $\delta \in T_1 \Rightarrow h(\delta) < \delta$ as $\mathrm{cf}(\delta) \geq \kappa > |b_\delta|$. So for some $\gamma_* < \gamma$ the set $T_2' := \{\delta \in T_1 : h(\delta) = \gamma_*\}$ is stationary and by $(*)$ for some c the set $T_2 := \{\delta \in T_2' : b_\delta \cap \delta = c\}$ is stationary. $\square_{2.13}$

2.14 Conclusion. If $\lambda > \kappa > \aleph_0, \lambda$ and κ are regular cardinals and $[\kappa < \mu < \lambda \Rightarrow \mathrm{cov}(\mu, \kappa, \kappa, 2) < \lambda]$ <u>then</u> $\{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\} \in \check{I}[\lambda]$.

Proof. Use $\mu(3)$ of 2.6.

2.15 Claim. *Let $(*)_{\mu,\lambda,\kappa}$ mean: if $a_i \in [\lambda]^{<\kappa}$ for $i \in S$ and $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \kappa\}$ is stationary, <u>then</u> for some $b \in [\lambda]^{<\kappa}$ the set $\{i \in S : a_i \cap i \subseteq b\}$ is stationary. Let $(*)^-_{\mu,\lambda,\kappa}$ be defined similarly but $\{i \in S : a_i \subseteq b\}$ only unbounded.*
<u>Then</u> for $\aleph_0 < \kappa < \lambda < \mu$ regular we have:

$$\begin{aligned} &\mathrm{cov}(\lambda, \kappa, \kappa, 2) < \mu \Rightarrow (*)_{\mu,\lambda,\kappa} \Rightarrow (*)^-_{\mu,\lambda,\kappa} \\ &\qquad \Rightarrow (\forall \lambda')[\kappa < \lambda' \leq \lambda \ \& \ \mathrm{cf}(\lambda') < \kappa \Rightarrow \mathrm{pp}_{<\kappa}(\lambda') < \mu]. \end{aligned}$$

Remark. So it is conceivable that the $\Rightarrow$ are $\Leftrightarrow$. See [Sh 430, §3].

Proof. Straightforward. $\square_{2.15}$

<u>Exercise</u>: Generalize to the following filter.

Let $\theta = \mathrm{cf}(\theta) \geq \kappa = \mathrm{cf}(\kappa)$ and $S_* \subseteq [\theta]^{<\kappa}$ be stationary. For any set X of cardinality $\geq \theta$ we define a filter $\mathscr{D}^1_{S_*}[X]$ as follows: $Y \in \mathscr{D}_{S_*}[X]$ <u>iff</u> $Y \subseteq [X]^{<\kappa}$ and for any χ large enough there is $\mathbf{x} \in \mathscr{H}(\chi)$ such that if $\langle N_\alpha, f_\alpha : \alpha \leq \theta \rangle$ satisfy $\circledast$ below, then for some $S' \in \mathscr{D}_{<\kappa}(\theta)$ for every $u \in S_* \cap S'$ we have:

if $\mathbf{x} \in f''_\theta(u)$ <u>then</u> $f''_\theta(u) \in Y$, when:

$\circledast$ (a) $N_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$

(b) N_α is $\prec$-increasing continuous

(c) $\|N_\alpha\| < |\alpha|^+ + \theta$

(d) $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$ if $\alpha < \theta$

(e) can add $\langle \kappa, \theta, X, S_* \rangle \in N_0$.

§3 Nice Filters Revisited

This generalizes [Sh 386] (and see there).
See [Sh 410, §5] on this generalization of normal filters.

3.1 Convention. 1) $\mathbf{n}$ is a niceness context; we use κ, FILL, etc., for $\kappa_{\mathbf{n}}$, $\mathrm{Fil}_{\mathbf{n}} = \mathrm{FIL}(\mathbf{n})$ when dealing from the content.

3.2 Definition. We say the $\mathbf{n}$ is a niceness context or a κ-niceness context or a (κ,μ)-niceness context if it consists of the following objects satisfying the following conditions:

- (a) κ is a regular uncountable cardinal
- (b) $I \subseteq {}^{\omega>}\omega$ is non-empty $\triangleleft$-downward closed with no $\triangleleft$-maximal member[2] default value is $\{0_n : n < \omega\}$
- (c) let μ be $> \kappa$ and $\langle \mathscr{Y} : i < \kappa\rangle$ is a sequence of pairwise disjoint sets and $\mathscr{Y} \cup \{\mathscr{Y}_i : i < \omega_1\}$ so $i < \omega_1 \Rightarrow |\mathscr{Y}|, |\mathscr{Y}_i|$
- (d) the function ι with domain $\mathscr{Y}$ is defined by $\iota(y) = i$ when $y \in \mathscr{Y}_i$
- (e) $\mathbf{e}$ is a set of equivalence relations e on $\mathscr{Y}$ refining $\bigcup\limits_{i<\omega_1} \mathscr{Y}_i \times \mathscr{Y}_i$ with $< \mu^*$ equivalence classes, each class of cardinality $|\mathscr{Y}|$
- (f) for $e \in \mathbf{e}$, $\mathrm{FIL}(e) = \mathrm{FIL}(e,\mathbf{n})$ is a set of D such that:
  - (α) D is a filter on $\mathscr{Y}/e$,
  - (β) for any club C of κ we have $\bigcup\limits_{i\in C} \mathscr{Y}_i/e \in D$,
  - (γ) normality: if $X_i \in D$ for $i < \omega_1$ <u>then</u> the following set belongs to D: $\{(\delta,j)/e : (\delta,j) \in \mathscr{Y}, \delta \text{ limit and } i < \delta \Rightarrow (\delta,j) \in X_i\}$
- (g) $\mathrm{Suc} \in \{(D_1,D_2) : e(D_1) \leq e(D_2)\}$.

Remark. For $\mathbf{e}$ an important case is when it is a singleton $\{\cup\{\mathscr{Y}_i \times \mathscr{Y}_i : i < \kappa\}\}$, so we are dealing with normal filters on the old case.

[2]For $\mathscr{T}$ the two interesting cases are $\mathscr{T} = {}^{\omega>}\omega$ and $\mathscr{T} = \{<>\}$ and ${}^{\omega>}\{0\}$. The default value will be ${}^{\omega>}\omega$.

3.3 Definition. Let $\mathbf{n}$ be a κ-niceness context.
1) We say $e_1 \le e_2$ if e_2 refines e_1. If not said otherwise, every e is from $\mathbf{e}$. Let $\mathbf{e}_\mu$ be the set of all such equivalence relations with $< \mu$ equivalence classes. Let $\iota(x/e) = \iota(x)$.
2) FIL $=$ FIL$(\mathbf{n})$ is $\bigcup\limits_{e \in \mathbf{e}}$ FIL$(e, \mathbf{n})$. For $D \in$ FIL, let $e = e[D]$ be the unique $e \in \mathbf{e}$ such that $D \in$ FIL$(e, \mathbf{n})$.
3) For $D \in$ FIL(e) let $D^{[*]} = \{X \subseteq \mathscr{Y} : X^{[*]} \in D\}$; see (5) below.
4) For $D \in$ FIL$(\mathbf{n})$ and $e(1) \ge e(D)$, let $D^{[e(1)]} = \{X \subseteq \mathscr{Y}/e(1) : X^{[*]} \in D^{[*]}\}$, see (5) below.
5) For $A \subseteq \mathscr{Y}/e, A^{[*]} = \{(x/e) : (x/e) \in A\}$, and for $e(1) \ge e$ let $A^{[e(1)]} = \{y/e(1) : y/e \in A\}$.

3.4 Definition. 1) For $D \in$ FIL$(e, \mathbf{n})$, let D^+ be $\{Y \subseteq \mathscr{Y}/e : Y \ne \emptyset \bmod D\}$.
2) $\mathbf{n}$ is 1-closed if $D \in$ FIL$(\mathbf{n}), A \in D^+ \Rightarrow D + A \in$ FIL$(\mathbf{n})$.
3) $\mathbf{n}$ is 0-closed if for every $D_1 \in$ FIL$_\mathbf{n}$ and $A \in D_1^+$ there is $D_2 \in$ FIL$_2$ such that $(D_1 + A) \in (D_2) \subseteq D_2$.
4) A niceness context $\mathbf{n}$ is full <u>if</u>

- (a) for every $e \in \mathbf{e}_\mathbf{n}$, every filter on $\mathscr{Y}_\mathbf{n}/e$ which is normal (with respect to the function $\iota_\mathbf{n}$) belong to FIL$_\mathbf{n}(e)$.

4A) A niceness content $\mathbf{n}$ is semi-full when: for every $e_1 \in \mathbf{e}_\mathbf{n}$ and $D_1 \in$ FIL$_\mathbf{n}(e_1)$ and $e_2, e_1 \le e_2 \in \mathbf{e}_\mathbf{n}$ and $\mathscr{A} \subset \mathscr{P}(\mathscr{Y}_\mathbf{n}/e_2)$ lift$(W) \in$ FIL(e_2) whenever

$(*)_{e_1,e_2,D_1,W}$
- (a) $e_1 \le e_2$ in $\mathbf{e}_\mathbf{n}$
- (b) $D_1 \in$ FIL$_n(e_2)$
- (c) $\mu \ge 2^{(\mathscr{Y}/e_2)}$ (or more ???)
- (d) $W \subseteq [\mu]^{\le \aleph_0}$ is stationary
- (e) $D_2 =$ lift$(W, D_1^{[e_2]})$ is normal (i.e. $\emptyset \in$ lift(W, D_1)).

5) A niceness context $\mathbf{n}$ is <u>thin</u> when

$$\begin{aligned} \mathrm{Suc}_\mathbf{n} = \{(D_1, D_2) : & D_1 = D_2 \in \mathrm{FIL}_\mathbf{n} \text{ and} \\ & D_2 = D_1^{[e_1]} + A \text{ for some } A \in (D_1^{[e_1]})^+\}. \end{aligned}$$

6) A niceness context $\mathbf{n}$ is thick if: Suc$_\mathbf{n} = \{(D_1, D_2) : D_1, D_2 \in$ FIL$_\mathbf{n}, e(D_1) \le e(D_2)$ and $D_1^{[e_2]} \subseteq D_2$ and if $\mu = 2^{|\mathscr{Y}_\mathbf{n}/e_2)}, W_1 \subseteq [\mu]^{\le \aleph_0}$ is stationary and lift$(W, D_1) = D_1$ then for some stationary $W_2 \subseteq W_1$ we have lift$(W_2, D_2) = D_2\}$.

Remark. 1) On lift see Definition 3.17, HERE??
2) We can use more freedom in the higher objects.

3.5 Claim. *Assume*

(a) *the κ-niceness context is thick*
(b) $D_1 \in \mathrm{FIL}_{\mathbf{n}}(e_1)$
(c) $e_1 \le e_2 \in \mathbf{e_d}$
(d) *for each* $y \in \mathscr{Y}_{\mathbf{n}}/e_1, \langle z_{y,\varepsilon} : \varepsilon < \varepsilon_y \rangle$ *list* $\{z/e_2 : z \in y_1\}, d_{y,\varepsilon}$ *is a κ-complete filter on* ε_y
(e) $D_2 \in \mathrm{FIL}_{\mathbf{n}}(e_2)$
(f) *if* $A \in D_2$ *then* $\{y \in \mathscr{Y}_{\mathbf{n}}/e_1 : \{\varepsilon < \varepsilon_y : z_{y,\varepsilon} \in A\} \in d_{y,\varepsilon}\}$ *belongs to* D_1.

Then $D_2 \in \mathrm{Suc}_{\mathbf{n}}(D_1)$.

Discussion: We may consider allowing player I, in the beginning of each move to choose W_n as above.

3.6 Definition. (0) For $f : \mathscr{Y}/e \to X$ let $f^{[*]} : \mathscr{Y} \to X$ be $f^{[*]}(x) = f(x/e)$. We say $f : \mathscr{Y} \to X$ is supported by e if it has the form $g^{[*]}$ for some $g : \mathscr{Y}/e \to X$. If $e_1, e_2 \in \mathbf{e}$ and $f_\ell : \mathscr{Y}/e_\ell \to X$ for $\ell = 1, 2$ then: we say $f_1 = f_2^{[e_1]}$ if $f_1^{[*]} = f_2^{[*]}$. Writing $f^{[*]}$ for $f \in {}^{\omega_1}X$ we identify $\{i\}$, $i < \omega_1$ with $\mathscr{Y}_i$.
(1) Let $F_c(\mathscr{T}, e) = F_c(\mathscr{T}, e, \mathscr{Y})$ be the family of $\bar{g}$, a sequence of the form $\langle g_\eta : \eta \in u \rangle$, $u \in f_c(\mathscr{T})$ = the family of non-empty finite subsets of ${}^{\omega >}\omega$ closed under taking initial segments, and for each $\eta \in u$ we have $g_\eta \in {}^{\mathscr{Y}}\mathrm{Ord}$ is supported by e. Let $\mathrm{Dom}(\bar{g}) = u$, $\mathrm{Range}(\bar{g}) = \{g_\eta : \eta \in u\}$. We let $e = e(\bar{g})$, for the minimal possible e assuming it exists and we shall say $g_\eta <_D g_\nu$ instead $g_\eta <_{D^{[*]}} g_\nu$ and not always distinguish between $g \in {}^{\mathscr{Y}/e}\mathrm{Ord}$ and $g^{[*]}$ in an abuse of notation.
(2) We say $\bar{g}$ is decreasing for D or D-decreasing (for $D \in \mathrm{FIL}(e, I)$) if $\eta \triangleleft \nu \Rightarrow g_\nu <_D g_\eta$.
(3) If $u = \{<>\}$, $g = g_{<>}$ we may write g instead $\langle g_\eta : \eta \in u \rangle$.

3.7 Definition. 1) For $e \in \mathbf{e}, D \in \mathrm{FIL}(e)$ and D-decreasing $\bar{g} \in F_c(\mathscr{T}, e)$ we define a game $\eth^*(D, \bar{g}, e) = \eth^*(D, \bar{g}, e, \mathbf{n})$. In the nth move (stipulating $e_{-1} = e$, $D_{-1} = D, \bar{g}_{-1} = \bar{g}$):
the case $\mathbf{n}$ is then

> player I chooses $e_n \ge e_{n-1}$ and $A_n \subseteq \mathscr{Y}/e_n$, $A_n \ne \emptyset \mod D_{n-1}^{[e_n]}$ and he chooses $\bar{g}^n \in F_c(\mathscr{T}, e_n)$ extending $\bar{g}_{n-1}$ (i.e. $\bar{g}^{n-1} = \bar{g}^n \restriction \mathrm{Dom}(\bar{g}_{n-1})), \bar{g}^n$ supported by e_n and $\bar{g}^n$ is $(D_n^{[e_n]} + A_n)$-decreasing, player II chooses $D_n \in \mathrm{FIL}(e_n)$ extending $D_{n-1}^{[e_n]} + A_n$.

<u>In the general case</u>:

Player I chooses e_n and $D_{n,1} \in \mathrm{Duc}_{\mathbf{n}}(D_{n-1})$ and let $e_n = e(D_{n-1})$ and he chooses $\bar{g}^n \in F \subset (\mathscr{T}, e(D_{n-1})$ which is extending $\bar{g}^{n-1}$ then $\eta \in \mathrm{Dom}(\bar{g}^n)$ (i.e. $\bar{g}^{n-1} = \bar{g}^n \restriction \mathrm{Dom}(\bar{g}^{n-1}), \bar{g}^n$ supported by $e(D_{n,1})$ and $\bar{g}^n$ is $D_{n,1}$-decreasing.

Player II chooses $D_n = D_{n,2} \in \mathrm{FIL}(\mathbf{e}_n)$ extending $D_{n,1}$.

In the end, the second player wins if $\bigcup\limits_{n<\omega} \mathrm{Dom}(\bar{g}^n)$ has no infinite branch.

2) Let $\bar{\gamma}$ be such that $\mathrm{Dom}(\bar{\gamma}) = \mathrm{Dom}(\bar{g})$ and each γ_η is an ordinal decreasing with η. Now $\Game^{\bar{\gamma}}(D, \bar{g}, e)$ is defined similarly to $\Game^*(D, \bar{g}, e)$ but the second player has in addition, to choose an ordinal α_η for $\eta \in \mathrm{Dom}(\bar{g}^n) \setminus \bigcup\limits_{\ell<n} \mathrm{Dom}(\bar{g}^\ell)$ such that $[\eta \triangleleft \nu \;\&\; \nu \in \mathrm{Dom}(\bar{g}^{n-1}) \Rightarrow \alpha_\nu < \alpha_\eta]$ we let $\alpha_\eta = \gamma_\eta$ for $\eta \in \mathrm{Dom}(\bar{g})$.
3) $w\Game^*(D, \bar{g}, e)$ and $w\Game^{\bar{\gamma}}(D, \bar{g}, e)$ are defined similarly but e is not changed during a play. (If e.g. $\mathbf{e} = \{e\}$ then this makes not difference.)
4) If $\bar{\gamma} = \langle \gamma_{<>} \rangle$, $\bar{g} = \langle g_{<>} \rangle$ we write $\gamma_{<>}$ instead $\bar{\gamma}$, $g_{<>}$ instead $\bar{g}$.
5) If $E \subseteq \mathrm{FIL}$ the games $\Game^*_E$, $\Game^{\bar{\gamma}}_E$ are defined similarly, but player II can choose filters only from E (so we naturally assume to have $A \in D^+, D \in E \Rightarrow D + A \in E$).

3.8 Remark. Denote the above games $\Game^*_0, \Game^{\bar{\gamma}}_0, w\Game^*_0$. Another variant is
3) For $e \in \mathbf{e}, D \in \mathrm{FIL}(e)$ and D-decreasing $\bar{g} \in F_c(\mathscr{T})$ we define a game $\Game^*_1(D, \bar{g}, e)$. We stipulate $e_{-1} = e$, $D_{-1} = D$.

In the nth move first player chooses $e_n, e_{n-1} \le e_n \in \mathscr{T}$ and $D'_n \in \mathrm{FIL}(e_n)$ and D'_n-decreasing $\bar{g}^n$ extending $\bar{g}^{n-1}$ such that $(D_{n-1} + A_n)^{[e_n]} \subseteq D_n$ and:

- $(*)$ for some $A_n \subseteq \mathscr{Y}/e_{n-1}$, $A_n \ne \emptyset \bmod D_{n-1}$ we have:
  - (i) D'_n is the normal filter on $\mathscr{Y}/e_n$ generated by $(D_{n-1} + A_n)^{[e_n]} \cup \{A^n_\zeta : \zeta < \zeta^*_n\}$ where for some $\langle C_\zeta : \zeta < \zeta_n \rangle$ we have:
  - (a) each C_ζ is a club of ω_1,
  - (b) if $\zeta_\ell < \zeta^*_n$ for $\ell < \omega$, $i \in \bigcap\limits_{\ell<\omega} C_{\zeta_\ell}$, $x \in \mathscr{Y}/e_{n-1}$, and $\iota(x) = i$, then for some $x' \in \mathscr{Y}/e_n$, we have $x' \subseteq x$, $x' \in \bigcap\limits_{\ell<\omega} A^n_{\zeta_\ell}$.

The first player also chooses $\bar{g}^n$ extending $\bar{g}^{n-1}, D'_n$-decreasing. Then second player chooses D_n such that $D'_n \subseteq D_n \in \mathrm{FIL}(e_n)$.
2) We define $\Game^{\bar{\gamma}}_1(D, \bar{g}, e)$ as in (2) using $\Game^*_1$ instead of $\Game^*_0$.
3) If player II wins, e.g. $\Game^{\bar{\gamma}}_E(D, \bar{f}, e)$ this is true for
$E' =: \{D' \in G : \text{ player II wins } \Game^{\bar{\gamma}}_{E^*}(D', \bar{f}, e)\}$.

3.9 Definition. 1) We say $D \in$ FIL is nice to $\bar{g} \in F_c(\mathscr{T}, e, \mathscr{Y})$, $e = e(D)$, if player II wins the game $\Game^*(D, \bar{g}, e)$ (so in particular $\bar{g}$ is D-decreasing, $\bar{g}$ supported by e).
2) We say $D \in$ FIL is nice <u>if</u> it is nice to $\bar{g}$ for every $\bar{g} \in F_c(\mathscr{T}, e)$.
3) We say D is nice to α <u>if</u> it is nice to the constant function α. We say D is nice to $g \in {}^\kappa\mathrm{Ord}$ <u>if</u> it is nice to $g^{[e(D)]}$.
4) "Weakly nice" is defined similarly but e is not changed.
5) Above replacing D by $\mathbf{n}$ means: for every $D \in \mathrm{FIL}_{\mathbf{n}}$.

3.10 Remark. "Nice" in [Sh 386] is the weakly nice here, but

- (a) we can use $\mathbf{n}$ with $\mathbf{e}_{\mathbf{n}} = \{e\}$
- (b) formally they act on different objects; but if $xey \Leftrightarrow \iota(x) = \iota(y)$ we get a situation isomorphic to the old one.

3.11 Claim. *Let $D \in$ FIL and $e = e(D)$.*
1) If D is nice to f, $f \in F_c(\mathscr{T}, e), g \in F_c(\mathscr{T}, e)$ and $g \le f$ <u>then</u> D is nice to f.
2) If D is nice to f, $e = e(D) \le e(1) \in \mathbf{e}$ then $D^{[e(1)]}$ is nice to $f^{[e(1)]}$.
3) The games from 3.7(2) are determined and winning strategies do not need memory.
4) D is nice to $\bar{g}$ <u>iff</u> D is nice to $g_{<>}$ (when $\bar{g} \in F_c(\mathscr{T}, e)$ is D-decreasing).
5) If $\mathbf{e} \subseteq \mathbf{e}$ and for simplicity $\bigcup\limits_{i<\omega_1} \{i\} \times \mathscr{Y}_i \in \mathbf{e}$ and for every $e \in \mathbf{e}, e \le e(1) \in \mathbf{e}$ for some permutation π of $\bar{\mathscr{Y}}$ (i.e. a permuation of $\mathscr{Y}$ mapping each $\mathscr{Y}_i\,(i < \omega_1)$ onto itself) (and $\mathbf{n}$ is full for simplicity) we have $\pi(e) = e, \pi(e(1)) \le e(2) \in \mathbf{e}$ <u>then</u> we can replace $\mathbf{e}$ by $\mathbf{e}$.
6) For $\mathbf{e} = \mathbf{e}_\mu$ (where $\mu \le \mu^*$) there is $\mathbf{e}$ as above with: $|\mathbf{e}|$ countable if μ is a successor cardinal ($> \aleph_1$), $|\mathbf{e}| = \mathrm{cf}(\mu)$ if μ is a limit cardinal.

Proof. Left to the reader. (For part (4) use 3.12(2) below).

3.12 Claim. *1) Second player wins $\Game^*(D, \bar{g}, e)$ <u>iff</u> for some $\bar{\gamma}$ second player wins $\Game^{\bar{\gamma}}(D, \bar{g}, e)$.*
2) If second player wins $\Game^\gamma(D, f, e)$ then for any D-decreasing $\bar{g} \in F_c(\mathscr{T}, e), \bar{g}$ supported by e and $\bigwedge\limits_{\eta,y} g_\eta(y) \le f(y)$, the second player wins in $\Game^{\bar{\gamma}}(D, \bar{g}, e)$, when we let

$$\gamma_\eta = \gamma + [\max\{(\ell g(\nu) - \ell g(\eta) + 1) : \nu \textit{ satisfies } \eta \trianglelefteq \nu \in \ \mathit{Dom}(\bar{g})\}].$$

3) If u_1, $u_2 \in F_c(\mathscr{T}), h : u_1 \to u_2$ satisfies $[\eta\nu \Leftrightarrow h(\eta)h(\nu)]$ and for $\ell = 1,2$ we have $\bar{g}^\ell \in F_c(\mathscr{T}, e_2), g^1_\eta \geq g^2_{h(\eta)}$ (for $\eta \in u_1$), $\bar{\gamma}^\ell = \langle \gamma^\ell_\eta : \eta \in u_\ell \rangle$ is a $\triangleleft$-decreasing sequence of ordinals, $\gamma^2_\eta \geq \gamma^2_{h(\eta)}$ and the second player wins in $\eth^{\bar{\gamma}^2}(D, \bar{g}^2, e)$ then the second player wins in $\eth^{\bar{\gamma}^1}(D, \bar{g}^1, e)$.

Proof. 1) The "if part" is trivial, the "only if part" [FILL] is as in [Sh 386].
2), 3) Left to the reader.

The following is a consequence of a theorem of Dodd and Jensen [DoJe81]:

3.13 Theorem. *If λ is a cardinal, $S \subseteq \lambda$ <u>then</u>:*
(1) $\mathbf{K}[S]$, the core model, is a model of $ZFC + (\forall \mu \geq \lambda) 2^\mu = \mu^+$.
(2) If in $\mathbf{K}[S]$ there is no Ramsey cardinal $\mu > \lambda$ (or much weaker condition holds) <u>then</u> ($\mathbf{K}[S], \mathbf{V}$) satisfies the μ-covering lemma for $\mu \geq \lambda + \aleph_1$; i.e. if $B \in \mathbf{V}$ is a set of ordinals of cardinality $\leq \mu$ then there is $B' \in \mathbf{K}[S]$ satisfying $B \subseteq B'$ and $\mathbf{V} \models |B'| \leq \mu$.
(3) If $\mathbf{V} \models (\exists \mu \geq \lambda)(\exists \kappa)[\mu^\kappa > \mu^+ > 2^\kappa]$ <u>then</u> in $\mathbf{K}[S]$ there is a Ramsey cardinal $\mu > \lambda$.

3.14 Lemma. *Suppose*

- (a) *$\mathbf{n}$ is a semi-full niceness content thin or medium $\kappa = \aleph_1$*
- (b) *$f^* \in {}^\kappa\mathrm{Ord}$, $\lambda > \lambda_0 =: \sup\{(2^{|\mathscr{Y}/e|^{\aleph_0}}) : e \in \mathbf{e_n}\}$*
- (c) *for every $A \subseteq \lambda_0$, in K there is a Ramsey cardindal $> \lambda_0$, <u>then</u> for every filter $D \in \mathrm{FIL}_{\mathbf{n}}(e)$ is nice to f^*.*

Remark. 1) The point in the proof is that via forcing we translate the filters from $\mathrm{FIL}(e, \mathscr{Y})$ to normal filters on κ [for higher κ's cardinal restrictions are better].
2) At present we do not care too much what is the value of λ_0, i.e., equivalently, how much we like the set S to code.
Saharon: compare with [Sh:g, V], i.e., improve as there! But if we use $\mathbf{e} = \{e\}$, the proofs are more similar to [Sh:g, V] we can consider just $\mathrm{Levy}(\aleph_1), |D|)$, now in some proofs we may consider filters generated by $|\mathrm{pcf}(\mathfrak{a})|$ set $|\mathfrak{a}| < aleph_\omega$.

First Proof. Without loss of generality $(\forall i) f(i) \geq 2$. Let $S \subseteq \lambda_0$ be such that $[\alpha < \mu \ \& \ A \subseteq 2^{|\alpha|^{\aleph_0}} \Rightarrow A \in \mathbf{L}[S]], \mathbf{e} \in \mathbf{L}[S]$ (see 3.11(6)) and: if $g \in {}^\kappa\mathrm{Ord}, (\forall i <$

$\kappa_1)g(i) \leq f(i)$ then $g \in \mathbf{L}[S]$ (possible as $\prod\limits_{i<\omega_1} |f(i)+1| \leq \lambda_0$. We work for awhile in $\mathbf{K}[S]$. In $\mathbf{K}[S]$ there is a Ramsey cardinal $\mu > \lambda_0$ (see 3.13(3)). Let in $\mathbf{K}[S]$. Let

$$\begin{aligned} Y_0 = \{X : & X \subseteq \mu, X \cap \kappa \text{ a countable ordinal } > 0, \{\kappa, \lambda_0\} \subseteq X, \\ & \text{moreover } X \cap \lambda_0 \text{ is countable}\}. \end{aligned}$$

Let

$$Y_* = Y_1 = \{X \in Y_0 : X \text{ has order type } \geq f(X \cap \kappa)\}.$$

Now for $g \in {}^{\kappa}\mathrm{Ord}$ such that $\bigwedge\limits_{i<\omega_1} g(i) < f(i)$ let $\hat{g}$ be the function with domain Y_1, $\hat{g}(X) =$ the $g(X \cap \kappa)$-th member of X.

Let $D_* = \{A_i : \kappa \leq i \leq 2^{|\mathscr{Y}/e|}\}$ and we arrange $\langle A_i^D : \kappa \leq i < 2^{|\mathscr{Y}/e|}\rangle \in \mathbf{L}[S]$, (as $\mathscr{Y}/e$ has cardinality $< \mu^*$, so $2^{|\mathscr{Y}/e|} \leq \lambda_0$).

Let J be the minimal fine normal ideal on Y (in $\mathbf{K}[S]$) to which $Y \backslash Y_D$ belongs where

$$Y_D = \{X : X \in Y_* \text{ and } i \in (\kappa, 2^{|\mathscr{Y}/e|}) \cap X \Rightarrow X \cap \omega_1 \in A_i\}.$$

Clearly it is a proper filter as $\mathbf{K}[S] \models$ "μ is a Ramsey cardinal".

3.15 Observation. Assume

- (a) $\mathbb{P}$ is a proper forcing notion of cardinality $\leq |\alpha|^{\aleph_0}$ for some $\alpha < \mu^*$ (or just $\mathbb{P}, MAC(\mathbb{P}) \in \mathbf{K}[S]$ and $\{X \in Y_1 : X \cap (MAC(\mathbb{P})|$ is countable$\} \in = Y_*$ mod J where $MAC(\mathbb{P})$ is the set of maximal antichains of $\mathbb{P}$) and let $J^{\mathbb{P}}$ be the normal fine ideal which J generates in $\mathbf{V}^{\mathbb{P}}$.

(1) F-positiveness is preserved; i.e. if $X \in \mathbf{K}[S], X \subseteq Y_1, F \in$ FIL and $\mathbf{V} \models$ "$X \neq \emptyset$ mod F" then $\Vdash_{\mathbb{P}}$ "$X \neq \emptyset$ mod $F^{\mathbb{P}}$.
(2) Moreover, if $\mathbb{Q} \lessdot \mathbb{P}$, ($\mathbb{Q}$ proper and) $\mathbb{P}/\mathbb{Q}$ is proper then forcing with $\mathbb{P}/\mathbb{Q}$ preserve $F^{\mathbb{Q}}$-positiveness.

Continuation of the proof of 3.14.
Case 1: $\mathbf{e} = \{e\}$. Here only 3.16(1) is needed and then it is as in the old case.

Case 2: General.

Let $\mathscr{P}(\mathscr{Y}/e) = \{A^e_\zeta : \zeta < 2^{|\mathscr{Y}/e|}\}$.

Now we describe a winning strategy for the second player. In the side we choose also $(p_n, \Gamma_n, \underset{\sim}{f}_n)$, $\bar\gamma^n, \underset{\sim}{W}_n$ such that[3] (where e_n, A_n are chosen by the second player):

$(A)(i)$ $\mathbb{P}_n = \prod\limits_{\ell \le n} \mathbb{Q}_\ell$ where $\mathbb{Q}_\ell$ is Levy$(\aleph_1, \mathscr{Y}/e_n)$ (we could use iterations, too, here it does not matter).

(ii) $p_n \in \mathbb{P}_n$

(iii) p_n increasing in n

(iv) $\underset{\sim}{f}_n$ is a $\mathbb{P}_n$-name of a function from ω_1 to $\mathscr{Y}/e_n$

(v) $p_n \Vdash_{\mathbb{P}_n}$ "$\underset{\sim}{f}_n(i) \in \mathscr{Y}_i/e_n$"

(vi) $p_{n+1} \Vdash$ "$\underset{\sim}{f}_{n+1}(i) \le \underset{\sim}{f}_n(i)$ for every $i < \omega_1$",

(vii) $\underset{\sim}{f}_n$ is given naturally — it can be interpreted as the generic object of $\mathbb{Q}_n$ except trivialities.

$(B)(i)$ $\bar\gamma^n, \bar g^n$ have the same domain, $\gamma^n_\eta < \mu$

(ii) $p_n \Vdash_{\mathbb{P}_n}$ "$\underset{\sim}{W}_n \subseteq Y_D$, $\underset{\sim}{W}_{n+1} \subseteq \underset{\sim}{W}_n$"

(iii) $\bar\gamma^n = \bar\gamma^{n+1} \restriction \mathrm{Dom}(\bar\gamma^n)$, $\mathrm{Dom}(\bar\gamma^n) = \mathrm{Dom}(\bar g^n)$ and $\bar\gamma^n$ is $\triangleleft$-decreasing

(iv) $p_n \Vdash_{\mathbb{P}_n}$ "$\{X \in Y_D :$ for $\ell \in \{0, ..., n\}, \underset{\sim}{f}_\ell(X \cap \omega_1) \in A_\ell$ and $\bigwedge\limits_{\eta \in \mathrm{Dom}(\bar g^n)} \hat g_\eta(X) = \gamma_\eta$ and for $\ell \in \{-1, 0, ..., n-1\}, \zeta \in X \cap 2^{|\mathscr{Y}/e_\ell|}$ we have: $A^{e_\ell}_\zeta \in D_\ell \Rightarrow \underset{\sim}{f}_\ell(X \cap \omega_1) \in A^{e_\ell}_\zeta\} \supseteq \underset{\sim}{W}_n \ne \emptyset \bmod F^{\mathbb{P}_n}$"

(v) $\bar g^n = \bar g^{n+1} \restriction \mathrm{Dom}(\bar g^n)$ [difference]

$(C)(i)$ $D_n = \{Z \subseteq \mathscr{Y}/e_n : p_n \Vdash_{\mathbb{P}_n}$ "$\{X \in J_D : \underset{\sim}{f}_n(X \cap \omega_1) \notin Z\} = \emptyset \bmod (D^{\mathbb{P}_n}_n + \underset{\sim}{W}_n)$"$\}$

(ii) $\bar g^n$ is D_n-decreasing. [Saharon: diff]

Note that $D_n \in \mathbf{K}[S]$, so every initial segment of the play (in which the second player uses this strategy) belongs to $\mathbf{K}[S]$.
By $(B)(iii)$ this is a winning strategy. $\square_{3.14}$

[3] For the forcing notions actually used below by the homogeneity of the forcing notion the value of p_n is immaterial

Recall all normal filters on $\mathscr{Y}/e$ belong to FIL(e).

Alternate: We split the proof to a series of claims and definitions.

3.16 Definition. 1) $W_* = \{u \subseteq \mu : \text{otp}(u) \geq f^*(u \cap w_1)$ and $u \cap \lambda$ is countable$\}$.
2) Let J be the following ideal on Y_0:

$W \in J$ iff for some model M on μ with countable vocabulary (with Skolem function) we have

$$W_* \supseteq W \subseteq \{w \in W_* : w = c\ell_M(w)\}.$$

3) For $g \in \prod_{i<\kappa}(f(i)+1))$ let $\hat{g}$ be the function with domain Y_* and $\hat{g}(A)$ is the $g(i)$-the member of A.
4) For $W \in J^+$ let $\text{proj}(W) = \{A \subseteq w_1 : \{w \in W : w \cap w_1 \notin A\} \in J\}$.

3.17 Fact. 1) $Y_* \notin J$.
2) J is a fine normal filter on W_* (and $W_* \notin J$) in fact the ideal of non-stationary subsets of W_*.
3) $Y_{\bar{A}} \in J^+$ if $\bar{A} = \langle A_i : i < 0\rangle, 2^{\aleph_1}$ list the subset of some normal filter D on ω_1 (see 3.23's proof.
4) If $\bar{A}', \bar{A}''$ list the same normal filter on w_1 then $Y_{\bar{A}'} = Y_{\bar{A}'}$ mod J.
5) For $g \in \prod_{i<\omega}(f^*(i)+1), \hat{g}$ is well defined, is a choice function of Y_*.
6) If $g_1 <_D g_2$ then $\hat{g}_1 \restriction J_D < \hat{g}_2 \restriction J_D$ mod $J + Y_*$.

Proof. 1) As μ is a Ramsey cardinal $> \lambda_0$.
2) By the definitions.
3) Easy.

3.18 Claim. *Assume $\mathbb{Q}$ is an $\aleph_1$-complete forcing notion with $\leq \lambda_0$ maximal antichains.*
1) Forcing with $\mathbb{Q}$ preserves all our assumptions:

- (a) *μ is a Ramsey cardinal$^+$*
- (b) *W_* is a family of subsets of μ such that* $\text{otp}(w) \geq f(w \cap \omega_1)$ *and J, defined above, is a fine normal ideal on Y_* satisfying 3.17(3)...then we can forget (a).*

2) Forcing with $\mathbb{Q}$ preserves "$y \in J^+$" (i.e. if $W \in J^+$ then $\Vdash_{\mathbb{Q}}$ "$W \in J^+$".

Proof. Easy, fill.

3.19 Definition. Assume $e \in \mathbf{e_n}$ and $D \in \mathrm{FIL}_{\mathbf{n}}(e)$.
1) $\mathbb{Q} = \mathbb{Q}_e = \{f : f$ is a function with domain a countable ordinal such that $i \in \mathrm{Dom}(f) \Rightarrow f(i) \in \mathscr{Y}_i^{\mathbf{n}}\}$.
2) $\underset{\sim}{f}_e$ is the $\mathbb{Q}$-name $\cup\{f : f \in \underset{\sim}{G}_{\mathbb{Q}_e}\}$.

3) Let $D/\underset{\sim}{f}_e$ be the $\mathbb{Q}_e$-name of $\{A \subseteq \omega_1$: for every $B \in D$ for stationarily many $i < \omega_1, \underset{\sim}{f}_e(i) \in B\}$ and $\mathrm{nor}(D, \underset{\sim}{f}_e)$ the normal filter which $D/\underset{\sim}{f}_e$ generates.

4) For $W \in J^+$ let $\mathrm{lift}(W, D) = \{A \subseteq \mathscr{Y}/e$ for some $B \in D :\Vdash_{\mathbb{Q}_e}$ "$\{w \in W : \underset{\sim}{f}_e(w \cap \omega_1) \in B \backslash A \in J$" (note that we have enough homogeneity for $\mathbb{Q}_e$.

3.20 Claim. *Assume $e \in \mathbf{e_n}$ and $D \in \mathrm{FIL}_{\mathbf{n}}(e)$.*
1) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}/\underset{\sim}{f}_e$ is a normal filter on ω_1", (i.e. $w_1 \notin \underset{\sim}{D}$).

2) $|\mathbb{Q}_e| \le |\mathscr{Y}^{\mathbf{n}}/e|^{\aleph_0}$ so $Z^{|\mathbb{Q}_e|} \le \lambda_0$ hence $\mathbb{Q}_e$ has $\le \lambda_0$ maximal antichains; in fact, equality holds as we have demand $|\mathscr{Y}/e| = |\cup\{\mathscr{Y}_i : i \in [i_0, \omega_1)\}/e|$ for every $e \in \mathbf{e}$.
3) Combine scite3.2A(4) + 3.19 - FILL.

3.21 Definition. 1) We say that $\mathfrak{x} = (e, D, \bar{g}, \bar{\alpha}, f, W)$ is a good position (in the content of proving 3.14) if

- (a) $e \in \mathbf{e_n}$
- (b) $D \in \mathrm{FIL}_{\mathbf{n}}(e)$
- (c) $\bar{g} = \langle g_\eta : \eta \in u\rangle \in \mathrm{Fc}(\mathscr{T}, e)$, so $u = u^{\mathfrak{x}}$
- (d) $\bar{\alpha} = \langle \alpha_\eta : \eta \in u\rangle, \alpha_\eta < \mu$
- (e) $p \in \mathbb{Q}_e$
- (f) $W = \{w \in W^* : \hat{g}_\eta(w) = \alpha_\eta$ for $\eta \in u\} \in J^+$
- (g) $p \Vdash_{\mathbb{Q}_e}$ "$W^{\mathfrak{x}} \cap W_{D,f_e} \in J^+$" and $\mathrm{proj}(W^{\mathfrak{x}} \cap W_{D,f_e}) = D$ $\mathrm{nor}(D, f_e)$ [FILL].

3.22 Observation. 1) If $\mathfrak{x} = (e, D, \bar{g}, \bar{\alpha}, p, \underset{\sim}{W})$ is a good position then

- (a) $\bar{\alpha}$ is decreasing
- (b) $D_{\underset{\sim}{W}}$.

3.23 Claim. *If $e \in \mathbf{e_n}, D \in \mathrm{FIL}_{\mathbf{n}}(e)$ and $\bar g = \langle g_\eta : \eta \in u \rangle \in \mathrm{Fc}(\mathscr{T}, e)$ and $g_\eta \le f[e]$ for every $\eta \in$ Dom$(\bar g)$ <u>then</u> we can find a good position $\mathfrak{x}$ with $\bar g^{\mathfrak{x}} = e^{\mathfrak{x}} = e, \bar g^{\mathfrak{x}} = g$ and $D \subseteq D^{\mathfrak{x}}$.*

Proof. Let $\mathbf{G} \in \mathbb{Q}_e$ be generic over $\mathbf{V}$ and $f_e = \underset{\sim}{f}_e[G]$. So in $\mathbf{V}[\mathbf{G}]$ the set $W_{D,\underset{\sim}{f}_e[\mathbf{G}]}$ belongs to J^+ (by 3.17(3)), i.e., let $\langle A^{D_1}_\zeta : \zeta < \zeta^* \rangle$ list D_1 and $W, D, f_e = \{w \in W$: if $\zeta \in w \cap \zeta^*$ then $f_e(i) = \underset{\sim}{f}_e[\mathbf{G}](i) \in \underset{\sim}{A}_\zeta\}$.

Also $\hat g_\eta$ defined in 3.16(3) is a choice function on W_{D,f_e} (see 3.17(4)), so as J is a normal ideal and u finite, we can find $\bar\alpha = \langle \alpha_\eta : \eta \in u \rangle$ such that $W = \{w \in W_{D,f_e} : \hat g_\eta(w) = \alpha_\eta$ for $\eta \in u\}$ belongs to J^+. As all this holds in $\mathbf{V}[\mathbf{G}]$. So $\bar\alpha$ there is a condition $p \in \mathbb{Q}_e$ which forces this, and we are done.

3.24 Claim. *Assume that*

(*a*) $\mathfrak{x}_1 = (e_1, D_1, \bar g_1, \bar\alpha_1, p, \underset{\sim}{W}_1)$ *is a good position*

(*b*) $\bar g_2 = \langle g^2_\eta : \eta \in u_2 \rangle \in \mathrm{Fc}(\mathscr{T}, \mathbf{n})$ *and* $\bar g_2 \restriction u_1 = \bar g_2$

(*c*) $e_1 \le e_2$ *in* $\mathbf{e}_n$ *and* $D_2 \in$ *FIL*$_{\mathbf{n}}(e_2)$ *or just* $\mathscr{A} \subseteq \mathscr{P}(\mathscr{Y}_{\mathbf{n}}/e_2), \mathscr{A} = \{A_\zeta : \zeta < \zeta^*\}$

(*d*) $p_1 \Vdash_{\mathbb{Q}_{e_1}}$ "$\{w \in \underset{\sim}{W}_1 : \mathscr{Y}_{w \cap w_1} \nsubseteq \cup\{A_\zeta : \zeta \in \zeta^* \cap w\}\}$ *does not belong to* $J^{\mathbf{V}[\mathbb{Q}_{e_1}]}$".

<u>Then</u> we can find a good position $\mathfrak{x}_2$ such that $e^{\mathfrak{x}_2} = e_2, \bar g^{\mathfrak{x}_2} = \bar g^2$ and $D_2 \subseteq D^{\mathfrak{x}_2}$.

Proof. Let $\mathbf{G}$ be a subset of $\mathbb{Q}_{e_1[\mathfrak{x}_1]}$ generic over $\mathbf{V}$ such that $p^{\mathfrak{x}_1} \in \mathbf{G}_1$. Now $\mathbb{Q}_{e_2}$ is an $\aleph_1$-complete forcing of cardinality $\le |\mathscr{Y}_{\mathbf{n}}/e_2|^{\aleph_0} \le \lambda_0$ and $\mathbb{Q}_{e_1}$ is $\aleph_1$-complete $|\mathbb{Q}_{e_1}| \le |\mathscr{Y}_{\mathbf{n}}/e_1|^{\aleph_0} \le |\mathscr{Y}_{\mathbf{n}}/e_2|^{\aleph_0} \le \lambda_0$, so $\mathbb{Q}_{e_2}$ satisfies the same conditions in $\mathbf{V}[\mathbf{G}_1]$ (if λ_0 is no longer a cardinal it does not matter).

Note that by assumption (c)

⊛ in $\mathbf{V}[\mathbf{G}_1], \mathbb{Q}_{e_2} \Vdash$ "the set $\{\underset{\sim}{W}^1_2 =: \{w \in \underset{\sim}{W}_1[\mathbf{G}_1]$: the set $((\underset{\sim}{f}_{e_1}[\mathbf{G}_1])(w \cap \omega_1))^{[e_2]} \in \mathscr{Y}_{w \cap \omega_1}/e_2$ is not included in $\cup\{A_\zeta : \zeta \in w\}\}$ is stationary (i.e. $\notin J$)".

We continue as in the previous claim.

3.25 Claim. *If clauses (a) + (b) of 3.23 holds, <u>then</u> a sufficient condition for clause (c) is*

(c)' FILL.

3.26 Proof of 3.14. During the play, the player II chooses also a good position $\mathfrak{x}_n$ and maintains $\bar{g}^{\mathfrak{x}_n} = \bar{g}_n, \bar{\alpha}^{\mathfrak{x}_n} = \bar{\alpha}$.

3.27 Remark. 1) From the proof, instead $\mathbf{K}[S] \models$ "λ is Ramsey", $\mathbf{K}[S] \models$ "$\mu \to (\alpha)^{<\omega}_{\lambda_0}$ for $\alpha < \lambda_0$" is enough for showing for 3.14.
2) Also if $\prod\limits_{i<\omega_1} (|f(i)|+1) < \mu_0, [\alpha < \mu_0 \Rightarrow |\alpha|^{\aleph_0} < \mu_0]$, it is enough: $S \subseteq \alpha < \mu_0 \Rightarrow$ in $\mathbf{K}[S]$ there is $\mu \to (\alpha)^{<\omega}_2$.

3.28 Theorem. *Assume* $\mathbf{n}$ *is a* κ*-niceness context. Let* $D^* \in$ *FIL*$(e, \mathscr{Y})$ *be a normal ideal on* $\mathscr{Y}_{\mathbf{n}}/e$*. If for every* $f : \mathscr{Y} \to (\sup\{\mathrm{Suc}(D') : D' \in \mathrm{FIL}_{\mathbf{n}}\})^+$ *supported by some* $e \in \mathbf{e}_{\mathbf{n}}$*.* $D^*_{\mathbf{n}}$ *is nice to* f*, <u>then</u> for every* $f \in {}^\kappa\mathrm{Ord}$*,* $\mathbf{n}$ *is nice to* f*.*

Proof. By determinacy of the games (and the LS argument).

3.29 Remark. 0) The value $|\mathrm{FIL}_{\mathbf{e}}|$ really should be an upper bound.
1) So, the existence of $\mu, \mu \to (\alpha)^{<\omega}_{\aleph_0}$ for every $\alpha < (\sum\limits_{\chi<\mu} \chi^\kappa)^+$, is enough for "$D^*$ is nice".
2) If there is a nice D's in the plays from 3.7, the second player winning strategy can be chosen such that all subsequent filters are nice: just by renaming have $g_{<>}$ constant large enough. [Saharon: diff]

3.30 Claim. *In claim 3.14 we can omit "*$\kappa_{\mathbf{n}} = \aleph_1$*".*

Proof. Let $\mathbb{P} =$ Levy$(\aleph_0, \kappa_{\mathbf{n}})$. Now

- $(*)$ also in $\mathbf{V}^{\mathbb{P}}$ the object $\mathbf{n}$ is a successor content, if we do not distinguish between $D \in$ $\mathrm{FIL}_{\mathbf{n}}$ and $\{A \in \mathbf{V}^{\mathbb{P}} : A \subseteq \mathscr{Y}/e(D)$ and $(\exists B \in D)(B \subseteq A)\}$.

3.31 Conclusion.: Let $\lambda_0 = (\sup\{|\mathrm{Suc}_{\mathbf{n}}(D')| : D' \in \mathrm{FIL}_{\mathbf{n}}\})^+ \cup \{2^{|\mathscr{Y}/e|^{<\kappa}} : e \in \mathbf{e}_{\mathbf{n}}\})^+, \mu^* \geq \aleph_2$; if for every $S \subseteq \lambda_0$ there is a Ramsey cardinal in $\mathbf{K}[S]$ above λ_0 then $\mathbf{n}$ is nice.

Proof. By 3.14, 3.28.

3.32 Concluding Remark. 1) We could have used other forcing notions, not $\mathrm{Levy}(\kappa, |\mathscr{Y}/e_n|)$. E.q., if $\kappa = \aleph_1, \mu = \kappa^+$ we could use finite iterations of the forcing of Baumgartner to add a club of ω_1, by finite conditions. (So this forcing notion has cardinality $\aleph_1$). Then in 3.14 we can weaken the demands on $\lambda_0 : \lambda_0 = \sum_{\chi<\mu_0} 2^\chi + \prod_{i<\omega_1} |1+f(i)| + |\mathbf{e}|$, hence also in 3.31, $\lambda_0 = \sum_{\chi<\mu_*} 2^\chi$ is O.K.
2) Concerning $|\mathbf{e}|$ remember 3.11(5),(6).
3) Similarly to (1). If $\theta < \mu \Rightarrow \mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$ then by 2.6 we can use forcing notions of Todorcevic for collapsing $\theta < \mu$ which has cardinality $< \mu$.
4) If we want to have $\lambda_0 =: \prod_{i<\omega_1} |f(i) + 2|$ (or even $T_D(f+2)$), we can get this by weakening further the first player letting him choose only A_n which are easily definable from the $\bar{g}^{n-1}$, we shall return to it in a subsequent paper.

§4 Ranks

4.1 Convention. 1) Like 3.2 and:
2) $\bar{g}^* \in F_c(\mathscr{T}, e^*, \mathscr{Y}), \eta^* \in$ Dom$(\bar{g}^*), \nu^*$ an immediate successor of η^* not in Dom g^*, $D^* \in$ FIL$(e^*, \mathscr{Y})$ is such that in $\Game^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins (all constant for this section). FIL$^*(e)$ will be the set of $D \in$ FIL$(e, \mathscr{Y})$ such that $e \geq e^*$, $(D^*)^{[e]} \subseteq D$ and in $\Game^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins. (So actually FIL$(e^*, \mathscr{Y})$ depends on $D^*, \bar{g}^*, e^*$, too).

4.2 Definition. 1) $\mathrm{rk}^5_D(f)$ for $D \in$ FIL$^*(e, \mathscr{Y})$, $f \in {}^{\mathscr{Y}/e}\mathrm{Ord}$, $f <_D \bar{g}^*_{\eta^*}$ will be: the minimal ordinal α such that for some D_1, e_1, $\bar{\gamma}^1$ we have $D^{[e_1]} \subseteq D_1 \in$ FIL$(e_1, \mathscr{Y})$, $\bar{\gamma}^1 = \bar{\gamma}^{*\frown}\langle \nu^*, \alpha\rangle$ (i.e. $\mathrm{dom}(\bar{\gamma}^1) = (\mathrm{dom}(\bar{\gamma}^*)) \cup \{\nu^*\}$, $\bar{\gamma}^1 \restriction \mathrm{dom}(\bar{\gamma}^*) = \bar{\gamma}^*$, $\gamma^1_{\nu^*} = \alpha$) and in $\Game^{\bar{\gamma}^1}(D, \bar{g}^{*\frown} < \nu^*, f >)$ second player wins and ∞ if there is no such α.
2) $\mathrm{rk}^4_D(f)$ is $\sup\{\mathrm{rk}^5_{D+A}(f) : A \in D^+\}$.

4.3 Claim. *1) $\mathrm{rk}^5_D(f)$ is (under the circumstances of 4.1, 4.2) an ordinal $< \gamma^*_{\eta^*}$.*
2) $\mathrm{rk}^4_D(f)$ is an ordinal $\leq \gamma^*_{\eta^*}$.

4.4 Claim. *If $D \in \mathrm{FIL}^*(e, \mathscr{Y}), h <_D f <_D g^*_{\eta^*}$ then $\mathrm{rk}^5_D(h) < \mathrm{rk}^5_D(f)$.*

Proof. Let e_1, D_1 witness $\mathrm{rk}^5_D(f) = \alpha$ so $e(D) \leq e_1$, $D \subseteq D_1 \in$ FIL$^*(e_1)$ and in $G^{\bar{\gamma}\frown<\nu^*,\alpha>}(D_1, \bar{g}^{*\frown} < \nu^*, f >, e)$ second player wins. We play for the first player: $e = e_1$, $A_0 = \mathscr{Y}/e_1$, $\bar{g}^0 = \bar{g}^{*\frown}\langle \nu^*, f\rangle^\frown\langle \nu^{*\frown} < 0 >, g\rangle$, now the first player should be able to answer say e_2, D_2, $\bar{\gamma}^2$. So $\gamma^2_{\nu^{*\frown}<0>} < \gamma^2_{\nu^*} = \alpha$, and by 3.12(3), we know that in $G^{\bar{\gamma}'}(D_2, \bar{g}^{*\frown} < \nu^*, g >, e_2)$ where $\bar{\gamma}' = \bar{\gamma}^\frown\langle \nu^*, \gamma^2_{\nu^{*\frown}<0>}\rangle$, second player wins. $\square_{4.4}$

4.5 Claim. *Let $e \geq e^*, D \in \mathrm{FIL}^*(e, \mathscr{Y})$.*
1) For $e \geq e(D), A \in (D^{[e]})^+, f \in {}^{\mathscr{Y}/e}\mathrm{Ord}$, $f <_D g^*_{\eta^*}$ we have:

$$\mathrm{rk}^5_D(f) \leq \mathrm{rk}^5_{D^{[e]}+A}(f) \leq \mathrm{rk}^4_{D^{[e]}+A}(f) \leq \mathrm{rk}^4_D(f).$$

2) If $e_2 \geq e_1 \geq e(D), f_\ell \in {}^{\mathscr{Y}}\mathrm{Ord}$ is supported by e_ℓ, $f_1 \leq_D f_2 <_D g^*_{\eta^*}$ <u>then</u> $\mathrm{rk}^\ell_D(f_1) \leq \mathrm{rk}^\ell_D(f_2)$ for $\ell = 4, 5$.

Proof. Left to the reader.

§5 More on Ranks and Higher Objects

5.1 Convention.

(a) μ^* is a cardinal $> \aleph_1$ (using $\aleph_1$ rather than an uncountable regular κ is to save parameters)

(b) $\mathscr{Y}$ a set of cardinality $\sum\limits_{\kappa<\mu_*} \kappa$

(c) ι a function from $\mathscr{Y}$ onto ω_1, $|\iota^{-1}(\{\alpha\})| = |\mathscr{Y}|$ for $\alpha < \omega$,

(d) Eq the set of equivalence relation e on $\mathscr{Y}$ such that:

- (α) $yez \Rightarrow \iota(y) = \iota(z)$
- (β) each equivalence class has cardinality $|\mathscr{Y}|$
- (γ) e has $< \mu^*$ equivalence classes

(e) D denotes a normal filter on some $\mathscr{Y}/e(e \in \text{ Eq})$, we write $e = e(D)$. The set of such D's is FIL($\mathscr{Y}$).

(f) E denotes a set of D's as above, such that:

- (α) for some $D = \text{ Min } E \in E$ $(\forall D')[D' \in E \Rightarrow (e, D) \le (e(D'), D')]$
- (β) if $D \in E$, $A \subseteq \mathscr{Y}/e_1, e_1 \ge e(D)$, $A \ne \emptyset$ mod D then $D^{[e_1]} + A \in E$

(g) $E^{[e]} =: \{D \in E : e(D) = e\}$

(h) $\mathscr{E}$ denotes a set of E's as above, such that:

- (α) there is $E = \text{ Min } \mathscr{E} \in \mathscr{E}$ satisfying $(\forall E')(E' \in E \Rightarrow E' \subseteq E)$
- (β) if $D \in E \in \mathscr{E}$ then $E_{[D]} = \{D' : D' \in E \text{ and } (e(D), D) \le (e(D'), D')\} \in \mathscr{E}$.

5.2 Definition. 1) We say E is λ-divisible when: for every $D \in E$, and Z, a set of cardinality $< \lambda$ there is D's such that:

(α) $D' \in E$

(β) $(e(D), D) \le (e(D'), D')$

(γ) $\mathbf{j} : \mathscr{Y}/e(D') \to Z$

(δ) for every function $h : \mathscr{Y}/e(D) \to Z$ we have $\{y/e(D') : h(y/e(D)) = (y/e(D'))\} \ne \emptyset$ mod D'.

2) We say E has λ-sums when: for every $D \in E \in \mathscr{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda\rangle$ of subsets of $\mathscr{Y}/e(D)$ there is $Z^* \subseteq \mathscr{Y}/(e/(D)$, such that: $Z^* \cap Z_\zeta = \emptyset$ mod D and: [if $(e(D), D) \le (e', D'), e' = e(D'), D' \in E_{[D]}$ and $\bigwedge\limits_\zeta Z_\zeta^{[e']} = \emptyset$ mod D' then $Z^* \in D'$].
3) We say E has weak λ-sum if for every $D \in E(\in \mathscr{E})$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda\rangle$ of subsets of $\mathscr{Y}/e(D)$ there is D^*, $D^* \in E_{[D]}$ such that:

- (α) if $(e(D), D) \le (e', D'), D' \in E_{[D]}$ and $Z_\zeta = \emptyset$ mod D' for $\zeta < \zeta^*$ and $e(D^*) \le e(D')$ then $D^* \subseteq D'$ (more exactly ${D^*}^{[*]} \subseteq D^{[*]}$ and)
- (β) $Z_\zeta = \emptyset$ mod D^* for $\zeta < \zeta^*$.

4) If $\lambda = \mu^*$ we omit it. We say $\mathscr{E}$ is λ-divisible if every $E \in \mathscr{E}$ has. We say $\mathscr{E}$ has weak λ-sums if: [rest diff] for every $E \in \mathscr{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda\rangle$ of subsets of $\mathscr{Y}/e(E)$ there is E^*, $E^* \in \mathscr{E}_{[E]}$ such that:

- (α) if $(e(E), E) \le (e', E')$, $E' \in \mathscr{E}$ and $Z_\zeta = \emptyset$ mod $\mathrm{Min}(E')$ for $\zeta < \zeta^*$ and $e(E^*) \le e(E')$ then $E^* \subseteq E'$
- (β) $Z_\zeta = \emptyset$ mod $\mathrm{Min}(E^*)$ for $\zeta < \zeta^*$.

We now define variants of the games from §3.

5.3 Definition. For a given $\mathscr{E}$, for every $E \in \mathscr{E}$:
1) We define a game $G_2^*(E, \bar g)$. In the $n-th$ move first player chooses $D_n \in E_{n-1}$ (stipulating $E_{-1} = E$) and choose $\bar g_n \in F_c({}^\omega\omega, e(D_n), \mathscr{Y})$ extending $\bar g_{n-1}$ (stipulating $\bar g_{-1} = \bar g$) such that $\bar g_n$ is D_n-decreasing. Then the second player chooses E_n, $(E_{n-1})_{[D_n]} \subseteq E_n \in \mathscr{E}$.

In the end the second player wins if $\bigcup\limits_{n<\omega}$ Dom $\bar g_n$ has no infinite branch.

2) We define a game $G_2^{\bar\gamma}(E, \bar g)$ where $\mathrm{Dom}(\bar\gamma) = \mathrm{Dom}(\bar g)$, each γ_η an ordinal, $[\eta \triangleleft \nu \Rightarrow \gamma_\eta > \gamma_\nu]$ similarly to $G_2^*(D, \bar g)$ but the second player in addition chooses an indexed set $\bar\gamma_n$ of ordinals, $\mathrm{Dom}(\bar\gamma_n) = \mathrm{Dom}(\bar g_n)$, $\bar\gamma_n \restriction \mathrm{Dom}(\bar\gamma_{n-1}) = \bar\gamma_{n-1}$ and $[\eta \triangleleft \nu \Rightarrow \gamma_{n,\eta} > \gamma_{n,\nu}]$.

5.4 Definition. 1) We say $\mathscr{E}$ is nice to $\bar g \in F_c(\mathscr{T}, e, \mathscr{Y})$ if for every $E \in \mathscr{E}$ with $e \le e(E)$ the second player wins the game $\Game_2^*(E, \bar g)$.
2) We say $\mathscr{E}$ is nice if it is nice to $\bar g$ whenever $E \in \mathscr{E}$, $e \le e(E)$, $\bar g \in F_c(\mathscr{T}, e)$, $\bar g$ is (Min E)-decreasing, we have: $\mathscr{E}_{[E]}$ is nice to $\bar g$.
3) If $\mathrm{Dom}(\bar g) = \{<>\}$ we write $g_{<>}$ instead $\bar g$.
4) We say $\mathscr{E}$ is nice to α if it is nice to the constant function α.

5.5 Claim. *1) If $\mathscr{E}$ is nice to f, $f \in F_c(\mathscr{T}, e, \mathscr{Y}), g \in F_c(\mathscr{T}, e, \mathscr{Y})$, $g \leq f$ then $\mathscr{E}$ is nice to f.*
2) The games from 5.4 are determined, and the winning side has winning strategy which does not need memory.
3) The second player wins $G_2^*(E, \bar{g})$ iff for some $\bar{\gamma}$ second player wins $G_2^{\bar{\gamma}}(E, g)$.
4) If the second player wins $G_2^{\gamma}(E, f)$, $\bar{g} \in F_c(\mathscr{T}, e(E)) g_\eta \leq f$ for $\eta \in \mathrm{Dom}(\bar{g})$ then the second player wins in $G_2^{\bar{\gamma}}(E, \bar{g})$ when we let

$$\gamma_\eta = \gamma + \max\{(\ell g(\nu) - \ell g(\eta) + 1) : \nu \text{ satisfies } \eta \trianglelefteq \nu \in \mathrm{Dom}(\bar{g})\}.$$

5.6 Lemma. *Suppose $f_0 \in {}^{(\mathscr{Y}/e)}Ord, e \in$ Eq and $\lambda_0 =: \sup\{\prod_{x \in Y} \mathscr{Y}_e(f_0^{[e]}(x) + 1 : e$ satisfies $e_0 \leq e \in \mathbf{e}\}$.*
1) If there is a Ramsey cardinal $\geq \cup\{f(x) + 1 : x \in \mathrm{Dom}(f_0)\}$ then there is a μ^*-divisible $\mathscr{E}$ nice to f_0 having weak μ^*-sums.
2) If for every $A \subseteq \lambda_0$ there is in $\mathbf{K}[A_0]$ a Ramsey cardinal $> \lambda_0$, then there is a μ^*-divisible $\mathscr{E}$ which has weak μ^*-sums and is nice to f.
3) In part 2 if $\lambda_0 = 2^{<\mu_0}$ then there is a μ^*-divisible nice $\mathscr{E}$ which has weak μ^*-sums.

5.7 Remark. This enables us to pass from "$\mathrm{pp}_{\Gamma(\theta,\aleph_1)}$ large" to "$\mathrm{pp}_{\mathrm{normal}}$ is large".

Proof. 1) Define $f_1 \in {}^{(\aleph_1)}\mathrm{Ord}, f_1(i) = \sup\{f_0(y/e) : \iota(y) = i\}$, let λ be such that: $\lambda \to (\sup\{f_1(i))_2^{<\omega} : i < \aleph_1\}$ (or just $\emptyset \notin D_n^*$ - see below) let $\lambda_n = (\lambda^{\mu^*})^{+n}$,

$$I_n = \{s : s \subseteq \lambda_n, s \cap \omega_1 \text{ a countable ordinal}\}$$

$$J_n = \{s \in I_n : s \cap \lambda \text{ has order type } \geq f_0(s \cap \omega_1)\}.$$

Let D_n^* be the minimal fine normal filter on J_n.

Let for $n < \omega$ and $e \in$ Eq, $H_{n,e} = \{h : h$ a function from J_n into $\mathscr{Y}/e$ such that $\iota(h(s)) = s \cap \omega_1\}$.

Let $\mathbb{P}_n = \{p : p \subseteq J_n, p \neq \emptyset \bmod D_n^*\}, \mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$ and for $p \in \mathbb{P}$ let $n(p)$ be the unique n such that $p \in \mathbb{P}_n$.

Let $p \leq q$ (in $\mathbb{P}$) if $n(p) \leq n(q)$ and $\{s \cap \lambda_{n(p)} : s \in q\} \subseteq p$.
Now for every $e \in$ Eq, $n < \omega$, $p \in P_n$, $h \in H_{n,e}$ we let:

$$D_p^{n,e,h} = \{A \subseteq \mathscr{Y}/e : h^{-1}(A) \supseteq p \bmod D_{n(p)}^*\}$$

$$E^{n,e,h}_p = \{D^{n^1,e^1,h^1}_q : p \le q \in P, n^1 = n(q) \text{ and } (n^1,e^1,h^1) \ge (n,e,h)\}$$

where $(n^1,e^1,h^1) \ge (n,e,h)$ means: $n \le n^1 < \omega$, $e \le e^1 \in$ Eq, $h^1 \in H_{n^1,e^1}$ and for $s \in J_{(n^1)}$, $h^1(s)^{[e]} = h(s \cap \lambda_n)$ and we define $(p^1,n^1,e^1,h^1) \ge (p,n,e,h)$ similarly. Let

$$\mathscr{E}^{n,e,h}_p = \{E^{n^1,e^1,h^1}_q : p \le q \in P, n^1 = n(q), (n^1,e^1,h^1) \ge (n,e,h)\}.$$

Note: $(p^1,n^1,e^1,h^1) \ge (p,e,n,h)$ implies $D^{n^1,e^1,h^1}_{p^1} \supseteq D^{n,e,h}_p$, $E^{n^1,e^1,h^1}_{p^1} \subseteq E^{n,e,h}_p$ and $\mathscr{E}^{n^1,e^1,h^1}_{p^1} \subseteq \mathscr{E}^{n,e,h}_p$. Now any $\mathscr{E} = \mathscr{E}^{n,e,h}_p (p \in P)$ is as required.

A new point is "$\mathscr{E}$ is μ^*-divisible". So suppose $E \in \mathscr{E} = \mathscr{E}^{n,e,h}_p$ so $E = E^{n^1,e^1,h^1}_q$ for some $(q,n^1,e^1,h^1) \ge (p,n,e,h)$. Let Z be a set of cardinality $< \mu^*$, so $(\lambda_{n^1})^{|Z|} = \lambda_{n_1}$; let $\{h_\zeta : \zeta < \zeta^* = |\mathscr{Y}/e_1|^{|Z|} \le 2^\mu \le \lambda_{n^1}\}$ list all function h from $\mathscr{Y}/e_1$ to Z. Let $\langle S_\zeta : \zeta < |\mathscr{Y}/e_1|^{|Z|}\rangle$ list a sequence of pairwise disjoint stationary subsets of $\{\delta < \lambda_{n^1+1} : \mathrm{cf}(\delta) = \aleph_0\}$. Let $e_2 \in$ Eq be such that $e_1 \le e_2$ and for every $y \in \mathscr{Y}$, $\{z/e_2 : z e_1 y\} = \{x(y/e,t) : t \in Z\}$, we let q_2, $q \le q_2 \in P$ be: $q_2 = \{s \in J_{n^1+1} : s \cap \lambda_{n^1} \in q$ and $\sup s \in \bigcup\limits_\zeta S_\zeta\}$, lastly we define $h^2 : J_{n^1+1} \to \mathscr{Y}/e_1$ by: $h^2(s) = x(h^1(s \cap \lambda_{n^1}), h_\zeta(s \cap \lambda_{n^1}))$ if $s \in q_2$, $\sup s \in S_\zeta$ (for $s \in J_{n^1+1} \backslash q_2$ it does not matter). The proof that q_2, e_2, h^2 are as required is as in [RuSh 117] and more specifically [Sh 212]. As for proving "$\mathscr{E}^{n,e,h}_p$ has weak μ^*-sums" the point is that the family of fine normal filters on μ has μ^*-sum.
2) Similar to 3.14(and 3.11(5),(6)).
3) Similar to [Sh 386, 1.7]. $\square_{5.6}$

§6 Hypotheses: Weakening of GCH

We define some hypotheses; except the first we do not know now whether their negations are consistent with ZFC.

6.1 Definition. We define a series of hypothesis:

(A) $\mathrm{pp}(\lambda) = \lambda^+$ for every singular λ.
(B) If $\mathfrak{a}$ is a set of regular cardinals, $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$ <u>then</u> $|\mathrm{pcf}(\mathfrak{a})| \le |\mathfrak{a}|$.
(C) If $\mathfrak{a}$ is a set of regular cardinals, $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$ <u>then</u> $\mathrm{pcf}(\mathfrak{a})$ has no accumulation point which is inaccessible (i.e. λ inaccessible $\Rightarrow \sup(\lambda \cap \mathrm{pcf}(\mathfrak{a}) < \lambda)$.
(D) For every λ, $\{\mu < \lambda : \mu$ singular and $\mathrm{pp}(\mu) \ge \lambda\}$ is countable.
(E) For every λ, $\{\mu < \lambda : \mu$ singular and $\mathrm{cf}(\mu) = \aleph_0$ and $\mathrm{pp}(\mu) \ge \lambda\}$ is countable.
(F) For every λ, $\{\mu < \lambda : \mu$ singular of uncountable cofinality, $\mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu) \ge \lambda\}$ is finite.
$(D)_{\theta,\sigma,\kappa}$ For every λ, $\{\mu < \lambda : \mu > \mathrm{cf}(\mu) \in [\sigma, \theta)$ and $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu) \ge \lambda\}$ has cardinality $< \kappa$.
$(A)_\Gamma$ If $\mu > \mathrm{cf}(\mu)$ then $\mathrm{pp}_\Gamma(\mu) = \mu^+$ (or in the definition of $\mathrm{pp}_\Gamma(\mu)$ the supremum is on the empty set).
$(B)_\Gamma, (C)_\Gamma$ Similar versions (i.e. use pcf_Γ).

We concentrate on the parameter free case.

6.2 Claim. *: In 6.1, we have:*

(1) $(A) \Rightarrow (B) \Rightarrow (C)$
(2) $(A) \Rightarrow (D) \Rightarrow (E)$, $(A) \Rightarrow (F)$
(3) $(E) + (F) \Rightarrow (D) \Rightarrow (B)$. *[Last implication — by the localization theorem [Sh 371, §2]]*
(4) *if* $(\forall \mu)(\mu > \mathrm{cf}(\mu) = \aleph_0$ *the hypothesis (A) of 6.1 holds. [Why? By [Sh:g, xx].]*

6.3 Theorem. *Assume Hypothesis 6.1(A).*
1) For every $\lambda > \kappa$,

$$\mathrm{cov}(\lambda, \kappa^+, \kappa^+, 2) = \begin{cases} \lambda^+ & \text{if } \mathrm{cf}(\lambda) \le \kappa \\ \lambda & \text{if } \mathrm{cf}(\lambda) > \kappa. \end{cases}$$

2) For every $\lambda > \kappa = \mathrm{cf}(\kappa) > \aleph_0$, *there is a stationary* $S \subseteq [\lambda]^{\le\kappa}, |S| = \lambda^+$ *if* $\mathrm{cf}(\lambda) \le \kappa$ *and* $|S| = \lambda$ *if* $\mathrm{cf}(\lambda) > \kappa$.

3) For μ singular, there is a tree with $\mathrm{cf}(\mu)$ *levels each level of cardinality* $< \mu$*, and with* $\geq \mu^+(\mathrm{cf}(\mu))$*-branches.*
4) If $\kappa \leq \mathrm{cf}(\mu) < \mu \leq 2^\kappa$ *<u>then</u> there is an entangled linear order* $\mathscr{T}$ *of cardinality* μ^+*.*

Proof. 1) By [Sh 400, §1].
2) By part (1) and 2.6.
3, 4) By [Sh 355, §4].

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