ADVANCES IN CARDINAL ARITHMETIC

SH420

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§1 $I[\lambda]$ is quite large

[If $\text{cf}\kappa = \kappa, \kappa^+ < \text{cf}\lambda = \lambda$ then there is a stationary subset $S$ of $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ in $I[\lambda]$. Moreover, we can find $\bar{C} = \{C_\delta : \delta \in S\}$, $C_\delta$ a club of $\lambda$, $\text{otp}(C_\delta) = \kappa$, guessing clubs and for each $\alpha < \lambda$ we have: $\{C_\delta \cap \alpha : \alpha \in \text{nacc} C_\delta\}$ has cardinality $< \lambda$.]

§2 Measuring $\mathcal{I}_{<\kappa}(\lambda)$

[We prove that e.g. there is a stationary subset of $\mathcal{I}_{<\aleph_1}(\lambda)$ of cardinality $\text{cf}(\mathcal{I}_{<\aleph_1}(\lambda), \subseteq)$.]

§3 Nice filters revisited

[We prove the existence of nice filters when instead being normal filters on $\omega_1$ they are normal filters with larger domains, which can increase during a play. They can help us transfer situation on $\aleph_1$-complete filters to normal ones].

§4 Ranks

[We reconsider ranks and niceness of normal filters, such that we can pass say from $\text{pp}_{\Gamma(\aleph_1)}(\mu)$ (where $\text{cf}\mu = \aleph_1$) to $\text{pp}_{\text{normal}}(\mu)$.]

§5 More on ranks and higher objects

§6 Hypotheses

[We consider some weakenings of G.C.H. and their consequences. Most have not been proved independent of ZFC.]
§1 \( I[\lambda] \) is Quite Large and Guessing Clubs

On \( I[\lambda] \) see [Sh 108], [Sh 88a], [Sh 351, §4] (but this section is self-contained; see Definition 1.1 and Claim 1.3 below). We shall prove that for regular \( \kappa, \lambda \), such that \( \kappa^+ < \lambda \), there is a stationary \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) in \( I[\lambda] \). We then investigate “guessing clubs” in (ZFC).

1.1 Definition. For a regular uncountable cardinal \( \lambda \), \( I[\lambda] \) is the family of \( A \subseteq \lambda \) such that \( \{ \delta \in A : \delta = \text{cf}(\delta) \} \) is not stationary and for some \( \langle P_\alpha : \alpha < \lambda \rangle \) we have:

(a) \( P_\alpha \) is a family of \( < \lambda \) subsets of \( \alpha \)
(b) for every limit \( \alpha \in A \) of cofinality \( < \alpha \) there is \( x \subseteq \alpha \), \( \text{otp}(x) < \alpha = \sup(x) \) such that \( \zeta < \alpha \Rightarrow x \cap \zeta \in \{ P_\gamma : \gamma < \alpha \} \).

1.2 Observation. In Definition 1.1 we can weaken (b) to:

for some club \( E \) of \( x \) for every limit \( \alpha \in A \cap E \) of cofinality \( < \alpha \).

Proof. Just replace \( P_\alpha \) by \( \{ x \cap \alpha : x \in \cup \{ P_\beta : \beta \leq \text{Min}(E \setminus (\alpha + 1)) \} \} \).

We know (see [Sh 108], [Sh 88a] or below)

1.3 Claim. Let \( \lambda > \aleph_0 \) be regular.
1) \( A \in I[\lambda] \) if (note: by (c) below the set of inaccessibles in \( A \) is not stationary and) there is \( \langle C_\alpha : \alpha < \lambda \rangle \) such that:

(a) \( C_\alpha \) is a closed subset of \( \alpha \)
(b) if \( \alpha^* \in \text{nacc}(C_\alpha) \) then \( C_\alpha^* = C_\alpha \cap \alpha \) (nacc stands for “non-accumulation”)
(c) for some club \( E \) of \( \lambda \), for every \( \delta \in A \cap E \), we have: \( \text{cf}(\delta) < \delta \) and \( \delta = \sup(C_\delta) \), and \( \text{cf}(\delta) = \text{otp}(C_\delta) \)
(d) \( \text{nacc}(C_\alpha) \) is a set of successor ordinals.

2) \( I[\lambda] \) is a normal ideal.

Proof. 1) The “if” part:
Assume \( \langle C_\beta : \beta < \lambda \rangle \) satisfy (a), (b), (c) with a club \( E \) for (c). For each limit \( \alpha < \lambda \) choose a club \( e_\alpha \) of order type \( \text{cf}(\alpha) \). We define, for \( \alpha < \lambda \):

(420)
\[ \mathcal{P}_\alpha = \{ C_\beta : \beta \leq \alpha \} \cup \{ e_\beta : \beta \leq \alpha \} \cup \{ e_\gamma \cap \alpha : \gamma \leq \text{Min}(E \setminus (\alpha + 1)) \}. \]

It is easy to check that \( \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle \) exemplify “\( A \in I[\lambda] \)”.

The “only if” part:

Let \( \mathcal{P} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle \) exemplify “\( A \in I[\lambda] \)” (by Definition 1.1). Without loss of generality

\[(*)\] if \( C \in \mathcal{P}_\alpha \), and \( \zeta \in C \) then \( C \setminus \zeta \in \mathcal{P}_\alpha \) and \( C \cap \zeta \in \mathcal{P}_\alpha \)

For each limit \( \beta < \lambda \) let \( e_\beta \) be a club of \( \beta \) satisfying \( \text{otp}(e_\beta) = \text{cf}(\beta) \) and \( \text{cf}(\beta) < \beta \Rightarrow \text{cf}(\beta) < \text{min}(e_\beta) \). Let \( \langle \gamma_i : i < \lambda \rangle \) be strictly increasing continuous, each \( \gamma_i \) a non-successor ordinal \( < \lambda \), \( \gamma_0 = 0 \), and \( \gamma_{i+1} - \gamma_i \geq \aleph_0 + | \bigcup_{\alpha \leq \gamma_i} \mathcal{P}_\alpha | + | \gamma_i | \)

and \( \gamma_i \in A \Rightarrow \text{cf}(\gamma_i) < \gamma_i \).

(Why? Let \( E' \) be a club of \( \lambda \) such that \( \gamma \in E \cap A \Rightarrow \text{cf}(\gamma) < \gamma \), and then choose \( \gamma_i \in E \) by induction on \( i < \lambda \).)

Let \( F_i \) be a one to one function from \( \bigcup_{\alpha \leq \gamma_i} \mathcal{P}_\alpha \times \gamma_i \) into \( \{ \zeta + 1 : \gamma_i < \zeta + 1 < \gamma_{i+1} \} \).

Now we choose \( C_\alpha \subseteq \alpha \) as follows. First, for \( \emptyset = 0 \) let \( C_\alpha = \emptyset \). Second, assume \( \alpha \) is a successor ordinal, let \( i(\alpha) \) be such that \( \gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1} \). If \( \alpha \notin \text{Rang}(F_{i(\alpha)}) \), let \( C_\alpha = \emptyset \). If \( \alpha = F_{i(\alpha)}(x, \beta) \) hence necessarily \( x \in \bigcup_{\epsilon \leq \gamma_{i(\alpha)}} \mathcal{P}_\epsilon \) and \( x, \beta \) are unique. Let \( C_\alpha \) be the closure (in the order topology) of \( C_\alpha^- \), which is defined as:

\[ \{ F_j(x \cap \zeta, \beta) : \text{the sequence } (j, \zeta, \beta) \text{ satisfies } (*)_j^{x, \beta} \text{ below} \} \] where

\[ x, \beta \]

\[ (i) \] \( \text{otp}(x \cap \zeta) \in e_\beta \),

\[ (ii) \] \( j < i(\alpha) \) is minimal such that \( x \cap \zeta \in \bigcup_{\epsilon \leq \gamma_j} \mathcal{P}_\epsilon \)

\[ (iii) \] if \( \xi \in x \cap \zeta \), \( \text{otp}(x \cap \xi) \in e_\beta \) then

\[ (\exists j(1) < j)[x \cap \xi \in \bigcup_{\epsilon \leq \gamma_{j(1)}} \mathcal{P}_\epsilon] \]

\[ (iv) \] \( \beta < \text{Min}(x) \).
Third, for $\alpha < \lambda$ limit, choose $C_\alpha$: if possible, $nacc(C_\alpha)$ is a set of successor ordinals, $C_\alpha$ is a club of $\alpha$, $[\beta \in nacc(C_\alpha) \Rightarrow C_\beta = \beta \cap C_\alpha]$; if this is impossible, let $C_\delta = \emptyset$.

Lastly, let $C_0 = \emptyset$ and let $E = \{ \gamma_i : i \text{ is a limit ordinal } < \lambda \}$.

Now we can check the condition in 1.3(1).

Note that for $\alpha$ successor $C_\alpha^- = nacc(C_\alpha)$.

**Clause (a):** $C_\alpha$ a closed subset of $\alpha$.

If $\alpha = 0$ trivial as $C_\alpha = \emptyset$ and if $\alpha$ is a limit ordinal, this is immediate by the definition. So let $\alpha$ be a successor ordinal, hence, by the choice of $\langle \gamma_i : i < \lambda \rangle$ as an increasing continuous sequence of nonsuccessor ordinals with $\gamma_0 = 0$, clearly $\iota(\alpha)$ is well defined, $\gamma_i(\alpha) < \alpha < \gamma_i(\alpha)+1$. Now if $\alpha \notin \text{Rang}(F_{i(\alpha)})$ then $C_\alpha = \emptyset$ and we are done so for some $x, \beta$ we have $\alpha = F_{i(\alpha)}(x, \beta)$ hence necessarily $x \in \bigcup_{\epsilon \leq \gamma_i(\alpha)} \mathcal{P}_\epsilon$ and $\beta < \gamma_i(\alpha)$. By the definition of $C_\alpha$ (the closure in the order topology on $\alpha$, of the set of $C_\alpha^-$ i.e. the set of $F_j(x \cap \zeta, \beta)$ for the pair $(j, \zeta)$ satisfying $\mathbb{X}_{j,\zeta}^{x,\beta}$ it suffices to show $C_\alpha^- \subseteq \alpha$, i.e.

\[(*) \text{ if the pair } (j, \zeta) \text{ satisfies } \mathbb{X}_{j,\zeta}^{x,\beta} \text{ then } F_j(x \cap \zeta, \beta) < \alpha.\]

So assume $(j, \zeta)$ satisfies $\mathbb{X}_{j,\zeta}^{x,\beta}$ but by clause (iii) we know that $j < i(\alpha)$ and so $\text{Rang}(F_j) \subseteq \gamma_{j+1} \subseteq \gamma_i(\alpha) < \alpha$ as required.

**Clause (b):** If $\alpha^* \in nacc(C_\alpha)$ then $C_{\alpha^*} = C_{\alpha} \cap \alpha^*$.

If it is enough to show $C_{\alpha^*} = \alpha^* \cap C_{\alpha^-}$ and as $C_{\alpha^-} = nacc(C_\alpha)$, we have $\alpha^* \in C_{\alpha^-}$.

As $\alpha^* \in C_{\alpha^-}$ necessarily for some $\zeta, j$ satisfying $\mathbb{X}_{j,\zeta}^{x,\beta}$ we have $\alpha^* = F_j(x \cap \zeta, \beta)$. By the choice of $F_j$ necessarily $\alpha^*$ is a successor ordinal and $\gamma_j < \alpha^* < \gamma_{j+1}$.

Now any member $\alpha(1)$ of $\alpha^* \cap C_{\alpha^-}$ has the form $F_{j(1)}(x \cap \zeta(1), \beta)$ with $j(1), \zeta(1)$ satisfying $\mathbb{X}_{j,\zeta}^{x,\beta}$; clearly $\gamma_{j(1)} < \alpha(1) = F_{j(1)}(x \cap \zeta(1), \beta) < \gamma_{j(1)+1}$ and $\gamma_j < \alpha^* = F_j(x \cap \zeta, \beta) < \gamma_{j+1}$. But $\alpha(1) < \alpha^*$ (being in $\alpha^* \cap C_{\alpha^-}$) so necessarily $j(1) + 1 \leq j$.

So $j(1), \zeta(1)$ satisfy $(i) - (v)$ with $x$ replaced by $x \cap \zeta$, i.e., satisfy $\mathbb{X}_{j,\zeta}^{x,\beta}$; recall by $\alpha^* = F_j(x \cap \zeta, \beta)$, so $F_{j(x)}(x \cap \zeta(1), \beta) \in C_{\alpha^-}$. So $\alpha^* \cap C_{\alpha^-} \subseteq C_{\alpha^-}$; similarly $C_{\alpha^-} \subseteq \alpha^* \cap C_{\alpha^-}$, so we get the desired equality.

**Clause (c):** We shall show that $E = \{ \gamma_i : i \text{ is a limit ordinal } < \lambda \}$ is as required in closed (c).

Clearly $E$ is a club of $\lambda$. So assume that $\delta \in A \cap E$ we should prove: $\text{cf}(\delta) < \delta, \delta = \text{sup}(C_\delta), \text{cf}(\delta) = \text{otp}(C_\delta)$.

Now $\delta \in E \cap A \Rightarrow \delta > \text{cf}(\delta)$ holds as we assume $\gamma_i \in A \Rightarrow \text{cf}(\gamma_i) < \gamma_i$. As $\delta \in E$, by $E$’s definition for some limit ordinal $i(*)$ we have $\delta = \gamma_i(*)$. By the choice of $C_\delta$ it is enough to find a set $C$ closed unbounded in $\delta$ of order type $\text{cf}(\delta)$ such that $\alpha \in nacc(C) \Rightarrow \alpha$ successor & $C_\alpha = C \cap \alpha$. 

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By the choice of $\mathcal{P}$, for some $x \subseteq \delta$, $\text{otp}(x) < \delta = \sup(x)$ and \( \bigwedge_{\zeta < \delta} x \cap \zeta \in \bigcup_{\gamma < \delta} \mathcal{P}_\gamma \).

By (*) above also $\xi \in x$ \& $S \in x \setminus \xi \Rightarrow x \cap \zeta \setminus \xi \in \bigcup_{\gamma < \delta} \mathcal{P}_\gamma$ so without loss of generality $\text{otp}(x) < \text{Min}(x)$. Let $\beta = \text{otp}(x)$, so we know that $\beta$ is a limit ordinal, moreover $\text{cf}(\beta) = \text{cf}(\delta)$. Remember $e_\beta$ is a club of $\beta$ of order type $\text{cf}(\beta)$ which is $\text{cf}(\delta)$. Let

$$y =: \{ \zeta \in x : \text{otp}(x \cap \zeta) \in e_\beta \}.$$

Clearly $y$ is a subset of $x$ of order type $\text{otp}(e_\beta) = \text{cf}(\delta)$. Define $h : y \rightarrow i(*)$ by $h(\zeta) = \text{Min}\{ j : x \cap \zeta \in \bigcup \mathcal{P}_\varepsilon \}$, so by (*) we know that $h$ is non-decreasing, and by the choice of $x$, \( \bigwedge_{\zeta \in y} \gamma_{h(\zeta)} < \delta \), equivalently \( \bigwedge_{\zeta \in y} h(\zeta) < i(*) \).

Let $z = \{ \zeta \in y : \text{for every } \xi \in y \cap \zeta \text{ we have } h(\xi) < h(\zeta) \}$. Let $C^- = \{ F_{h(\zeta)}(x \cap \zeta, \beta) : \zeta \in z \}$; it satisfies: $C^- \subseteq \delta = \sup \delta$ and it is easy to check, as in the proof of clause (c) that $[\alpha \in C^- \Rightarrow C^- = C^- \cap \alpha]$. So by the choice of $C^-$ its closure in $\delta$ is as required.

Clause (d): $\text{nacc}(C_\alpha)$ is a set of successor ordinals.

Check.

Remark. 1) We could also strengthen (*) to make $z \cap \zeta \in \mathcal{P}_{h(\zeta)}$.

2) By Definition 1.1 we know that $I[\lambda]$ is an ideal; by 1.3(1) we know that $I[\lambda]$ includes the ideal of non-stationary subsets of $\lambda$. By the last phrase and Definition 1.1, clearly $I[\lambda]$ is normal.

\[ \square_{1.3} \]

1.4 Claim. If $\kappa, \lambda$ are regular, $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \}, S \subseteq I[\lambda], S$ stationary, $\kappa^+ < \lambda$ then we can find $\mathcal{P} = \{ \mathcal{P}_\alpha : \alpha < \lambda \}$ such that for $\delta(*) =: \kappa$ we have:

\( \oplus_{\mathcal{P}_S} \)

(i) $\mathcal{P}_\alpha$ is a family of closed subsets of $\alpha, | \mathcal{P}_\alpha | < \lambda$

(ii) $\text{otp}(C) \leq \delta(*)$ for $C \in \bigcup_{\alpha} \mathcal{P}_\alpha$

(iii) for some club $E$ of $\lambda$, we have:

$[\alpha \notin E \Rightarrow \mathcal{P}_\alpha = \emptyset]$ and

$[\alpha \in E \Rightarrow (\forall C \in \mathcal{P}_\alpha)(\text{otp}(C) \leq \delta(*))]

[\alpha \in E \setminus (S \cap \text{acc}(E)) \Rightarrow (\forall C \in \mathcal{P}_\alpha)[\text{otp}(C) < \delta(*)]]

[\alpha \in S \cap \text{acc}(E) \Rightarrow (\exists C \in \mathcal{P}_\alpha)(\text{otp}(C) = \delta(*))]

[\alpha \in S \cap \text{acc}(E) \& C \in \mathcal{P}_\alpha \& \text{otp}(C) = \delta(*) \Rightarrow \alpha = \sup(C)]$
(iv) \( C \in \mathcal{P}_\alpha \land \beta \in nacc(C) \Rightarrow \beta \cap C \in \mathcal{P}_\beta \)

(v) for any club \( E' \) of \( \lambda \) for some \( \delta \in S \cap E' \) and \( C \in \mathcal{P}_\delta \) we have \( C \subseteq E' \land otp(C) = \delta(\ast) \).

Proof. Let \( \langle C_\alpha : \alpha < \lambda \rangle \) witness “\( S \in I[\lambda] \)” be as in 1.3(1); without loss of generality \( otp(C_\alpha) \leq \delta(\ast) \). For any club \( E \), consisting of limit ordinals for simplicity, let us define \( \mathcal{P}_E^\alpha \) by induction on \( \alpha < \lambda \):

\[
\mathcal{P}_E^\alpha = \{ \alpha \cap g\ell(C_\beta, E) : \alpha \in E \text{ and } \alpha \leq \beta < \text{Min}(E \setminus (\alpha + 1)) \}
\cup \{ C \cup \{ \beta \} : \beta \in E \cap \alpha, C \in \mathcal{P}_E^\beta \text{ and } otp(C) < \delta(\ast) \}
\]

where

\[
g\ell(C_\beta, E) = \{ \sup(E \cap (\gamma + 1)) : \gamma \in C_\beta \text{ and } \gamma > \text{Min}(E) \}.
\]

Note that \( |\mathcal{P}_E^\alpha| \leq |\text{Min}(E \setminus (\alpha + 1))| < \lambda \).

We can prove that for some club \( E \) of \( \lambda \) the sequence \( \langle \mathcal{P}_E^\alpha : \alpha < \lambda \rangle \) is as required except possibly clause (v) which can be corrected gotten by a right of \( E \) (just by trying successively \( \kappa^+ \) clubs \( E_\zeta \) (for \( \zeta < \kappa^+ \)) decreasing with \( \zeta \), see [Sh 365]). Note that clause (iv) guaranteed by demanding \( E \) to consist of limit ordinals only and the second set in the union defining \( \mathcal{P}_E^\alpha \). \( \square_{1.4} \)

The following lemma gives sufficient condition for the existence of “quite large” stationary sets in \( I[\lambda] \) of almost any fixed cofinality.

1.5 Lemma. Suppose

(i) \( \lambda > \kappa > \aleph_0, \lambda \) and \( \kappa \) are regular

(ii) \( \mathcal{P} = \langle \mathcal{P}_\alpha : \alpha < \kappa \rangle, \mathcal{P}_\alpha \) a family of \( < \lambda \) closed subsets of \( \alpha \)

(iii) \( I_\mathcal{P} = \{ S \subseteq \kappa : \text{for some club } E \text{ of } \kappa \text{ for no } \delta \in S \cap E \text{ is there a club } C \text{ of } \delta, \text{ such that } C \subseteq E \text{ and } [\alpha \in nacc(C) \Rightarrow C \cap \alpha \subseteq \bigcup_{\beta < \alpha} \mathcal{P}_\beta] \} \text{ is a proper ideal on } \kappa. \)

Then there is \( S^* \in I[\lambda] \) such that for stationarily many \( \delta < \lambda \) of cofinality \( \kappa \), \( S^* \cap \delta \) is stationary in \( \delta \), moreover for some club \( E \) of \( \delta \) of order type \( \kappa \)

\[
\{ otp(\alpha \cap E) : \alpha \in E \setminus S^* \} \in I_\mathcal{P}.
\]
1.6 Remark. 1) The “for stationarily many” in the conclusion can be strengthened to: a set whose complement is in the ideal defined in [Sh 371, §2].

2) So if $\kappa^\sigma < \lambda$ then we can have $\{i < \kappa : \text{cf}(i) = \sigma\} \in I_\mathcal{P}$.

Proof. Let $\chi$ be regular large enough, $N^*$ be an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi)$ of cardinality $\lambda$ such that $(\lambda + 1) \subseteq N^*$, $\mathcal{P} \in N$. Let $\bar{C} = \langle C_i : i < \lambda \rangle$ list $N^* \cap \{A \subseteq \lambda : |A| < \kappa\}$ and let

$$S^* = \{\delta : \delta < \lambda : \text{cf}(\delta) < \kappa \text{ and for some } A \subseteq \delta \text{ satisfying } \delta = \text{sup}(A), \text{ we have } \text{otp}(A) < \kappa \text{ and } (\forall \alpha < \delta) [A \cap \alpha \in \{C_i : i < \delta\}]\}.$$

Clearly $S^* \in I[\lambda]$; so we should only find enough $\delta < \lambda$ of cofinality $\kappa$ as required in the conclusion of 1.5. So let $E^*$ be a club of $\lambda$ and we shall prove that such $\delta \in E^*$ exists. We can choose $M_\zeta$ by induction on $\zeta \leq \kappa$ such that:

(a) $M_\zeta \prec (\mathcal{H}(\chi), \in, <_\chi)$

(b) $\|M_\zeta\| < \lambda, M_\zeta \cap \lambda$ an ordinal

(c) $M_\zeta$ is increasing continuous

(d) $N, \kappa, \mathcal{P}, \bar{C}, E^*$ belongs to $M_0$

(e) $\langle M_\zeta : \epsilon \leq \zeta \rangle \in M_{\zeta + 1}$.

Let $\delta_\zeta = \sup(M_\zeta \cap \lambda)$, clearly $\delta_\zeta \in E^*$ for every $\zeta \leq \kappa$ and $\langle \delta_\zeta : \zeta \leq \kappa \rangle$ is a (strictly) increasing continuous, so $\delta =: \delta_\kappa$ has cofinality $\kappa$. Hence there is a (strictly) increasing continuous sequence $\langle \alpha_\zeta : \zeta < \kappa \rangle \in N^*$ with limit $\delta$, and clearly $E = \{\zeta < \kappa : \alpha_\zeta = \delta_\zeta \text{ and } \zeta \text{ is a limit ordinal}\}$ is a club of $\kappa$. We know that

$$T =: \{\zeta < \kappa : \zeta \in E \text{ and for some club } C \text{ of } \zeta, C \subseteq E \text{ and } \bigwedge_{\epsilon \ll \zeta} [C \cap \epsilon \subseteq \bigcup_{\xi \ll \zeta} \mathcal{P}_\xi]\}.$$

is stationary; moreover, $\kappa \setminus T \in I_\mathcal{P}$ (see assumption (iii)) and clearly $T \subseteq E$. Clearly it suffices to show

$$(\ast) \quad \zeta \in T \Rightarrow \delta_\zeta \in S^*.$$
Suppose \( \zeta \in T \), so there is \( C \), a club of \( \zeta \) such that \( C \subseteq E \) and \( \bigwedge [C \cap \epsilon \in \bigcup_{\xi \leq \zeta} P_{\xi}] \).

Let \( C^* = \{ \delta_\epsilon : \epsilon \in C \} \), so \( C^* \) is a club of \( \delta_\zeta \) of order type \( \leq \zeta < \kappa \) (which is \( < \delta_0 \leq \delta_\zeta \)). It suffices to show for \( \xi \in C \) that \( \{ \delta_\epsilon : \epsilon \in \xi \cap C \} \subseteq \{ C_i : i < \delta_\zeta \} \).

For this end we shall show

\[
\begin{align*}
(\alpha) & \ \{ \delta_\epsilon : \epsilon \in C \cap \xi \} \subseteq \{ C_i : i < \lambda \} \\
(\beta) & \ \{ \delta_\epsilon : \epsilon \in C \cap \xi \} \subseteq M_{\xi+1}.
\end{align*}
\]

This suffices as \( \langle C_i : i < \lambda \rangle \in M_0 < M_{\xi+1} \) and \( M_{\xi+1} \cap \{ C_i : i < \lambda \} = \{ C_i : i \in \lambda \cap M_{\xi+1} \} = \{ C_i : i < \delta_{\xi+1} \} \).

**Proof of** \( (\alpha) \). Remember \( \langle \alpha_\epsilon : \epsilon < \kappa \rangle \in N^* \). Also \( \tilde{\mathcal{P}} = \langle \mathcal{P}_\epsilon : \epsilon < \kappa \rangle \in N^* \) hence \( \bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon \subseteq N^* \) (as \( \kappa < \lambda, |\mathcal{P}_\epsilon| < \lambda, \lambda + 1 \subseteq N \), \( \tilde{\mathcal{P}} \in N^* \) so now for \( \xi \in C \) we have \( C \cap \xi \in \bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon \); hence \( C \cap \xi \in N^* \). Together \( \{ \alpha_\epsilon : \epsilon \in \xi \cap C \} \in N^* \); as \( \epsilon \in C \Rightarrow \epsilon \in E \Rightarrow \alpha_\epsilon = \delta_\epsilon \) (as \( C \subseteq E \) and the definition of \( E \)), and the definition of \( \langle C_i : i < \lambda \rangle \), we are done.

**Proof of** \( (\beta) \). We know \( \tilde{\mathcal{P}} \in M_0 \); as \( |\mathcal{P}_\epsilon| < \lambda, \kappa < \lambda \) clearly \( |\bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon| < \lambda \) so as \( M_\epsilon \cap \lambda \) is an ordinal, clearly \( \bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon \subseteq M_0 \). So for \( \epsilon < \zeta \) we have \( C \cap \epsilon \in \bigcup_{\gamma < \zeta} \mathcal{P}_\gamma \subseteq M_0 \subseteq M_{\xi+1} \). As \( \langle M_i : i \leq \xi \rangle \in M_{\xi+1} \) clearly \( \langle \delta_i : i \leq \xi \rangle \in M_{\xi+1} \) hence by the previous sentence also \( \langle \delta_i : i \in C \cap \xi \rangle \in M_{\xi+1} \), as required. \( \square_{1.5} \)

1.7 **Conclusion.** If \( \kappa, \lambda \) are regular, \( \kappa^+ < \lambda \) then there is a stationary \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) in \( I[\lambda] \).

**Proof.** If \( \lambda = \kappa^+ \) - use [Sh 351, 4.1]. So assume \( \lambda > \kappa^+ \). By [Sh 351, 4.1] the pair \( (\kappa, \kappa^+) \) satisfies the assumption of 1.4 for \( S = \{ \delta < \kappa^+ : \text{cf}(\delta) = \kappa \} \); (i.e. \( \kappa, \lambda \) there stands for \( \kappa, \kappa^+ \) here). Hence the conclusion of 1.4 holds for some \( \tilde{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa^+ \rangle \), \( |\mathcal{P}_\alpha| < \kappa^+ \). Now apply 1.5 with \( (\kappa^+, \lambda) \) here standing for \( (\kappa, \lambda) \) there (we have just proved \( I_{\tilde{\mathcal{P}}} \) is a proper ideal, so assumption (ii) holds). Note:

\[
\begin{align*}
(\ast) & \ \{ \delta < \kappa^+ : \text{cf}(\delta) = \kappa \} \notin I_{\tilde{\mathcal{P}}}.
\end{align*}
\]
Now the conclusion of 1.5 (see the moreover and choice of \( P \) i.e. (*) ) gives the desired conclusion.

\[ \square_{1.7} \]

1.8 Conclusion. If \( \lambda > \kappa \) are uncountable regular, \( \kappa^+ < \lambda \), then for some stationary \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) and some \( \bar{P} = \langle P_\alpha : \alpha < \lambda \rangle \) we have: \( \oplus P_\alpha^* \) from the conclusion of 1.4 holds.

**Proof.** As \( \kappa \) is regular apply 1.7 and then 1.4. \( \square_{1.8} \)

Now 1.8 was a statement I have long wanted to know, still sometimes we want to have "\( C_\delta \subseteq E, \text{otp}(C) = \delta(\ast) \)", \( \delta(\ast) \) not a regular cardinal. We shall deal with such problems.

1.9 Claim. Suppose

(i) \( \lambda > \kappa > \aleph_0 \), \( \lambda \) and \( \kappa \) are regular cardinals

(ii) \( \bar{P}_\ell = \langle \bar{P}_{\ell,\alpha} : \alpha < \kappa \rangle \) for \( \ell = 1, 2 \), where \( \bar{P}_{1,\alpha} \) is a family of \( < \lambda \) closed subsets of \( \alpha \), \( \bar{P}_{2,\alpha} \) is a family of \( \leq \lambda \) clubs of \( \alpha \) and \( [C \in \bar{P}_{2,\alpha} \& \beta \in C \Rightarrow C \cap \beta \in \bigcup_{\gamma < \alpha} \bar{P}_{1,\gamma}] \)

(iii) \( I_{\bar{P}_1,\bar{P}_2} =: \{ S \subseteq \kappa : \text{for some club } E \text{ of } \kappa \text{ for no } \delta \in S \cap E \text{ is there } C \in \bar{P}_{2,\alpha}, C \subseteq E \} \) is a proper ideal on \( \kappa \).

Then we can find \( \bar{P}_\ell^* = \langle \bar{P}_{\ell,\alpha}^* : \alpha < \lambda \rangle \) for \( \ell = 1, 2 \) such that:

(A) \( \bar{P}_{1,\alpha}^* \) is a family of \( < \lambda \) closed subsets of \( \alpha \)

(B) \( \beta \in \text{nacc}(C) \& C \in \bar{P}_{1,\alpha}^* \Rightarrow C \cap \beta \in \bar{P}_{1,\beta}^* \)

(C) \( \bar{P}_{2,\delta}^* \) is a family of \( \leq \lambda \) clubs of \( \delta \) (for \( \delta \) limit \( < \lambda \) such that) \( [\beta \in \text{nacc}(C) \& C \in \bar{P}_{2,\delta}^* \Rightarrow C \cap \beta \in \bar{P}_{1,\beta}^*] \)

(D) for every club \( E \) of \( \lambda \) for some strictly increasing continuous sequence \( \langle \delta_\zeta : \zeta \leq \kappa \rangle \) of ordinals \( < \lambda \) we have \( \{ \zeta < \kappa : \zeta \text{ limit, and for some } C \in \bar{P}_{2,\zeta} \text{ we have:} \}

\{ \delta_\epsilon : \epsilon \in C \} \subseteq \bar{P}_{2,\delta_\zeta}^* \) (hence \( [\xi \in \text{nacc}(C) \Rightarrow \{ \delta_\epsilon : \epsilon \in C \cap \xi \} \subseteq \bar{P}_{1,\delta_\xi}^*] \) \( \equiv \kappa \mod I_{\bar{P}_1,\bar{P}_2} \)

(E) we have \( e_\delta \) a club of \( \delta \) of order type \( \text{cf}(\delta) \) for any limit \( \delta < \lambda \); such that for any \( C \in \bigcup_{\alpha < \lambda} \bar{P}_{2,\alpha}^* \) for some \( \delta < \lambda, \text{cf}(\delta) = \kappa \) and \( C' \in \bigcup_{\beta < \kappa} \bar{P}_{2,\beta} \) we have

\( C = \{ \gamma \in e_\delta \cap \gamma \in C' \} \).
Proof. Same proof as 1.5. (Note that without loss of generality \( C \in \mathcal{P}_{1,\alpha} \) & \( \beta < \alpha < \kappa \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta} \)).

1.10 Conclusion. If \( \delta(*) \) is a limit ordinal and \( \lambda = \text{cf}(\lambda) > |\delta(*)|^+ \) then we can find \( \bar{\mathcal{P}}^*_\ell = \{ \mathcal{P}_{1,\alpha}^* : \alpha < \lambda \} \) for \( \ell = 1, 2 \) and stationary \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*)) \} \) such that:

\[
\begin{align*}
\oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*} \lambda, \delta(*) & \quad \mathcal{P}_{1,\alpha}^* \text{ is a family of } < \lambda \text{ closed subsets of } \alpha \text{ each of order type } < \delta(*) \hfill (A) \\
\beta \in \text{nacc}(C) \land C \in \mathcal{P}_{1,\alpha}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}^* & \quad (B) \\
\mathcal{P}_{2,\delta}^* \text{ is a family of } \leq \lambda \text{ clubs of } \delta \text{ (yes, maybe } = \lambda) \text{ of order type } \delta(*) \text{, and } \beta \in \text{nacc}(C) \land C \in \mathcal{P}_{2,\delta}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}^* & \quad (C) \\
\text{for every club } E \text{ of } \lambda \text{ for some } \delta \in E \cap S, \text{ cf}(\delta) = \text{cf}(\delta(*)) \text{ and there is } C \in \mathcal{P}_{2,\beta}^* \text{ such that } C \subseteq E. & \quad (D)
\end{align*}
\]

Proof. If \( \lambda = |\delta(*)|^++ \) (or any successor of regulars) use [Sh:e, ChIII,6.4](2) or [Sh 365, 2.14](2)((c)+(d)).

If \( \lambda > |\delta(*)|^++ \) let \( \kappa = |\delta(*)|^++ \) and let \( S_1 = \{ \delta < \kappa^+ : \text{cf}(\delta) = \text{cf}(\delta(*)) \} \); applying the previous sentence we get \( \bar{\mathcal{P}}^*_1, \bar{\mathcal{P}}^*_2 \) satisfying \( \oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*, S_1}^{\kappa^+, \delta(*)} \), hence satisfying the assumption of 1.9 so we can apply 1.9. \( \square_{1.10} \)

1.11 Definition. \( +\oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*, S}^{\lambda, \delta(*)} \) is defined as in 1.10 except that we replace \( (C) \) by

\( (C)^+ \mathcal{P}_{2,\delta}^* \) is a family of \( < \lambda \) clubs of \( \delta \) of order type \( \delta(*) \).

1.12 Remark. Note that if \( \mathcal{P}_\alpha = \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}, \ |\mathcal{P}_{2,\alpha}| \leq 1, \mathcal{P}_{1,\alpha} = \{ C \in \mathcal{P}_\alpha : \text{otp}(C) < \delta(*) \}, \mathcal{P}_{2,\alpha} = \{ C \in \mathcal{P}_\alpha : \text{otp}(C) = \delta(*) \} \) then \( +\oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*, S}^{\lambda, \delta(*)} \equiv +\oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*, S}^{\lambda, \delta(*)} \) mod.

1.13 Claim. Suppose \( \lambda = \text{cf}(\lambda) > |\delta(*)|^+, \delta(*) \text{ a limit ordinal, additively indecomposable (i.e. } \alpha < \delta(*) \Rightarrow \alpha + \alpha < \delta(*) \), \( +\oplus_{\mathcal{P}_1^*, \mathcal{P}_2^*, S}^{\lambda, \delta(*)} \) from 1.10 and

\( (*) \alpha \in S \Rightarrow |\mathcal{P}_{2,\alpha}| \leq |\alpha| \).
(Note: A non-stationary subset of $S$ does not count; e.g. for $\lambda$ successor cardinal the $\alpha$ with $|\alpha|^+ < \lambda$. Note: $+\oplus_{\mathcal{P}_1,\mathcal{P}_2}^{\lambda,\delta(\ast)}$ holds by (\ast) and if $\lambda$ is successor then $+\oplus_{\mathcal{P}_1,\mathcal{P}_2}^{\lambda,\delta(\ast)}$ suffice).

Then for some stationary $S_1 \subseteq S$ and $\mathcal{P} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ we have: $\mathcal{P}_\alpha \subseteq \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}$ and:

\begin{itemize}
  \item[(i)] $\mathcal{P}_\alpha$ is a family of closed subsets of $\alpha$, $|\mathcal{P}_\alpha| < \lambda$
  \item[(ii)] $\text{otp}(C) < \delta(\ast)$ if $C \in \mathcal{P}_\alpha, \alpha \notin S_1$
  \item[(iii)] if $\alpha \in S_1$ then: $\mathcal{P}_\alpha = \{C_\alpha\}, \text{otp}(C_\alpha) = \delta(\ast)$, $C_\alpha$ a club of $\alpha$ disjoint to $S_1$
  \item[(iv)] $C \in \mathcal{P}_\alpha$ & $\beta \in \text{acc}(C) \Rightarrow \beta \cap C \in \mathcal{P}_\beta$
  \item[(v)] for any club $E$ of $\lambda$ for some $\delta \in S_1$ we have $C_\delta \subseteq E$.
\end{itemize}

1.14 Remark. Note there are two points we gain: for $\alpha \in S_1$, $\mathcal{P}_\alpha$ is a singleton (similarly to 1.4 where we have $(\exists \leq 1^C \in \mathcal{P}_\delta)[\text{otp}(C) = \delta(\ast)])$, and an ordinal $\alpha$ cannot have a double role $- C_\alpha$ a guess (i.e. $\alpha \in S_1$) and $C_\alpha$ is a proper initial segment of such $C_\delta$. When $\delta(\ast)$ is a regular cardinal this is easier.

Proof. Let $\mathcal{P}_{2,\alpha} = \{C_{\alpha,i} : i < \alpha\}$ (such a list exists as we have assumed $|\mathcal{P}_{2,\alpha}| \leq |\alpha|$, we ignore the case $\mathcal{P}_{2,\alpha} = \emptyset$). Now

\begin{itemize}
  \item[$\ast_0$] for some $i < \lambda$ for every club $E$ of $\lambda$ for some $\delta \in S \cap E$ we have $C_{\delta,i} \setminus E$ is bounded in $\alpha$
    [Why? If not, for every $i < \lambda$ there is a club $E_i$ of $\lambda$ such that for no $\delta \in S \cap E$ is $C_{\delta,i} \setminus E$ bounded in $\alpha$. Let $E^* = \{j < \lambda : j \text{ a limit ordinal, } j \in \bigcap_{i<j} E_i\}$, it is a club of $\lambda$, hence for some $\delta \in S \cap E^*$ and $C \in \mathcal{P}_{2,\delta}$ we have $C \subseteq E^*$. So for some $i < \alpha, C = C_{\delta,i}$, so $C \subseteq E^* \subseteq E_i \cup i$ hence $C_{\delta,i} \setminus E_i$, contradicting the choice of $E_i$.]
  \item[$\ast_1$] for some $i < \lambda$ and $\gamma < \delta(\ast)$, letting $C_\delta = : C_{\delta,i} \setminus \{\xi \in C_{\delta,i} : \text{otp}(\xi \cap C_{\delta,i}) < \gamma\}$ we have: for every club $E$ of $\lambda$ for some $\delta \in S \cap E$ we have: $C_\delta \subseteq E$
    [Why? Let $i(\ast)$ be as in ($\ast_0$), and for each $\gamma < \delta(\ast)$ suppose $E_\gamma$ exemplify the failure of ($\ast_1$) for $i(\ast)$ and $\gamma$, now $\bigcap_{\gamma < \delta(\ast)} E_\gamma$ is a club of $\lambda$ exemplifying the failure of ($\ast_0$) for $i(\ast)$ contradiction. So for some $\gamma < \delta(\ast)$ we succeed.]
  \item[$\ast_2$] Without loss of generality $|\mathcal{P}_{2,\alpha}| \leq 1$, so let $\mathcal{P}_{2,\alpha} = \{C_\alpha\}$
    [Why? Let $i, \gamma$ and $C_\delta$ (for $\delta \in S$) be as in ($\ast_1$) and use $\mathcal{P}_{1,\alpha}' = \{C \setminus \{\xi \in C : \text{otp}(\xi \cap C) < \gamma\} : C \in \mathcal{P}_{1,\alpha}\}, \mathcal{P}_{2,i}' = \{C_\delta\}$]
(∗)$_3$ for some $h : \lambda \to |\delta(*)|^{+}$, for every $\alpha \in S$ we have $h(\alpha) \notin \{h(\beta) : \beta \in C_\alpha\}$

[Why? Choose $h(\alpha)$ by induction on $\alpha$.]

(∗)$_4$ for some $\beta < |\delta(*)|^{+}$ for every club $E$ of $\lambda$, for some $\delta \in S \cap h^{-1}(\{\beta\}), C_\delta \subseteq E$

[Why? If for each $\beta$ there is a counterexample $E_\beta$ then $\bigcap\{E_\beta : \beta < |\delta(*)|^{+}\}$ is a counterexample for (∗)$_2$.]

Now we have gotten the desired conclusion. \hfill □$_{1.13}$

1.15 Claim. If $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}, S \in I_\lambda, \kappa^{+} < \lambda = \text{cf}(\lambda)$, then for some stationary $S_1 \subseteq S$ and $\bar{\mathcal{P}}_1$ we have $\ast \otimes \lambda,\delta(*) \bar{\mathcal{P}}_1,S_1$.

Proof. Same proof as 1.4 (plus (∗)$_3$, (∗)$_4$ in the proof of 1.10). \hfill □$_{1.15}$

1.16 Claim. Assume $\lambda = \mu^{+}$, $|\delta(*)| < \mu$ and $\text{cf}(\delta(*)) \neq \text{cf}(\mu)$.

Then we can find stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*))\}$ and $\bar{\mathcal{P}}$ such that $\ast \otimes \lambda,\delta(*) \bar{\mathcal{P}}_1,S_1$.

Remark. This strengthens 1.10.


By [Sh:e, Ch.III,6.4](2), [Sh 365, 2.14](2)((c)+(d)).

Case β. μ singular.

Let $\theta =: \text{cf}(\mu), \sigma =: |\delta(*)|^{+} + \theta^{+}$ and $\mu = \sum_{\zeta < \theta} \mu_\zeta, \langle A_{\alpha,\zeta} : \zeta < \theta \rangle$ strictly increasing, $\mu_0 > \sigma$ and for each $\alpha < \lambda$ let $\alpha = \bigcup_{\zeta < \theta} A_{\alpha,\zeta}, \langle A_{\alpha,\zeta} : \zeta < \theta \rangle$ increasing, $|A_{\alpha,\zeta}| \leq \mu_\zeta$.

By 1.8 there is a sequence $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ and stationary $S_1 \subseteq \{\delta < \lambda : \text{cf}(\delta) = \sigma\}$ such that $\otimes_{\mathcal{P}_1,S_1}$ of 1.4 holds. Let $\bigcup \{\mathcal{P}_\alpha : \alpha < \lambda\} \cup \{\emptyset\}$ be $\{C_\alpha : \alpha < \lambda\}$ such that $C_\alpha \subseteq \alpha, [\alpha \in S_1 \Rightarrow C_\alpha \in \mathcal{P}_\alpha \ & \ \text{otp}(C_\alpha) = \sigma] \text{ and } [\alpha \notin S_1 \Rightarrow \text{otp}(C_\alpha) < \sigma]$. For some club $E_\beta^{\ast}$ of $\lambda, [\alpha \in E_\beta^{\ast} \Rightarrow \bigcup_{\beta < \alpha} \mathcal{P}_\beta = \{C_\beta : \beta < \alpha\}]$.

Looking again at $\otimes_{\mathcal{P}_1,S_1}$, we can assume $S_1 \subseteq E_\beta^{\ast} \ & \ (\forall \delta)[\delta \in S_1 \Rightarrow C_\delta \subseteq E_\beta^{\ast}]$, hence

\[(*) \ \delta \in S_1 \ & \ \alpha \in \text{nacc} \ C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \text{Min}(C_\delta \setminus (\alpha + 1))\}.
\]
So as we can replace every $C_\alpha$ by $\{ \beta \in C_\alpha : \otp(C_\alpha \cap \beta) \}$ is even, without loss of generality [because we can replace every $C_\alpha$ by $\{ \beta \in C_\alpha : \otp(\beta \cap C_\alpha) \}$ is even, without loss of generality (check)]

\[ (*)^+ \delta \in S_1 \& \alpha \in \nacc C_\delta \Rightarrow \alpha \cap C_\delta \in \{ C_\beta : \beta < \alpha \}. \]

Without loss of generality $[\beta \in A_\alpha, \zeta \Rightarrow C_\beta \subseteq A_\alpha, \zeta]$ (just note $|C_\beta| \leq \sigma < \mu_\zeta$) and $\alpha \in A_\beta, \zeta \Rightarrow A_\alpha, \zeta \subseteq A_\beta, \zeta$. For $\alpha \in S_1$ let $C_\alpha = \{ \beta, \alpha, \epsilon : \epsilon < \sigma \}$ be minimal such that $C_\alpha \cap \beta_\alpha, \epsilon + 1 = C_{\beta, \alpha, \epsilon}$ (exists as $\delta \in S_1 \Rightarrow C_\delta \subseteq E_1^* \}.$ Without loss of generality every $C_\alpha$ is an initial segment of some $C_\beta, \beta \in S_1$ (if not, we redefine it as $\emptyset$).

\[ (*)_1 \] there are $\gamma = \gamma(*) < \theta$ and stationary $S_2 \subseteq S_1$ such that for every club $E$ of $\lambda$, for some $\delta \in S_2$ we have: $C_\delta \subseteq E$, and for arbitrarily large $\epsilon < \sigma, \beta_{\delta, \epsilon}^* \in A_{\beta_{\delta, \epsilon}^*, \gamma}$.

[Why? If not, for every $\gamma < \theta$ there is a club $E_\gamma$ of $\lambda$ exemplifying the failure of $(*)_1$ for $\gamma$. Let $E = \bigcap_{\gamma < \theta} E_\gamma \cap E_1^*$, so $E$ is a club of $\lambda$, hence

\[ S' =: \{ \delta : \delta < \lambda, \delta \in S_1(\text{so } \text{cf}(\delta) = \sigma) \} \]

is a stationary subset of $\lambda$. For each $\delta \in S'$ and $\epsilon < \sigma$ for some $\gamma = \gamma(\delta, \epsilon) < \theta$ we have $\beta_{\delta, \epsilon}^* \in A_{\beta_{\delta, \epsilon}^*, \gamma}$, but as $\sigma = \text{cf}(\sigma) \neq \text{cf}(\theta) = \theta$ for some $\gamma(\delta)$,

\[ \{ \epsilon < \sigma : \epsilon \gamma(\delta, \epsilon) = \gamma(\delta) \} \]

is unbounded in $\sigma$. But $\delta \in E_\gamma(\delta)$, contradiction.]

\[ (*)_2 \] Without loss of generality: if $\beta \in \nacc(C_\alpha), \alpha < \lambda$ then $\exists \xi \in A_{\beta, \gamma(*)}\} \beta > \xi > \text{sup}(\beta \cap C_\alpha) \& \beta \cap C_\alpha = C_\xi]$. 

[Why? Define $C'_\alpha$ for $\alpha < \lambda:

\[ C'_\alpha = \{ \beta : \beta \in \nacc(C_\alpha) \} \exists \xi \in A_{\beta, \gamma(*)} \beta > \xi \geq \text{sup}(\beta \cap C_\alpha) \& \beta \cap C_\alpha = C_\xi] \}

C'_\alpha$ is: $\emptyset$ if $\alpha \in S_2$, $\alpha > \text{sup}(C'_\alpha) \& \alpha \cap \text{closure of } C'_\alpha \text{ otherwise.}$ Now $\langle C_\alpha : \alpha < \lambda \rangle$ can be replaced by $\langle C'_\alpha : \alpha < \lambda \rangle.$]

\[ (*)_3 \] For some $\gamma_1 = \gamma_1(*) < \theta$ for every club $E$ of $\lambda$ for some $\delta \in E : \text{cf}(\delta) = \text{cf}(\delta(\ast))$, and there is a club $e$ of $\delta$ satisfying: $e \subseteq E, \otp(e) = \delta(\ast)$, and for arbitrarily large $\beta \in \nacc(e)$ we have $e \cap \beta \in \{ C_\xi : \xi \in A_{\delta, \gamma_1} \}.$

[Why? If not, for each $\gamma_1 < \theta$ there is a club $E_{\gamma_1}$ of $\lambda$ for which there is no $\delta$ as required. Let $E =: \bigcap_{\gamma_1 < \theta} E_{\gamma_1}$, so $E$ is a club of $\lambda$ hence for some $\alpha \in \text{acc}(E) \cap S_2, C_\alpha \subseteq E$. Letting again $C_\alpha = \{ \beta, \alpha, \epsilon : \epsilon < \sigma \}$ (increasing), $C_\alpha \cap \beta_{\alpha, \epsilon} = C_{\delta, \beta_{\alpha, \epsilon}}$ where $\beta_{\delta, \epsilon} \in A_{\beta_{\delta, \epsilon}, \gamma(*)}$ clearly $\delta =: \beta_{\alpha, \delta(*)}, e = \{ \beta_{\delta, \epsilon} : \}$
\[ \epsilon < \delta(*) \] satisfies the requirements except the last. As \( \text{cf}(\delta(*)) \neq \text{cf}(\mu) \), for some \( \gamma_1(*) < \theta, \gamma_1(*) \geq \gamma(*) \) and \( \{ \epsilon < \delta(*): \beta_{\epsilon, \epsilon}^\ast \in A_{\beta_{\epsilon, \epsilon}, \gamma_1(*)} \} \) is unbounded in \( \delta(*) \). Clearly \( \delta =: \beta_{\alpha, \delta(*)}, e =: C_\alpha \cap \delta \) satisfies the requirement. Now this contradicts the choice of \( E_{\gamma_1(*)} \).

\((*)_4\) For some club \( E^a \) of \( \lambda \), for every club \( E^b \subseteq E^a \) of \( \lambda \), for some \( \delta \in E^b \) we have:

\(\begin{align*}
(\text{a}) & \quad \text{cf}(\delta) = \text{cf}(\delta(*)) \\
(\text{b}) & \quad \text{for some club } e \text{ of } \delta: e \subseteq E^b, \text{otp}(e) = \delta(*), \text{and for arbitrarily large } \\
& \quad \beta \in \text{nacc}(e) \text{ we have } e \cap \beta \in \{ C_\xi : \xi \in A_{\delta, \gamma_1(*)} \} \\
(\text{c}) & \quad \text{for every } \beta \in A_{\delta, \gamma_1(*)} \text{ we have: } C_\beta \subseteq E^a \Rightarrow C_\beta \subseteq E^b \text{ (we could have} \\
& \quad \text{demanded } C_\beta \cap E^a = C_\beta \cap E^b). \\
\end{align*}\)

[Why? If not we choose \( E_i \) for \( i < \mu^+_{\gamma_1(*)} \) by induction on \( i, [j < i \Rightarrow E_i \subseteq E_j] \), \( E_i \) a club of \( \lambda \), and \( E_{i+1} \) exemplify the failure of \( E_i \) as a candidate for \( E^a \). So \( \bigcap_i E_i \) is a club of \( \lambda \) hence by \((*)_3\) there are \( \delta \) and \( e \) as there. Now \( \{ \beta \in A_{\delta, \gamma_1(*)} : C_\beta \subseteq E_i \} : i < \mu^+_{\gamma_1(*)} \) is a decreasing sequence of subsets of \( A_{\delta, \gamma_1(*)} \) of length \( \mu^+_{\gamma_1(*)} \); and \( | A_{\delta, \gamma_1(*)} | \leq \mu_{\gamma_1(*)} \), hence it is eventually constant. So for every \( i \) large enough, \( \delta \) contradicts the choice of \( E_{i+1} \).]

\[ * * * \]

Let \( S = \{ \delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*)), \text{and there is a club } e = e_\delta \text{ of } \delta \text{ satisfying:} \]
\( e \subseteq E^a, \text{otp}(e) = \delta(*), \alpha \in \text{nacc}(e) \Rightarrow e \cap \alpha \in A_{\alpha, \gamma(*)} \) and for arbitrarily large \( \beta \in \text{nacc}(e) \) we have \( e \cap \beta \in \{ C_\xi : \xi \in A_{\delta, \gamma(*)} \} \} \).

So \( S \) is stationary, let for \( \delta \in S, C_\delta^* \) be an \( e \) as above. For \( \alpha < \lambda \) let \( \mathcal{P}_{1, \alpha} = \{ C_\beta : \beta \leq \alpha, \beta \in A_{\alpha, \gamma_2(*)} \} \)

\((*)_5(\text{a})\) for every club \( E \) of \( \lambda \), for some \( \delta \in S, C_\delta^* \subseteq E \)

\(\begin{align*}
(\text{b}) & \quad C_\delta^* \text{ is a club of } \delta, \text{otp}(C_\delta^*) = \delta(*) \\
(\text{c}) & \quad \text{if } \beta \in \text{nacc } C_\delta^* (\delta \in S) \text{ then } C_\delta^* \cap \beta \in \mathcal{P}_{1, \beta} \\
(\text{d}) & \quad | \mathcal{P}_{1, \beta} | \leq \mu_{\gamma(*)}, \mathcal{P}_{1, \beta} \text{ is a family of closed subsets of } \beta \text{ of order type } < \delta(*), \\
\end{align*}\)

[Why? This is what we have proved in \((*)_4\); noting that in \((*)_4\) in \( (b), (e) \) is not uniquely determined, but by \( (c) \) every “reasonable” candidate is O.K.]

Now repeating \((*)_3, (*)_4\) of the proof of 1.13, and we finish. \( \Box_{1.16} \)
1.17 Claim. 1) Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$, $\aleph_0 < \text{cf}(\delta(*)) = \text{cf}(\mu)(< \mu)$; then we can find stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*))\}$ and $\mathcal{P}$ such that $^\lambda \otimes \mathcal{P}_{\lambda}^S$, except when:

$$\oplus \text{for every regular } \sigma < \mu, \text{we can find } h : \sigma \rightarrow \text{cf}(\mu) \text{ such that for no } \delta, \epsilon \text{ do we have: if } \delta < \sigma, \text{cf}(\delta) = \text{cf}(\mu), \epsilon < \text{cf}(\mu) \text{ then } \{\alpha < \delta : h(\alpha) < \epsilon\} \text{ is not a stationary subset of } \delta.$$ 

2) In 1.16 and 1.17(1) we can have $\mu > \mu$. 
3) If 1.17(2) if $\mu$ is strong limit we can have $|\mathcal{P}_\alpha| \leq 1$ for each $\alpha$.

Remark. Compare with [Sh 186, §3].

Proof. Left to the reader (reread the proof of 1.16 and [Sh 186, §3].

1.18 Claim. 1) Let $\kappa$ be regular uncountable and we have global choice (or restrict ourselves to $\lambda < \lambda^*$). We can choose for each regular $\lambda > \kappa^+$, $\mathcal{P}_\lambda^\lambda = (\mathcal{P}_\alpha^\lambda : \alpha < \lambda)$ (assuming global choice) such that:

(a) for each $\lambda$, $\mathcal{P}_\alpha^\lambda$ is a family of $\leq \lambda$ of closed subsets of $\alpha$ of order type $< \kappa$.

(b) if $\chi$ is regular, $F$ is the function $\lambda \rightarrow \mathcal{P}_\lambda^\lambda$ (for $\lambda$ regular $< \chi$), $\aleph_0 < \kappa = \text{cf}(\kappa), \kappa^+ < \chi, x \in \mathcal{H}(\chi)$ then we can find $\bar{N} = \langle N_i : i \leq \kappa\rangle$, an increasing continuous chain of elementary submodels of $(\mathcal{H}(\chi), \in, <, \chi, F), \langle N_j : j \leq i\rangle \in N_{i+1}, \|N_i\| = \aleph_0 + |i|, x \in N_0$ such that:

$(*)$ if $\kappa^+ < \theta = \text{cf}(\theta) \in N_i$, then for some club $C$ of $\sup(N_\kappa \cap \theta)$ of order type $\kappa$; for any $j^1_1 < j < \kappa$ we have: $C \cap \sup(N_j \cap \theta) \in N_{j+1}, \text{otp}(C \cap \sup(N_j \cap \theta)) = j$.

2) We can above have $|\mathcal{P}_\alpha^\lambda| < \lambda$.

Proof. 1) Let $\langle C_\alpha : \alpha \in S\rangle$ be such that $S \subseteq \{\alpha \leq \kappa^+ : \text{cf}(\alpha) \leq \kappa\}$ is stationary, $\text{otp}(C_\alpha) \leq \kappa$, $[\beta \in C_\alpha \Rightarrow C_\beta = \beta \cap C_\alpha], C_\alpha$ a closed subset of $\alpha$, $[\alpha \text{ limit } \Rightarrow \alpha = \text{sup}(C_\alpha)], \{\alpha \in S : \text{cf}(\alpha) = \kappa\}$ stationary, and for every club $E$ of $\kappa^+$ there is $\delta \in S, \text{cf}(\delta) = \kappa, C_\delta \subseteq E$. For $i \in \kappa^+ \backslash S$ let $C_i = \emptyset$. Now for every regular $\lambda > \kappa^+$ and $\alpha \leq \lambda$, let $e_\alpha^\lambda \subseteq \alpha$ be a club of $\alpha$ of order type $\text{cf}(\alpha)$. For $\lambda$ as above and for $\alpha \leq \lambda$ limit let $\mathcal{P}_\alpha^\lambda = \{i \in e_\delta : i < \alpha, \text{otp}(e_\delta \cap i) \in C_\beta : \delta < \lambda \text{ has cofinality } \kappa^+, \text{and } \beta \in S\}$. Given $x \in H(\chi)$, we choose by induction on $i < \kappa^+, M_i, N_i$ such that:
$N_i \prec M_i \prec \langle \mathcal{H}(\chi), \in, \subset^*, F \rangle$

$\|M_i\| = |i| + \aleph_0$

$\|N_i\| = |C_i| + \aleph_0$

$M_i(i < \kappa^{++})$ is increasing continuous

$x \in M_0,$

$\langle M_j : j \leq i \rangle \in M_{i+1}$

$N_i$ is the Skolem Hull of $\{\langle N_j : j \in C_\zeta \rangle : \zeta \in C_i \}.$

We leave the checking to the reader.

2) We imitate the proof of 1.5. $\square_{1.18}$
§2 Measuring $[\lambda]^{<\kappa}$

We prove here that two natural ways to measure $\mathcal{I}_{<\kappa}(\lambda)$ for $\kappa$ regular uncountable, give the same cardinal: the minimal cardinality of a cofinal subset; i.e. its cofinality (i.e. $\text{cov}(\lambda, \kappa, \kappa, 2)$) and the minimal cardinality of a stationary subset. The theorem is really somewhat stronger: for appropriate normal ideal on $\mathcal{I}_{<\kappa}(\lambda)$, some member of the dual filter has the right cardinality.

The problem is natural and I did not trace its origin, but until recent years it seems (at least to me) it surely is independent, and find it gratifying we get a clean answer. I thank P. Matet and M. Gitik of reminding me of the problem.

We then find applications to $\Delta$-systems and largeness of $\check{\mathcal{I}}_{\lambda}$.

2.1 Definition. 1) Let $(\bar{C}, \bar{\mathcal{P}}, Z) \in \mathcal{P}^*[\theta, \kappa]$ when:

(i) $\aleph_0 < \kappa = \text{cf}(\kappa) < \theta = \text{cf}(\theta)$,
(ii) $S \subseteq \theta, S$ is stationary
(iii) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ (and we shall write $S = S(\bar{C})$), $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S \rangle$, $Z = \langle <\mathcal{P}_\delta : \delta \in S \rangle$
(iv) $C_\delta$ is an unbounded subset of $\delta$, (not necessarily closed)
(v) $\text{id}^a(\bar{C})$ is a proper ideal (i.e. for every club $E$ of $\theta$ for some $\delta \in S, C_\delta \subseteq E$)
(vi) $\bigwedge \delta \in S \text{otp}(C_\delta) < \kappa$, (hence $[\delta \in S \Rightarrow \text{cf}(\delta) < \kappa]$)
(vii) (a) $\mathcal{P}_\delta$ is a family of bounded subsets of $C_\delta$, directed by the partial order $<\mathcal{P}_\delta$ which is a partial order on $\mathcal{P}_* = \{x \cap \alpha : x \in \mathcal{P}_\delta \text{ for some } \delta \in S \text{ and } \alpha < \theta \}$ satisfying

\[ y < \mathcal{P}_\delta z \Rightarrow y \subseteq z, \text{ (but see parts (1A),(1B))} \]
(\beta) $\bigcup_{x \in \mathcal{P}_\delta} x = C_\delta$, and $|\mathcal{P}_\delta| < \kappa$
(viii) for some\(^1\) list $\langle b^*_i : i < \theta \rangle$ of $\bigcup_{\alpha \in S} \mathcal{P}_\alpha \cup \{\emptyset\}$ satisfying $b^*_i \subseteq i$ we have: for every $\alpha \in S$ we have $\mathcal{P}_\alpha \subseteq \{b^*_j : j < \alpha\}$
(ix) for $x \in \bigcup_{\delta \in S} \mathcal{P}_\delta$ we have the set $\mathcal{P}_x := \{y \in \bigcup_{\delta \in S} \mathcal{P}_\delta : y < \mathcal{P}_\delta x\}$ has cardinality $< \kappa$.

\(^1\)a sufficient condition is:
(viii)\(^+\) for every $\alpha < \theta$ the set $\mathcal{P}^*_\alpha := \{a \cap \alpha : \text{ for some } \delta \in S \text{ we have } \alpha < \delta \in S, a \in \mathcal{P}_\delta \text{ and } \alpha \in C_\delta\}$ has cardinality $< \theta$ or at least
1A) If each $\prec_{P_\delta}$ is inclusion we may omit it.

1B) If $\prec_*$ is a partial order of $\bigcup_{\delta \in S} P_\delta$ and $\delta \in S \Rightarrow \prec_{P_\delta} = \prec_* |_{P_\delta}$ then we may write $\prec_*$ instead of $Z$.

2) $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$, if $(\bar{C}, \bar{P}) \in \mathcal{T}^*[\theta, \kappa]$ where $\delta \in S(\bar{C}) \Rightarrow P_\delta = \{ C_\delta \cap \alpha : \alpha \in C_\delta \}$.

2.2 Claim. 1) If $\theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \sigma = \mathrm{cf}(\sigma)$, then there is $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$ such that:

$$\{ \delta \in S(\bar{C}) : \mathrm{cf}(\delta) = \sigma \} \neq \emptyset \mod id^a(\bar{C}).$$

2) If $S \subseteq \{ \delta < \theta : \mathrm{cf}(\delta) < \kappa \}$ is stationary, $\bar{C}$ an $S$-club system, $|C_\delta| < \kappa$, and $\mathrm{id}^a(\bar{C})$ a proper ideal, then $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$.

3) In (2) if in addition for each $\alpha < \theta$ we have $|\{ C_\delta \cap \alpha : \alpha \in C_\delta, \delta \in S \}| < \theta$ then $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$.

4) If $\theta$ is a successor of regular then in part (2) we can demand $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ and each $C_\delta$ closed.

5) If $\theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \sigma = \mathrm{cf}(\sigma)$, then there is $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ such that:

$$\{ \delta \in S(\bar{C}) : \mathrm{cf}(\delta) = \sigma \} \neq \emptyset \mod id^a(\bar{C}).$$

6) If $\theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa) > \sigma = \mathrm{cf}(\sigma)$ and $S \in \mathcal{I}[\theta]$ is stationary then there is $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ such that $S(\bar{C}) = S$.

Proof. 1) Let $S_0 \subseteq \{ \delta < \theta : \mathrm{cf}(\delta) = \sigma \}$ be stationary, $C_\delta^0$ a club of $\delta$ of order type $\sigma$ for every $\delta \in S_0$. By [Sh 365, §2], for some club $E$ of $\theta$ letting $S = S_0 \cap \mathrm{acc}(E)$ and letting, for $\delta \in S, C_\delta = g(\{ \alpha \cap E : \alpha \in C_\delta^0 \})$ we have $S \notin \mathrm{id}^a(\langle C_\delta : \delta \in S_0 \rangle)$, now use part (2).

2) Check.

3) Check.

4) By [Sh 351, §4], [Sh:e, Ch.IV.3.4](2) or [Sh 365, 2.14](2)((c)+(d)) but see [Sh:E12].

5) By 1.7 and 1.15 (so we use the non-accumulation points).

6) Similarly. □

2.2 Remember (see [Sh 52, §3]).
2.3 Definition. 1) $\mathcal{D}_\kappa$ is the filter generated by the family of clubs of $\kappa$.
2) $\mathcal{D}_{<\kappa}^*(\lambda)$ is the filter on $[\lambda]^{<\kappa}$ defined by:

\[ X \in \mathcal{D}_{<\kappa}^*(\lambda) \iff \text{there is a function } F \text{ with domain the set of sequences}\]

of length $<\kappa$ with elements from $[\lambda]^{<\kappa}$ and $F$ is into $[\lambda]^{<\kappa}$ such that: if $a_\zeta \in [\lambda]^{<\kappa}$ for $\zeta < \kappa$, is $\subseteq$-increasing continuous and for each $\zeta < \kappa$ we have $F((\ldots, a_\zeta, \ldots))_{\zeta \leq \zeta} \subseteq a_{\zeta + 1}$ then $\{\zeta < \kappa : a_\zeta \in X\} \in \mathcal{D}_\kappa$.

Similarly

2.4 Definition. For $\lambda \geq \theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0$, $(\bar{C}, \mathcal{P}) \in \mathcal{F}^*[\theta, \kappa]$ we define a filter $\mathcal{D}_{(C,\mathcal{P})}(\lambda)$ on $[\lambda]^{<\kappa}$; (letting, e.g. $\chi = \beth_{\omega+1}(\lambda)$):

\[ Y \in \mathcal{D}_{(C,\mathcal{P})}(\lambda) \forall Y \subseteq [\lambda]^{<\kappa} \text{ and for some } x \in \mathcal{H}(\chi), \text{ for every } \langle N_\alpha, N_\alpha^* : \alpha < \theta, a \in \bigcup \mathcal{P}_\delta \rangle \text{ satisfying } \otimes \text{ below, also there is } A \in \text{id}^a(\bar{C}) \text{ such that: } \delta \in S(\bar{C}) \setminus A \Rightarrow \]

\[ \bigcup_{\delta \in S} N_\alpha^* \cap \lambda \in Y \text{ where, letting } \mathcal{P} = \bigcup \{ \mathcal{P}_\delta : \delta \in S \}, \]

\[ \otimes(i) \ N_\alpha < (\mathcal{H}(\chi), \varepsilon, <^*_\chi) \]

\[ (ii) \ ||N_\alpha|| < \theta, \]

\[ (iii) \ \langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1} \]

\[ (iv) \ \langle N_\alpha : \alpha < \theta \rangle \text{ is increasing continuous} \]

\[ (v) \ N_\alpha^* < (\mathcal{H}(\chi), \varepsilon, <^*_\chi) \text{ for } a \in \bigcup \mathcal{P}_\delta \]

\[ (vi) \ ||N_\alpha^*|| < \kappa, N_\alpha^* \cap \kappa \text{ an initial segment of } \kappa \]

\[ (vii) \ b \subseteq a \text{ (both in } \bigcup_{\delta \in S} \mathcal{P}_\delta \text{ implies } N_\alpha^* \nsubseteq N_a^* \]

\[ (viii) \text{ if } \alpha \in a \in \bigcup_{\delta \in S} \mathcal{P}_\delta \text{ then } \langle N_\beta, N_\beta^* : \beta \leq \alpha, b \subseteq a, b \in \{b_i^* : i \leq \alpha \} \subseteq \mathcal{P} \text{ belongs to } N_a^* \]

\[ (ix) \ \langle N_\beta, N_\beta^* : \beta \leq \alpha, b \subseteq a \alpha + 1, b \in \{b_i^* : i \leq \alpha + 1 \} \subseteq \mathcal{P} \rangle \text{ belongs to } N_{\alpha+1} \]

\[ (x) \ a \nsubseteq N_a^* \text{ and } \alpha \in a \Rightarrow \alpha \cap a \nsubseteq N_a^* \]

\[ (xi) \ a \subseteq a \in \mathcal{P} \text{ implies } N_a^* \in N_{\alpha+1} \text{ (follows from (ix) by clause (viii) of Definition 2.1(1))} \]

\[ (xii) \ a \in \mathcal{P}_\delta \& \delta \in S \& \alpha < \theta \Rightarrow x \in N_a^* \& x \in N_\alpha. \]

Clearly
2.5 Claim. 1) If $\chi > \lambda^{<\kappa}$ then $H(\chi)$ can serve, and $x = (Y, \lambda, \bar{C}, \bar{P})$ is enough.  
2) $D(\bar{C}, \bar{P})(\lambda)$ is a (non-trivial) fine ($<\kappa$)-complete filter on $[\lambda]^{<\kappa}$ when $(\bar{C}, \bar{P}) \in \mathcal{F}^*[\theta, \kappa]$, $\lambda \geq \theta$, hence it extends $\mathcal{D}_{<\kappa}(\lambda)$. (Remember $\text{id}^a(\bar{C})$ is a proper ideal).

Proof. Should be clear. □

2.6 Theorem. Suppose $\lambda > \theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0$ and $\theta = \kappa^+$. Then the following four cardinals are equal for any $(\bar{C}, \bar{P}) \in \mathcal{F}^*[\theta, \kappa]$, recalling there are such $(\bar{C}, \bar{P})$ by 2.2:

$\mu(0) = \text{cf}([\lambda]^{<\kappa}, \subseteq)$

$\mu(1) = \text{cov}(\lambda, \kappa, \kappa, 2) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\kappa}, \text{and for every } a \subseteq \lambda, |a| < \kappa \text{ there is } b \in \mathcal{P} \text{ satisfying } a \subseteq b\}$

$\mu(2) = \text{Min}\{|S| : S \subseteq [\lambda]^{<\kappa} \text{ is stationary}\}$

$\mu(3) = \mu(\bar{C}, \bar{P}) = \text{Min}\{|Y| : Y \in \mathcal{D}(\bar{C}, \bar{P})(\lambda)\}$.

2.7 Remark. 0) We thank M. Shioya for asking for a correction of an inaccuracy in the proof in a meeting in the summer of 1999 in which we answer him; this and other minor changes are done here. I thank P. Komjath for helpful comments and S. Garti for help in proofreading.

1) It is well known that if $\lambda > 2^{<\kappa}$ then the equality holds as they are all equal to $\lambda^{<\kappa}$.

2) This is close to “strong covering”.

3) Note that only $\mu(3)$ has $(\bar{C}, \bar{P})$ in its definition, so actually $\mu(3)$ does not depend on $(\bar{C}, \bar{P})$, recalling that by Claim 2.2 we know that $\mathcal{F}^*[\theta, \kappa]$ is not empty.

4) $\mu(0), \mu(1)$ are equal trivially.

2.8 Remark. 0) We can concentrate on the case $(\bar{C}, \bar{P}) \in \mathcal{F}^1[\theta, \kappa]$ or $\mathcal{F}^0[\theta, \kappa]$. This somewhat simplifies and is enough.

1) We can weaken in Definition 2.1(1) demand (ix) as follows:

(ix)' there is a sequence $\langle a_i, \mathcal{P}^*_i : i < \lambda \rangle$ such that

(a) $|a_i| < \kappa$, $\mathcal{P}^*_i$ is a family of $<\kappa$ subsets of $a_i$

(b) for every $\delta \in S$ and $x \in \mathcal{P}_\delta$ for some $i < \delta$, $a_i = x$ and $(\forall b)[b \in \mathcal{P}_\delta \& b \subseteq a \Rightarrow b \in \mathcal{P}^*_i]$. 


In this case 2.6, 2.7(4) (and 2.5) remain true and we can strengthen 2.2.
2) We can even use $\mathcal{D}_\delta$ with another order (not $\subseteq$).

\textit{Proof.} Clearly $\lambda \leq \mu(0) = \mu(1) \leq \mu(2) \leq \mu(3)$ (the last — by 2.5(2)). So we shall finish by proving $\mu(3) \leq \mu(1)$, and let $\mathcal{D}$ exemplify $\mu(1) = \text{cov}(\lambda, \kappa, \kappa, 2)$. Let $S = S(C)$, etc.

Let $\chi$ be e.g. $\beth_3(\lambda)^+$ and let $M^*_\chi$ be the model with universe $\lambda + 1$ and all functions definable in $(\mathcal{H}(\chi), \in, <^*_\lambda, \lambda, \kappa, \mu(1))$. Let $M^*$ be an elementary submodel of $(\mathcal{H}(\chi), \in, <^*_\chi)$ of cardinality $\mu(1)$ such that $\mathcal{D} \in M^*$, $M^*_\chi \in M^*$, $(C, \mathcal{D}) \in M^*$ and $\mu(1) + 1 \subseteq M^*$ hence $\mathcal{D} \subseteq M^*$. It is enough to prove that $M^* \cap [\lambda]^{< \kappa}$ belongs to $\mathcal{D}(\bar{C}, \mathcal{P})(\lambda)$.

So let $N_i$ (for $i < \theta$), $N^*_x$ (for $x \in \bigcup_{\delta \in S} \mathcal{D}_\delta$) be such that: they satisfy $\otimes$ of Definition 2.4 for $x := (M^*_\chi, M^*, \mathcal{D}, \lambda, \kappa, (C, \mathcal{D}))$ so it belongs to every $N_\alpha$, $N^*_\chi$. It is enough to prove that $\{\delta \in S : [\lambda]^{< \kappa} \cap \bigcup_{x \in \mathcal{D}_\delta} N^*_x \in M^*\} = \theta$ mod $\otimes$ of $(\bar{C})$. For $i \in S$ clearly $x \subseteq y$ (or $x < \mathcal{D}_i$, $y$) $\Rightarrow N^*_x < N^*_y$ and $\mathcal{D}_i$ is directed (by the partial order $\subseteq$ or $<_{\mathcal{D}_i}$, recalling clause (vii) of $\otimes$ of Definition 2.4) hence $N^*_i := \bigcup\{N^*_x : x \in \mathcal{D}_i\}$ is $< (\mathcal{H}(\chi), \in, <^*_\chi)$ and even $< N^*_{i+1}$ and $N^*_i$ has cardinality $< \kappa$ (as $|\mathcal{D}_i| < \kappa$ and each $N^*_x$ has cardinality $< \kappa$ and $\kappa$ is regular) and we have to show that $\{i \in S : [\lambda]^{< \kappa} \cap N^*_i \in M^*\} = \theta$ mod $\otimes(\bar{C})$.

For each $i \in S$ by the choice of $\mathcal{D}$, there is a set $a_i$ such that $N^*_i \cap \lambda = (\bigcup_{y \in \mathcal{D}_i} N^*_y) \cap \lambda \subseteq a_i \in \mathcal{D}$; so as $\mathcal{D}$ and $\{N^*_y : y \in \mathcal{D}_i\}$ belong to $N_{i+1}$, see clause (ix) of Definition 2.4 without loss of generality $a_i \in N_{i+1}$. Let $a_i := \text{Reg} \cap a_i \cap \lambda^+ \setminus \theta^+$, so $a_i$ is a set of $< \kappa$ regular cardinals $\geq \theta^+$ and $a_i \in N_{i+1}$ too, so there is a generating sequence $(b_i[a_i]) : \lambda \in \text{pcf}(a_i))$ as in [Sh:g, VII,2.6] = [Sh 371, 2.6], without loss of generality it is definable from $a_i$ in $(\mathcal{H}(\chi), \in, <^*_\chi)$ say the $<^*_\chi$-first such object). Also $a_i \in \mathcal{D}_i \subseteq M^*$ and $\text{Reg}, \lambda^+, \theta^+ \in M^*$ so $a_i \in M^*$. As $a_i \in N_{i+1}$ we have $(b_i[a_i]) : \lambda \in \text{pcf}(a_i)) \in N_{i+1} \cap M^*$, and also there is $(f_{\alpha, \delta}^\alpha : \alpha < \theta, \delta \in \text{pcf}(a_i))$ as in [Sh:g, VIII,1.2] = [Sh 371, 1.2], and again without loss of generality it belongs to $N_{i+1} \cap M^*$. As max $\text{pcf}(a_i) \leq \text{cov}(\lambda, \kappa, \kappa, 2) = \mu(1)$ (first inequality by [Sh:g, II,5.4] = [Sh 355, 5.4]) clearly each $f_{\alpha, \delta}^\alpha \in M^*$.

Let $\otimes h$ be the function with domain $a := \bigcup_{i < \theta} a_i$ defined by $h(\sigma) = \sup(\sigma \cap \bigcup_{i < \theta} N_i)$.

So by [Sh:g, VIII,2.3](1) = [Sh 371, 2.3](1)
\( \odot_2 \) if \( i \in S \) then \( h \upharpoonright a_i \) has the form \( \text{Max}\{f^{a_i}_{\partial_\ell, \alpha_\ell} : \ell < n\} \) for some \( n < \omega, \partial_\ell \in \text{pcf}(a_\ell) \) and \( \alpha_\ell < \partial_\ell \) for \( \ell < n \)

\( \odot_3 \) if \( i \in S \) then \( h \upharpoonright a_i \) belongs to \( M^* \)

and obviously (as \( \sigma \in a_i \) \& \( i < j_1 < j_2 \) \( \Rightarrow \) \( \sup(\sigma \cap N_{j_1}) < \sup(\sigma \cap N_{j_2}) \))

\( \odot_4 \) \( \sigma \in \text{Dom}(h) \) \( \Rightarrow \) \( \text{cf}(h(\sigma)) = \theta. \)

Let \( e \) be a definable function in \( (\mathscr{H}(\chi), \in, <^*, \lambda, \kappa) \) with \( \text{Dom}(e) = \lambda + 1 \) such that \( e(\alpha) = e_\alpha \) is a club of \( \alpha \) of order type \( \text{cf}(\alpha) \), enumerated as \( \langle e_\alpha(\zeta) : \zeta < \text{cf}(\alpha) \rangle. \)

Now for each \( \sigma \in \bigcup_{i < \theta} a_i \) let

\( \odot_5 \) \( E_\sigma = \{ i < \theta : (\forall \zeta < \theta)[e_{h(\sigma)}(\zeta) \in N_i \Leftrightarrow \zeta < i], i \) is a limit ordinal and \( \sup(N_i \cap \sigma) = \sup\{e_{h(\sigma)}(\zeta) : \zeta < i\}\}. \)

Clearly \( E_\sigma \) is a club of \( \theta \), hence (on \( \langle b_j^* : j < \theta \rangle \), see clause (viii) of Definition 2.1)

\[ E = \{ \delta < \theta : \delta \text{ is a limit ordinal and } \sigma \in \bigcup\{a_i : i < \delta\} \subseteq \text{Reg} \cap \lambda^+ \setminus \theta^+ \Rightarrow \delta \in \text{acc}(E_\sigma) \text{ and } N_\delta \cap \theta = \delta \} \]

is a club of \( \theta \). For each \( \delta \in E \cap S \) such that \( C_\delta \subseteq E \), let \( \delta^* = \sup(\kappa \cap N^*_y) = \sup(\kappa \cap \bigcup_{y \in \mathcal{P}_\delta} N^*_y) \) so \( \delta^* < \kappa \), and we define by induction on \( n \in \omega \) models \( M_{y, \delta, n} \) for every \( y \in \mathcal{P}_\delta \).

First, \( M_{y, \delta, 0} \) is the Skolem Hull in \( M^*_\lambda \) of \( \{ i : i \in y \} \cup (N'_y \cap \kappa) \).

Second, \( M_{y, \delta, n+1} \) is the Skolem Hull in \( M^*_\lambda \) of \( M_{y, \delta, n} \cup \{ e_{h(\sigma)}(\zeta) : \sigma \in (\text{Reg} \cap \lambda^+ \setminus \theta^+) \cap M_{y, \delta, n} \text{ and } \zeta \in y \} \).

Now we note

\[ (*_0) \text{ if } y \in \{ b_i^* : i < \zeta \}, \zeta \in C_\delta \text{ and } \delta \in E \text{ then } N^*_y \subseteq N_\zeta \text{ hence } N^*_y \prec N_\zeta. \]

[Why? By clause (ix) of \( \odot \) of Definition 2.4 we have \( N^*_y \subseteq N_\zeta \) so \( \|N^*_y\| \in N_j \); as \( \|N^*_y\| < \kappa < \theta \) and \( N_\zeta \cap \theta \in \theta \) as \( \zeta \in C_\delta \subseteq E \) we have \( N^*_y \subseteq N_\zeta \) hence \( N^*_y \prec N_\zeta \).]

\[ (*_1) \text{ if } \zeta \in E(\subseteq \theta) \text{ and } \sigma \in \text{ Reg} \cap N_\zeta \cap \lambda^+ \setminus \theta^+ \text{ then } e_{h(\sigma)}(\zeta) = \sup(N_\zeta \cap \sigma). \]

[Why? By the choice of \( E \).]

\[ (*_2) \text{ assume } \delta \in S \text{ satisfies } \delta \in E, \text{ moreover } C_\delta \subseteq E; \text{ if } y \in \mathcal{P}_\delta \text{ and } \sigma \in N^*_y \cap \text{ Reg } \lambda^+ \setminus \theta^+ \text{ then } (h(\sigma) \text{ has cofinality } \theta, \text{ the sequence } \langle e_{h(\sigma)}(\zeta) : \zeta < \theta \rangle \text{ is increasing with limit } h(\sigma) \text{ and):} \]

\[ (i) \text{ if } y \in \{ b_i^* : i < \zeta \} \text{ and } \zeta \in C_\delta \text{ then } \sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta) \]
(ii) if $y \in \{b^*_i : i < \zeta\}, \zeta \in z \in \mathcal{P}_\delta$ and $y < \mathcal{P}_\delta z$ then $y \in N^*_y, N^*_y \leq N^*_z$ and $e_{h(\sigma)}(\zeta) \in N^*_z$

(iii) \{e_{h(\sigma)}(\zeta) : \zeta \in C_\delta\} is a subset of $N'_\delta = \bigcup_{z \in \mathcal{P}_\delta} N^*_z$

(iv) the set above is an unbounded subset of $N^*_\delta \cap \sigma$.

[Why? Clause (i): So we assume $\zeta \in C_\delta$ and $y \in \{b^*_i : i < \zeta\}$.

By $(*)_0$ (and recall that $\delta \in E$) we have $N^*_y < N^*_\zeta$. By the definition of $E_\sigma$ as $\sigma \notin N^*_y \wedge \zeta \in E$ clearly $\zeta \in E_\sigma$ hence $\sup(N^*_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$ by $(*)_1$.

Clause (ii): So assume $y \in \{b^*_i : i < \zeta\}, \zeta \in z$ and $y < \mathcal{P}_\delta z$ (so $y, z \in \mathcal{P}_\delta$) hence

$\mathcal{P}_{z, \zeta} = \{x \in \bigcup_{\alpha \in S} \mathcal{P}_\alpha : x \subseteq z \cap \zeta\}$ has cardinality $< \kappa$ and $z \cap \zeta \in N^*_z$ by clause (x) of 2.4, so $\mathcal{P}_{z, \zeta} = \{x \in \bigcup_{\alpha \in S} \mathcal{P}_\alpha : x \subseteq z \cap \zeta\} \subseteq N^*_z$, so (as $N^*_y \cap \zeta \in \kappa$, $|\mathcal{P}_{z, \zeta}| < \kappa$) clearly $\mathcal{P}_{z, \zeta} \subseteq N^*_z$ hence $y \in N^*_z$. By clause (viii) of $\otimes$ of Definition 2.4 it follows that $N^*_y \leq N^*_z$. But $||N^*_y|| < \kappa \wedge N^*_y \cap \zeta \in \kappa$ hence $N^*_y \leq N^*_z$ so $N^*_z < N^*_y$. But $\sigma \in N^*_y$ hence $\sigma \notin N^*_y$. Also $N^*_y \in N^*_y$ as $\zeta \in z \subseteq N^*_y$ recalling (viii) of 2.4 hence $e_{h(\sigma)}(\zeta) = \sup(N^*_\zeta \cap \sigma) \in N^*_z$ recalling $(*)_1$ so we have shown all clauses of (ii).

Clause (iii): So let $\zeta \in C_\delta$; by clause (vii)(\beta) of Definition 2.1 we know that $C_\delta = \{y : y \in \mathcal{P}_\delta\}$ hence for some $y_1 \in \mathcal{P}_\delta$ we have $\zeta \notin y_1$. By clause (x) of $\otimes$ from Definition 2.4 we have $y_1 \subseteq N^*_y$ hence $\zeta \notin N^*_y$. Also we are assuming in $(*)_2$ that $\sigma \in N^*_y, y \in \mathcal{P}_\delta$, so recalling $\mathcal{P}_\delta$ is directed, we can find $y_2 \in \mathcal{P}_\delta$ which is a common $\subseteq$-upper bound of $y, y_1$ hence $N^*_y < N^*_y, N^*_y < N^*_y$ hence $\sigma, \zeta \in N^*_y$.

By the choice of the function $e$ and the model $M^*_\lambda$, clearly $e(=,-)$ is a function of $M^*_\lambda$, but the object $x$ belongs to $N^*_y$ and by its choice this implies that $e \in N^*_y$. By clause (viii) of 2.4 recalling $\zeta \in N^*_y$ we know that $\zeta \notin N^*_y$ but $\sigma \in N^*_y$ hence $\sup(N^*_\zeta \cap \sigma) \in N^*_y$. But we are assuming in $(*)_2$ that $C_\delta \subseteq E$ and, see above, $\zeta \notin C_\delta$ so $\zeta \in E$ and $\zeta \in C_\delta \subseteq N^*_\zeta, \sigma \in N^*_y \subseteq N^*_y \subseteq N^*_\zeta$ so $\sup(N^*_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$ so by the previous sentence $e_{h(\sigma)}(\zeta) \in N^*_y$, hence $e_{h(\sigma)}(\zeta) \in \{N^*_y : x \in \mathcal{P}_\delta\} = N'_\delta$ as required.

Clause (iv): By clause (iii) it is $\subseteq N'_\delta$, and by the choice of the function $e$ it is $\subseteq \sigma$ hence it is $\subseteq N'_\delta \cap \sigma$. Now $N'_\delta = \{N^*_z : z \in \mathcal{P}_\delta\}$ and $z \in \mathcal{P}_\delta \Rightarrow N^*_z < N^*_z$ by $(*)_0$ hence $N'_\delta \subseteq N^*_\delta$. Now we know that $e_{h(\sigma)}(\zeta) > \delta$ is increasing with limit $e_{h(\sigma)}(\delta) = \sup(N^*_\zeta \cap \sigma)$ hence is unbounded in it and even $e_{h(\sigma)}(\zeta) : \zeta \in C_\delta$ is an unbounded subset of $e_{h(\sigma)}(\delta)$ and it is included in $N'_\delta$ as required.

So $(*)_2$ indeed holds.

Now (A), (B), (C), (D), (E) below clearly suffice to finish.
(A) (a) for $\delta \in S, y \in \mathcal{P}_\delta$ and $n < \omega$ we have $M_{y,\delta,n} \subseteq N'_\delta = \bigcup_{z \in \mathcal{P}_\delta} N^*_z$.

[Why? We prove this by induction on $n$. First assume $n = 0$, $M_{y,\delta,n}$ is the Skolem hull of $y \cup (N'_\delta \cap \kappa)$ in the model $M^*_\lambda$, well defined as $y \subseteq \lambda$ hence $y \subseteq M^*_\lambda$ and $N' \cap \kappa \subseteq \kappa \subseteq \lambda$. As $y \subseteq N^*_y \subseteq N'_\delta$ and $M^*_\lambda \subseteq N^*_y \subseteq N'_\delta$ clearly $M_{y,\delta,n} \subseteq N'_\delta$.]

Second, assume $n = m + 1$ and $M_{y,\delta,m} \subseteq N'_\delta$. Now $M_{y,\delta,n}$ in the Skolem hull of $M_{y,\delta,m} \cup \{e_{h(\sigma)}(\zeta) : \sigma \in M_{y,\delta,m} \cap \text{Reg} \cap (\lambda^+ \setminus \theta^+) \}$ and $\zeta \in y$, so it is enough to show that: if $\sigma \in M_{y,\delta,m}$ (hence $\sigma \in N'_\delta$) and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \theta^+$ and $\zeta \in y$ then $e_{h(\sigma)}(\zeta) \in N'_\delta$. But by (2)(iii) this holds.

(b) for $z \subseteq y$ in $\mathcal{P}_\delta$ we have $M_{z,\delta,n} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on $n < \omega$.]

(c) for $y \in \mathcal{P}_\delta$ and $m \leq n$ we have $M_{y,\delta,m} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on $n$.]

(d) $M'_\delta := \cup \{M_{y,\delta,n} : y \in \mathcal{P}_\delta \text{ and } n < \omega \}$ is $\prec N'_\delta$.

[Why? By the above.]

(e) if $\zeta \in z$ (hence $\zeta \in C_\delta \subseteq E$), $\{y, z\} \subseteq \mathcal{P}_\delta$, $\sup(y) < \zeta, y \prec \mathcal{P}_\delta z$ and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \theta^+$ then: $\sigma \in N^*_y \prec N^*_\zeta \Rightarrow e_{h(\sigma)}(\zeta) = \sup(\sigma \cap N^*_\zeta) \in N^*_z$.

[Why? By (2)(i) + (ii) this holds.]

(B) We can also prove that $(M_{y,\delta,n} : n < \omega, y \in \mathcal{P}_\delta)$ is definable in $(H(\chi), \in, <^*_\chi)$ from the parameters $\delta, M^*_\lambda, (\bar{C}, \bar{P})$ and $h \upharpoonright a_i$, all of them belong to $M^*_\lambda$, hence the sequence, and $M'_\delta = \cup \{M_{y,\delta,n} : n < \omega, y \in \mathcal{P}_\delta \}$, belong to $M^*_\lambda$.

(C) $M'_\delta \cap \text{Reg} \cap (\theta, \lambda^+)$ is a subset of $a_\delta$.

[Why? Use (A)(a) and definition of $a_i, a_i$].]

(D) if $\sigma \in M'_\delta$ and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \kappa$ then $\sigma \cap M'_\delta$ is unbounded in $\sigma \cap N'_\delta$.

[Why? When $\sigma > \theta$ use (2)(iii), (iv). For $\sigma = \theta$ we have $N'_\delta \cap \theta \subseteq N_\delta \cap \theta = \delta$ as $\delta \in E$ and $C_\delta \subseteq \delta = \sup(C_\delta)$ so it is enough to show $C_\delta \subseteq N'_\delta$, but $C_\delta$ is equal to $\bigcup_{y \in \mathcal{P}_\delta} y$. For $\sigma = \kappa$ see the choice of $M_{y,\delta,0}$. So as $\theta = \kappa^+$ we are done.]

(E) $M'_\delta \cap \lambda = N'_\delta \cap \lambda$. 


2.9 Definition. Assume \( \theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0, (\bar{C}, \mathcal{P}) \in \mathcal{F}^*[\theta, \kappa] \) and \( X \) is a set, of cardinality \( \geq \theta \) for simplicity and let \( \chi \) be large enough. We define a filter \( \mathcal{D}_{(\bar{C}, \mathcal{P})}[X] \) on \( [X]^{<\kappa} \) as the set of \( Y \subseteq [X]^{<\kappa} \) such that for some \( \mathbf{x} \in \mathcal{H}(\chi) \), for every sequence \( \langle N_\alpha, N_\alpha^* : \alpha < \theta, \alpha \in \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle \) satisfying \( \otimes \) below, there is \( A \in \text{id}^*(\bar{C}) \) such that \( \mathbf{x} \in \bigcup_{\mathcal{P}_\delta} N_\alpha^* \) and \( \delta \in S(\bar{C}) \setminus A \Rightarrow \bigcup_{\alpha \in \mathcal{P}_\delta} N_\alpha^* \cap [X]^{<\kappa} \in Y \) where

\[ \otimes \text{ as in Definition 2.4 omitting } \mathbf{x} \in N_\alpha. \]

2.10 Claim. Let \( (\bar{C}, \mathcal{P}) \in \mathcal{F}^*[\theta, \kappa] \).
1) Any \( \chi \) such that \( \mathcal{P}(X) \subseteq \mathcal{H}(\chi) \) can serve in Definition 2.9, and \( \mathbf{x} = Y \) can serve.
2) If \( X_1, X_2 \) are sets of cardinality \( \lambda \geq \chi \) and \( f \) is a one-to-one function from \( X_1 \) onto \( X_2 \), then \( f \) maps \( \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_1) \) onto \( \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_2) \).
3) If \( X_1 \subseteq X_2 \) has cardinality \( \geq \theta \) then \( Y \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_1) \Rightarrow \{ u \in [X_2]^{<\kappa} : u \cap X_1 \in Y \} \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_2) \) and \( Y \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_2) \Rightarrow \{ u \cap X_1 : u \in Y \} \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X_1) \).
4) For any set \( X \) of cardinality \( \geq \kappa \), really \( \mathcal{D}_{(\bar{C}, \mathcal{P})}(X) \) is a fine normal filter on \( X \), i.e.:

(a) fine: \( t \in X \Rightarrow \{ u \in [X]^{<\kappa} : t \in u \} \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X) \)

(b) normal: if \( Y_t \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X) \) for \( t \in X \) then \( Y \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(X) \), when \( Y := \Delta \{ Y_t : t \in X \} = \{ u \in [X]^{<\kappa} : u \neq \emptyset \text{ and } t \in u \Rightarrow u \in Y \} \).

Proof. 1),2) Easy.
3) The “fine” is trivial and for normal let \( \mathbf{x}_t \) be a witness for \( Y_t \in \mathcal{D}_{(\bar{C}, \mathcal{P})}[X] \) now \( \mathbf{x} = \langle \mathbf{x}_t : t \in X \rangle \) witness that \( Y \in \mathcal{D}_{(\bar{C}, \mathcal{P})}[X] \).

2.11 Claim. Let \( (\bar{C}, \mathcal{P}) \in \mathcal{F}^*[\theta, \kappa] \).
1) \( \mathcal{D}_{(\bar{C}, \mathcal{P})}(\lambda) \supseteq \mathcal{D}_{(\bar{C}, \mathcal{P})}[\lambda] \).
2) In 2.6 we can replace \( \mathcal{D}_{(\bar{C}, \mathcal{P})}(\lambda) \) by \( \mathcal{D}_{(\bar{C}, \mathcal{P})}[\lambda] \).
3) Assume that \( \text{cf}(\lambda) \geq \kappa \) and \( \beta < \alpha \Rightarrow \lambda > \text{cov}(|\beta|, \kappa, \kappa, 2) \). Then there is \( S \in \mathcal{D}_{(\bar{C}, \mathcal{P})}(\lambda) \) such that \( \alpha < S \Rightarrow \lambda > |\{ u \in S : u \subseteq \alpha \}|. \)
Proof. 1) Trivial.
2) Repeat the proof, the change is minor.
3) We can find \( \mathcal{Q} = \{ u_i : i < \lambda \} \subseteq [\lambda]^{<\kappa} \) which is cofinal such that \( (\forall \alpha < \lambda)(\exists \beta)(\alpha \leq \beta < \lambda \land \{ u_i : i < \beta, u_i \subseteq \alpha \}) \) is cofinal in \([\alpha]^{<\kappa}\).

2.12 Remark. In 2.6 we can replace \( \theta = \kappa^+ \) by \( \theta > \kappa \sigma > \sigma \) and \( \alpha < \theta \Rightarrow |\alpha|^{<\sigma} < \theta \) and \( \delta \in S(\bar{C}) \Rightarrow \text{cf}(\delta) = \sigma \).

Proof. Fill.

2.13 Conclusion. Suppose \( \lambda > \kappa > \aleph_0 \) are regular cardinals and \( (\forall \mu < \lambda)[\text{cov}(\mu, \kappa, \kappa, 2) < \lambda] \).

1) If for \( \alpha < \lambda \), \( a_\alpha \) is a subset of \( \lambda \) of cardinality \( < \kappa \) and \( S \in \mathcal{D}_{<\kappa}(\lambda) \) and \( T_1 \subseteq \{ \delta < \lambda : \text{cf}(\delta) \geq \kappa \} \) is stationary, then we can find a stationary \( T_2 \subseteq T_1, c \subseteq \lambda \) and \( \{ b_\delta : \delta \in T_2 \} \) such that:

\[
a_\delta \subseteq b_\delta \in S \text{ for } \delta \in T_2
\]

\[
b_\delta \cap \delta = c \text{ for } \delta \in T_2.
\]

2) If in addition \( (\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\kappa^+, \kappa] \) and \( S \in (\mathcal{D}(\bar{C}, \bar{\mathcal{P}})(\lambda))^+ \) then part (1) holds for this \( S \).

Remark. See on this and on 2.15 Rubin Shelah [RuSh 117, 4.12, pg.76] and [Sh 371, §6]. There we do not know that \( (\forall \mu < \lambda)[\text{cov}(\mu, \kappa, \kappa, 2) < \lambda] \) implies (as proved here) that

\( \exists \lambda, \kappa \) for each \( \alpha < \lambda \) we can find \( S_\alpha \) a stationary \( S_\alpha \subseteq [\alpha]^{<\lambda} \) of cardinality \( < \lambda \); moreover such that \( \{ \alpha \} \cup u : u \in S_\alpha, \alpha < \lambda \} \subseteq [\lambda]^{<\kappa} \) is stationary, (if \( \lambda \) is a successor cardinal, the moreover follows. So the assumption there seems just what was used now. So we could just quote.

Proof. 1) By part (2).

2) For each \( \alpha < \lambda \) let \( S_\alpha \in \mathcal{D}(\bar{C}, \bar{\mathcal{P}})[\alpha] \) be of cardinality \( \text{cov}(|\alpha|, \kappa, \kappa, 2) \).

Let \( S = \{ u \in [\lambda]^{<\kappa} : \text{if } \alpha \in u \setminus \kappa^+ \text{ then } u \cap \alpha \in S_\alpha \} \), so by 2.10 we know that \( S \in \mathcal{D}(\bar{C}, \bar{\mathcal{P}})[\lambda] \); and by 2.11(3) without loss of generality

\[
(\ast) \alpha < \lambda \Rightarrow \{ u \in S : u \subseteq \alpha \} \text{ has cardinality } < \lambda.
\]
Now for each \( \alpha < \lambda \) let \( b_\alpha \in S \) be such that \( a_\alpha \subseteq b_\alpha \), clearly exist and let \( h : T_1 \to \lambda \) be defined by \( h(\delta) = \text{sup}(b_\delta \cap \delta) \) so \( \delta \in T_1 \Rightarrow h(\delta) < \delta \) as \( \text{cf}(\delta) \geq \kappa > |b_\delta| \). So for some \( \gamma_* < \gamma \) the set \( T_2' := \{ \delta \in T_1 : h(\delta) = \gamma_* \} \) is stationary and by (*) for some \( c \) the set \( T_2 := \{ \delta \in T_2' : b_\delta \cap \delta = c \} \) is stationary. \( \square_{2.13} \)

2.14 Conclusion. If \( \lambda > \kappa > \aleph_0 \), \( \lambda \) and \( \kappa \) are regular cardinals and \( [\kappa < \mu < \lambda] \Rightarrow \text{cov}(\mu, \kappa, \kappa, 2) < \lambda \) then \( \{ \delta < \lambda : \text{cf}(\delta) < \kappa \} \in \mathcal{I}[\lambda] \).

Proof. Use \( \mu(3) \) of 2.6.

2.15 Claim. Let \( (*)_{\mu, \lambda, \kappa} \) mean: if \( a_i \in [\lambda]^{<\kappa} \) for \( i \in S \) and \( S \subseteq \{ \delta < \mu : \text{cf}(\delta) = \kappa \} \) is stationary, then for some \( b \in [\lambda]^{<\kappa} \) the set \( \{ i \in S : a_i \cap i \subseteq b \} \) is stationary. Let \( (*)_{\mu, \lambda, \kappa}^{-} \) be defined similarly but \( \{ i \in S : a_i \subseteq b \} \) only unbounded. Then for \( \aleph_0 < \kappa < \lambda < \mu \) regular we have:

\[
\text{cov}(\lambda, \kappa, \kappa, 2) < \mu \Rightarrow (**)_{\mu, \lambda, \kappa} \Rightarrow (**)_{\mu, \lambda, \kappa}'
\]

\[
\Rightarrow (\forall \lambda') [\kappa < \lambda' \leq \lambda \& \text{cf}(\lambda') < \kappa \Rightarrow \text{pp}_{<\kappa}(\lambda') < \mu].
\]

Remark. So it is conceivable that the \( \Rightarrow \) are \( \Leftrightarrow \). See [Sh 430, §3].

Proof. Straightforward. \( \square_{2.15} \)

Exercise: Generalize to the following filter.

Let \( \theta = \text{cf}(\theta) \geq \kappa = \text{cf}(\kappa) \) and \( S_* \subseteq [\theta]^{<\kappa} \) be stationary. For any set \( X \) of cardinality \( \geq \theta \) we define a filter \( \mathcal{D}_1^{S_*}[X] \) as follows: \( Y \in \mathcal{D}_1^{S_*}[X] \) if \( Y \subseteq [X]^{<\kappa} \) and for any \( \chi \) large enough there is \( x \in \mathcal{H}(\chi) \) such that if \( (N_\alpha, f_\alpha : \alpha \leq \theta) \) satisfy \( \otimes \) below, then for some \( S' \in \mathcal{D}_{<\kappa}(\theta) \) for every \( u \in S_* \cap S' \) we have:

- if \( x \in f''_\theta(u) \) then \( f''_\theta(u) \in Y \), when:
  - (a) \( N_\alpha < (\mathcal{H}(\chi), \in, < \chi) \)
  - (b) \( N_\alpha \) is \( \prec \)-increasing continuous
  - (c) \( \|N_\alpha\| < |\alpha| + \theta \)
  - (d) \( \langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1} \) if \( \alpha < \theta \)
  - (e) can add \( \langle \kappa, \theta, X, S_* \rangle \in N_0 \).
This generalizes [Sh 386] (and see there).
See [Sh 410, §5] on this generalization of normal filters.

3.1 Convention. 1) n is a niceness context; we use κ, FILL, etc., for κ_n, Fil_n = FIL(n) when dealing from the content.

3.2 Definition. We say the n is a niceness context or a κ-niceness context or a (κ, µ)-niceness context if it consists of the following objects satisfying the following conditions:

(a) κ is a regular uncountable cardinal
(b) I ⊆ ω>ω is non-empty ≺-downward closed with no ≺-maximal member\(^2\) default value is \(\{0_n : n < \omega\}\)
(c) let µ be > κ and \(\mathcal{Y} : i < \kappa\) is a sequence of pairwise disjoint sets and \(\mathcal{Y} \cup \{\mathcal{Y}_i : i < \omega_1\}\) so \(i < \omega_1 \Rightarrow |\mathcal{Y}|, |\mathcal{Y}_i|\)
(d) the function \(i\) with domain \(\mathcal{Y}\) is defined by \(i(y) = i\) when \(y \in \mathcal{Y}_i\)
(e) \(e\) is a set of equivalence relations \(e\) on \(\mathcal{Y}\) refining \(\bigcup_{i<\omega_1} \mathcal{Y}_i \times \mathcal{Y}_i\) with \(<\mu^*\) equivalence classes, each class of cardinality \(|\mathcal{Y}|\)
(f) for \(e \in e\), FIL(e) = FIL(e, n) is a set of \(D\) such that:
   (α) \(D\) is a filter on \(\mathcal{Y}/e\),
   (β) for any club \(C\) of \(\kappa\) we have \(\bigcup_{i \in C} \mathcal{Y}_i/e \in D\),
   (γ) normality: if \(X_i \in D\) for \(i < \omega_1\) then the following set belongs to \(D\):
      \(\{(\delta, j)/e : (\delta, j) \in \mathcal{Y}, \delta\ \text{limit and} \ i < \delta \Rightarrow (\delta, j) \in X_i\}\)
   (γ) \(\text{Suc} \in \{(D_1, D_2) : e(D_1) \leq e(D_2)\}\).

Remark. For \(e\) an important case is when it is a singleton \(\{\mathcal{Y}_i \times \mathcal{Y}_i : i < \kappa\}\), so we are dealing with normal filters on the old case.

\(^2\)For \(\mathcal{T}\) the two interesting cases are \(\mathcal{T} = \omega>\omega\) and \(\mathcal{T} = \{<>\}\) and \(\omega>\{0\}\). The default value will be \(\omega>\omega\).
3.3 Definition. Let $n$ be a $\kappa$-niceness context.
1) We say $e_1 \leq e_2$ if $e_2$ refines $e_1$. If not said otherwise, every $e$ is from $e$. Let $e_n$ be the set of all such equivalence relations with $< \mu$ equivalence classes. Let $i(x/e) = i(x).
2) FIL = FIL(n) is $\bigcup_{e \in e} FIL(e, n)$. For $D \in FIL$, let $e = e[D]$ be the unique $e \in e$ such that $D \in FIL(e, n).
3) For $D \in FIL(e)$ let $D^{[s]} = \{X \subseteq \mathcal{Y} : X^{[*]} \in D\}$; see (5) below.
4) For $D \in FIL$ and $e(1) \geq e(D)$, let $D^{[e(1)]} = \{X \subseteq \mathcal{Y} / e(1) : X^{[*]} \in D^{[s]}\}$, see (5) below.
5) For $A \subseteq \mathcal{Y} / e, A^{[*]} = \{(x/e) : (x/e) \in A\}$, and for $e(1) \geq e$ let $A^{[e(1)]} = \{y/e(1) : y/e \in A\}.

3.4 Definition. 1) For $D \in FIL(e, n)$, let $D^+$ be $\{Y \subseteq \mathcal{Y} / e : Y \neq \emptyset \mod D\}.
2) n is 1-closed if $D \in FIL(n), A \in D^+ \Rightarrow D + A \in FIL(n).
3) n is 0-closed if for every $D_1 \in FIL$ and $A \in D_1^+$ there is $D_2 \in FIL_2$ such that $(D_1 + A) \in (D_2) \subseteq D_2.
4) A$ niceness context $n$ is full if
   
   (a) for every $e \in e_n$, every filter on $\mathcal{Y}_n / e$ which is normal (with respect to the function $i_n$) belong to $FIL_n(e).

   4A) A niceness content $n$ is semi-full when: for every $e_1 \in e_n$ and $D_1 \in FIL_n(e_1)$ and $e_2, e_1 \leq e_2 \in e_n$ and $\mathcal{A} \subset \mathcal{P} (\mathcal{Y}_n / e_2)$ lift$(W) \in FIL(e_2)$ whenever

   $$(*)_{e_1, e_2, D_1, W} \begin{cases}
   (a) & e_1 \leq e_2 \text{ in } e_n \smallskip
   (b) & D_1 \in FIL_n(e_2) \smallskip
   (c) & \mu \geq 2^{(\mathcal{Y} / e_2)} \text{ (or more ???)} \smallskip
   (d) & W \subseteq [\mu]^{< \aleph_0} \text{ is stationary} \smallskip
   (e) & D_2 = \text{lift}(W, D_1^{[e_2]}) \text{ is normal (i.e. } \emptyset \in \text{lift}(W, D_1)).
   \end{cases}

5) A niceness context $n$ is thin when

   $$\text{Suc}_n = \{(D_1, D_2) : D_1 = D_2 \in FIL_n \text{ and } D_2 = D_1^{[e_1]} + A \text{ for some } A \in (D_1^{[e_1]})^+\}.$$

6) A niceness context $n$ is thick if: Suc$_n = \{(D_1, D_2) : D_1, D_2 \in FIL_n, e(D_1) \leq e(D_2) \text{ and } D_1^{[e_2]} \subseteq D_2 \text{ and if } \mu = 2^{[\mathcal{Y}_n / e_2]}, W_1 \subseteq [\mu]^{< \aleph_0} \text{ is stationary and lift}(W, D_1) = D_1 \text{ then for some stationary } W_2 \subseteq W_1 \text{ we have lift}(W_2, D_2) = D_2\).
Remark. 1) On lift see Definition 3.17, HERE??
2) We can use more freedom in the higher objects.

3.5 Claim. Assume

(a) the \( \kappa \)-niceness context is thick
(b) \( D_1 \in \text{FIL}_n(e_1) \)
(c) \( e_1 \leq e_2 \in e_d \)
(d) for each \( y \in \mathcal{Y}_n/e_1 \), \( \langle z_{y, \varepsilon} : \varepsilon < \varepsilon_y \rangle \) list \( \{ z/e_2 : z \in y_1 \} \), \( d_{y, \varepsilon} \) is a \( \kappa \)-complete filter on \( \varepsilon_y \)
(e) \( D_2 \in \text{FIL}(e_2) \)
(f) if \( A \in D_2 \) then \( \{ y \in \mathcal{Y}_n/e_1 : \varepsilon < \varepsilon_y : z_{y, \varepsilon} \in A \} \in d_{y, \varepsilon} \) belongs to \( D_1 \).

Then \( D_2 \in \text{Suc}_n(D_1) \).

Discussion: We may consider allowing player I, in the beginning of each move to choose \( W_n \) as above.

3.6 Definition. (0) For \( f : \mathcal{Y}/e \to X \) let \( f^{[e]} : \mathcal{Y} \to X \) be \( f^{[e]}(x) = f(x/e) \). We say \( f : \mathcal{Y} \to X \) is supported by \( e \) if it has the form \( g^{[e]} \) for some \( g : \mathcal{Y}/e \to X \).

If \( e_1, e_2 \in e \) and \( f_\ell : \mathcal{Y}/e_\ell \to X \) for \( \ell = 1, 2 \) then: we say \( f_1 = f_2^{[e_1]} \) if \( f_1^{[e]} = f_2^{[e]} \).

Writing \( f^{[e]} \) for \( f \in \omega^1 X \) we identify \( \{ i \}, i < \omega_1 \) with \( \mathcal{Y}_i \).

(1) Let \( F_c(\mathcal{F}, e) = F_c(\mathcal{F}, e, \mathcal{Y}) \) be the family of \( \bar{g} \), a sequence of the form \( \langle g_\eta : \eta \in u \rangle \), \( u \in F_c(\mathcal{F}) = \) the family of non-empty finite subsets of \( \omega^\omega \) closed under taking initial segments, and for each \( \eta \in u \) we have \( g_\eta \in \mathcal{Y} \) for each \( \eta \in u \) we have \( g_\eta \in \mathcal{Y} \) for each \( \eta \in u \) we have \( g_\eta \in \mathcal{Y} \) and for each \( \eta \in u \) we have \( g_\eta \in \mathcal{Y} \). Let \( \text{Dom}(\bar{g}) = u \), \( \text{Range}(\bar{g}) = \{ g_\eta : \eta \in u \} \). We let \( e = e(\bar{g}) \), for the minimal possible \( e \) assuming it exists and we shall say \( g_\eta <_D g_\nu \) instead \( g_\eta <_D g_\nu \) and not always distinguish between \( g \in \mathcal{Y}/e \) and \( g^{[e]} \) in an abuse of notation.

(2) We say \( \bar{g} \) is decreasing for \( D \) or \( D \)-decreasing (for \( D \in \text{FIL}(e, I) \)) if \( \eta < \nu \Rightarrow g_\eta <_D g_\nu \).

(3) If \( u = \{ < > \} \), \( g = g_{< >} \) we may write \( g \) instead \( \langle g_\eta : \eta \in u \rangle \).

3.7 Definition. 1) For \( e \in e, D \in \text{FIL}(e) \) and \( D \)-decreasing \( \bar{g} \in F_c(\mathcal{F}, e) \) we define a game \( \mathcal{O}^*(D, \bar{g}, e) = \mathcal{O}^*(D, \bar{g}, e, n) \). In the nth move (stipulating \( e_{-1} = e \), \( D_{-1} = D, \bar{g}_{-1} = \bar{g} \)):

- player I chooses \( \epsilon_n \geq \epsilon_{n-1} \) and \( A_n \subseteq \mathcal{Y}/e_n \), \( A_n \neq \emptyset \) mod \( D_{n-1}^{[e_n]} \) and he chooses \( \bar{g}^n \in F_c(\mathcal{F}, e_n) \) extending \( \bar{g}_{n-1} \) (i.e. \( \bar{g}^n = \bar{g}^n \)) \( \text{Dom}(\bar{g}_{n-1}), \bar{g}^n \) supported by \( e_n \) and \( \bar{g}^n \) is \( (D_{n-1}^{[e_n]} + A_n) \)-decreasing,
- player II chooses \( D_n \in \text{FIL}(e_n) \) extending \( D_{n-1}^{[e_n]} + A_n \).
In the general case:
Player I chooses $e_n$ and $D_{n,1} \in \text{Duc}_n(D_{n-1})$ and let $e_n = e(D_{n-1})$ and he chooses $\bar{g}^n \in F \subset (\mathcal{F}, e(D_{n-1})$ which is extending $\bar{g}^{n-1}$ then $\eta \in \text{Dom}(\bar{g}^n)$ (i.e. $\bar{g}^{n-1} = \bar{g}^n \upharpoonright \text{Dom}(\bar{g}^{n-1})$, $\bar{g}^n$ supported by $e(D_{n-1})$ and $\bar{g}^n$ is $D_{n,1}$-decreasing.
Player II chooses $D_n = D_{n,2} \in \text{FIL}(e_n)$ extending $D_{n,1}$.
In the end, the second player wins if $\bigcup_{n<\omega} \text{Dom}(\bar{g}^n)$ has no infinite branch.

2) Let $\bar{\gamma}$ be such that $\text{Dom}(\bar{\gamma}) = \text{Dom}(\bar{g})$ and each $\gamma_\eta$ is an ordinal decreasing with $\eta$. Now $\mathcal{O}^\bar{\gamma}(D, \bar{g}, e)$ is defined similarly to $\mathcal{O}^*(D, \bar{g}, e)$ but the second player has in addition, to choose an ordinal $\alpha_\eta$ for $\eta \in \text{Dom}(\bar{g}^n) \setminus \bigcup_{\ell<n} \text{Dom}(\bar{g}^\ell)$ such that $\eta < \nu$ & $\nu \in \text{Dom}(\bar{g}^{n-1}) \Rightarrow \alpha_\nu < \alpha_\eta$ we let $\alpha_\eta = \gamma_\eta$ for $\eta \in \text{Dom}(\bar{g})$.

3) $w\mathcal{O}^*(D, \bar{g}, e)$ and $w\mathcal{O}^\bar{\gamma}(D, \bar{g}, e)$ are defined similarly but $e$ is not changed during a play. (If e.g. $e = \{e\}$ then this makes no difference.)

4) If $\bar{\gamma} = \langle \gamma_{<\eta} \rangle$, $\bar{g} = \langle g_{<\eta} \rangle$ we write $\gamma_{<\eta}$ instead $\bar{\gamma}$, $g_{<\eta}$ instead $\bar{g}$.

5) If $E \subseteq \text{FIL}$ the games $\mathcal{O}^*_E, \mathcal{O}^\bar{\gamma}_E$ are defined similarly, but player II can choose filters only from $E$ (so we naturally assume to have $A \in D^+ \wedge D \in E \Rightarrow D+A \in E$).

3.8 Remark. Denote the above games $\mathcal{O}^*_0, \mathcal{O}^\bar{\gamma}_0, w\mathcal{O}^*_0$. Another variant is

3) For $e \in e, D \in \text{FIL}(e)$ and $D$-decreasing $\bar{g} \in F_c(\mathcal{F})$ we define a game $\mathcal{O}^*_1(D, \bar{g}, e)$.

We stipulate $e_{-1} = e, D_{-1} = D$.

In the nth move first player chooses $e_n, e_{n-1} \leq e_n \in \mathcal{F}$ and $D'_n \in \text{FIL}(e_n)$ and $D'_n$-decreasing $\bar{g}^n$ extending $\bar{g}^{n-1}$ such that $(D_{n-1} + D_0)^{(e_n)} \subseteq D_n$ and:

(*) for some $A_n \subseteq \mathcal{Y}/e_n-1, A_n \neq \emptyset \mod D_{n-1}$ we have:

(i) $D'_n$ is the normal filter on $\mathcal{Y}/e_n$ generated by $(D_{n-1} + A_0)^{(e_n)} \cup \{A_\zeta : \zeta < e_n \}$ where for some $\langle C_\zeta : \zeta < \zeta_n \rangle$ we have:

(a) each $C_\zeta$ is a club of $\omega_1$,

(b) if $\zeta_\ell < \zeta_n$ for $\ell < \omega$, $i \in \bigcap_{\ell<\omega} C_{\zeta_\ell}$, $x \in \mathcal{Y}/e_{n-1}$, and $i(x) = i$, then for some $x' \in \mathcal{Y}/e_n$, we have $x' \subseteq x$, $x' \in \bigcap_{\ell<\omega} A_\zeta^\zeta$.

The first player also chooses $\bar{g}^n$ extending $\bar{g}^{n-1}, D'_n$-decreasing. Then second player chooses $D_n$ such that $D'_n \subseteq D_n \in \text{FIL}(e_n)$.

2) We define $\mathcal{O}^\bar{\gamma}_1(D, \bar{g}, e)$ as in (2) using $\mathcal{O}^*_1$ instead of $\mathcal{O}^*_0$.

3) If player II wins, e.g. $\mathcal{O}^\bar{\gamma}_E(D, \bar{f}, e)$ this is true for $E' =: \{D' \in G : \text{player II wins } \mathcal{O}^\bar{\gamma}_E(D', \bar{f}, e)\}$. 
3.9 Definition. 1) We say $D \in \text{FIL}$ is nice to $\bar{g} \in F_c(\mathcal{F}, e, \mathcal{Y})$, $e = e(D)$, if player II wins the game $\mathcal{O}^*(D, \bar{g}, e)$ (so in particular $\bar{g}$ is $D$-decreasing, $\bar{g}$ supported by $e$).
2) We say $D \in \text{FIL}$ is nice if it is nice to $\bar{g}$ for every $\bar{g} \in F_c(\mathcal{F}, e)$.
3) We say $D$ is nice to $\alpha$ if it is nice to the constant function $\alpha$. We say $D$ is nice to $g \in \kappa$ Ord if it is nice to $g[\varepsilon(D)]$.
4) “Weakly nice” is defined similarly but $e$ is not changed.
5) Above replacing $D$ by $n$ means: for every $D \in \text{FIL}_n$.

3.10 Remark. “Nice” in [Sh 386] is the weakly nice here, but

(a) we can use $n$ with $e_n = \{e\}$

(b) formally they act on different objects; but if $xey \iff \iota(x) = \iota(y)$ we get a situation isomorphic to the old one.

3.11 Claim. Let $D \in \text{FIL}$ and $e = e(D)$.
1) If $D$ is nice to $f$, $f \in F_c(\mathcal{F}, e)$, $g \in F_c(\mathcal{F}, e)$ and $g \leq f$ then $D$ is nice to $f$.
2) If $D$ is nice to $f$, $e = e(D) \leq e(1) \in e$ then $D^{[e(1)]}$ is nice to $f^{[e(1)]}$.
3) The games from 3.7(2) are determined and winning strategies do not need memory.
4) $D$ is nice to $\bar{g}$ iff $D$ is nice to $g_{<\omega}$ (when $\bar{g} \in F_c(\mathcal{F}, e)$ is $D$-decreasing).
5) If $e \subseteq e$ and for simplicity $\bigcup_{i<\omega_1} \{i\} \times \mathcal{Y}_i \in e$ and for every $e \in e$, $e \leq e(1) \in e$ for some permutation $\pi$ of $\mathcal{Y}$ (i.e. a permutation of $\mathcal{Y}$ mapping each $\mathcal{Y}_i (i < \omega_1)$ onto itself) (and $n$ is full for simplicity) we have $\pi(e) = e, \pi(e(1)) \leq e(2) \in e$ then we can replace $e$ by $e$.
6) For $e = e_\mu$ (where $\mu < \mu^*$) there is $e$ as above with: $|e|$ countable if $\mu$ is a successor cardinal ($\succ \aleph_1$), $|e| = \text{cf}(\mu)$ if $\mu$ is a limit cardinal.

Proof. Left to the reader. (For part (4) use 3.12(2) below).

3.12 Claim. 1) Second player wins $\mathcal{O}^*(D, \bar{g}, e)$ iff for some $\gamma$ second player wins $\mathcal{O}^{\gamma}(D, \bar{g}, e)$.
2) If second player wins $\mathcal{O}^{\gamma}(D, f, e)$ then for any $D$-decreasing $\bar{g} \in F_c(\mathcal{F}, e)$, $\bar{g}$ supported by $e$ and $\bigwedge_{\eta,y} g_\eta(y) \leq f(y)$, the second player wins in $\mathcal{O}^{\gamma}(D, \bar{g}, e)$, when we let

$$\gamma_\eta = \gamma + \max\{\ell g(\nu) - \ell g(\eta) + 1 : \nu \text{ satisfies } \eta \leq \nu \in \text{Dom}(\bar{g})\}.$$
3) If \( u_1, u_2 \in F_c(\mathcal{T}), h : u_1 \to u_2 \) satisfies \( [\eta \leftrightarrow h(\eta)h(\nu)] \) and for \( \ell = 1, 2 \) we have \( \gamma^\ell \in F_c(\mathcal{T}, e_2), g^{\eta}_\ell \geq g^{\nu}_\ell \) (for \( \eta \in u_1 \)), \( \gamma^\ell = (x^\ell : x \in u_1) \) is a \( \prec \) -decreasing sequence of ordinals, \( \gamma_1^2 \geq \gamma_2^2 \) and the second player wins in \( \mathcal{D}^{\forall^2}(D, \bar{g}^2, e) \) then the second player wins in \( \mathcal{D}^{\forall^1}(D, \bar{g}^1, e) \).

**Proof.** 1) The “if part” is trivial, the “only if part” \([\text{FILL}]\) is as in [Sh 386]. 2), 3) Left to the reader.

The following is a consequence of a theorem of Dodd and Jensen [DoJe81]:

**3.13 Theorem.** If \( \lambda \) is a cardinal, \( S \subseteq \lambda \) then:

1) \( K[S] \), the core model, is a model of \( \text{ZFC} + (\forall \mu \geq \lambda)2^\mu = \mu^+ \).
2) If in \( K[S] \) there is no Ramsey cardinal \( \mu > \lambda \) (or much weaker condition holds) then \( (K[S], V) \) satisfies the \( \mu \)-covering lemma for \( \mu \geq \lambda + \aleph_1 \); i.e. if \( B \in V \) is a set of ordinals of cardinality \( \mu \) then there is \( B' \in K[S] \) satisfying \( B \subseteq B' \) and \( V \models |B'| \leq \mu \).
3) If \( V = (\exists \mu \geq \lambda)(\exists \kappa)[\mu^\kappa > \mu^+ > 2^\kappa] \) then in \( K[S] \) there is a Ramsey cardinal \( \mu > \lambda \).

**3.14 Lemma.** Suppose

(a) \( n \) is a semi-full niceness content thin or medium \( \kappa = \aleph_1 \)
(b) \( f^* \in {}^\kappa \text{Ord}, \lambda > \aleph_0 \Rightarrow \sup\{(2|^\kappa/e|^\aleph_0) : e \in e_n\} \)
(c) for every \( A \subseteq \aleph_0 \), in \( K \) there is a Ramsey cardinal \( > \lambda_0 \), then for every filter \( D \in \text{FIL}_n(e) \) is nice to \( f^* \).

**Remark.** 1) The point in the proof is that via forcing we translate the filters from \( \text{FIL}(e, \mathcal{Y}) \) to normal filters on \( \kappa \) [for higher \( \kappa \)’s cardinal restrictions are better].
2) At present we do not care too much what is the value of \( \lambda_0 \), i.e., equivalently, how much we like the set \( S \) to code.
Saharon: compare with [Sh:g, V], i.e., improve as there! But if we use \( e = \{e\} \), the proofs are more similar to [Sh:g, V] we can consider just Levy(\( \aleph_1 \)), |\( D | \), now in some proofs we may consider filters generated by \( |\text{pcf}(a)| \) set \( |a| < \text{aleph}_\omega \).

**First Proof.** Without loss of generality \( (\forall i)f(i) \geq 2 \). Let \( S \subseteq \lambda_0 \) be such that \( [\alpha < \mu \& A \subseteq 2^{\alpha}|\aleph_0 \Rightarrow A \in L[S], e \in L[S] \) (see 3.11(6)) and: if \( g \in {}^\kappa \text{Ord}, (\forall i < \)}
\[ \kappa_1 g(i) \leq f(i) \] then \( g \in L[S] \) (possible as \( \prod_{i<\omega_1} |f(i) + 1| \leq \lambda_0 \). We work for awhile in \( K[S] \). In \( K[S] \) there is a Ramsey cardinal \( \mu > \lambda_0 \) (see 3.13(3)). Let in \( K[S] \).

Let

\[ Y_0 = \{ X : X \subseteq \mu, X \cap \kappa \text{ a countable ordinal} > 0, \{ \kappa, \lambda_0 \} \subseteq X, \text{ moreover } X \cap \lambda_0 \text{ is countable} \}. \]

Let

\[ Y_s = Y_1 = \{ X \in Y_0 : X \text{ has order type } \geq f(X \cap \kappa) \}. \]

Now for \( g \in {}^{\text{Ord}} \subseteq \mu \), \( g(i) \leq f(i) \) let \( \hat{g} \) be the function with domain \( Y_1 \), \( \hat{g}(X) = g(X \cap \kappa) \)-th member of \( X \).

Let \( D_s = \{ A_i : \kappa \leq i \leq 2^{\|Y/e\|} \} \) and we arrange \( \langle A_i^D : \kappa \leq i < 2^{\|Y/e\|} \rangle \in L[S] \), (as \( Y/e \) has cardinality \( < \mu^* \), so \( 2^{\|Y/e\|} \leq \lambda_0 \)).

Let \( J \) be the minimal fine normal ideal on \( Y \) (in \( K[S] \)) to which \( Y \setminus Y_D \) belongs where

\[ Y_D = \{ X : X \in Y_s \text{ and } i \in (\kappa, 2^{\|Y/e\|} \cap X \Rightarrow X \cap \omega_1 \in A_i \}. \]

Clearly it is a proper filter as \( K[S] \models " \mu \text{ is a Ramsey cardinal}" \).

3.15 Observation. Assume

(a) \( P \) is a proper forcing notion of cardinality \( \leq |\alpha|^{\aleph_0} \) for some \( \alpha < \mu^* \) (or just \( P, MAC(P) \in K[S] \) and \( X \in Y_1 : X \cap (MAC(P)) \text{ is countable} \) \( e := Y_s \text{ mod } J \) where \( MAC(P) \) is the set of maximal antichains of \( P \) and let \( J^P \) be the normal fine ideal which \( J \) generates in \( V^P \).

(1) \( F \)-positiveness is preserved; i.e. if \( X \in K[S], X \subseteq Y_1, F \in \text{ FIL and } V \models "X \neq \emptyset \text{ mod } F" \) then \( \models_P "X \neq \emptyset \text{ mod } F^P" \).

(2) Moreover, if \( Q \ll P, (Q \text{ proper and) } P/Q \text{ is proper then forcing with } P/Q \text{ preserve } F^Q \text{-positiveness.} \)

Continuation of the proof of 3.14.

Case 1: \( e = \{ e \} \). Here only 3.16(1) is needed and then it is as in the old case.
Case 2: General.

Let $\mathcal{P}(\mathcal{Y}/e) = \{A_\zeta^e : \zeta < 2^{||Y/e||}\}$.

Now we describe a winning strategy for the second player. In the side we choose also $(p_n, \Gamma_n, f_n, \tilde{\gamma}^n, W_n)$ such that\(^3\) (where $e_n, A_n$ are chosen by the second player):

**(A)**

1. $\mathbb{P}_n = \prod_{\ell \leq n} \mathcal{Q}_\ell$ where $\mathcal{Q}_\ell$ is Levy($\aleph_1, \mathcal{Y}/e_n$)

   (we could use iterations, too, here it does not matter).
2. $p_n \in \mathbb{P}_n$
3. $p_n$ increasing in $n$
4. $f_n$ is a $\mathbb{P}_n$-name of a function from $\omega_1$ to $\mathcal{Y}/e_n$
5. $p_n \Vdash "f_n(i) \in \mathcal{Y}/e_n"$
6. $p_{n+1} \Vdash "f_{n+1}(i) \leq f_n(i) \text{ for every } i < \omega_1"$
7. $f_n$ is given naturally — it can be interpreted as the generic object of $\mathcal{Q}_n$ except trivialities.

**(B)**

1. $\tilde{\gamma}^n, \bar{g}^n$ have the same domain, $\gamma^n < \mu$
2. $p_n \Vdash "W_n \subseteq Y_D, W_{n+1} \subseteq W_n"$
3. $\tilde{\gamma}^n = \tilde{\gamma}^{n+1} \upharpoonright \text{Dom}(\tilde{\gamma}^n), \text{Dom}(\tilde{\gamma}^n) = \text{Dom}(\bar{g}^n)$ and $\tilde{\gamma}^n$ is $\zeta$-decreasing
4. $p_n \Vdash "\{X \in Y_D : \text{ for } \ell \in \{0, \ldots, n\}, f_\ell(X \cap \omega_1) \in A_\ell \text{ and } \bigwedge_{\eta \in \text{Dom}(g^n)} \bar{g}_\eta(X) = \gamma_\eta \text{ and for } \ell \in \{-1, 0, \ldots, n-1\}, \zeta \in X \cap 2^{||\mathcal{Y}/e_\ell||} \text{ we have: } A^{f_\ell^e}_\zeta \subseteq D_\ell \Rightarrow f_\ell(X \cap \omega_1) \in A^{f_\ell^e}_\zeta \} \supseteq W_n \neq \emptyset \text{ mod } \mathbb{P}_n"$
5. $\bar{g}^n = \bar{g}^{n+1} \upharpoonright \text{Dom}(\bar{g}^n)$ [difference]

**(C)**

1. $D_n = \{Z \subseteq \mathcal{Y}/e_n : p_n \Vdash "\{X \in J_D : f_n(X \cap \omega_1) \notin Z\} = \emptyset \text{ mod } (D^{\mathbb{P}_n}_{e_n} + W_n)"\}$
2. $\bar{g}^n$ is $D_n$-decreasing. [Saharon: diff]

Note that $D_n \in \mathbf{K}[S]$, so every initial segment of the play (in which the second player uses this strategy) belongs to $\mathbf{K}[S]$. By (B)(iii) this is a winning strategy. \(\blacksquare_{3.14}\)

\(^3\)For the forcing notions actually used below by the homogeneity of the forcing notion the value of $p_n$ is immaterial.
Recall all normal filters on \( \mathcal{Y} / e \) belong to \( \text{FIL}(e) \).

Alternate: We split the proof to a series of claims and definitions.

3.16 Definition. 1) \( W_* = \{ u \subseteq \mu : \text{otp}(u) \geq f^*(u \cap \omega_1) \text{ and } u \cap \lambda \text{ is countable} \} \).

2) Let \( J \) be the following ideal on \( Y_0 \):

\[
W \in J \text{ iff for some model } M \text{ on } \mu \text{ with countable vocabulary (with Skolem function) we have }
\]

\[
W_* \supseteq W \subseteq \{ w \in W_* : w = \ell_M(w) \}.
\]

3) For \( g \in \prod_{i<\kappa} (f(i) + 1) \) let \( \hat{g} \) be the function with domain \( Y_* \) and \( \hat{g}(A) \) is the \( g(i) \)-the member of \( A \).

4) For \( W \in J^+ \) let \( \text{proj}(W) = \{ A \subseteq \omega_1 : \{ w \in W : w \cap \omega_1 \notin A \} \in J \} \).

3.17 Fact. 1) \( Y_* \notin J \).

2) \( J \) is a fine normal filter on \( W_* \) (and \( W_* \notin J \)) in fact the ideal of non-stationary subsets of \( W_* \).

3) \( Y_{\hat{A}} \in J^+ \) if \( \hat{A} = \langle A_i : i < 0 \rangle \), \( 2^{\aleph_1} \) list the subset of some normal filter \( D \) on \( \omega_1 \) (see 3.23’s proof).

4) If \( \hat{A}', \hat{A}'' \) list the same normal filter on \( \omega_1 \) then \( Y_{\hat{A}'} = Y_{\hat{A}''} \mod J \).

5) For \( g \in \prod_{i<\omega} (f^*(i) + 1) \), \( \hat{g} \) is well defined, is a choice function of \( Y_* \).

6) If \( g_1 \leq_D g_2 \) then \( \hat{g}_1 \models J_D < \hat{g}_2 \models J_D \mod J + Y_* \).

Proof. 1) As \( \mu \) is a Ramsey cardinal \( > \lambda_0 \).

2) By the definitions.

3) Easy.

3.18 Claim. Assume \( Q \) is an \( \aleph_1 \)-complete forcing notion with \( \leq \lambda_0 \) maximal antichains.

1) Forcing with \( Q \) preserves all our assumptions:

(a) \( \mu \) is a Ramsey cardinal \( ^+ \)

(b) \( W_* \) is a family of subsets of \( \mu \) such that \( \text{otp}(w) \geq f(w \cap \omega_1) \) and \( J \), defined above, is a fine normal ideal on \( Y_* \) satisfying 3.17(3)...then we can forget (a).
Forcing with $Q$ preserves “$y \in J^+$” (i.e. if $W \in J^+$ then $\forces Q \{ W \in J^+ \}$.

Proof. Easy, fill.

3.19 Definition. Assume $e \in e_n$ and $D \in \text{FIL}_n(e)$.
1) $Q = Q_e = \{ f : f$ is a function with domain a countable ordinal such that $i \in \text{Dom}(f) \Rightarrow f(i) \in \mathcal{Y}^n \}$. 
2) $f_e$ is the $Q$-name $\cup \{ f : f \in G_{Q_e} \}$.
3) Let $D/f_e$ be the $Q_e$-name of $\{ A \subseteq \omega_1 :$ for every $B \in D$ for stationarily many $i < \omega_1, f_e(i) \in B \}$ and nor($D, f_e$) the normal filter which $D/f_e$ generates.
4) For $W \in J^+$ let lift($W, D$) = $\{ A \subseteq Y/e \text{ for some } B \in D : f_e(w \cap \omega_1) \in B \setminus A \in J^+ \}$ (note that we have enough homogeneity for $Q_e$.

3.20 Claim. Assume $e \in e_n$ and $D \in \text{FIL}_n(e)$.
1) $\forces Q \{ D/f_e $ is a normal filter on $\omega_1$, (i.e. $w_1 \notin D)$. 
2) $|Q_e| \leq |\mathcal{Y}^n/e|^{\aleph_0}$ so $Z[Q_e] \leq \lambda_0$ hence $Q_e$ has $\leq \lambda_0$ maximal antichains; in fact, equality holds as we have demand $|\mathcal{Y}/e| = | \cup \{ \mathcal{Y}^i : i \in [i_0, \omega_1) \}/e |$ for every $e \in e$. 
3) Combine scite3.A(4) + 3.19 - FILL.

3.21 Definition. 1) We say that $r = (e, D, \bar{g}, \bar{\alpha}, f, W)$ is a good position (in the content of proving 3.14) if

(a) $e \in e_n$
(b) $D \in \text{FIL}_n(e)$
(c) $\bar{g} = \langle g_\eta : \eta \in u \rangle \in \text{Fc}(\mathcal{Y}, e)$, so $u = u^x$
(d) $\bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle, \alpha_\eta < \mu$
(e) $p \in Q_e$
(f) $W = \{ w \in W^r : \hat{g}_\eta(w) = \alpha_\eta \text{ for } \eta \in u \} \in J^+$
(g) $p \forces Q_e \{ W^r \cap W_{D, f_e} \in J^+ \}$ and proj($W^r \cap W_{D, f_e}) = D \text{ nor}(D, f_e)$ [FILL].

3.22 Observation. 1) If $r = (e, D, \bar{g}, \bar{\alpha}, p, W)$ is a good position then

(a) $\bar{\alpha}$ is decreasing
(b) $D_W$. 
3.23 Claim. If \( e \in e_n, D \in \text{FIL}_n(e) \) and \( \bar{g} = \langle g_\eta : \eta \in u \rangle \in \text{Fc}(\mathcal{T}, e) \) and \( g_\eta \leq f[e] \) for every \( \eta \in \text{Dom}(\bar{g}) \) then we can find a good position \( \bar{r} \) with \( \bar{g}^\bar{r} = e^\bar{r} = e, \bar{g}^\bar{r} = g \) and \( D \subseteq D^\bar{r} \).

Proof. Let \( G \in \mathbb{Q}_e \) be generic over \( V \) and \( f_e = f_e[G] \). So in \( V[G] \) the set \( W_{D,f_e[G]} \) belongs to \( J^+ \) (by 3.17(3)), i.e., let \( \langle A^D_\zeta : \zeta < \zeta^* \rangle \) list \( D_1 \) and \( W, D, f_e = \{ w \in W : \) if \( \zeta \in w \cap \zeta^* \) then \( f_e(i) = f_e[G][i] \in A_\zeta \} \). Also \( \bar{g}_\eta \) defined in 3.16(3) is a choice function on \( W_{D,f_e} \) (see 3.17(4)), so as \( J \) is a normal ideal and \( u \) finite, we can find \( \bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle \) such that \( W = \{ w \in W_{D,f_e} : \bar{g}_\eta(w) = \alpha_\eta \text{ for } \eta \in u \} \) belongs to \( J^+ \). As all this holds in \( V[G] \). So \( \bar{\alpha} \) there is a condition \( p \in \mathbb{Q}_e \) which forces this, and we are done.

3.24 Claim. Assume that

(a) \( \bar{r}_1 = (e_1, D_1, \bar{g}_1, \bar{\alpha}_1, p, W_1) \) is a good position

(b) \( \bar{g}_2 = \langle g^2_\eta : \eta \in u_2 \rangle \in \text{Fc}(\mathcal{T}, n) \) and \( \bar{g}_2 \restriction u_1 = \bar{g}_2 \)

(c) \( e_1 \leq e_2 \) in \( e_n \) and \( D_2 \in \text{FIL}_n(e_2) \) or just \( A \subseteq \mathcal{P}(\mathcal{Y}_n/e_2), A = \{ A_\zeta : \zeta < \zeta^* \} \)

(d) \( p_1 \Vdash_{\mathbb{Q}_e} \{ w \in W_1 : \mathcal{Y}_{w \cap w_1} \not\subseteq \cup\{ A_\zeta : \zeta < \zeta^* \cap w \} \text{ does not belong to } J^{V[Q_{e_1}]} \} \)

Then we can find a good position \( \bar{r}_2 \) such that \( e^\bar{r}_2 = e_2, \bar{g}^\bar{r}_2 = \bar{g}^2 \) and \( D_2 \subseteq D^\bar{r}_2 \).

Proof. Let \( G \) be a subset of \( \mathbb{Q}_{e_1[\mathbb{F}]} \) generic over \( V \) such that \( p_{\mathbb{F}} \in G_1 \). Now \( \mathbb{Q}_{e_2} \) is an \( \aleph_1 \)-complete forcing of cardinality \( \leq |\mathcal{Y}_n/e_2|^{|\mathbb{F}|} \leq \lambda_0 \) and \( \mathbb{Q}_{e_1} \) is \( \aleph_1 \)-complete \( |\mathbb{Q}_{e_1}| \leq |\mathcal{Y}_n/e_1|^{|\mathbb{F}|} \leq |\mathcal{Y}_n/e_2|^{|\mathbb{F}|} \leq \lambda_0 \), so \( \mathbb{Q}_{e_2} \) satisfies the same conditions in \( V[G_1] \) (if \( \lambda_0 \) is no longer a cardinal it does not matter).

Note that by assumption (c)

\[ \mathbb{Q}_{e_2} \Vdash \text{"the set } \{ W^1_2 = \{ w \in W_1[G_1] : \text{the set } ((f_e[G_1])(w \cap \omega_1))^{e_2}[\mathcal{Y}_{w \cap \omega_1}/e_2] \text{ is not included in } \cup\{ A_\zeta : \zeta \in w \} \text{ is stationary (i.e. } \not\in J)" \}. \]

We continue as in the previous claim.
3.25 Claim. If clauses (a) + (b) of 3.23 holds, then a sufficient condition for clause (c) is 
(c)' FILL.

3.26 Proof of 3.14. During the play, the player II chooses also a good position \( x_n \) and maintains \( \bar{g} = g_n, \bar{\alpha} = \bar{\alpha} \).

3.27 Remark. 1) From the proof, instead \( K[S] \models \lambda \) is Ramsey", \( K[S] \models \mu \rightarrow (\alpha)^{<\omega}_{\lambda_0} \) for \( \alpha < \lambda_0 \)" is enough for showing for 3.14.
2) Also if \( \prod_{i<\omega} (|f(i)| + 1) < \mu_0, [\alpha < \mu_0 \Rightarrow |\alpha|^{N_0} < \mu_0] \), it is enough: \( S \subseteq \alpha < \mu_0 \Rightarrow \) in \( K[S] \) there is \( \mu \rightarrow (\alpha)^{<\omega}_{2}. \)

3.28 Theorem. Assume \( n \) is a \( \kappa \)-niceness context. Let \( \mathcal{D}_n \in \mathcal{FIL}(e, \forall) \) be a normal ideal on \( \forall n/e \). If for every \( f : \forall \rightarrow (\text{sup}\{\text{Suc}(D') : D' \in \mathcal{FIL}_n\})^+ \) supported by some \( e \in e_n \), \( \mathcal{D}_n^* \) is nice to \( f \), then for every \( f \in \kappa\text{Ord}, \mathcal{n} \) is nice to \( f \).

Proof. By determinacy of the games (and the LS argument).

3.29 Remark. 0) The value \( |\mathcal{FIL}_e| \) really should be an upper bound.
1) So, the existence of \( \mu, \mu \rightarrow (\alpha)^{<\omega}_{\lambda_0} \) for every \( \alpha < \lambda_0 \), is enough for “\( \mathcal{D}^* \) is nice”.
2) If there is a nice \( \mathcal{D}^* \)’s in the plays from 3.7, the second player winning strategy can be chosen such that all subsequent filters are nice: just by renaming have \( g_{<>} \) constant large enough. [Saharon: diff]

3.30 Claim. In claim 3.14 we can omit “\( \kappa_n = N_1 \)”.

Proof. Let \( \mathbb{P} = \text{Levy}(N_0, \kappa_n) \). Now

\( (*) \text{ also in } \mathbb{V}^\mathbb{P} \text{ the object } n \text{ is a successor content, if we do not distinguish between } D \in \mathcal{FIL}_n \text{ and } \{A \in \mathbb{V}^\mathbb{P} : A \subseteq \forall / e(D) \text{ and } (\exists B \in D)(B \subseteq A)\}. \)
3.31 Conclusion. Let \( \lambda_0 = (\sup\{|\text{Suc}_n(D')| : D' \in \text{FIL}_n\})^+ \cup \{2^{|\mathcal{V}/e|^{<\kappa}} : e \in e_n\}^+ \), \( \mu^* \geq \aleph_2 \); if for every \( S \subseteq \lambda_0 \) there is a Ramsey cardinal in \( K[S] \) above \( \lambda_0 \) then \( n \) is nice.


3.32 Concluding Remark. 1) We could have used other forcing notions, not \( \text{Levy}(\kappa, |\mathcal{V}/e_n|) \). E.q., if \( \kappa = \aleph_1, \mu = \kappa^+ \), we could use finite iterations of the forcing of Baumgartner to add a club of \( \omega_1 \), by finite conditions. (So this forcing notion has cardinality \( \aleph_1 \)). Then in 3.14 we can weaken the demands on \( \lambda_0 : \lambda_0 = \sum_{\chi^{<\mu_0}} 2^{\chi} + \prod_{i<\omega_1} |1+f(i)|+|e| \), hence also in 3.31, \( \lambda_0 = \sum_{\chi^{<\mu^*}} 2^{\chi} \) is O.K.

2) Concerning \( |e| \) remember 3.11(5),(6).

3) Similarly to (1). If \( \theta < \mu \Rightarrow \text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu \) then by 2.6 we can use forcing notions of Todorcevic for collapsing \( \theta < \mu \) which has cardinality \( < \mu \).

4) If we want to have \( \lambda_0 =: \prod_{i<\omega_1} |f(i)+2| \) (or even \( T_D(f+2) \)), we can get this by weakening further the first player letting him choose only \( A_n \) which are easily definable from the \( g^{n-1} \), we shall return to it in a subsequent paper.
4.1 Convention. 1) Like 3.2 and:
2) \( \bar{g}^* \in F_c(\mathcal{T}, e^*, \mathcal{A}) \), \( \eta^* \in \text{Dom}(\bar{g}^*) \), \( \nu^* \) an immediate successor of \( \eta^* \) not in \( \text{Dom}(g^*) \), \( D^* \in \text{FIL}(e^*, \mathcal{A}) \) is such that in \( \mathcal{O}^{\gamma_1^*}(D^*, \bar{g}^*, e^*) \) second player wins (all constant for this section). \( \text{FIL}^*(e) \) will be the set of \( D \in \text{FIL}(e, \mathcal{A}) \) such that \( e \geq e^* \), \( (D^*)^{[e]} \subseteq D \) and in \( \mathcal{O}^{\gamma^*_2}(D^*, \bar{g}^*, e^*) \) second player wins. (So actually \( \text{FIL}(e^*, \mathcal{A}) \) depends on \( D^*, \bar{g}^*, e^* \), too).

4.2 Definition. 1) \( \text{rk}^5_D(f) \) for \( D \in \text{FIL}^*(e, \mathcal{A}) \), \( f \in \mathcal{A}/e \text{Ord} \), \( f \prec_D \bar{g}^*_\eta \) will be: the minimal ordinal \( \alpha \) such that for some \( D_1, e_1 \), \( \tilde{\gamma}^1 \) we have \( D_1^{[e_1]} \subseteq D_1 \in \text{FIL}(e_1, \mathcal{A}) \), \( \tilde{\gamma}^1 = \bar{g}^* (\nu^*, \alpha) \) (i.e. \( \text{dom}(\tilde{\gamma}^1) = (\text{dom}(\gamma^*)) \cup \{ \nu^* \} \), \( \tilde{\gamma}^1 \) \( \text{dom}(\gamma^*) = \bar{g}^*, \gamma^*_1 = \alpha \) and in \( \mathcal{O}^{\gamma^*_2}(D, \bar{g}^*, \nu^*, f) \) second player wins and \( \infty \) if there is no such \( \alpha \).
2) \( \text{rk}^4_D(f) \) is \( \sup \{ \text{rk}^5_{D^+A}(f) : A \in D^+ \} \).

4.3 Claim. 1) \( \text{rk}^5_D(f) \) is (under the circumstances of 4.1, 4.2) an ordinal \( < \gamma^*_\eta \).
2) \( \text{rk}^4_D(f) \) is an ordinal \( \leq \gamma^*_\eta \).

4.4 Claim. If \( D \in \text{FIL}^*(e, \mathcal{A}) \), \( h \prec_D f \prec_D \bar{g}^*_\eta \) then \( \text{rk}^5_D(h) < \text{rk}^5_D(f) \).

Proof. Let \( e_1, D_1 \) witness \( \text{rk}^5_D(f) = \alpha \) so \( e(D) \leq e_1 \), \( D \subseteq D_1 \in \text{FIL}^*(e_1) \) and in \( G^{\nu^*, \alpha} (\bar{g}^*, \nu^*, f) \) second player wins. We play for the first player: \( e = e_1 \), \( A_0 = \mathcal{A}/e_1 \), \( \bar{g}^0 = \bar{g}^* (\nu^*, f) \) \( \nu^*, \nu^* < 0 >, g \) now the first player should be able to answer say \( e_2, D_2, \tilde{\gamma}^2 \). So \( \gamma^*_2 < \mathcal{O}^* > \leq \gamma^*_\nu^* = \alpha \) and by 3.12(3), we know that in \( G^{\gamma_*^2}(D_2, \bar{g}^*, \nu^*, f) \geq e_2 \) where \( \gamma_*^2 = \gamma^*(\nu^*, \nu^*_2, \nu^*_2 < 0 >) \) second player wins. \( \square_{4.4} \)

4.5 Claim. Let \( e \geq e^*, D \in \text{FIL}^*(e, \mathcal{A}) \).
1) For \( e \geq e(D) \), \( A \in (D)^{[e]} \), \( f \in \mathcal{A}/e \text{Ord} \), \( f \prec_D \bar{g}^*_\eta \) we have:
\[
\text{rk}^5_D(f) \leq \text{rk}^5_{D[\bar{g}^*]+A}(f) \leq \text{rk}^4_{D[\bar{g}^*]+A}(f) \leq \text{rk}^4_D(f).
\]
2) If \( e_2 \geq e_1 \geq e(D) \), \( f_\ell \in \mathcal{A}/\text{Ord} \) is supported by \( e_\ell \), \( f_1 \leq_D f_2 \prec_D \bar{g}^*_\eta \) then \( \text{rk}^\ell_D(f_1) \leq \text{rk}^\ell_D(f_2) \) for \( \ell = 4, 5 \).

Proof. Left to the reader.
§5 More on Ranks and Higher Objects

5.1 Convention.

(a) \( \mu^* \) is a cardinal > \( \aleph_1 \) (using \( \aleph_1 \) rather than an uncountable regular \( \kappa \) is to save parameters)
(b) \( \mathcal{Y} \) a set of cardinality \( \sum_{\kappa < \mu^*} \kappa \)
(c) \( \iota \) a function from \( \mathcal{Y} \) onto \( \omega_1 \), \( |\iota^{-1}(\{\alpha\})| = |\mathcal{Y}| \) for \( \alpha < \omega \),
(d) Eq the set of equivalence relation \( e \) on \( \mathcal{Y} \) such that:
   (\( \alpha \)) \( y e z \Rightarrow \iota(y) = \iota(z) \)
   (\( \beta \)) each equivalence class has cardinality \( |\mathcal{Y}| \)
   (\( \gamma \)) \( e \) has \( < \mu^* \) equivalence classes
(e) \( D \) denotes a normal filter on some \( \mathcal{Y} / e \in \text{Eq} \), we write \( e = e(D) \). The set of such \( D \)'s is \( \text{FIL}(\mathcal{Y}) \).
(f) \( E \) denotes a set of \( D \)'s as above, such that:
   (\( \alpha \)) for some \( D = \text{Min} E \in E (\forall D'[D' \in E \Rightarrow (e, D) \leq (e(D'), D')] \)
   (\( \beta \)) if \( D \in E, A \subseteq \mathcal{Y} / e_1, e_1 \geq e(D), A \neq \emptyset \mod D \) then \( D[e_1] + A \in E \)
(g) \( E[e] = \{D \in E : e(D) = e\} \)
(h) \( \mathcal{E} \) denotes a set of \( E \)'s as above, such that:
   (\( \alpha \)) there is \( E = \text{Min} \mathcal{E} \in \mathcal{E} \) satisfying (\( \forall E' \))(\( E' \in E \Rightarrow E' \subseteq E \))
   (\( \beta \)) if \( D \in E \in \mathcal{E} \) then \( E[D] = \{D' : D' \in E \text{ and } (e(D), D) \leq (e(D'), D')\} \in \mathcal{E} \).

5.2 Definition. 1) We say \( E \) is \( \lambda \)-divisible when: for every \( D \in E \), and \( Z \), a set of cardinality \( < \lambda \) there is \( D \)'s such that:

(\( \alpha \)) \( D' \in E \)
(\( \beta \)) \( (e(D), D) \leq (e(D'), D') \)
(\( \gamma \)) \( j : \mathcal{Y} / e(D') \rightarrow Z \)
(\( \delta \)) for every function \( h : \mathcal{Y} / e(D) \rightarrow Z \) we have \( \{y/e(D') : h(y/e(D)) = (y/e(D'))\} \neq \emptyset \mod D' \).
2) We say $E$ has $\lambda$-sums when: for every $D \in E \in \mathcal{E}$ and sequence $(Z_\zeta : \zeta < \zeta^* < \lambda)$ of subsets of $\mathcal{Y}/e(D)$ there is $Z^* \subseteq \mathcal{Y}/(e(D))$, such that: $Z^* \cap Z_\zeta = \emptyset \mod D$ and: if $(e(D), D) \leq (e', D'), \gamma = e(D'), D' \in E[\mathcal{Y}]$ and $\bigwedge_{\zeta} Z[\zeta] = \emptyset \mod D'$ then $Z^* \subseteq D'$.

3) We say $E$ has weak $\lambda$-sums if for every $D \in E(\in \mathcal{E})$ and sequence $(Z_\zeta : \zeta < \zeta^* < \lambda)$ of subsets of $\mathcal{Y}/e(D)$ there is $D^*$, $D^* \in E[\mathcal{Y}]$ such that:

\begin{itemize}
  \item[(α)] if $(e(D), D) \leq (e', D'), D' \in E[\mathcal{Y}]$ and $Z_\zeta = \emptyset \mod D'$ for $\zeta < \zeta^*$ and $e(D^*) \leq e(D')$ then $D^* \subseteq D'$ (more exactly $D^{\gamma[\zeta]} \subseteq D^{\gamma}$) and
  \item[(β)] $Z_\zeta = \emptyset \mod D^*$ for $\zeta < \zeta^*$.
\end{itemize}

4) If $\lambda = \mu^*$ we omit it. We say $\mathcal{E}$ is $\lambda$-divisible if every $E \in \mathcal{E}$ has. We say $\mathcal{E}$ has weak $\lambda$-sums if: [rest diff] for every $E \in \mathcal{E}$ and sequence $(Z_\zeta : \zeta < \zeta^* < \lambda)$ of subsets of $\mathcal{Y}/e(E)$ there is $E^*$, $E^* \in \mathcal{E}[E]$ such that:

\begin{itemize}
  \item[(α)] if $(e(E), E) \leq (e', E')$, $E' \in \mathcal{E}$ and $Z_\zeta = \emptyset \mod \text{Min}(E')$ for $\zeta < \zeta^*$ and $e(E^*) \leq e(E')$ then $E^* \subseteq E'$
  \item[(β)] $Z_\zeta = \emptyset \mod \text{Min}(E^*)$ for $\zeta < \zeta^*$.
\end{itemize}

We now define variants of the games from §3.

### 5.3 Definition

For a given $\mathcal{E}$, for every $E \in \mathcal{E}$:

1) We define a game $G^*_2(E, \bar{g})$. In the $n$-th move first player chooses $D_n \in E_{n-1}$ (stipulating $E_{n-1} = E$) and choose $g_n \in F_c(\omega, E_{n-1})$ extending $g_{n-1}$ (stipulating $\bar{g}_{n-1} = \bar{g}$) such that $g_n$ is $D_n$-decreasing. Then the second player chooses $E_n$, $(E_{n-1})[p_n] \subseteq E_n \in \mathcal{E}$.

In the end the second player wins if $\bigcup_{n<\omega} \text{Dom}(\bar{g}_n)$ has no infinite branch.

2) We define a game $G^*_2(E, \bar{g})$ where $\text{Dom}(\bar{g}) = \text{Dom}(\bar{g})$, each $\gamma_n$ an ordinal, $[\eta \prec \nu \Rightarrow \gamma_\eta > \gamma_\nu]$ similarly to $G^*_2(D, \bar{g})$ but the second player in addition chooses an indexed set $\bar{\gamma}_n$ of ordinals, $\text{Dom}(\bar{\gamma}_n) = \text{Dom}(\bar{g}_n)$, $\bar{\gamma}_n \upharpoonright \text{Dom}(\bar{\gamma}_{n-1}) = \bar{\gamma}_{n-1}$ and $[\eta \prec \nu \Rightarrow \gamma_{n, \eta} > \gamma_{n, \nu}]$.

### 5.4 Definition

1) We say $\mathcal{E}$ is nice to $\bar{g} \in F_c(\mathcal{S}, e, \mathcal{Y})$ if for every $E \in \mathcal{E}$ with $e \leq e(E)$ the second player wins the game $G^*_2(E, \bar{g})$.

2) We say $\mathcal{E}$ is nice if it is nice to $\bar{g}$ whenever $E \in \mathcal{E}$, $e \leq e(E)$, $\bar{g} \in F_c(\mathcal{S}, e)$, $\bar{g}$ is $(\text{Min}(E))$-decreasing. we have: $E[\mathcal{E}]$ is nice to $\bar{g}$.

3) If $\text{Dom}(\bar{g}) = \{<>\}$ we write $g_{<>}$ instead $\bar{g}$.

4) We say $\mathcal{E}$ is nice to $\alpha$ if it is nice to the constant function $\alpha$.  


5.5 Claim. 1) If $\mathcal{E}$ is nice to $f$, $f \in F_c(\mathcal{F}, e, \mathcal{Y}), g \in F_c(\mathcal{F}, e, \mathcal{Y}), g \leq f$ then $\mathcal{E}$ is nice to $f$.
2) The games from 5.4 are determined, and the winning side has winning strategy which does not need memory.
3) The second player wins $G_2^* (E, g)$ if for some $\gamma$ second player wins $G_2^* (E, g)$. 
4) If the second player wins $G_2^* (E, f), \bar{g} \in F_c(\mathcal{F}, e(\mathcal{E})) g_\eta \leq f$ for $\eta \in Dom(\bar{g})$ then the second player wins in $G_2^* (E, \bar{g})$ when we let

$$\gamma_\eta = \gamma + \max\{\ell g(\nu) - \ell g(\eta) + 1 : \nu \text{ satisfies } \eta \subseteq \nu \in Dom(\bar{g})\}.$$ 

5.6 Lemma. Suppose $f_0 \in (\mathcal{Y}/e) Ord, e \in Eq$ and $\lambda_0 = : \sup \{\prod_{x \in Y} \mathscr{Y}_x(f_0^{|x|}(x) + 1 : e$ satisfies $e_0 \leq e \in e\}.$
1) If there is a Ramsey cardinal $\geq \bigcup\{f(x) + 1 : x \in Dom(\bar{f}_0)\}$ then there is a $\mu^*$-divisible $\mathcal{E}$ nice to $f_0$ having weak $\mu^*$-sums.
2) If for every $A \subseteq \lambda_0$ there is in $K[A_0]$ a Ramsey cardinal $> \lambda_0$, then there is a $\mu^*$-divisible $\mathcal{E}$ which has weak $\mu^*$-sums and is nice to $f$.
3) In part 2 if $\lambda_0 = 2^{<\mu_0}$ there is a $\mu^*$-divisible nice $\mathcal{E}$ which has weak $\mu^*$-sums.

5.7 Remark. This enables us to pass from “pp$_{\Gamma(\theta, N_1)}$ large” to “pp$_{\text{normal}}$ is large”.

Proof. 1) Define $f_1 \in (\mathcal{N}_1) Ord, f_1(i) = \sup\{f_0(y/e) : \iota(y) = i\}$, let $\lambda$ be such that: $\lambda \to (\sup\{f_1(i)\}^\omega : i < \mathcal{N}_1)$ (or just $\emptyset \notin D^*_n$ - see below) let $\lambda_n = (\lambda^{<\mathcal{N}_1})^{+n},$

$$I_n = \{s : s \subseteq \lambda_n, s \cap \omega_1 \text{ a countable ordinal}\}$$

$$J_n = \{s \in I_n : s \cap \lambda \text{ has order type } \geq f_0(s \cap \omega_1)\}.$$ 

Let $D^*_n$ be the minimal fine normal filter on $J_n$.

Let for $n < \omega$ and $e \in Eq, H_{n,e} = \{h : h \text{ a function from } J_n \text{ into } \mathcal{Y}/e \text{ such that } \iota(h(s)) = s \cap \omega_1\}.$

Let $\mathbb{P}_n = \{p : p \subseteq J_n, p \not= \emptyset \text{ mod } D^*_n\}, \mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ and for $p \in \mathbb{P}$ let $n(p)$ be the unique $n$ such that $p \in \mathbb{P}_n$.

Let $p \leq q$ (in $\mathbb{P}$) if $n(p) \leq n(q)$ and $\{s \cap \lambda_{n(p)} : s \in q\} \subseteq p$.

Now for every $e \in Eq, n < \omega, p \in P_n, h \in H_{n,e}$ we let:

$$D^*_{p,e,h} = \{A \subseteq \mathcal{Y}/e : h^{-1}(A) \supseteq p \text{ mod } D^*_{n(p)}\}.$$
\[ E^{n,e,h}_p = \{ D^{n,e,h}_q : p \leq q \in P, n^1 = n(q) \text{ and } (n^1, e^1, h^1) \geq (n, e, h) \} \]

where \( (n^1, e^1, h^1) \geq (n, e, h) \) means: \( n \leq n^1 < \omega, e \leq e^1 \in \text{Eq}, h^1 \in H_{n^1, e^1} \) and for \( s \in J_{(n^1), h^1(s)[e]} = h(s \cap \lambda_n) \) and we define \( (p^1, n^1, e^1, h^1) \geq (p, n, e, h) \) similarly. Let

\[ E^{n,e,h}_p = \{ E^{n,e,h}_q : p \leq q \in P, n^1 = n(q), (n^1, e^1, h^1) \geq (n, e, h) \}. \]

Note: \( (p^1, n^1, e^1, h^1) \geq (p, n, e, h) \) implies \( D^{n^1,e^1,h^1}_p \supseteq D^{n,e,h}_p, E^{n^1,e^1,h^1}_p \subseteq E^{n,e,h}_p \) and \( E^{n^1,e^1,h^1}_p \subseteq E^{n,e,h}_p \). Now any \( \mathcal{E} = E^{n,e,h}_p (p \in P) \) is as required.

A new point is “\( \mathcal{E} \) is \( \mu^* \)-divisible”. So suppose \( E \in \mathcal{E} = E^{n,e,h}_p \) so \( E = E^{n^1,e^1,h^1}_p \) for some \( (q, n^1, e^1, h^1) \geq (p, n, e, h) \). Let \( Z \) be a set of cardinality < \( \mu^* \), so \( (\lambda_{n^1})^{\card Z} = \lambda_{n^1} \); let \( \{ h_{\zeta} : \zeta < \zeta^+ = \card \mathcal{Y}/e_1 \mid \card Z \leq 2^\mu \leq \lambda_{n^1} \} \) list all function \( h \) from \( \mathcal{Y}/e_1 \) to \( Z \). Let \( \langle S_\zeta : \zeta < \card \mathcal{Y}/e_1 \mid \card Z \rangle \) list a sequence of pairwise disjoint stationary subsets of \( \{ \delta < \lambda_{n^1+1} : \cf(\delta) = \aleph_0 \} \). Let \( e_2 \in \text{Eq} \) be such that \( e_1 \leq e_2 \) and for every \( y \in \mathcal{Y}, \{ z/e_2 : ze_2y \} = \{ x(y/e,t) : t \in Z \} \), we let \( q_2, q \leq q_2 \in P \) be: \( q_2 = \{ s \in J_{n^1+1}, s \cap \lambda_{n^1} = q \} \) and \( \sup s \in \bigcup S_\zeta \), lastly we define \( h^2 : J_{n^1+1} \rightarrow \mathcal{Y}/e_1 \) by:

\[ h^2(s) = x(h^1(s \cap \lambda_{n^1}), h^1(s \cap \lambda_{n^1})) \text{ if } s \in q_2, \text{ sup } s \in S_\zeta \text{ (for } s \in J_{n^1+1}/q_2 \text{ it does not matter). The proof that } q_2, e_2, h^2 \text{ are as required is as in [RuSh 117] and more specifically [Sh 212]. As for proving } \mu^*-\text{sums} \text{ the point is that the family of fine normal filters on } \mu \text{ has } \mu^*-\text{sum.} \]

2) Similar to 3.14(and 3.11(5),(6)).
3) Similar to [Sh 386, 1.7].

\( \square_{5.6} \)
§6 Hypotheses: Weakening of GCH

We define some hypotheses; except the first we do not know now whether their negations are consistent with ZFC.

6.1 Definition. We define a series of hypothesis:

(A) \( \text{pp}(\lambda) = \lambda^+ \) for every singular \( \lambda \).

(B) If \( a \) is a set of regular cardinals, \( |a| < \text{Min}(a) \) then \( |\text{pcf}(a)| \leq |a| \).

(C) If \( a \) is a set of regular cardinals, \( |a| < \text{Min}(a) \) then \( \text{pcf}(a) \) has no accumulation point which is inaccessible (i.e. \( \lambda \) inaccessible \( \Rightarrow \sup(\lambda \cap \text{pcf}(a) < \lambda) \).

(D) For every \( \lambda \), \( \{\mu < \lambda : \mu \) singular and \( \text{pp}(\mu) \geq \lambda \} \) is countable.

(E) For every \( \lambda \), \( \{\mu < \lambda : \mu \) singular and \( \text{cf}(\mu) = \aleph_0 \) and \( \text{pp}(\mu) \geq \lambda \} \) is countable.

(F) For every \( \lambda \), \( \{\mu < \lambda : \mu \) singular of uncountable cofinality, \( \text{pp}(\text{cf}(\mu))(\mu) \geq \lambda \} \) is finite.

(D) for all \( \theta, \sigma, \kappa \) For every \( \lambda \), \( \{\mu < \lambda : \mu > \text{cf}(\mu) \in [\sigma, \theta) \) and \( \text{pp}(\text{cf}(\mu))(\mu) \geq \lambda \} \) has cardinality \( < \kappa \).

(A) If \( \mu > \text{cf}(\mu) \) then \( \text{pp}(\mu) = \mu^+ \) (or in the definition of \( \text{pp}(\mu) \) the supremum is on the empty set).

(B), (C) Similar versions (i.e. use \( \text{pcf}(\mu) \)).

We concentrate on the parameter free case.

6.2 Claim. In 6.1, we have:

1) \( (A) \Rightarrow (B) \Rightarrow (C) \)

2) \( (A) \Rightarrow (D) \Rightarrow (E), (A) \Rightarrow (F) \)

3) \( (E) + (F) \Rightarrow (D) \Rightarrow (B) \). [Last implication — by the localization theorem [Sh 371, §2]]

4) if \( \forall \mu)(\mu > \text{cf}(\mu) = \aleph_0 \) the hypothesis (A) of 6.1 holds. [Why? By [Sh:g, xx].]


1) For every \( \lambda > \kappa \),

\[
\text{cov}(\lambda, \kappa^+, \kappa^+, 2) = \begin{cases} 
\lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa \\
\lambda & \text{if } \text{cf}(\lambda) > \kappa.
\end{cases}
\]

2) For every \( \lambda > \kappa = \text{cf}(\kappa) > \aleph_0 \), there is a stationary \( S \subseteq [\lambda]^{<\kappa}, |S| = \lambda^+ \) if \( \text{cf}(\lambda) \leq \kappa \) and \( |S| = \lambda \) if \( \text{cf}(\lambda) > \kappa \).
3) For $\mu$ singular, there is a tree with $\text{cf}(\mu)$ levels each level of cardinality $< \mu$, and with $\geq \mu^+(\text{cf}(\mu))$-branches.

4) If $\kappa \leq \text{cf}(\mu) < \mu \leq 2^\kappa$ then there is an entangled linear order $\mathcal{T}$ of cardinality $\mu^+$.

Proof. 1) By [Sh 400, §1].
2) By part (1) and 2.6.
3, 4) By [Sh 355, §4].
REFERENCES.


