THE COMBINATORICS OF COMBINATORIAL CODING BY A REAL

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ABSTRACT. We lay the combinatorial foundations for [5] by setting up and proving the essential properties of the coding apparatus for singular cardinals. We also prove another result concerning the coding apparatus for inaccessible cardinals.

§0. INTRODUCTION.

In this paper, we lay the combinatorial foundations for the work of [5]. For the most part, this involves setting up the coding apparatus for singular cardinals, and proving its essential properties, most notably the result about the existence of supercoherent sequences, Lemma 3 (the Lemma of (1.4) of [5]). The sole exception occurs in (11.2), where, as promised in (2.1.1) of [5], we show that we can assume some additional properties for the system of b_α, with *card* α inaccessible.

The combinatorial apparatus for singular cardinals is based on our work in Part I, where, working in L, we prove that the "*Squarer Scales*" principle holds. This is Theorem 1, below; the proof stretches across §§1 - 6. This material is based on (and improves) that of [2]; [2] bears the same relationship to the material of [1], which is where many of the basic ideas of this construction made their first published appearance. §2, in particular,

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reviews the constuctions of §§1 - 2 of [2], without proofs. In §7, we quote a "classical" result of Jensen, from [3] which, again in L, gives a square system on singular cardinals. The last section of Part I hints at things to come in that it steps outside of L to remark that the methods of §§1 - 6 allow us to build "local versions" of the combinatorial system of (1.2) between τ and $\tau^{+\omega}$ working in $L[X_\tau]$, where $X_\tau \subseteq \tau^+$. As noted there, the X_τ we have in mind are the $A \cap \tau^{+\omega}$, where A is as given by Lemma 3 of [5].

In Part II, we assume that $V = L[A]$, for this A (and that $0^\sharp$ does not exist). We show, in §9, how to transfer the combinatorial systems of Part I to V. This culminates in (9.4), where we define a **fine system of squares and pseudo-scales** to be one which satisfies properties (A) - (D); these are restatements of similarly labelled items of (1.2) of [5]. We observe (the crucial fact having already been noted in (9.3.2)) that the system obtained in (9.1) - (9.3) is indeed a fine system. This is Corollary 2, below.

Of course, it is here that we make essential use of the Covering Lemma. This is used to guarantee that L "gets the successors of singular cardinals (cardinals of V) right", but also that our L-scales remain something close enough to V-scales. Ostensibly, what is required for Part II is that if d is a club subset of κ, a singular limit of limit cardinals, such that $o.t.\ d < inf\ d$, and $g \in V$ is a function with domain d such that for $\lambda \in d$, $g(\lambda) < \lambda^+$, then there is a function f in our L-scale such that for sufficiently large $\lambda \in d$, $g(\lambda) < f(\lambda)$. In fact, something more is needed for the result of (11.1), namely that the preceding holds when d is any Easton set. While it is "folklore" that this follows from the Covering Lemma when $0^\sharp$ does not exist, it is tempting, but false, to think that this remains true without restriction to an Easton set, as the referee pointed out. This theme of restriction to an Easton set is also implicit in §10 (the restriction to "controlled" cardinals, see (10.1.2) and (10.3)).

In §10, we prove Lemma 3, below, the Lemma of (1.4) of [5], which states, roughly, that for the system of §9, there are enough supercoherent sequences. This is the centerpiece of this paper, and, in many ways of [5] as well, as the whole approach to [5], the precise formulation of the definition of the forcing conditions, for example, was driven by the plan of using Lemma 3 to underly the proof of distributivity.

Lemma 3 is proved in two stages, first, by proving, in (10.2), that there are enough strongly coherent sequences, and then, in (10.3) - (10.5), that if $(\mathcal{N}_i | i \leq \theta)$ is strongly coherent then $(\mathcal{N}_{\omega i} |\ \leq \theta)$ is supercoherent. The arguments of (10.3) - (10.5) use the most intricate properties of the system of §§1 - 6. In §11, we close by proving two other, smaller results needed in [5]: in (11.1), we prove the Proposition of (1.5) of [5] which plays an important role in the proof, in (4.3) of [5] that the "very tidy" conditions are dense, and in (11.2) we prove the result mentioned above about the b_α

for α such that *card* α is inaccessible.

Before stating Theorem 1, Corollary 2 and Lemma 3, we should say that our notation is intended to either be standard or have a clear meaning, or is introduced as needed. It may, however, be worth pointing out that we use the same definitions of $U(\kappa)$, for κ a limit cardinal, as in [5]; for singular κ this is reintroduced in (9.2); for inaccessible κ this is reintroduced in (11.2). One instance where notation is required to do double duty is S_α; throughout most of Part I, this is the notion introduced in (1.1), but on two occasions in the proof of (3.1), explicitly noted when they occur, the same notation refers to Jensen's auxiliary hierarchy of [3].

We turn now to the statements of our main results.

Theorem 1. *($V = L$) The Squarer Scales principle of (1.2), below, holds.*

Corollary 2. *($0^\sharp$ does not exist and $V = L[A]$, where A is as given by Lemma 3 of* [3]*) There is fine system of squares and pseudo-scales, i.e., one satisfying (A) - (D) of (9.4).*

Lemma 3. *(... as in Corollary 2 ...) The system of Corollary 2 has the additional property that whenever $\mathcal{M}$, ν and θ are as in (10.1), below, and $C \subseteq [H_{\nu^+}]^\theta$ is club then there is super $\mathcal{M}$-coherent $(\mathcal{N}_i | i \leq \theta)$, with each $|\mathcal{N}_i| \in C$.*

PART I: LIFE IN L.

In Part I, comprising §§1 - 8, we develop the L-combinatorics summarized in the *Squarer Scales* principle of §1. This is a strengthening of the Squared Scales principle from [2]. In §2 we review material from §§1, 2 of [2]. In §3 we pause to give a more explicit (and perhaps clearer) development of certain crucial ideas implicit in §§2, 3 of [2]; we then return to reviewing the material of §3 of [2]. In §4 we introduce a new fine structure parameter, and prove some its important properties. Finally in §5, we rework the construction of §§2 - 3 of [2] based on this new parameter, and we prove the important lemmas which are the analogues of those of §4 of [2]. This culminates, in §6, in the proof of:

Theorem 1. *In L, Squarer Scales holds.*

In §7, we recall Jensen's construction from [3] of a square system defined on ordinals, which, in L, are singular cardinals. Finally, in §8, we note that the techniques of §§1 - 6 allow us to construct "local versions" of the squares and scales obtained there. More precisely, if $\tau = \aleph_2$, or τ is a limit cardinal, if $X_\tau \subseteq \tau^+$ is such that, letting $\mu = \tau^{+\omega}$, $H_\mu = L_\mu[X_\tau]$, then, in $L[X_\tau]$, working as in §§1 - 6, we construct a scale between μ and μ^+, and for cardinals, λ, with $\tau < \lambda \leq \mu$, a square system between λ and λ^+, which will enjoy all the properties of the "global" system constructed in L. In fact, we will not really need all of the properties, here, and notably, not the Condensation Coherence properties, but the construction is the same, and they fall out anyway. Of course, the X_τ we have in mind are the $A \cap \tau^{+\omega}$, where A is as guaranteed by [4], and as in Lemma 3 of [5].

As in [2], it will simplify notation if we assume, throughout §§1 - 7, that $\mathbf{V} = \mathbf{L}$. As there, however, this is purely a matter of notational convenience.

§1. SQUARER SCALES.

We state *Squarer Scales*, and point out how it is stronger than the principle of [2]. We state the strengthened principle in a notation designed to be suggestive of that of [5] rather than in the notation of [2]. Thus, we write $\tilde{f}_{\omega\nu}$ where Φ^ν was used in [2], etc. We have however, kept the same organization of items as in (4.11), of [2]. The principal difference in the principles is that our (B)(5) is stronger than that of (4.11) of [2], as our (B)(5) handles the $g \in \mathcal{S}(\kappa)$ (see below) and not just subfunctions of such g whose domains are cofinal subsets of κ of small cardinality. We need some preliminary definitions, which carry over to the rest of Part I.

(1.1) Definition. *S will denote the class of ordinals, ν, such that there is $\omega < \alpha < \omega\nu$ for which $J_\nu \models$ "α is the largest cardinal". For $\nu \in S$, α_ν is the unique such α. S_α is $\{\nu \in S | \alpha_\nu = \alpha\}$. For limit cardinals, κ, $\mathcal{S}(\kappa)$ is the set of functions, g, such that dom g is a final segment of the uncountable cardinals smaller than κ, and for $\overline{\kappa} \in$ dom g, $g(\overline{\kappa}) \in (\overline{\kappa}, \overline{\kappa}^+)$. As usual, if $f,\ g \in \mathcal{S}(\kappa)$, $f <^* g$ if for some κ_0, $\omega < \kappa_0 < \kappa$ for all cardinals, $\overline{\kappa}$ with $\kappa_0 \leq \overline{\kappa} < \kappa$, $\tilde{f}(\overline{\kappa}) < \tilde{g}(\overline{\kappa})$ and $f \leq^* g$ iff the final "$<$" is replaced by "$\leq$".*

We should note that the above is the "official" definition of S_α, but that in §3, below, we use this notation for a different notion. This will noted when it occurs.

(1.2) THE PRINCIPLE.

There is a sequence $(C_{\omega\nu} | \nu \in S)$, and for each limit cardinal, κ, a sequence $(\tilde{f}_{\omega\nu} | \nu \in S_\kappa$ & o.t. $C_{\omega\nu} < \kappa)$ such that:

(A) For all $\nu \in S$, letting $\alpha = \alpha_\nu$:
  (1) $C_{\omega\nu}$ is a closed subset of $\{\omega\tau | \tau \in S_\alpha\} \cap \omega\nu$; *sup* $C_{\omega\nu} < \omega\nu \Rightarrow$ *cf* $\omega\nu = \omega$,
  (2) $\omega\gamma \in C_{\omega\nu} \Rightarrow C_{\omega\gamma} = \omega\gamma \cap C_{\omega\nu}$,
  (3) o.t. $C_{\omega\nu} \leq \alpha$, and if α is a singular cardinal, then for sufficiently large $\nu \in S_\alpha, <$ holds.

(B) For all limit cardinals, κ, all $\nu \in S_\kappa$, *o.t.* $C_{\omega\nu} < \kappa \Rightarrow \tilde{f}_{\omega\nu} \in \mathcal{S}(\kappa)$ and:

(1) $\overline{\kappa} \in dom\ \tilde{f}_{\omega\nu} \Rightarrow (\tilde{f}_{\omega\nu}(\overline{\kappa})$ is a limit ordinal and $\overline{\nu} \in S_{\overline{\kappa}}$, where $\omega\overline{\nu} = \tilde{f}_{\omega\nu}(\overline{\kappa}))$,
(2) $\overline{\kappa} \in dom\ \tilde{f}_{\omega\nu} \Rightarrow (\forall\omega\lambda \in C_{\omega\nu})\overline{\kappa} \in dom\ \tilde{f}_{\omega\lambda}$, and $\tilde{f}_{\omega\lambda}(\overline{\kappa}) \leq \tilde{f}_{\omega\nu}(\overline{\kappa})$,
(3) $(\forall\tau \in S_\kappa \cap \nu) o.t.\ C_{\omega\tau} < \kappa \Rightarrow \tilde{f}_{\omega\tau} <^* \tilde{f}_{\omega\nu}$,
(4) Suppose that $sup\ C_{\omega\nu} = \omega\nu$ & $\overline{\kappa} \in dom\ \tilde{f}_{\omega\nu}$. If $(\tilde{f}_{\omega\lambda}(\overline{\kappa})|\omega\lambda \in C_{\omega\nu})$ is not eventually constant then $\tilde{f}_{\omega\nu}(\overline{\kappa}) = sup\ \{\tilde{f}_{\omega\lambda}(\overline{\kappa})|\omega\lambda \in C_{\omega\nu}\}$,
(5) if κ is singular, then whenever $g \in \mathcal{S}(\kappa)$, there is $\nu_0 \in S_\kappa$ such that $o.t.\ C_{\omega\nu_0} < \kappa$ and $g <^* \tilde{f}_{\omega\nu_0}$.

(C) For limit cardinals, κ, & $\nu \in S_\kappa$, if $o.t.\ C_{\omega\nu} < \kappa$ and $\overline{\kappa} \in dom\ \tilde{f}_{\omega\nu}$, then, letting $\omega\overline{\nu} = \tilde{f}_{\omega\nu}(\overline{\kappa})$ and $\Phi = \{\tilde{f}_{\omega\lambda}(\overline{\kappa})|\omega\lambda \in C_{\omega\nu}\}$:
(1) Φ is a final segment of $C_{\omega\overline{\nu}}$ (we take this to include the case where $C_{\omega\overline{\nu}}$ is bounded in $\omega\overline{\nu}$ and $\Phi = \emptyset$),
(2) $\tilde{f}_{\omega\overline{\nu}} = \tilde{f}_{\omega\nu}|\overline{\kappa}$,
(3) $\Phi \in J_{\overline{\beta}}$, whenever $J_{\overline{\beta}} \models$ "$\omega\overline{\nu}$ is not a cardinal".

Remark. *We only use the scales for κ which are singular cardinals, but the construction gives them for inaccessibles as well. In §9, we ignore the scales for inaccessibles.*

§2. REVIEW OF §§1 - 2 OF [2].

(2.1) THE COLLAPSING STRUCTURES.

(2.1.1) Definition. *For $\nu \in S$, if $\omega\nu$ is not a cardinal, $\beta(\nu)$ is the least $\beta \geq \nu$ such that $J_{\beta+1} \models$ "$\omega\nu$ is not a cardinal".*

Let $\beta = \beta(\nu)$; then, for some n there is f, which is Σ_{n+1}-definable over J_β (in parameters from J_β) and f is a map onto $\omega\nu$ from a subset of a smaller ordinal.

(2.1.2) Definition. *$n(\nu)$ is the least n such that there is such an f which is Σ_{n+1}-definable over J_β (in parameters from J_β). Let $n = n(\nu)$; then $\rho(\nu)$ is ρ^n_β, the n^{th}-projectum of β, $A(\nu) = A^n_\beta =$ the n^{th}-master-code of β, and setting $\rho = \rho(\nu)$, $A = A(\nu)$, $\mathcal{A}(\nu) = (J_\rho, \in, A)$. It can be shown that $\rho^{n+1}_\beta \leq \alpha_\nu$ and $\nu \leq \rho$, so that for some finite set of ordinals, $p \subseteq \omega\rho$, all elements of J_ρ are Σ_1- definable in $\mathcal{A}(\nu)$ (i.e., are unique solutions in $\mathcal{A}(\nu)$ of Σ_1-formulas in one free variable) using parameters from $\alpha_\nu \cup p$.*

We abbreviate this last assertion by writing: $J_\rho = h``(\omega \times (\alpha_\nu \cup p))$, where $h = h_{\mathcal{A}(\nu)} = h_\nu$ is the canonical Σ_1-Skolem function for $\mathcal{A}(\nu)$. We let $p(\nu)$ be the least such p with respect to the lexicographic ordering of the *decreasing enumeration of* p. *Then,* $\mathcal{A}^+(\nu) = (\mathcal{A}(\nu),\ p(\nu))$ ($p(\nu)$ *is a new individual constant). This is the collapsing structure for* ν.

An important and useful fact is provided by Corollary (1.8) of [2]: if $n = 0$ then there is a largest cardinal γ in the sense of J_β, and $p(\nu) \not\subseteq \gamma$; further, if $(X,\ \in) \prec_{\Sigma_1} (J_\beta,\ \in)$ and $p(\nu) \in X$, then $\gamma \in X$.

(2.2) CLOSING THE CLASS OF COLLAPSING STRUCTURES.

We close off the class of collapsing structures under transitive collapses of (constructible) rudimentarilly-closed substructures.

Definition. $\mathcal{O}^+ := \{\mathcal{A}^+(\nu) | \nu \in S,\ \omega\nu$ *is not a cardinal*$\}$. *We let* $(\mathcal{B},\ q) \in \tilde{\mathcal{O}}^+$ *iff* $|\mathcal{B}|$ *is transitive and for some* $(\mathcal{A},\ p) \in \mathcal{O}^+$, $(\mathcal{B},\ q)$ *is isomorphic to a (constructible)* $rud(\mathcal{A})$*-closed substructure of* $(\mathcal{A},\ p)$.

Thus, if $(\mathcal{B},\ q) \in \tilde{\mathcal{O}}^+$, $\mathcal{B}$ is amenable and of the form $(J_{\rho'},\ \in,\ A')$. Further, $\tilde{\mathcal{O}}^+$ is closed for taking transitive collapses of constructible $rud(\mathcal{B})$-closed substructures of $(\mathcal{B},\ q) \in \tilde{\mathcal{O}}^+$; in particular, it is closed under amenable initial segments and transitive collapses of constructible Σ_1-elementary substructures.

(2.3) A SQUARE SYSTEM ON $\tilde{\mathcal{O}}^+$.

For $s = (\mathcal{A},\ p) = (J_\rho,\ \in,\ A,\ p) \in \tilde{\mathcal{O}}^+$, a closed subset, $\tilde{C}_s \subseteq \omega\rho$ is constructed; $\tilde{C}_s$ is cofinal in $\omega\rho$ if $cf\ \omega\rho > \omega$. Crucial in the definition and structure of $\tilde{C}_s$ are the sets $\Delta(\xi,\ s)$ for $\xi < \omega\rho$, where $\omega\delta \in \Delta(\xi,\ s)$ iff $\omega\delta < \omega\rho$ and for some β, $\omega\delta = sup\ h_s``(\omega \times (\beta \cup \{\xi\}))$. Recall that for a set X of ordinals, X' is the set of limit points of X, below $sup\ X$. First, consider $\Delta(0,\ s)$: if this is empty, $cf\ \omega\rho = \omega$ and $\tilde{C}_s = \emptyset$. If this is cofinal in $\omega\rho$, then $\tilde{C}_s = (\Delta(0,\ s))'$; of course, if $\Delta(0,\ s)$ is cofinal in $\omega\rho$, then $\tilde{C}_s$ is cofinal in $\omega\rho$ if $cf\ \omega\rho > \omega$. The remaining case is when $\Delta(0,\ s)$ has a largest element, $\omega\delta$. Then, for some β, $\omega\delta = sup(OR \cap h_s``(\omega \times \beta))$, but $\omega\rho = sup(OR \cap h_s``(\omega \times (\beta + 1)))$; note that this can occur even if $cf\ \omega\rho > \omega$, since we must consider all the unique solutions in s of Σ_1 formulas $\phi(\nu_0,\ \xi_1,\ \cdots,\ \xi_n,\ \beta)$, where $\xi_1,\ \cdots,\ \xi_n < \beta$; so all we have for certain is that $cf\ \omega\rho \leq cf\ \beta$.

In this case, we set $\beta = \beta^0_s$, $\delta = \delta^1_s (\delta^0_s = 0$, for all $s)$. We have the same trichotomy for $\Delta(\omega\delta^1_s,\ s)$: if $\Delta(\omega\delta^1_s,\ s) = \emptyset$, $cf\ \omega\rho = \omega$ and in this case, $\tilde{C}_s = (\Delta(0,\ s))'$; if $\Delta(\omega\delta^1_s,\ s)$ is cofinal in $\omega\rho$ then $\tilde{C}_s = (\Delta(0,\ s))' \cup (\Delta(\omega\delta^1_s,\ s))'$. Finally, if $\Delta(\omega\delta^1_s,\ s)$ has a largest element, $\omega\delta$, then we have $\beta = \beta^1_s$, $\delta = \delta^2_s$ such that $\omega\delta^2_s$ is the largest element of $\Delta(\omega\delta^1_s,\ s)$, $\omega\delta^2_s = sup(OR \cap h_s``(\omega \times (\beta \cup \{\omega\delta^1_s\})))$ and $\omega\rho = sup(OR \cap h_s``(\omega \times ((\beta + 1) \cup \{\omega\delta^1_s\})))$.

The crucial observation, proved in (2.40) of [2], is that, in this case, $\beta^0_s > \beta^1_s$. Thus, the process terminates after a finite number, $m_s \geq 1$, of steps; in all cases, $\tilde{C}_s = \bigcup\{(\Delta(\omega\delta^i_s,\ s))'|i < m_s\}$. If $i = m_{s-1},\ \delta = \delta^i_s$, then $\tilde{C}_s$ has a (possibly empty) final segment, $(\Delta(\omega\delta,\ s))'$ and if $cf\ \omega\rho > \omega$ then $(\Delta(\omega\delta,\ s))'$ is cofinal in $\omega\rho$, since otherwise $\Delta(\omega\delta,\ s)$ would have a largest element (the other possibilities are eliminated by the cofinality hypothesis), which is impossible since the process terminates after m_s steps.

It is not really necessary to "thin out" by taking only the limit points of the $\Delta(\omega\delta^i_s,\ s)$, but this slightly facilitates the proof of the coherence property of the $\tilde{C}_s$: if $\omega\delta \in \tilde{C}_s$, then, setting $s' = s|J_\delta,\ \tilde{C}_{s'} = \tilde{C}_s \cap \omega\delta$. As an important preliminary step it is shown that if, for $t \in \tilde{\mathcal{O}}^+$, we let $a_t = \{\omega\delta^i_t|i < m_t\}$, then, for all $s \in \tilde{\mathcal{O}}^+$ and all $\omega\delta \in \tilde{C}_s$, letting $s' = s|J_\delta,\ a_{s'} = a_s \cap \omega\delta$. Of course, if we chose not to thin out, then the coherence property would hold for $\omega\delta \in (\tilde{C}_s)'$, and we could, by choosing constructible cofinal ω-sequences in the appropriate cases, guarantee that $\tilde{C}_s$ is <u>always</u> cofinal in $\omega\rho$. Jensen has taken this approach in [1], where the cofinal ω-sequences are chosen in a canonical and natural fashion.

§3. A CLOSER LOOK AT THE $X_{\mu',\ 0,\ s}$, AND §3 OF [2].

We prove three Lemmas related to the structure of the $X_{\mu',\ 0,\ s}$. The first, in (3.1), guarantees that when μ is a singular cardinal, $\nu \in S_\mu$ and ν is sufficiently large that $\mathcal{A}(\nu) \models$ "μ is singular", then, letting $s = \mathcal{A}^+(\nu)$, for some $\mu' < \mu,\ X_{\mu',\ 0,\ s}$ is cofinal in $\omega\rho$. This is certainly well known to fine-structure experts, but was never stated explicitly in [2]. For completeness, we give it here. Some of the ideas involved in (3.1) and (3.2) appear in the proofs of (4.1) and (4.3) of [2].

The second Lemma, in (3.2) shows that when μ' is as guaranteed by (3.1), then, under two additional, mild assumptions, $\tilde{C}_s \subseteq X_{\mu',\ 0,\ s}$. The third Lemma, (3.3), explores what occurs when $X_{\mu',\ 0,\ s}$ is not cofinal in $\omega\rho$. Essentially, it shows that if $s' = s|\delta'' \in \tilde{\mathcal{O}}^+$, then, at least as far as $X_{s',0,\mu'}$ is concerned, we can assume without loss of generality that either $\delta'' = \rho(s)$ or that $\delta'' \in X_{\mu',\ 0,\ s}$. These Lemmas will be heavily used in §§5, 6, below.

(3.1) Lemma. *Assume that μ is a singular cardinal, $\nu \in S_\mu$ and ν is sufficiently large that $\mathcal{A}(\nu) \models$ "μ is singular", and let $s = \mathcal{A}^+(\nu)$. Then, for some $\mu' < \mu$, $X_{\mu',\ 0,\ s}$ is cofinal in $\omega\rho(\nu)$.*

Proof. Let $f : a \rightarrow_{onto} \omega\nu$ be $\Sigma_1(\mathcal{A}(\nu))$ in parameters $\vec{y} \in J_{\rho(\nu)}$. Suppose, e.g., that ϕ is a Σ_1 formula such that $\zeta = f(\xi) \Leftrightarrow \mathcal{A}(\nu) \models \phi(\zeta,\ \xi,\ \vec{y})$. Let θ be the Σ_0 formula such that ϕ is $\exists v_0\theta(v_0,\ \zeta,\ \xi,\ \vec{y})$. Let $\theta'(\eta,\ \xi)$ be:

$$\vec{y} \in S_\eta \wedge (\exists v_0 \in S_\eta)(\exists\zeta < \eta)\theta(v_0,\ \zeta,\ \xi,\ \vec{y}).$$

In the above, S_η is the η^{th} stage in Jensen's auxiliary hierarchy, not the notion defined in (1.1), above. Note that if $\theta'(\eta,\ \xi)$ and

$\eta < \eta^*$, then $\theta'(\eta^*\ \xi)$. Let $g(\xi) \simeq$ the least η such that $\theta'(\eta,\ \xi)$. Thus, g is Σ_0.

Well known arguments (involving the downward extension of embeddings Lemma) then show:

$$(\forall \mu' \leq \mu)[\omega\nu = sup\ f``(a \cap \mu') \Leftrightarrow \omega\rho = sup\ g``(a \cap \mu')].$$

Thus, if there is $\mu' < \mu$ with $\omega\rho = sup\ g``(a \cap \mu')$, (3.1) holds, so, towards a contradiction, assume that $g``(a \cap \mu')$ is bounded in μ for all $\mu' < \mu$. For such μ', let $\sigma(\mu') = sup\ g``(a \cap \mu')$. Also, let $g_{\mu'} = g|\mu'$, so $g_{\mu'} \subseteq \mu' \times \sigma(\mu')$ and $g_{\mu'}$ is definable over $S_{\sigma(\mu')}$ (Jensen's auxiliary hierarchy again).

This makes it clear that each $g_{\mu'} \in J_{\rho(\nu)}$, and, in fact that $\mu' \mapsto g_{\mu'}$ is $\Sigma_1(\mathcal{A}(\nu))$ in parameters $\overrightarrow{y}$. But then, the same holds for $\mu' \mapsto \sigma(\mu')$, and, denoting this last function by σ, σ is non-decreasing with domain μ. Now, let $g^* \in J_{\rho(\nu)}$ be a map of a subset of some $\mu' < \mu$ cofinally into μ. Then, the function $\sigma \circ g^*$ is $\Sigma_1(\mathcal{A}(\nu))$ and maps a subset of μ' cofinally into $\omega\rho(\nu)$, contradiction. This completes the proof.

Remark. *If $\mu,\ \nu,\ s$ are as in (3.1), then (3.1) clearly gives that o.t. $\Delta(0,\ s) < \mu$, and therefore, for all $\xi \in a_s$, o.t. $\Delta(\xi,\ s) < \mu$. But then, clearly o.t. $\tilde{C}_s < \mu$.*

(3.2) Lemma. *If $\mu,\ \nu$, etc., are as in (3.1), then whenever $\mu' < \mu$ is as guaranteed by (3.1), $(a_s \subseteq X_{\mu',\ 0,\ s}$ & o.t. $\tilde{C}_s \leq \mu') \Rightarrow \tilde{C}_s \subseteq X_{\mu',\ 0,\ s}$.*

Proof. Let $f_{\mu',\ 0,\ s} = (\overline{s},\ |f|,\ s)$, where $|f|$ is the inverse of the transitive collapsing map for $s|X_{\mu',\ 0,\ s}$. We first argue in the case where $\tilde{C}_s$ is cofinal in $\omega\rho$. Then, applying (2.31)(b) of [2] to $f_{\mu',\ 0,\ s}$, we get that $\tilde{C}_{\overline{s}}$ is cofinal in $\omega\overline{\rho}$, where $\overline{\rho} = \rho(\overline{s})$. But then, since *range* $|f|$ is cofinal in $\omega\rho$, in fact, $|f|``\tilde{C}_{\overline{s}}$ is cofinal in $\omega\rho$, and by (2.31)(c) of [2], $|f|``\tilde{C}_{\overline{s}} \subseteq \tilde{C}_s$, so $X_{\mu',\ 0,\ s} \cap \tilde{C}_s$ is cofinal in $\omega\rho$. But then, let $\omega\delta \in X_{\mu',\ 0,\ s} \cap \tilde{C}_s$. Since $\tilde{C}_{s|\delta}$ is an initial segment of $\tilde{C}_s$, *o.t.* $\tilde{C}_{s|\delta} < \mu'$. Finally, by (2.25) of [2] (whose statement contains a typo; the statement should read: "$\cdots$, $\delta < \rho(s)$ and $s|\delta \in \tilde{\mathcal{O}}^+$ then $\cdots$ "), it easily follows that $\tilde{C}_{s|\delta} \in X_{\mu',\ 0,\ s}$. But then, since *o.t.* $\tilde{C}_{s|\delta} < \mu'$, in fact $\tilde{C}_{s|\delta} = \tilde{C}_s \cap \omega\delta \subseteq X_{\mu',\ 0,\ s}$. Thus, arbitrarily large initial segments of $\tilde{C}_s$ are included in $X_{\mu',\ 0,\ s}$.

This completes the proof when $\tilde{C}_s$ is cofinal in $\omega\rho$. When $\tilde{C}_s = \emptyset$, there is nothing to prove. So, suppose that $\tilde{C}_s$ has a greatest element. Since $X_{\mu',\ 0,\ s}$ is cofinal in $\omega\rho$, it follows from (2.31) and (2.38) of [2] that if $f_{\mu',\ 0,\ s}$ is as above, then $\tilde{C}_{\overline{s}}$ has a largest element and that $|f|(max\ \tilde{C}_{\overline{s}}) = max\ \tilde{C}_s$. Then, arguing as above, and appealing, once again, to (2.31) and also (2.38) of [2], the conclusion is clear.

(3.3)

In dealing with the situation where $X_{\mu',\ 0,\ s}$ is not cofinal in $\omega\rho(s)$, it will facilitate some of the arguments to replace μ' by $\mu'+1$, so that, letting $f = |f_{\mu'+1,\ 0,\ s}|,\ f(\mu',\ \mu'+1) = (\mu',\ \mu'+1)$. This also is faithful to the context in which we shall apply this material, in §§5, 6, below. We adopt the same notation as in (3.1) and (3.2), but with $\delta = \delta(f_{\mu'+1,\ 0,\ s}) < \rho(s)$.

Lemma. *Whenever $\delta'' \le \rho(s)$ and $s|\delta'' \in \tilde{\mathcal{O}}^+$, there is $\delta^* \in \{\rho(s)\} \cup X_{\mu'+1,\ 0,\ s}$ such that $s|\delta^* \in \tilde{\mathcal{O}}^+$ and $|f_{\mu'+1,\ 0,\ s|\delta''}| = |f_{\mu'+1,\ 0,\ s|\delta^*}|$.*

Proof. Suppose, first, that $\delta \le \delta'' < \rho(s)$. By (2.23) of [2], $f = |f_{\mu'+1,\ 0,\ s|\delta''}|$. Next, suppose that $f(\omega\overline{\delta}^*) = \omega\delta^* > sup\ f``\omega\overline{\delta}^*$, where $\overline{s}|\overline{\delta}^* \in \tilde{\mathcal{O}}^+$. Let $g = (\overline{s}|\overline{\delta}^*,\ f|J_{\overline{\delta}^*},\ s|\delta^*)$. By (2.32) of [2], $s|\delta^* \in \tilde{\mathcal{O}}^+$ and $f|J_{\overline{\delta}^*} : \overline{s}|\overline{\delta}^* \to_{\Sigma_\omega} s|\delta^*$. But then, we clearly have that $X_{\mu'+1,\ 0,\ \overline{s}|\overline{\delta}^*}$ is cofinal in $\omega\overline{\delta}^*$ iff $X_{\mu'+1,\ 0,\ s|\delta^*}$ is cofinal in $\omega\delta^*$. However, since $X_{\mu'+1,\ 0,\ s|\delta^*} \subseteq X_{\mu'+1,\ 0,\ s} \cap J_{\delta^*}$, and $sup\ f``\omega\overline{\delta}^* = sup\ (X_{\mu'+1,\ 0,\ s} \cap J_{\delta^*})$, clearly $X_{\mu'+1,\ 0,\ \overline{s}|\overline{\delta}^*}$ is not cofinal in $\omega\overline{\delta}^*$, i.e., $\delta(f_{\mu'+1,\ 0,\ \overline{s}|\overline{\delta}^*}) < \overline{\delta}^*$. Let $\overline{\delta}' = \delta(f_{\mu'+1,\ 0,\ \overline{s}|\overline{\delta}^*})$, and let $\delta' = |f|(\overline{\delta}')$. By (2.30) of [2], $\delta' = \delta(f_{\mu'+1,\ 0,\ s|\delta^*})$.

Then, $\delta' < \delta(g) = sup\ f``\omega\overline{\delta}^*$. Finally, if $\delta' \le \delta'' < \delta^*$, applying (2.23) of [2], with $s|\delta^*$ in place of s, g in place of f, $\delta' = \delta(g)$ in place of $\delta(f)$ (in the notation of (2.23) of [2], $\delta = \delta(f)$), we have $|f_{\mu'+1,\ 0,\ s|\delta''}| = |f_{\mu'+1,\ 0,\ s|\delta^*}|$. This completes the proof.

(3.4) "PROJECTING" A TAIL OF $\tilde{C}_s$ TO A SUBSET OF $\omega\nu$.

In §3 of [2], $C_{\omega\nu}$ is defined, for $\nu \in S$, $\omega\nu$ not a cardinal. First, a final segment of $\tilde{C}_s$ is chosen, where $s = \mathcal{A}^+(\nu)$.

(3.4.1) Definition. *Let $\omega\delta \in \tilde{C}_s$, $\omega\delta > \alpha_\nu$, let $s' = s|J_\delta$, and let $Y = Y_{\delta,\ \nu} = h_{s'}``(\omega \times \alpha_\nu)$; then, $\omega\delta \in \hat{C}_\nu$ iff $\alpha_\nu \in Y$.*

It is shown in (3.2)(b) and (3.3) of [2] that if $\omega\rho = \omega\nu$, then $\hat{C}_\nu = \tilde{C}_s$. For $\omega\delta \in \hat{C}_\nu$, it is shown, in (3.2)-(3.4) of [2], that there is unique λ such that $\lambda \in S_\alpha$ and $\mathcal{A}^+(\lambda) \cong s'|Y_{\delta,\ \nu} = s|Y_{\delta,\ \nu}$.

(3.4.2) Definition. *For $\omega\delta \in \hat{C}_\nu$ we set $\lambda(\delta,\ \nu) =$ the unique $\lambda \in S_\alpha$ such that $\mathcal{A}^+(\lambda) \cong s'|Y_{\delta,\ \nu} = s|Y_{\delta,\ \nu}$.*

If $\rho = \nu$, then $\lambda = \delta$, as is shown in (3.3) of [2]. An important observation is made in (3.2)(a) of [2]: $Y_{\delta,\ \nu}$ is cofinal in $\omega\delta$.

(3.4.3) Definition. $C_{\omega\nu} = \{\omega\lambda(\delta,\ \nu) | \omega\delta \in \hat{C}_\nu\}$.

It is then shown in (3.6) - (3.8) that the C_ν have the correct properties, i.e., those of (A) of (1.2), above.

§4. A NEW PARAMETER.

Our main tool in proving the strengthened version, (B)(5) of (1.1), above, of the (B)(5) of [5], is a small but potentially quite useful Lemma, below, involving a new parameter which we now introduce. Then, in §5, we supply the arguments which replace those of §4 of [2], making the changes and improvements enabled by this Lemma.

(4.1) Definition. *Let* $\nu \in S,\ \rho = \rho(\nu),\ \mathcal{A} = \mathcal{A}(\nu),\ \alpha = \alpha_\nu,\ \alpha \le \tau \le \omega\rho$. *Let* $R_\nu(p,\ \tau)$ *be the property:* $p \in [\omega\rho]^{<\omega} \&\ h_{\mathcal{A}}``(\omega \times (\tau \cup p)) = J_\rho$. *Let* $P_\nu(p)$ *be the property* $R_\nu(p,\ \alpha)$*; let* $Q_\nu(p)$ *be the property* $R_\nu(p,\ \alpha+1)$. *So* $p(\nu) =$ *the least* p *such that* $P_\nu(p)$*, with respect to lexicographic order of the decreasing enumeration of finite subsets of* $\omega\rho$. *Analogously,* <u>*define*</u> $q(\nu) =$ *the least* q *such that* $Q_\nu(q)$*, with respect to the same ordering.*

Remarks.

(1) $p(\nu) \cap \alpha_\nu = \emptyset$; $q(\nu) \cap \alpha_\nu + 1 = \emptyset$,
(2) $q(\nu) = p(\nu) \Leftrightarrow \alpha_\nu \in h_{\mathcal{A}(\nu)}``(\omega \times (\alpha_\nu \cup q(\nu)))$,
(3) $\forall r(P_\nu(r) \Rightarrow Q_\nu(r))$, *so* $q(\nu) \le_* p(\nu)$,
(4) $P(\{\alpha_\nu\} \cup q(\nu))$; *thus* $p(\nu) \le_* \{\alpha_\nu\} \cup q_\nu$.

(4.2) Lemma. *Either* $p(\nu) = q(\nu)$ *or* $p(\nu) = q(\nu) \cup \{\alpha_\nu\}$.

Proof. Let $p = p(\nu),\ q = q(\nu),\ \alpha = \alpha_\nu$. Note that by Remark 3, if $p \neq q$, then there is $c \subset p$ which is a common final segment of p and q and either $c = q$ or else the largest member of $q \setminus c$ is less than the largest member of $p \setminus c$. However, by Remark 4, there is $d \subseteq q \cup \{\alpha\}$ which is a common final segment of p and $q \cup \{\alpha\}$, and if $d \neq p$, then $d \neq q \cup \{\alpha\}$ and the largest member of $p \setminus d$ is less than the largest member of $(q \cup \{\alpha\}) \setminus d$. In the latter case the largest member of $(q \cup \{\alpha\}) \setminus d$ must be greater than α so it is simply the largest member of $q \setminus d$, and we have a contradiction. Thus, we must have that $d = p$. If $\alpha \notin p$, then $p \subseteq q$, which is also impossible. Thus, $\alpha \in p$ and $p = q \cup \alpha$.

The main difference between the arguments in §§5, 6, below, and those of §4 of [2] is that for $s = \mathcal{A}^+(\nu)$, below, we use $X^*_{\mu',\ 0,\ s} = X_{\mu'+1,\ 0,\ s} = h_s``(\omega \times (\mu'+1))$, whereas, in §4 of [2], we used $X_{\mu',\ 0,\ s} = h_s``(\omega \times \mu')$. Of course, $X_{\mu',\ 0,\ s} \subseteq X^*_{\mu',\ 0,\ s}\ \&\ \mu' \in X^*_{\mu',\ 0,\ s}$. By the above Lemma, either $p(\nu) \neq q(\nu)$, in which case, we have $X_{\mu',\ 0,\ s} = X^*_{\mu',\ 0,\ s}$, or else $\alpha_\nu \notin p(\nu)$. One main observation is that none of this really depends on μ'.

§5. REWORKING §4 OF [2].

In this Section, we rework the material corresponding to (4.5) - (4.10) of [2]. There is no analogue of (4.8), however, because of our use of the $X^*_{\mu',\ 0,\ s}$. (5.1) corresponds to (4.5) of [2]. (5.2) corresponds to (4.7), of [2], in ideas, if not in statement. In (5.3), we define the $\tilde{f}_{\omega\nu}$ (the analogous definitions in [2] were (4.6) and (4.9)). (5.4) corresponds to (4.10) of [2] and establishes the Condensation Coherence property, (C)(1) of (1.2).

(5.1) Lemma. *If $\nu \in S_\mu$, μ is a limit cardinal, $s = \mathcal{A}^+(\nu)$, if $\mu' < \mu$, μ' is a cardinal and $f^*_{\mu',\,0,\,s} = (\overline{s},\ |f|,\ s)$, where $|f| : \overline{s} \to s|X^*_{\mu',\,0,\,s}$ is the inverse of the transitive collapsing map, then there is a unique $\overline{\nu} \in S_{\mu'}$, such that either $\omega\overline{\nu} = OR \cap |\overline{s}|$ or $\overline{s} \models$ "$\omega\overline{\nu}$ is a cardinal"; further, either $X_{\mu',\,0,\,s} = X^*_{\mu',\,0,\,s}$ & $\overline{s} = \mathcal{A}^+(\overline{\nu})$, or $X_{\mu',\,0,\,s} \neq X^*_{\mu',\,0,\,s}$, $\overline{s} = (\mathcal{A}(\overline{\nu}),\ q(\overline{\nu}))$ & $\mu' \in p(\overline{\nu}) \setminus q(\overline{\nu})$.*

Proof. The existence and uniqueness of $\overline{\nu}$ are immediate from the fact that $\mu' < sup\ X^*_{\mu',\,0,\,s}$. To get the remainder of the Lemma, we shall apply the downward extension of embeddings lemma to $|f|$. Let $n = n(\nu)$, and let $\overline{s} = (J_{\overline{\rho}},\ \overline{A},\ \overline{p})$. The downward extension of embeddings gives us a $\overline{\beta}$ and $\hat{f} :\ J_{\overline{\beta}} \to_{\Sigma_{n+1}} J_{\beta(\nu)}$, $|f| \subseteq \hat{f}$, such that $\overline{\rho} = \rho^n_{\overline{\beta}}$ and $\overline{A} = A^n_{\overline{\beta}}$. Since $h_{\overline{s}}$"$(\omega \times (\mu' + 1)) = J_{\overline{\rho}}$, as usual we have $\overline{\beta} = \beta(\overline{\nu})$ and $n \geq n(\overline{\nu})$.

For the reverse inequality, if $n = 0$, there is nothing to prove, so suppose $n > 0$. Then, if $n > n(\overline{\nu})$, exactly as in (3.1), (3.3) and (3.4) of [2], we would have $\rho(\overline{\nu}) \geq \rho^{n(\overline{\nu})+1}_{\overline{\beta}} \geq \overline{\rho}$, on the one hand, but $\rho^{n(\overline{\nu})+1}_{\overline{\beta}} \leq \mu' + 1 < \overline{\nu} \leq \overline{\rho}$, on the other hand, a contradiction.

Thus, $\overline{s} = (\mathcal{A}(\overline{\nu}),\ \overline{p})$. Of course, $|f|(\overline{p}) = p(\nu)$ and, by construction, $J_{\overline{\rho}} = h_{\mathcal{A}(\overline{\nu})}$"$(\omega \times ((\mu' + 1) \cup \overline{p}))$; i.e. $Q_{\overline{\nu}}(\overline{p})$ holds. If $q \in J_{\overline{\rho}}$, $q <_* \overline{p}$, and for some $i < \omega$ and $\vec{\xi} \in [(\mu' + 1)]^{<\omega}$, $\overline{p} = h_{\mathcal{A}(\overline{\nu})}(i,\ \vec{\xi},\ q)$, then $p = h_{\mathcal{A}(\nu)}(i,\ |f|(\vec{\xi}),\ |f|(q))$ and $|f|(q) <_* p$. This, however, contradicts the definition of $p = p(\nu)$ since $|f|(\vec{\xi}) = \vec{\xi} \in [\mu]^{<\omega}$ (recall that $|f|(\mu') = \mu'$). Thus, $\overline{p} = q(\overline{\nu})$. By (4.2), either $\overline{p} = p(\overline{\nu})$ and $J_{\overline{\rho}} = h_{\mathcal{A}(\overline{\nu})}$"$(\omega \times ((\mu' \cup \overline{p}))$, in which case $\mu' \in X_{\mu',\,0,\,s}$, so $X_{\mu',\,0,\,s} = X^*_{\mu',\,0,\,s}$ and $\overline{s} = \mathcal{A}^+(\overline{\nu})$, or $p(\overline{\nu}) \neq \overline{p}$, in which case $\mu' = \alpha_{\overline{\nu}} \notin \overline{p}$, $p(\overline{\nu}) = \{\alpha_{\overline{\nu}}\} \cup \overline{p} = \{\mu'\} \cup \overline{p}$. Then, $\mu' \notin h_{\mathcal{A}(\overline{\nu})}$"$(\omega \times (\mu' \cup \overline{p}))$, so $\mu' \in X^*_{\mu',\,0,\,s} \setminus X_{\mu',\,0,\,s}$, $\overline{s} = (\mathcal{A}(\overline{\nu}),\ q(\overline{\nu}))$. Note, here, that $\mu' \cup p(\overline{\nu}) = (\mu' + 1) \cup \overline{p} = (\mu' + 1) \cup q(\overline{\nu})$.

(5.2) Propostion. *Let ν, μ, s be as in (5.1). Assume that $\hat{C}_\nu \neq \emptyset$, and let $\omega\delta_0 = inf\ \hat{C}_\nu$. In addition to our hypotheses on μ' from (5.1), suppose further that $\mu \in X_{\mu'+1,\,0,\,s|\delta_0}$. Let $X^*_{\mu',\,0,\,s}$, $f^*_{\mu',\,0,\,s}$, $|f|$, $\overline{s}$, etc. be as in (5.1). Let $\overline{\nu}$ be as guaranteed by (5.1). Let $\omega\delta \in \hat{C}_\nu$. Let $Y = Y_{\delta,\,\nu}$ and let $\lambda = \lambda(\delta,\ \nu)$. Suppose that $\delta \in X^*_{\mu',\,0,\,s}$, and let $|f|(\overline{\delta}) = \delta$. Let $\overline{\lambda}$ be as guaranteed by (5.1) with λ in the place of ν. Then, $\overline{\lambda} = \lambda(\overline{\delta},\ \overline{\nu})$.*

Proof. Note that our additional hypothesis on μ' guarantees that the analogous statement holds for *any* $\omega\eta \in \hat{C}_\nu$ in place of $\omega\delta_0$. The only real difficulty in proving the Lemma is that, in general, $X^*_{\mu',\,0,\,s|\delta} \subset X^*_{\mu',\,0,\,s} \cap J_\delta$.

Let $\pi : (J_{\rho'},\ A',\ p') \to (s|\delta)|Y$ be the isomorphism, and let $s' = (J_{\rho'},\ A',\ p')$ so, by §3 of [2], $s' = \mathcal{A}^+(\lambda)$. As remarked after (3.4.2), $\pi``\omega\rho'$ is cofinal in $\omega\delta$. By (5.1), $\overline{s} = (\mathcal{A}(\overline{\nu}),\ q(\overline{\nu}))$, with the dichotomy of the conclusion of (5.1).

Clearly, $X^*_{\mu',\ 0,\ \overline{s}|\overline{\delta}} = |f|^{-1}[X^*_{\mu',\ 0,\ s|\delta}]$ and so, letting $\overline{Y} = X^*_{\mu',\ 0,\ \overline{s}|\overline{\delta}}$, $\overline{Y}$ is cofinal in $\omega\overline{\delta}$; this follows immediately from (2.30) and (3.2)(a) of [2]. Also, here we have $\mu' \in \overline{Y}$.

The following easy observation will be important in establishing (B)(4) and (C)(1) of (1.2), above; this will be done in (5.4), below:

(∗) $\overline{Y} = X_{\mu',\ 0,\ \overline{t}}$, where $\overline{t} = (\mathcal{A}(\overline{\nu}),\ p(\overline{\nu}))|J_{\overline{\delta}}$. So, depending on the dichotomy of (5.1), either $\overline{t} = \overline{s}|J_{\overline{\delta}}$ or $\overline{t} = (\mathcal{A}(\overline{\nu}),\ \{\mu'\} \cup \overline{p})|J_{\overline{\delta}}$.

It is clear from (∗) that $\omega\overline{\delta} \in \hat{C}_{\overline{\nu}}$, in either case.

Let $\overline{\pi} : (J_{\overline{\rho}'},\ \overline{A}',\ \overline{p}') \to (\overline{s}|\overline{\delta})|\overline{Y}$ be the inverse of the transitive collapse. So $\overline{s}' := (J_{\overline{\rho}'},\ \overline{A}',\ \overline{p}') = (\mathcal{A}(\overline{\lambda}),\ q(\overline{\lambda}))$, and either $p(\overline{\lambda}) = q(\overline{\lambda})$ or $p(\overline{\lambda}) = \{\mu'\} \cup q(\overline{\lambda})$. In either case, $(\overline{s}',\ |f| \circ \overline{\pi},\ s|\delta) = f^*_{\mu',\ 0,\ s|\delta}$. Then, clearly, *range* $|f| \circ \overline{\pi} \subseteq Y$, and $(\overline{s}',\ \pi^{-1} \circ |f| \circ \overline{\pi},\ \overline{s}) = f^*_{\mu',\ 0,\ \overline{s}}$. It then follows easily that $\overline{\lambda} = \lambda(\overline{\delta},\ \overline{\nu})$, as required.

(5.3) Definition. *Let $\mu,\ \nu,\ s$ be as in (5.1). Let $\mu_1^*(\nu) =$ the least uncountable cardinal, $\mu' < \mu$, such that for all $\omega\delta \in \hat{C}_\nu,\ \mu \in X^*_{\mu'+1,\ 0,\ s|\delta}$. Thus, if $\hat{C}_\nu = \emptyset,\ \mu_1^*(\nu) = \aleph_1$. Otherwise, as remarked at the beginning of the proof of (5.2), this is just the least μ' such that $\mu \in X^*_{\mu',0,s|\delta_0}$, where δ_0 is as in (5.2). For cardinals $\mu' \in [\mu_1^*(\nu),\ \mu)$, let $\tilde{f}_{\omega\nu}(\mu') = \omega\overline{\nu}$, where $\overline{\nu}$ is as guaranteed by (5.1).*

(5.4) Proposition. *(Condensation Coherence): If $\nu,\ \mu,\ s$, etc., are as in (5.1) and $\mu_1^*(\nu) \le \mu' < \mu,\ \mu'$ a cardinal, then letting $\omega\overline{\nu} = \tilde{f}_{\omega\nu}(\mu')$:*

(a) *for all $\omega\lambda \in C_{\omega\nu},\ \mu_1^*(\lambda) \le \mu'$,*

(b) *let $\Phi = \{\tilde{f}_{\omega\lambda}(\mu')|\omega\lambda \in C_{\omega\nu}\}$; then Φ is a final segment of $C_{\omega\overline{\nu}}$ (we take this to include the case where $C_{\omega\overline{\nu}}$ is bounded in $\omega\overline{\nu}$ and $\Phi = \emptyset$),*

(c) *if μ' is a limit cardinal, then $\mu' > \mu_1^*(\nu),\ \mu_1^*(\overline{\nu}) = \mu_1^*(\nu)$ and $\tilde{f}_{\omega\overline{\nu}} = \tilde{f}_{\omega\nu}|\mu'$.*

Proof. (a) is clear and (c) follows easily from (∗) of (5.2), above. For (b), let $s = \mathcal{A}^+(\nu) = (J_\rho,\ A,\ p)$, and let $f = f^*_{\mu',\ 0,\ s} = (\overline{s},\ |f|,\ s)$. Let $\delta = \delta(f)$. If *range* $|f|$ is not cofinal in $\omega\rho$, then as we have already noted in arguing for (3.3), above, $X^*_{\mu',\ 0,\ s} = X^*_{\mu',\ 0,\ s|\delta} \subseteq Y_{\delta,\ \nu}$, so composing with π^{-1}, the isomorphism between $s|\delta$ and $\mathcal{A}^+(\lambda(\delta,\ \nu))$, we transport the whole situation down to $\lambda^* = \lambda(\delta,\ \nu)$. Now, if (b) holds between λ^* and $\overline{\nu}$, as we shall argue that it will, we can use (3.3), above, to conclude that it holds between ν and $\overline{\nu}$, since (3.3) gives that $\Phi = \{\tilde{f}_{\omega\lambda}(\mu')|\omega\lambda \in C_{\omega\lambda^*}\}$.

Thus, we may assume that $\delta = \rho$, i.e. that $range\ |f| = X^*_{\mu',\ 0,\ s}$ is cofinal in $\omega\rho$. This allows us to appeal to (2.31) of [2] to conclude that, letting $\overline{s} = (J_{\overline{\rho}},\ \overline{A},\ \overline{p})$, $\tilde{C}_s$ is cofinal in $\omega\rho$ iff $\tilde{C}_{\overline{s}}$ is cofinal in $\omega\overline{\rho}$, that $\tilde{C}_s = \emptyset$ iff $\tilde{C}_{\overline{s}} = \emptyset$, and that if $\omega\delta$, $\omega\overline{\delta}$ are the maxima of $\tilde{C}_s$, $\tilde{C}_{\overline{s}}$, respectively, then $\delta = |f|(\overline{\delta})$. Now, since $\mu^*_1(\nu) \leq \mu'$, $\mu \in X^*_{\mu',\ 0,\ s}$, so let $\overline{\mu}$ be such that $\mu = |f|(\overline{\mu})$. Recalling the last clause of (5.1), above, it is then easy to see that:

(∗) $\hat{C}_\nu \neq \emptyset$ iff there is $\omega\overline{\delta} \in \tilde{C}_{\overline{s}}$ such that $\overline{\mu} \in h_{\mathcal{A}^+(\overline{\nu})|J_{\overline{\delta}}}$"$(\omega \times \mu')$.

Thus, if $\hat{C}_\nu = \emptyset$, then $C_{\omega\overline{\nu}}$ is bounded in $\omega\overline{\nu}$ and Φ is the empty final segment of $C_{\omega\overline{\nu}}$. So, for the remainder of the proof, we assume that $\hat{C}_\nu \neq \emptyset$.

Let $\omega\delta \in \hat{C}_\nu$ and let $\lambda = \lambda(\delta,\ \nu)$. By (3.3), above, we may suppose that, as in (5.2), $\delta = |f|(\overline{\delta})$. Adopt the notation of (5.2), above. We proved there that $\omega\overline{\delta} \in \hat{C}_{\overline{\nu}}$ and that $\tilde{f}_{\omega\lambda}(\mu') = \lambda(\overline{\delta},\ \overline{\nu})$, so $\tilde{f}_{\omega\lambda}(\mu') \in C_{\omega\overline{\nu}}$, for all $\omega\lambda \in C_{\omega\nu}$. It remains only to show that letting $W = \{\omega\overline{\delta} | \omega\delta \in \hat{C}_\nu\ \cap X^*_{\mu',\ 0,\ s}\}$, then W is a final segment of $\hat{C}_{\overline{\nu}}$. This, however, is clear, since $W = \hat{C}_{\overline{\nu}} \setminus \omega\overline{\delta}_0$, where $|f|(\omega\overline{\delta}_0) = inf\ \hat{C}_\nu \cap X^*_{\mu',\ 0,\ s}$.

(5.5) Remark. *We should point out that* $\tilde{f}_{\omega\nu}(\mu') = \beta(f^*_{\mu',\ 0,\ s})$.

§6. COMPLETING THE PROOF OF SQUARER SCALES.

(A) of (1.2) is immediate from the material of §§1 - 3 of [2], summarized in §§2, 3, above. (B)(1) is clear from construction. (B)(2) follows easily from the definition of μ^*_1, in (5.3), above, the remark about μ^*_1 in (5.3), above, prior to the definition of $\tilde{f}_{\omega\nu}$ and the proof of (3.3), above. (C)(1) and (C)(2) follow easily from (5.4). (C)(3) follows from (5.4) and the analogous statement about $C_{\omega\overline{\nu}}$, but the latter follows readily from (2.25) and (2.33) of [2].

We argue for (B)(4). Let $\omega\overline{\nu} = \tilde{f}_{\omega\nu}(\overline{\kappa})$. We should note that the hypothesis that Φ has limit order type will hold if $X^*_{\overline{\kappa},\ 0,\ s}$ is cofinal in $\omega\rho(\nu)$, by (3.2), above, where $s = \mathcal{A}^+(\nu)$. Let $f^*_{\overline{\kappa},\ 0,\ s} = (\overline{s},\ |f|,\ s)$. As in (5.1), $\overline{s} = (\mathcal{A}(\overline{\nu}),\ q(\overline{\nu}))$. Applying (C)(1), we have that Φ is a final segment of $C_{\omega\overline{\nu}}$. However, since Φ has limit order type, by hypothesis, it must therefore be cofinal in $\omega\overline{\nu}$.

It remains to verify the scale properties, (B)(3) and (B)(5). We first argue for (B)(5); we shall appeal to a part of its proof in arguing for (B)(3). So, let κ be singular and let $g \in \mathcal{S}(\kappa)$. Clearly there is $\nu_0 \in S_\kappa$ such that $g \in J_{\nu_0}$, and of course, taking ν_0 sufficiently large, we may suppose that $J_{\nu_0} \models$ "κ is singular". But then, as in the arguments for (3.1) and (3.2), above, $o.t.\ C_{\omega\nu_0} < \kappa$. Since $J_{\nu_0} \subseteq J_{\rho(\nu_0)}$, it will suffice to prove:

($*$) if κ is a singular cardinal, $\eta \in S_\kappa,\ o.t.\ C_{\omega\eta} < \kappa,\ g \in \mathcal{S}(\kappa) \cap J_{\rho(\eta)}$, then $g <^* \tilde{f}_\eta$.

Proof of ($*$). Let $s = \mathcal{A}^+(\eta)$ and let $\overline{\kappa} < \kappa$ be such that $g \in h_s``(\omega \times \overline{\kappa})$. Let $\overline{\kappa},\ \mu_1^*(\eta) \le \kappa' < \kappa$ be a cardinal. We shall argue that $g(\kappa') < \tilde{f}_\eta(\kappa')$. The main observation is that since $\kappa' \in X^*_{\kappa',\ 0,\ s}$, we also have $g(\kappa') \in X^*_{\kappa',\ 0,\ s}$. But then, since $s \models$ "*card* $g(\kappa') = \kappa'$", clearly $g(\kappa') + 1 \subseteq X^*_{\kappa',\ 0,\ s}$ and so $|f||(g(\kappa') + 1) = id|(g(\kappa') + 1)$. Thus, letting $f^*_{\kappa',\ 0,\ s} = (\overline{s},\ |f|,\ s),\ \overline{s} \models$ "*card* $g(\kappa') = \kappa'$" and so $g(\kappa') < \beta(f^*_{\kappa',\ 0,\ s}) = \tilde{f}_\eta(\kappa')$. The last equality is by (5.5), above. This completes the proof of ($*$) and therefore of (B)(5).

We finish by arguing for (B)(3). In view of ($*$), and since $J_\nu \subseteq J_{\rho(\nu)}$, it will clearly suffice to show that if $\tau \in S_\kappa \cap \nu,\ o.t.\ C_{\omega\tau} < \kappa$ then $\tilde{f}_{\omega\tau} \in J_\nu$. Now, under these hypotheses, it is clear that $\beta(\tau) < \nu$, and therefore that $\mathcal{A}^+(\tau) \in J_\nu$ and so, letting $s = \mathcal{A}^+(\tau),\ h_s \in J_\nu$. But then, the function $\kappa' \mapsto \beta(f^*_{\kappa',\ 0,\ s})$ is also an element of J_ν. Finally, in virtue of (5.5), $\tilde{f}_{\omega\tau}$ is a restriction of this function to the set of cardinals in a final segment of its domain and therefore $\tilde{f}_{\omega\tau} \in J_\nu$, as required.

§7. A Square on Singular L-Cardinals.

We simply recall that in [3], Jensen contructed a system $(\tilde{D}_\kappa | \kappa$ is a singular L-cardinal$)$ with the properties that $\tilde{D}_\kappa \subset \kappa$ is a club of κ such that $o.t.\tilde{D}_\kappa < min D_k$ and such that if $\lambda \in (\tilde{D}_\kappa)'$, then λ is a singular L-cardinal and $D_\lambda = D_\kappa \cap \lambda$.

§8. Local Systems in L[X$_\tau$].

Prior to (1.1), we outlined the thrust of this section, so we limit ourselves to the statement of the result.

Lemma. *Suppose that $\tau = \aleph_2$ or τ is a limit cardinal, and let $\mu = \tau^{+\omega}$. Suppose that $X_\tau \subseteq \tau^+$ is such that $H_\mu = L_\mu[X_\tau]$. Let $S^\tau = \bigcup\{S_\lambda | \tau < \lambda \le \mu$ and λ is a cardinal $\}$. Then, in $L[X_\tau]$, there are systems $(C_{\omega\nu} | \nu \in S^\tau)$ and $(\tilde{f}_{\omega\nu} | \nu \in S_\mu$ & $o.t.\ C_{\omega\nu} < \mu)$ which satisfy (A) - (C) of (1.2),except that, in addition, we require that if $\lambda \in dom\ \tilde{f}_{\omega\nu}$, then $\lambda > \tau$.*

Of course, the X_τ we have in mind are the $A \cap \tau^{+\omega}$.

Part II: Life In A Sharpless V.

In Part II, which comprises §§9 - 11, we transfer the combinatorial structures of Part I to a sharpless V, and prove the results required for [5], notably Corollary 2 and Lemma 3 (Lemma (1.4) of [5]). As in [5], we work in the context provided by [4], i.e., we assume that $0^\sharp$ does not exist and we work in $L[A]$, where $A \subseteq OR$ is such that for all uncountable cardinals κ, $H_\kappa = L_\kappa[A]$, such that $A = (A \cap \omega_2) \cup \bigcup\{A \cap (\kappa, \kappa^+) | \kappa \in \Lambda\}$, where Λ is the class of limit cardinals together with $\aleph_2$. Further, if $\kappa = \aleph_2$, or κ is inaccessible, then for $\delta \in (\kappa,\ \kappa^+)$, $(card\ \delta)^{L[A\cap\delta]} = \kappa$.

In §9, we show how to transfer the combinatorial systems of Part I to V, indicating briefly how the necessary modifications are performed. We culminate, in (9.4), with the definition of a **fine system of squares and pseudo-scales** and the observation that the system obtained in (9.1) - (9.3) is indeed a fine system. This proves Corollary 2 and corresponds to (1.2) of [5]. In §10 we prove Lemma 3. We finish, in §11, by proving two smaller results, used in (1.5) of [5] and (2.1.1) of [5].

In the remainder of this paper, notions such as "cardinal", "singular cardinal", etc., mean "cardinal in the sense of V", "singular cardinal in the sense of V", etc.

§9. From L to V.

(9.1) OBTAINING THE D_κ FROM THE $\tilde{D}_\kappa$.

First, for singular cardinals, μ of the form $\eta^{+\omega}$, we let Λ be as above, we let η^* be the unique member of Λ such that $\mu = (\eta^*)^{+\omega}$, and we define $D_\mu := \{\aleph_\tau \in (\eta^*, \mu) | \tau$ is odd$\}$.

So, assume that κ is a singular limit of limit cardinals. Let E_κ be the set of singular cardinals in $\tilde{D}_\kappa$, where $\tilde{D}_\kappa$ is as in §7. If $(E_\kappa)'$ is cofinal in κ, let $D^*_\kappa =$ $\kappa \cap (E_\kappa)'$ and set $\lambda \in I(\kappa)$ iff λ is a successor point of D^*_κ. If $(E_\kappa)'$ is bounded in κ, set $D^*_\kappa = \emptyset$, $I(\kappa) = \{\kappa\}$. Note that if $\lambda \leq \kappa$, $\lambda \in I(\kappa)$, then λ is a singular limit of limit cardinals and $\lambda \in I(\lambda)$. Also, note that if $\lambda \in I(\lambda)$, then $cf\ \lambda = \omega$. Thus, for all singular limits of limit cardinals, λ, such that $\lambda \in I(\lambda)$, choose $x(\lambda) = \{\lambda_j | j < \omega\}$, cofinal in λ, $(\lambda_j | j < \omega)$ increasing, such that:

1. **min $\tilde{D}_\lambda < \lambda_0$; whenever $\lambda' \in \lambda \cap E'$, $\lambda' < \lambda_0$,**
2. **for all $j < \omega$, there is $\delta(j)$, which is not a successor ordinal, such that $\lambda_j = \aleph_{\omega(\delta(j)+j+1)}$.**

Then, for all κ which are singular limits of limit cardinals, let $D_\kappa = D^*_\kappa \cup \bigcup\{x(\lambda) | \lambda \in I(\kappa)\}$.

Note that by construction, $(D_\kappa | \kappa$ a singular limit of limit cardinals$)$ has the usual coherence property; further, letting $\delta_\kappa = o.t.\ D_\kappa$, $\delta_\kappa \leq o.t.\tilde{D}_\kappa < min\ \tilde{D}_\kappa < min\ D_\kappa$, and letting $(\lambda^\kappa_i | i < \delta_\kappa)$ increasingly enumerate D_κ, if $\lambda^{\kappa_1}_{i_1} = \lambda^{\kappa_2}_{i_2}$, then $i_1 = i_2$. Further, note that for all λ, $\{\delta_\kappa | \lambda \in D_\kappa\} \subseteq \lambda$.

(9.2) MODIFYING THE C_α.

If κ is a singular cardinal, then, by Covering, $\kappa^+ = (\kappa^+)^L$, so that the system $(C_{\omega\nu}|\nu \in S_\kappa)$ is very close to being a square-system between κ and κ^+. In fact, in virtue of (3.1) and (3.2), above, except for an initial segment, I, of $\alpha \in S_\kappa$, we always have *o.t.* $C_\alpha < \kappa$. Recall that as in [5], for singular κ, we let $U(\kappa)$ be the set of multiples of κ^2 in $(\kappa,\ \kappa^+)$. Let ϕ_κ be the continuous order-isomorphism between $\{\omega\nu|\nu \in S_\kappa \setminus I\}$ and the set of limit multiples of κ^2 in $(\kappa,\ \kappa^+)$. We transfer the system to live on the latter set, via ϕ, by taking $C_{\phi(\alpha)} := \phi``C_\alpha$. Finally, the C_α constructed in Part I are not necessarily club: they have been thinned by removing successor points. These are restored, in a canonical way by recursion on the well-founded relation "$\alpha \in C_\beta$" by supplying cofinal ω-sequences above *sup* C_α to those α whose C_α is not cofinal. We have abused notation by using C_α to denote this modified system as well.

(9.3) MODIFYING THE $\tilde{f}_{\omega\nu}$.

There are several kinds of modifications we carry out. The first is to transfer the scales to live on the $(U(\kappa))'$, as we did for the squares, in (9.2). Here, it is a bit more complicated, since we must also transfer the values, via different continuous order-isomorphisms. Also, at least in the first few stages of the modifications, we continue to deal with certain L-cardinals which may not be cardinals of V.

So, if κ is an L-cardinal, we let ϕ_κ be the order-isomorphism of $\{\alpha \in S_\kappa | o.t.\ C_\alpha < \kappa\}$ to an initial segment, T_κ, of the set of limit multiples of κ^2. Note that if κ is actually a cardinal, then $T_\kappa = (U(\kappa))'$. Further, if κ is actually a singular cardinal, then ϕ_κ is as in (9.2). Finally, if κ is actually a regular cardinal, then ϕ_κ is only $<\ \kappa$-continuous but, as will be clear, that is all that is required.

(9.3.1) Definition. *Now, suppose that κ is actually a singular limit of limit cardinals. We define $\hat{f}_\eta$ for $\eta \in (U(\kappa))'$, with domain the set of L-cardinals between $\aleph_1$ and κ. Let $\alpha \in S_\kappa$ with o.t. $C_\alpha < \kappa$ be such that $\eta = \phi_\kappa(\alpha)$. First, suppose that $\lambda \in dom\ \tilde{f}_{\omega\alpha}$. We then set $\hat{f}_\eta(\lambda) := \phi_\lambda(\tilde{f}_{\omega\alpha}(\lambda))$. If $\aleph_1 < \lambda < \kappa$, λ is an L-cardinal and $\lambda \notin dom\ \tilde{f}_{\omega\alpha}$, we set $\hat{f}_\eta(\lambda) := \lambda^2\omega$.*

If $\tau = \aleph_2$ or τ is a limit cardinal and $\mu = \tau^{+\omega}$, the procedure is similar: for $\eta \in (U(\mu))'$, letting $\alpha \in S_\mu$ with o.t. $C_\alpha < \mu$ be such that $\eta = \phi_\mu(\alpha)$, if $\lambda \in dom\ \tilde{f}_{\omega\alpha}$, we set $\hat{f}_\eta(\lambda) := \phi_\lambda(\tilde{f}_{\omega\alpha}(\lambda)$, but we only extend the domain *to be the set of cardinals between τ and μ, again, using $\lambda^2\omega$ as the default value.*

Next, we must define the scale functions $\hat{f}_\eta$, for $\eta \in U(\kappa) \setminus (U(\kappa))'$, where κ is a singular cardinal. This is rather straightforward. First, if κ is a singular limit of limit cardinals, let $\lambda \in X$ $iff \lambda$ is an L-cardinal and $\aleph_1 < \lambda < \kappa$, so suppose that κ is an ω-successor. If $\kappa = \aleph_\omega$, let $\tau = \aleph_2$; otherwise, let τ be the unique limit cardinal with $\kappa = \tau^{+\omega}$. In both of these cases, let $\lambda \in X$ $iff \tau < \lambda < \kappa$ and λ is a cardinal. if $0 < n < \omega$ and $\eta = \kappa^2 n$, for all $\lambda \in X$, we let $\hat{f}_\eta(\lambda) := \lambda^2 2n$. Otherwise, let σ be a limit ordinal, $0 < n < \omega$, and suppose that $\eta = \kappa^2(\sigma + n)$. Then, for all $\lambda \in X$, we set $\hat{f}_\eta(\lambda) := \hat{f}_\sigma(\lambda) + \lambda^2 2n$.

(9.3.2) Remark. *It is easy to see that the transferred system of C_η and $\hat{f}_\eta$ for $\eta \in (U(card\ \eta))'$ satisfies the obvious analogues of (A) - (C) of (1.2), above. We shall use this observation in (9.4) and in §10, without additional comment.*

(9.3.3) Definition. *Finally, we define the f^*_η for $\eta \in U(\kappa)$, where κ is a singular cardinal. These are simply $\hat{f}_\eta|D_\kappa$, where D_κ is as given by (9.1).*

(9.4) A FINE SYSTEM.

We now define the notion of a **fine system of squares and pseudo-scales** as one which satisfies properties (A) - (D), below (these are restatements of the similarly labelled items of (1.2) of [5]). When this is done, it will be clear (by (9.3.2)) that since we are assuming that $0^\sharp$ does not exist and that $V = L[A]$, where A is as given by Lemma 3 of [5], the combinatorial system developped in (9.1) - (9.3) is a fine system of squares and pseudo-scales. This proves Corollary 2.

Definition. *A* **fine system of squares and pseudo-scales** *is a system $(D_\mu|\mu$ is a singular limit of limit cardinals), $(C_\alpha|\alpha \in (U(\kappa))' \cap \kappa^+ \& \kappa$ is a singular cardinal), $(f^*_\alpha|\alpha \in U(\kappa)$, $\&\kappa$ is a singular cardinal) satisfying the following properties (A) - (D).*

(A)

For singular cardinals, μ, D_μ is a club subset of the set of cardinals less than μ such that if μ is a limit of limit cardinals, then all members of D_μ are singular, while if (($\tau = \aleph_2$ or τ is a limit cardinal) and $\mu = \tau^{+\omega}$), then $\lambda \in D_\mu$ $iff (\tau < \lambda < \mu\ \& \lambda = \aleph_\xi$, where ξ is odd), and:

(1) *o.t.* $D_\mu <$ *min* D_μ,
(2) if λ is a limit point of D_μ, $D_\lambda = D_\mu \cap \lambda$.

(3) if $\lambda \in D_\mu$ is not a limit point of D_μ then λ is not a limit of limit cardinals.
(4) suppose that $\lambda \in D_{\kappa_i}$, $i = 1, 2$, and let j_i be such that λ is the j_i^{th} member of D_{κ_i}. Then, $j_1 = j_2$.

(B) For singular cardinals, κ, and $\alpha \in (U(\kappa))' \cap \kappa^+)$, C_α is a club subset of the set of even multiples of κ^2 below α, of order type less than κ, and such that if $\beta \in C_\alpha$ but is not a limit point of C_α, then β is not a limit point of $U(\kappa)$, and with the usual coherence property: if β is a limit point of C_α, $C_\beta = C_\alpha \cap \beta$.

(C) For singular cardinals, κ, and $\alpha \in U(\kappa))$, *dom* $f^*_\alpha = D_\kappa$, for $\lambda \in D_\kappa$, $f^*_\alpha(\lambda)$ is an even multiple of λ^2 and:
(1) if $\kappa < \alpha < \beta$, α, $\beta \in U(\kappa)$ then $f^*_\alpha <^* f^*_\beta$, i.e., for some $\lambda_0 < \kappa$, whenever $\lambda \in D_\kappa \setminus \lambda_0$, $f^*_\alpha(\lambda) < f^*_\beta(\lambda)$; further, if $\alpha \in C_\beta$, then the preceding holds for **all** $\lambda \in D_\kappa$,
(2) whenever g is a function with *dom* $g = D_\kappa$ and for all $\lambda \in D_\kappa$, $g(\lambda) < \lambda^+$, for some $\alpha \in U(\kappa)$, $g <^* f^*_\alpha$,
(3) if κ is a singular limit of limit cardinals, $\lambda \in D_\kappa$, $\alpha \in U(\kappa)$, $\alpha' = f^*_\alpha(\lambda)$ and $\lambda' \in D_\kappa \cap \lambda$, then $f^*_\alpha(\lambda') = f^*_{\alpha'}(\lambda')$, and if κ is not a limit of limit cardinals and α, $\beta \in U(\kappa)$, $\lambda \in D_\kappa$ and $f^*_\alpha(\lambda) = f^*_\beta(\lambda)$, then $f^*_\alpha|\lambda = f^*_\beta|\lambda$,
(4) for limit points, α, of $U(\kappa)$, and $\lambda \in D_\kappa$, $\Phi(\alpha, \lambda) := \{f^*_\beta(\lambda)|\beta \in C_\alpha\}$ is a final segment of $C_{f^*_\alpha(\lambda)}$; further, on a tail of D_κ, $\Phi(\alpha, \lambda)$ has limit order type.

We recall the observation made in (1.2) of [5] to the effect that even though the f^*_α are not defined when *card* α is a successor cardinal, nevertheless the property of the second clause of (3) allows us to define them in a conventional way so that we will then have the property of the first clause of (3), even for κ which are not limits of limit cardinals.

(D) Decodability of (A) - (C): For all singular κ, D_κ and the systems $(C_\alpha|\alpha < \kappa^+$ is a limit point of $U(\kappa))$, $(f^*_\alpha|\alpha \in U(\kappa))$ are canonically definable in $L[A \cap \kappa]$.

To make it completely clear why this follows from (9.3.2), it will be useful to give the correspondence between items of (B), (C), above, and the items of the *Squarer Scales* principle of (1.2). (B) corresponds to (A) of (1.2). (C)(1) corresponds to the conjunction of (B)(2) and (B)(3) of (1.2). (C)(2) corresponds to (B)(5) of (1.2). (C)(3) corresponds to (C)(2) of (1.2). (C)(4) corresponds to the conjunction of (B)(4) and (C)(1)of (1.2). (D) corresponds to (C)(3) of (1.2).

§10. THE EXISTENCE OF SUPER-COHERENT SEQUENCES.

In this section we prove Lemma 3 (Lemma (1.4) of [5]). This lemma states that there are "enough" super-coherent sequences. We do this by first showing, in (10.2), that there "enough" strongly coherent sequences, and then, in (10.5), showing that if $(\mathcal{N}_i | i \leq \theta)$ is strongly coherent then $(\mathcal{N}_{\omega i} | i \leq \theta)$ is super-coherent. The proofs of (10.3) - (10.5) exploit the most subtle combinatorial properties of the *Squarer Scales*. For convenience, we begin by restating the definitions of strongly coherent and super-coherent, and some preliminary related notions from (1.1) and (1.3) of [5]. Following (10.2) we lay out the plan for the proof carried out in (10.3) - (10.5).

(10.1) MODEL SEQUENCES AND COHERENCE.

Let $\theta > \aleph_1$ be regular. Let $\mathcal{M} = (H_{\nu^+}, \in, \cdots,)$, where ν is a singular cardinal, $\nu >> \theta$ and $(H_\nu, \in)$ models a sufficiently rich fragment of ZFC. Let $\sigma \leq \theta$ and let $(\mathcal{N}_i : i \leq \sigma)$ be an increasing continuous elementary tower of elementary substructures of $\mathcal{M}$.

(10.1.1) Definition. *We say that* $(\mathcal{N}_i | i \leq \sigma)$ *is* $(\mathcal{M}, \theta)$**-standard of length** $\sigma + 1$ *if, letting* $N_i := |\mathcal{N}_i|$, *for all* $i \leq \sigma$, *card* $N_i = \theta$, $\theta{+}1 \subseteq N_0$, *for* $i < \sigma$, $[N_{i+1}]^{<\theta} \subseteq N_{i+1}$ *and, if* i *is even,* $\mathcal{N}_i \in N_{i+1}$.

(10.1.2) Definition. *For such* $\theta > \aleph_1$ *and* $\mathcal{M} = (H_{\nu^+}, \in, \cdots)$, *suppose that* $\mathcal{N} \prec \mathcal{M}$, *where, letting* $N := |\mathcal{N}|$, *card* $N = \theta$, *and let* κ *be a cardinal with* $\theta \leq \kappa$, $\kappa \in N$. *Let* $\chi_{\mathcal{N}}(\kappa) = sup(N \cap (\kappa, \kappa^+))$.

Recall that an Easton set of ordinals is one which is bounded below any inaccessible cardinal. For such $\mathcal{N}$ *and singular cardinals,* κ, *with* $\theta < \kappa \leq \nu$, *we say that* κ *is* $\mathcal{N}-$ **controlled** *if there is an Easton set* d *with* $\kappa \in d \in N$.

We define $p\chi_{\mathcal{N}}$, *an analogue of* $\chi_{\mathcal{N}}$, *defined on all singular cardinals,* κ, *which are* $\mathcal{N}-$ *controlled. The definition makes sense for all cardinals* $\kappa \in [\theta, \nu]$, *but we will only use it for the singulars which are* $\mathcal{N}-$*controlled. If* $\kappa \in N$, *then of course* κ *is* $\mathcal{N}-$ *controlled and in this case,* $p\chi_{\mathcal{N}}(\kappa) := \chi_{\mathcal{N}}(\kappa)$. *Otherwise,* $p\chi_{\mathcal{N}}(\kappa) := sup\ (\kappa^+ \cap Sk_{\mathcal{M}}(\{\kappa\} \cup N))$.

The reason that we only consider controlled κ is that (10.3), below, gives an alternative characterization of $p\chi_{\mathcal{N}}(\kappa)$ which is central in proving (10.5). The alternative characterization is equivalent only for controlled κ. As we noted in [5], the restriction to such κ is benign, for our purposes.

"Characteristic" functions of a model $\mathcal{N}$ like $\chi_{\mathcal{N}}$ and $p\chi_{\mathcal{N}}$ often appear in the work of the first author in a slightly different formulation, defined to be "pressing down" functions: the value at a cardinal κ is the supremum **below** κ of some set of ordinals associated with $\mathcal{N}$. Thus, in this formulation, our $\chi_{\mathcal{N}}(\kappa)$ and $p\chi_{\mathcal{N}}(\kappa)$ would become values at κ^+ of these functions, and we would also have at our dispostion the corresponding suprema below limit cardinals. In this connection, see the second Remark, follwoing the proof of the Proposition in (10.3).

(10.1.3) Definition. *Suppose* $\kappa \in |\mathcal{M}|$, κ *is a singular cardinal,* $\mathcal{N}$, N *are as in (10.1.2), and* $\kappa \notin N$. *Let* $\mu_N(\kappa) =$ *the least ordinal,* $\xi \in N$, *such that* $\xi > \kappa$ *(clearly such exists, since* $\nu \in N$*). Clearly* $\mu_N(\kappa)$ *is a limit of limit cardinals and either* $\mu_N(\kappa)$ *is inaccessible, or* $\theta < cf\,\mu_N(\kappa) < \kappa$.

(10.1.4) Remark. *If* κ *is* $\mathcal{N}$*-controlled but* $\kappa \notin N$, *then* $\mu_N(\kappa)$ *is singular.*

To see this, suppose that $\mu > \kappa$, $\mu \in N$, with μ inaccessible. Since κ is $\mathcal{N}$- controlled, let $\kappa \in d \in N$ where d is an Easton set. Thus, $sup\ d \cap \mu < \mu$ and clearly $sup\ d \cap \mu \in N$. Now $\kappa \in d \cap \mu$, so $\kappa < sup\ d \cap \mu$. But then, it is easy to see that $\mu \neq \mu(\kappa)$, since if equality held, we would have $N \cap [\kappa,\ \mu) = \emptyset$, contradicting that $\kappa < sup\ d \in N \cap \mu$.

(10.1.5) Now, let $(\mathcal{N}_i|\ i \leq \theta)$ be $(\mathcal{M},\ \theta)$-standard of length $\theta + 1$. For $i \leq \theta$, let $\chi_i = \chi_{\mathcal{N}_i}$, $p\chi_i = p\chi_{\mathcal{N}_i}$. Let $\mathcal{N} = \mathcal{N}_\theta = \bigcup\{\mathcal{N}_i | i < \theta\}$, and let $\chi = \chi_\theta$, $p\chi = p\chi_\theta$, so $dom\ \chi = \bigcup\{dom\ \chi_i | i < \theta\}$, and for $\kappa \in dom\ \chi$, $\chi(\kappa) = sup\ \{\chi_i(\kappa) | \kappa \in N_i\}$. Also, for singular cardinals, $\kappa \in [\theta,\ \nu]$, which are $\mathcal{N}$-controlled, $p\chi(\kappa) = sup\ \{p\chi_i(\kappa) | i < \theta\ \&\ \kappa \text{ is } N_i\text{-controlled}\}$.

Let κ be a singular cardinal, $\kappa \in dom\ \chi$. Note that since $cf\ \theta = \theta > \omega$, there is a club $D \subseteq \theta$ such that for all $i \in D$, $\chi_i(\kappa) \in C_{\chi(\kappa)}$. This motivates the following.

Definition. *Let* $\mathcal{M}$, θ *be as above, and let* $(\mathcal{N}_i|\ i \leq \theta)$ *be* $(\mathcal{M},\ \theta)$*-standard of length* $\theta + 1$. *Let* $\mathcal{N} = \mathcal{N}_\theta$. *Let* N, N_i, χ, $p\chi$, χ_i, $p\chi_i$ *be as above.*

Let $\kappa \geq \theta$ *be a singular cardinal,* $\kappa \in N$. $(\mathcal{N}_i|i \leq \theta)$ *is* $\mathcal{M}$**-coherent at** κ *iff for all limit ordinals* $\delta \leq \theta$ *with* $\kappa \in N_\delta$, *there is a club* $D \subseteq \delta$ *such that for all* $i \in D$, $\chi_i(\kappa) \in C_{\chi_\delta(\kappa)}$. $(\mathcal{N}_i|i \leq \theta)$ *is* $\mathcal{M}$**-coherent** *if for all singular cardinals* $\kappa \in N \setminus \theta$, $(\mathcal{N}_i|i \leq \sigma)$ *is* $\mathcal{M}$*-coherent at* κ. $(\mathcal{N}_i|i \leq \theta)$ *is* **strongly** $\mathcal{M}$**-coherent** *iff for all* $i < \theta$ *and all singular cardinals* $\kappa \in N_i$, $\chi_i(\kappa) \in C_{\chi(\kappa)}$. *Finally,* $(\mathcal{N}_i|i \leq \theta)$ *is* **super** $\mathcal{M}$**-coherent** *iff* $(\mathcal{N}_i|i \leq \theta)$ *is strongly* $\mathcal{M}$*-coherent and for all limit ordinals,* $\sigma \leq \theta$ *and all singular cardinals,* κ *which are* $\mathcal{N}_\sigma$*-controlled, for sufficiently large* $i < \sigma$, $p\chi_i(\kappa) \in C_{p\chi_\sigma(\kappa)}$.

(10.1.6) Remark. *Let* $(\mathcal{N}_i|i \leq \theta)$ *be* $(\mathcal{M},\ \theta)$*-standard of length* $\theta + 1$. *For* $i \leq \theta$, *let* $\mu_i := \mu_{\mathcal{N}_i}$, *and let* $\mu := \mu_\theta$. *Note that if* $i < j$ *then* $dom\ \mu_j \subseteq dom\ \mu_i$ *and that if* $\kappa \in dom\ \mu_j$, *then* $\mu_j(\kappa) \leq \mu_i(\kappa)$. *Thus,* $dom\ \mu = \bigcap\{dom\ \mu_i | i < \theta\}$ *and for* $\kappa \in dom\ \mu$, $\mu(\kappa)$ *is the eventually constant value of the* $\mu_i(\kappa)$, $i < \theta$.

(10.2) Lemma. *Let* θ *be regular,* $\theta > \aleph_1$. *Let* $\nu > cf\ \nu >> \theta$ *be such that* $(H_\nu, \in) \models$ *a sufficiently rich fragment of ZFC. Let* $\mathcal{M} = (H_{\nu^+}, \in, \cdots)$. *Let* $C \subseteq [H_{\nu^+}]^\theta$ *be club. Then there's strongly* $\mathcal{M}$*-coherent* $(\mathcal{N}_i|i \leq \theta)$, *each* $|\mathcal{N}_i| \in C$.

Proof. Without loss of generality, we may assume that $X \in C \Rightarrow \mathcal{M}|X \prec \mathcal{M}$. We first build $(\mathcal{M}_j|j \leq \theta^+)$, an increasing continuous tower of elementary submodels of $\mathcal{M}$, each $|\mathcal{M}_j| \in C$, $\mathcal{M}_j \in |\mathcal{M}_{j+1}|$, $|\mathcal{M}_{j+1}|$ closed under sequences of length $< \theta$, for $j < \theta^+$. Let $\chi_j = \chi_{\mathcal{M}_j}$, $\chi = \chi_{\mathcal{M}_{\theta^+}}$ be as in (10.1.5).

For singular $\kappa > \theta$, $\kappa \in |\mathcal{M}_{\theta^+}|$, let $E(\kappa) \subseteq \theta^+$ be club such that $j \in E(\kappa) \Rightarrow \kappa \in |\mathcal{M}_j|$ and $\chi_j(\kappa) \in C_{\chi(\kappa)}$. For $i < \theta^+$, let
$E_i = \bigcap\{E(\kappa) | \kappa \in |\mathcal{M}_i|,\ \kappa > \theta,\ \kappa \text{ is singular}\}$, so each E_i is a club of θ^+. Let $E = \Delta_{i<\theta^+} E_i =$ the diagonal intersection of the E_i. Thus,
$j \in E \Rightarrow (\forall i < j)(\forall \kappa \in |\mathcal{M}_i|)\chi_i(\kappa) \in C_{\chi(\kappa)}$. Let $E^\theta = \{\alpha \in E | cf\ \alpha = \theta\}$ and let $E^* = E^\theta \cup ((E^\theta)' \cap \theta^+)$ and let $(j_i | i < \theta^+)$ be the increasing enumeration of E^*. Thus for all $i < \theta^+$, $cf\ j_{i+1} = \theta$. For $i \leq \theta$, let $\mathcal{N}_i = \mathcal{M}_{j_i}$. Then, $(\mathcal{N}_i | i \leq \theta)$ is strongly $\mathcal{M}$-coherent. All properties are clear from construction, except possibly that for $i < \theta$, $[|\mathcal{N}_{i+1}|]^{<\theta} \subseteq |\mathcal{N}_{i+1}|$. This, however, is an easy consequence of the fact that for successor ζ, $|\mathcal{M}_\zeta|$ is closed under sequences of length $< \theta$ and that $cf\ j_{i+1} = \theta$.

Discussion. *We are now in a position to lay out the ideas behind the proof, in (10.3) - (10.5), that if $(\mathcal{N}_i | i \leq \theta)$ is strongly $\mathcal{M}$-coherent then $(\mathcal{N}_{\omega i} | i \leq \theta)$ is super $\mathcal{M}$-coherent. Let $\sigma \leq \theta$ be a limit ordinal, and $\theta < \kappa < \nu$ be a singular cardinal. We say that σ is κ-good if κ is $\mathcal{N}_\sigma$-controlled. Now suppose that $\delta \leq \theta$ is a limit of limit ordinals, $\theta < \kappa < \nu$ is a singular cardinal and that δ is κ-good. Let $\eta = p\chi_{\mathcal{N}_\delta}(\kappa)$. Our aim is to show that for sufficiently large κ-good limit ordinals $\sigma < \delta$, $p\chi_{\mathcal{N}_\sigma}(\kappa) \in C_\eta$.*

If we "go up" to $\mu = \mu_{\mathcal{N}_\delta}$ and let $\eta' = \chi_{\mathcal{N}_\delta}(\mu)$, then, since $(\mathcal{N}_i | i \leq \theta)$ is strongly $\mathcal{M}$-coherent, we have that for $i < \delta$, $\chi_{\mathcal{N}_i}(\mu) \in C_{eta'}$. Is there some way of "projecting" this fact back down to "level κ"? One such way would be to evaluate the L-scale functions from "level μ" (the $\hat{f}'s$ at κ. And, in fact, by (C) (1) of (1.2), if we let $\eta^* = \hat{f}_{\eta'}(\kappa)$, $\{\hat{f}_\tau(\kappa) | \tau \in C_{\eta'}\}$ will be a final segment of C_{η^*}. But what is the relationship between η and η^*, and, for $i < \delta$, between $p\chi_{\mathcal{N}_i}(\kappa)$ and $\hat{f}_{\chi_{\mathcal{N}_i}(\mu)}(\kappa)$. The argument would be complete, if we knew we had equality in the first case, and equality in the second case for sufficiently large limit ordinals which are κ-good. This is exactly what will be proved in (10.4). (10.3) supplies a technical result underlying the argument of (10.4). In (10.5), we fill in the last few missing details of the above sketch, in the presence of the result of (10.4).

(10.3)

If κ is $\mathcal{N}$-controlled, set $g \in \mathcal{G}_\mathcal{N}$ iff $f \in |\mathcal{N}|$, f is a function, $dom\ f$ is a set of L-cardinals and for all $\tau \in dom\ f$, $\tau < f(\tau) < (\tau^+)^L$. We also set $\mathcal{G}^L_\mathcal{N} := \mathcal{G}_\mathcal{N} \cap L$.

Proposition. *If κ is $\mathcal{N}$-controlled, $p\chi_\mathcal{N}(\kappa) = sup\ \{f(\kappa) | f \in \mathcal{G}_\mathcal{N},\ \kappa \in dom\ f\} = sup\ \{f(\kappa) | f \in \mathcal{G}^L_\mathcal{N},\ \kappa \in dom\ f\}$.*

Proof. Clearly $p\chi_\mathcal{N}(\kappa) \geq sup\ \{f(\kappa) | f \in \mathcal{G}_\mathcal{N},\ \kappa \in dom\ f\} \geq sup\ \{f(\kappa) | f \in \mathcal{G}^L_\mathcal{N},\ \kappa \in dom\ f\}$, so we show that $p\chi_\mathcal{N}(\kappa) \leq sup\ \{f(\kappa) \in \mathcal{G}_\mathcal{N},\ \kappa \in$

dom $f\} \leq$ *sup* $\{f(\kappa) | f \in \mathcal{G}^L_{\mathcal{N}},\ \kappa \in$ *dom* $f\}$. Since κ is $\mathcal{N}$-controlled (this is the whole point of the notion), the last inequality is clear by covering, so we prove the first.

Let $\xi < \kappa^+$, ξ definable in $\mathcal{M}$, by ψ, from $x_1, \cdots, x_k \in |\mathcal{N}|$ and κ. Let $f(\tau) \simeq$ the least $\alpha < (\tau^+)^L$ such that $\mathcal{M} \models \psi(\alpha, x_1, \cdots, x_k, \tau)$, for L-cardinals τ. Clearly $f(\kappa) = \xi$ and for all $\eta \leq \nu$, $f|\eta \in |\mathcal{M}|$. Also, $\eta \mapsto f|\eta$ is $\mathcal{M}$-definable. Thus, if $\eta \in |\mathcal{N}|$, $f|\eta \in |\mathcal{N}|$. But clearly $\nu \in |\mathcal{N}|$. Thus $f|\nu \in |\mathcal{N}|$ and so $\xi = (f|\nu)(\kappa)$.

Remarks.

(1) We could also have defined $\mathcal{G}^E_{\mathcal{N}}$ to be the set of $f \in \mathcal{G}_{\mathcal{N}}$ such that dom f is an Easton set, and $\mathcal{G}^{L,E}_{\mathcal{N}}$ to be $\mathcal{G}^E_{\mathcal{N}} \cap L$, thereby "building in" the restriction to controlled κ.

(2) In connection with the alternative definition of the $\chi_{\mathcal{N}}$ and $p\chi_{\mathcal{N}}$ as "pressing down" functions, mentioned at the end of (10.1.2), the above Proposition remains true, with these alternative definitions, and the appropriately modified definition of the various $\mathcal{G}'s$: for $f \in \mathcal{G}_{\mathcal{N}}$ and $\kappa \in$ dom f, $f(\kappa)$ would be required to be less than κ.

(10.4) Suppose now that $\mathcal{M}$ is as in (10.2) and that $\mathcal{N}' \prec \mathcal{M}$, *card* $|\mathcal{N}'| = \theta$ and let $\chi = \chi_{\mathcal{N}'}$, $p\chi = p\chi_{\mathcal{N}'}$ Let κ be a singular cardinal which is $\mathcal{N}'$-controlled. Let $\mu(\kappa) = \mu_{\mathcal{N}'}(\kappa)$, so that, by (10.1.4), $\mu(\kappa)$ is a singular cardinal. Let $\mu = \mu(\kappa)$, let $\eta' = \chi(\mu)$, $\eta = p\chi(\kappa)$ and suppose that $C_{\eta'} \cap |\mathcal{N}'|$ is cofinal in η'. This will hold, in all cases of interest.

Lemma. $\eta = \hat{f}_{\eta'}(\kappa)$.

Proof. We will end up applying (1.2)(B)(4) (here, and in what follows, recall (9.3.2)!), so we must first show that here, we have the hypothesis that $(\hat{f}_\tau(\kappa) | \tau \in C_{\eta'})$ is not eventually constant. We begin with a number of easy observations, which we shall use at various places in the proof.

(1) For τ a limit ordinal in (μ, μ^+), $\hat{f}_\tau$ is canonically definable from τ in $\mathcal{M}$, so for $\tau \in C_{\eta'} \cap |\mathcal{N}'|$, $\hat{f}_\tau \in |\mathcal{N}'|$.

(2) if $f,\ g \in |\mathcal{N}'|$, where $f,\ g$ are functions with domain the set of uncountable L-cardinals $<\ \mu$ and $f <^* g$ then the least λ_0 such that $(\forall \lambda \geq \lambda_0) f(\lambda) < g(\lambda)$ is definable from $f,\ g$ and is therefore in $|\mathcal{N}'|$. So, since $\lambda_0 \in |\mathcal{N}'|$ and $\lambda_0 < \mu$, we must have $\lambda_0 < \kappa$.

We are now in a position to argue that $(\hat{f}_\tau(\kappa) | \tau \in C_{\eta'})$ is not eventually constant. We will do this by proving that for $\tau_1 < \tau_2$, both in $C_{\eta'} \cap |\mathcal{N}'|$, for all $\lambda \geq \kappa$, $\hat{f}_{\tau_1}(\lambda) < \hat{f}_{\tau_2}(\lambda)$. In particular, this means that the map from $C_{\eta'} \cap |\mathcal{N}'|$ to Φ, given by $\tau \mapsto \hat{f}_\tau(\kappa)$ is order preserving, so Φ has limit order type, as required, since clearly $C_{\eta'}$ does. So, suppose $\tau_1,\ \tau_2$ are as above.

Applying (1), we have that for $i = 1,\ 2,\ \hat{f}_{\tau_i} \in |\mathcal{N}'|$. But then, we have the desired conclusion, by applying (2), with $f = \hat{f}_{\tau_1},\ g = \hat{f}_{\tau_2}$.

As we have just proven, we have the hypotheses of (1.2)(B)(4), so, by (1.2)(B)(4), $\hat{f}_{\eta'}(\kappa) = sup\ \{\hat{f}_\tau(\kappa)|\tau \in C_{\eta'}\}$. Further, by (1.2)(B)(2), $sup\ \{\hat{f}_\tau(\kappa)|\tau \in C_{\eta'}\} = sup\ \{\hat{f}(\kappa)|\tau \in C_{\eta'} \cap |\mathcal{N}'|\}$. Again, by (1), if τ is as in (1), $\hat{f}_\tau(\kappa) < p\chi(\kappa)$ so finally, $\hat{f}_{\eta'}(\kappa) \leq \eta$.

Clearly $\eta = sup\ \{f(\kappa)|f \in |\mathcal{N}'| \cap L_{\mu^+}\} = sup\ \{f(\kappa)|f \in |\mathcal{N}'| \cap L_{\eta'}\}$. Thus, it suffices to show:

(∗): if $f \in |\mathcal{N}'| \cap L_{\eta'}$, *dom* f is the set of uncountable L-cardinals $< \mu$ and for $\lambda \in$ *dom* $f,\ f(\lambda) \in (\lambda, (\lambda^+)^L)$, then there's $\gamma \in |\mathcal{N}'| \cap L_{\eta'}$ such that
  (a) $f <^* \hat{f}_\gamma$, and
  (b) for all $\lambda \geq \kappa,\ f(\lambda) < \hat{f}_\gamma(\lambda)$.

Now, the existence of a $\gamma \in |\mathcal{N}'| \cap L_{\eta'}$ satisfying (a) is an easy consequence of $\mathcal{N}' \prec \mathcal{M}$ and the fact, which holds in $\mathcal{M}$, that $(\hat{f}_\xi|\xi \in (\mu, \mu^+))$ is an L-scale, by (B)(5) of (1.2). But then for such a $\gamma,\ \hat{f}_\gamma \in |\mathcal{N}'|$, and then (b) follows immediately from (2), with $g = \hat{f}_\gamma$.

(10.5) Lemma. *If $(\mathcal{N}_i|i \leq \theta)$ is strongly $\mathcal{M}$-coherent then $(\mathcal{N}_{\omega i}|i \leq \theta)$ is super $\mathcal{M}$-coherent.*

Proof. We fill in the details of the argument sketched in the Discussion following (10.2). We adopt the notation and terminology established there. Let $\delta \leq \theta$ be a limit of ordinals. Suppose that $\theta < \kappa < \nu$ and that δ is κ-good. By (10.1.6), there is $i_0 < \delta$ such that if $i_0 \leq i < \delta,\ \mu_{\mathcal{N}_i}(\kappa) = \mu_{\mathcal{N}_\delta}(\kappa)$. Let $\mu, \eta,\ \eta',\ \eta^*$ be as in the Discussion.

Now, let $\mathcal{N}' = \mathcal{N}_\delta$. Since $(\mathcal{N}_i|i \leq \theta)$ is strongly $\mathcal{M}$-coherent, it is easy to see that $C_{\eta'} \cap |\mathcal{N}'|$ is cofinal in η', so we have the hypotheses of (10.4). Thus, by the Lemma of (10.4), $\eta = \hat{f}_{\eta'}(\kappa)\ (=\ \eta^*)$.

Suppose, now that $i_0 < \sigma < \delta$, where σ is a κ-good limit ordinal. Since $i_0 < \sigma,\ \mu_{\mathcal{N}_\sigma}(\kappa) = \mu$. Therefore, we can apply (10.4), again, but with $\mathcal{N}' = \mathcal{N}_\sigma$; just as in the preceding paragraph, this give us that $p\chi_{\mathcal{N}_\sigma}(\kappa) = \hat{f}_{\chi_{\mathcal{N}_\sigma}(\mu)}(\kappa)$. The conclusion is now clear, as in the Discussion: $\{\hat{f}_\tau(\kappa)|\tau \in C_{\eta'}\}$ is a final segment of C_η, for all $i < \delta,\ \chi_{\mathcal{N}_i}(\mu) \in C_{\eta'}$, and for all κ-good limit ordinals, σ, with $i_0 < si < \delta,\ p\chi_{\mathcal{N}_\sigma}(\kappa) = \hat{f}_{\chi_{\mathcal{N}_\sigma}(\mu)}(\kappa)$, so for all sufficiently large κ-good limit ordinals, σ, with $i_0 < \sigma < \delta,\ p\chi_{\mathcal{N}_\sigma}(\kappa) \in C_\eta$, as required.

Now, clearly, combining (10.2) and (10.5), we have proved Lemma 3.

(10.6) We now expand somewhat on the proof of (10.4). We have already noted that $\mu(\kappa)$ is a limit of limit cardinals. Suppose first that κ is of the form $\lambda^{+\omega}$. Then, for all such λ, and all $\mathcal{N}' \prec \mathcal{M}$, $\kappa \notin |\mathcal{N}'| \Rightarrow [\lambda, \kappa] \cap |\mathcal{N}'| = \emptyset$. Thus, in this setting, for all $\lambda \in D_\kappa$, $\mu(\lambda) = \mu(\kappa)$.

If κ is a singular limit of limit cardinals and $\kappa \cap |\mathcal{N}'|$ is bounded in κ, then, on a tail of D_κ, $\mu(\lambda) = \mu(\kappa)$. Let us then examine the most difficult case, where κ is a singular limit of limit cardinals, $\kappa \notin |\mathcal{N}'|$, but $\kappa \cap |\mathcal{N}'|$ is cofinal in κ; note that if $|\mathcal{N}'|$ is closed for sequences of length $< \theta$ (as was the case, in the context of (10.5), taking $\mathcal{N}' = \mathcal{N}_\theta$), this means that $cf\ \kappa = \theta$. Note also that we may even have $D_\kappa \subseteq |\mathcal{N}'|$. Recall that, in this latter case, $p\chi(\lambda) = \chi(\lambda)$, for $\lambda \in D_\kappa$. Even if $\lambda \in D_\kappa \setminus |\mathcal{N}'|$, we still have $\mu(\lambda) < \kappa < \mu(\kappa)$.

Our principal aim is to show that one inequality of the Lemma of (10.4) remains true when we replace $\eta = p\chi(\kappa)$ by $\sigma = p\chi(\lambda)$ and $\hat{f}_{\eta'}(\kappa)$ by $\hat{f}_{\eta'}(\lambda)$, but maintain $\eta' = \chi(\mu(\kappa))$, instead of using $\sigma' = \chi(\mu(\lambda))$. Of course, by (10.4) with λ in place of κ, we <u>do</u> have $\sigma = \hat{f}_{\sigma'}(\lambda)$, and this is our point of departure in proving:

Lemma. *$\sigma \geq \hat{f}_{\eta'}(\lambda)$, on a tail of $\lambda \in D_\kappa$.*

Proof. We follow the proof of (10.4). Obtaining $\leq$ seems problematical since the proof of the analogue of $(*)$ of (10.4) does not seem to go over.

First, take λ sufficiently large that $\mu_1^*(\eta') < \lambda$, where $\mu_1^*(\eta')$ is as in (5.3). This is possible, since, in (10.4), we showed that $\mu_1^*(\eta') < \kappa$. Now, for such λ, the proof in (10.4), that $\hat{f}_{\eta'}(\kappa) \leq \eta$, goes over verbatim to show that $\hat{f}_{\eta'}(\lambda) \leq \sigma$.

§11. ODDS AND ENDS.

We close by providing the proofs of two small results needed for [5]. In (11.1) we prove the Proposition of (1.5) of [5] needed for the construction of the very tidy conditions. In (11.2) we show, as promised in (2.1.1) of [5], that, without loss of generality, the system of b_α for α which are multiples of *card* α, which is inaccessible, can be taken to be *tree-like*.

(11.1) Proposition. *Let $\theta > \aleph_1$ and let ν, $\mathcal{M}$ be as in (10.2). Let $d \subseteq [\theta, \nu)$ be an Easton set of cardinals, and let γ be a function with domain d such that for all $\kappa \in d$, $\gamma(\kappa) < \kappa^+$. Then, there is a function γ^* with domain d such that for all $\kappa \in d$ which are either singular or of the form $\aleph_\tau$, with $\tau > 1$ and odd, $\gamma^*(\kappa) > \gamma(\kappa)$ and such that for all singular $\kappa \in d$, letting $\alpha = \gamma^*(\kappa)$, $f^*_\alpha =^* \gamma^* | D_\kappa$. Further, if $\mathcal{N} \prec \mathcal{M}$ with $(\theta + 1) \cup \{\gamma\} \subseteq |\mathcal{N}|$, then $\gamma^* \in |\mathcal{N}|$.*

Proof. We first define a function γ_1 as follows: if $\theta \leq \kappa < \nu$, where κ is of the form $\aleph_{\alpha+\omega}$, we let $\gamma_1(\kappa) =$ the least $\eta \in (\gamma(\kappa), \kappa^+)$ such that η is a multiple of κ^2 and $\gamma|[\aleph_\alpha,\ \kappa) <^* \hat{f}_\eta$ (where we define $\gamma(\lambda)$ to be the usual default value, $\lambda^2 2$, for $\lambda \in [\aleph_\alpha,\ \kappa) \setminus d$). For regular cardinals $\lambda \in d \cap [\theta,\ \nu)$, we let $\gamma_1(\lambda) = max(\gamma(\lambda) + \lambda^2,\ \hat{f}_{\gamma_1(\lambda^{+\omega})}(\lambda))$. For all other $\kappa \in [\theta,\ \nu)$, we let $\gamma_1(\kappa) = \gamma(\kappa) + \kappa^2$.

To obtain γ^* from γ_1, we first define, by recursion on $n < \omega$, ordinals, $\nu_n,\ \eta_n$, and a function, f_n. We will have that if $\nu_n > \theta$, then $\nu_{n+1} < \nu_n$, so there will be $m < \omega$ such that $\nu_{m+1} \leq \theta < \nu_m$. We stop the recursion at this m.

Let $\nu_0 = \nu$. Having defined ν_n, if ν_n is a singular limit of limit cardinals with $\nu_n \in (\theta,\ \nu]$, we let $\eta_n \in [\gamma_1(\nu_n),\ \nu_n^+)$ be the least η which is a multiple of ν_n^2 such that:

$$(*)_n : \hat{f}_\eta >^* \gamma_1|d \cap \nu_n$$

and we let $f_n = \hat{f}_{\eta_n} \cup \{(\nu_n,\ \eta_n)\}$. Once again, this is possible by Covering, because we have taken the precaution of restricting to an Easton, d.

So, having defined $\nu_n,\ \eta_n,\ f_n$, satisfying $(*)_n$, we define:

$\nu^0_{n+1} =$ the least cardinal $\nu' \in [\theta,\ \nu_n)$ such that for singular $\kappa \in d \cap [\nu',\ \nu_n),\ \gamma_1(\kappa) < f_n(\kappa)$.

Having defined ν^i_{n+1}, if $\nu^i_{n+1} \leq \theta$, we set $\nu_{n+1} = \theta,\ m = n$ and we stop. If $\nu^i_{n+1} > \theta$ is a singular limit of limit cardinals, we set $\nu_{n+1} = \nu^i_{n+1}$. If $\nu^i_{n+1} > \theta$ is either a successor cardinal or of the form $\aleph_{\tau+\omega}$, we set $\nu^{i+1}_{n+1} =$ the largest limit of limit cardinals $< \nu^i_{n+1}$. Finally, if $\nu^i_{n+1} > \theta$ is inaccessible, we set $\nu^{i+1}_{n+1} = sup\ d \cap \nu^i_{n+1}$.

Clearly there is $i < \omega$ such that $\nu_{n+1} = \nu^i_{n+1}$ and either $\nu^i_{n+1} \leq \theta$ or ν^i_{n+1} is a singular limit of limit cardinals. In all cases, we let $a_n = \{\kappa \in (\nu_{n+1},\ \nu^0_{n+1}] | \kappa \text{ is a singular cardinal}\}$; note that $a_n \cap d$ is finite, and for all $\kappa \in a_n \cap d,\ \kappa$ is not a limit of limit cardinals. When $\nu_{n+1} > \theta$, we have $m > n$, and we continue, to define η_{n+1} and ν_{n+2}. Clearly $m < \omega$, i.e., for some $n,\ \nu_{n+1} \leq \theta$.

We now define γ^*:

if $\kappa \in d,\ \kappa$ is singular, $\kappa \notin\ \bigcup\{a_n | n \leq m\}$, we set $\gamma^*(\kappa) = f_n(\kappa)$, where n is such that $\nu_{n+1} < \kappa \leq \nu_n$. If $\kappa \in a_n$, where $n \leq m$, we let $\gamma^*(\kappa) = \gamma_1(\kappa)$. Finally, if $\lambda = \aleph_\tau$,with τ odd, $\lambda \in d$, we set $\kappa = \aleph_{\tau+\omega}$ and we set $\gamma^*(\lambda) = max(\gamma_1(\lambda),\ \hat{f}_{\gamma^*(\kappa)}(\lambda))$. For all other $\lambda \in d$, we set $\gamma^*(\lambda) = \gamma_1(\lambda)$.

It is clear that:

(#) If $\mathcal{M}$ is as in (10.1), $\mathcal{N}' \prec \mathcal{M},\ N' := |\mathcal{N}'|,\ \theta+1 \subseteq N',\ card\, N' = \theta,\ [N']^{<\,\theta} \subseteq N',\ d,\ \gamma \in N'$, then $\gamma^* \in N'$.

But then, clearly, γ^* is as required.

(11.2) GETTING "TREE-LIKE" $\mathbf{b}_\alpha$.

We begin by recalling some notions from the Introduction and (2.1) of [5]. First, recall that for inaccessible κ, $U(\kappa)$ is the set of multiples of κ in $(\kappa,\ \kappa^+)$. Let κ be inaccessible. Recall that a system, $(b_\alpha | \alpha \in U(\kappa))$ of almost-disjoint cofinal subsets of κ was called *decodable* if

$$(*) : \quad \text{for all } \theta \in (\kappa,\ \kappa^+),\ (b_\alpha | \alpha \leq \theta) \in L[A \cap \theta], \\ \text{and is "canonically definable" there.}$$

Recall that Corollary 4 of the Introduction of [5] gives that for all inaccessible κ, there is decodable $\vec{b} = (b_\alpha | \alpha \in U(\kappa))$ of cofinal almost-disjoint subsets of κ as above.

In (2.1.1) of [5], we defined $\mathcal{U} := \bigcup\{U(\kappa) | \kappa \text{ is inaccessible}\}$, and we considered the following additional property of the system $(b_\eta | \eta \in \mathcal{U})$ which we called **tree-like** :

whenever $\eta_1,\ \eta_2 \in \mathcal{U}$, if $\xi \in b_{\eta_1} \cap b_{\eta_2}$, then $b_{\eta_1} \cap \xi = b_{\eta_2} \cap \xi$.

We promised there, to show, here:

Lemma. *Without loss of generality, we can assume that $(b_\eta | \eta \in \mathcal{U})$ is tree-like and has the following additional property: $b_\eta = range\ g_\eta$, where g_η is a function, $dom\ g_\eta = \{\aleph_\tau | \aleph_\tau < card\ \eta\ \&\ \tau \text{ is an even successor ordinal}\}$; further, for all $\xi \in b_\eta$, ξ is a multiple of 4 but not of 8.*

Proof. This is actually a rather simple observation; for the record, the following is one way this can be achieved.

For inaccessible κ and $\alpha \in U(\kappa)$, and $\lambda < \kappa$ of the form $\aleph_\tau$, where τ is an even successor, let $\zeta_\alpha(\lambda)$ be the rank of $b_\alpha \cap \lambda$ in $<_{L[A \cap \lambda^+]}$, and let $g_\alpha(\lambda) =$ the $\zeta_\alpha(\lambda)^{th}$ η such that $\lambda < \eta < \lambda^+$ and η is a multiple of 4 but not of 8. Then let $b^*_\alpha = range\ g_\alpha$. It is clear that the b^*_α are decodable, since the b_α were, and that they have the desired tree-like property.

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