ON COHERENT SYSTEMS OF PROJECTIONS FOR $\aleph_1$-SEPARABLE GROUPS

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Abstract. It is proved consistent with either CH or $\neg$CH that there is an $\aleph_1$-separable group of cardinality $\aleph_1$ which does not have a coherent system of projections. It had previously been shown that it is consistent with $\neg$CH that every $\aleph_1$-separable group of cardinality $\aleph_1$ does have a coherent system of projections.

1 Introduction

An abelian group A is called $\aleph_1$-separable if every countable subset of A is contained in a countable free direct summand of A. An $\aleph_1$-separable group which is not free was first constructed by Griffith [3], extending a construction by Hill [4] for torsion groups. Such groups have been extensively studied, for example, in [6], [1], [7] and [2]. To show that a group A is $\aleph_1$-separable it suffices to produce an unbounded set of projections onto countable free subgroups, that is, a family $\{\pi_i: i \in I\}$ of functions $\pi_i: A \to H_i$ such that $\pi_i \circ \pi_i = \pi_i, H_i = \text{rge}(\pi_i)$ is a countable free group, and such that for every countable subset X of A, there is $i \in I$ with $X \subseteq H_i$. (In fact, the existence of such a family is obviously equivalent to saying that A is $\aleph_1$-separable.)

In most cases, the construction of an $\aleph_1$-separable group A yields a group with a stronger property: it has a *coherent unbounded system of projections*, i.e., a family $\{\pi_i : i \in I\}$ as above with the additional property that if $H_j \subseteq H_i$, then $\pi_j \circ \pi_i = \pi_j$. In fact, one cannot prove in ZFC that an $\aleph_1$-separable group of cardinality $\aleph_1$ fails to have this stronger property, because Mekler [7] has shown that PFA + $\neg$CH implies that every $\aleph_1$-separable group of cardinality $\aleph_1$ has this property (and more: it is in standard form).

It has also been shown that the question of whether an $\aleph_1$-separable group has a coherent system of projections (in an apparently stronger sense — "with respect to a filtration" — to be defined below), is relevant to the study of dual groups. Specifically, every $\aleph_1$-separable group, A, of cardinality $\aleph_1$ which has a coherent system of projections with respect to a filtration and is such that $\Gamma(A) \neq 1$ is a dual group. (See [2, XIV.3.1]. It is an open question whether it is provable in ZFC that every $\aleph_1$-separable group of cardinality $\aleph_1$ is a dual group.)

Thus it is a natural question to ask whether or not it is provable in ZFC that every $\aleph_1$-separable group (of cardinality $\aleph_1$) has a coherent system of projections. This is posed as an open question in [2]. Here we answer that question in the negative by showing that it is consistent both with CH and with $\neg$CH that there is an $\aleph_1$-separable group of cardinality $\aleph_1$ with no coherent unbounded system of projections. Moreover, such a group can be constructed to have any desired Gamma invariant (other than 0) and to be filtration-equivalent to an $\aleph_1$-separable group which *does* have a coherent system of projections.

2 Preliminaries

We will generally adhere to the terminology and notation of [2]. All groups referred to will be of cardinality at most $\aleph_1$. A *filtration* of an $\aleph_1$-separable group A is a continuous chain $\{A_\nu : \nu < \omega_1\}$ of subgroups of A such that $A_0 = 0$, $A = \bigcup_{\nu<\omega_1} A_\nu$, and for all $\nu < \omega_1$, $A_{\nu+1}$ is a countable free direct summand of A. A homomorphism $\pi : A \to A$ is a *projection* if $\pi^2 = \pi$; in that case, the image, H, of π is a direct summand of A.

Given an $\aleph_1$-separable group A and a filtration $\{A_\nu : \nu \in \omega_1\}$ of A, let

$$E \stackrel{\text{def}}{=} \{\nu \in \lim(\omega_1) : A_{\nu+1}/A_\nu \text{ is not free}\}.$$

Define $\Gamma(A) = \tilde{E}$, the equivalence class of E modulo the closed unbounded filter on $\mathcal{P}(\omega_1)$ (cf. [2, II.4.4 and IV.1.6]).

A *coherent system of projections with respect to the filtration* $\{A_\nu : \nu \in \omega_1\}$ of A is a family of projections $\{\pi_\nu : A \to A_\nu : \nu \notin E\}$ such that for all $\nu < \tau$ in $\omega_1 \setminus E$, $\pi_\nu \circ \pi_\tau = \pi_\nu$.

Clearly, $\{\pi_\nu : A \to A_\nu : \nu \notin E\}$ is a coherent unbounded system of projections, as defined in the Introduction. We do not know if, conversely, any $\aleph_1$-separable group which has a coherent unbounded system of projections also has a coherent system of projections with respect to a filtration.

We say that an $\aleph_1$-separable group A has *quotient type H* if A has a filtration $\{A_\nu : \nu \in \omega_1\}$ such that $A_{\nu+1}/A_\nu \cong H$ for all ν such that $A_{\nu+1}/A_\nu$ is not free. (See [2, p. 251].)

Let $\mathrm{succ}(\omega_1)$ (respectively, $\lim(\omega_1)$) denote the set of all successor (resp., limit) ordinals in ω_1.

3 Construction of a counterexample using $\diamondsuit$

For a prime p, $\mathbb{Q}^{(p)}$ denotes the subgroup of $\mathbb{Q}$ consisting of rationals whose denominators are a power of p.

THEOREM 1 *Assume $\diamondsuit_{\omega_1}(S)$, where S is a stationary set of limit ordinals $< \omega_1$. Let p be a prime. Then there exists an $\aleph_1$-separable group A of cardinality $\aleph_1$ such that $\Gamma(A) = \tilde{S}$, A is of quotient type $\mathbb{Q}^{(p)}$, and A has no coherent unbounded system of projections.*

PROOF. Let D be the $\mathbb{Q}$-vector space with basis $\{x_{\nu,n} : n \in \omega, \nu < \omega_1\} \cup \{y_\delta : \delta \in S\}$. Let D_α be the subspace of D generated by $\{x_{\nu,n} : n \in \omega, \nu < \alpha\} \cup \{y_\delta : \delta \in S \cap \alpha\}$. We shall define inductively subgroups A_α of D_α such that for all $\mu \geq \alpha$, $A_\mu \cap D_\alpha = A_\alpha$. At the same time, we will define homomorphisms $t_{\alpha\nu} : A_\alpha \to A_\nu$ for all successor ordinals $\nu < \alpha$. Our inductive construction will satisfy:

(1) for all successor ordinals ν and all $\gamma > \alpha > \nu$, A_ν is free and $t_{\alpha\nu} \restriction A_\nu$ is the identity (i.e., $t_{\alpha\nu}$ is a projection onto A_ν) and $t_{\gamma\nu} \restriction A_\alpha = t_{\alpha\nu}$;

(2) if $\alpha \notin S$, then $A_{\alpha+1}/A_\alpha$ is free and if $\alpha \in S$, then $A_{\alpha+1}/A_\alpha \cong \mathbb{Q}^{(p)}$.

When the construction is completed we will define $A = \cup_{\alpha<\omega_1} A_\alpha$ and

$$t_\nu = \cup_{\alpha<\omega_1} t_{\alpha\nu} : A \to A_\nu$$

for each successor ordinal $\nu < \omega_1$. We will carry out the construction so that the following properties will hold:

(I) for every projection $\pi : A \to H$ onto a countable subgroup H of A, there is a finite set $W_\pi \subseteq \mathrm{succ}(\omega_1)$ such that for all $a \in A$, if $t_\nu(a) = 0$ for all $\nu \in W_\pi$, then $\pi(a) = 0$.

(II) whenever W_0 and W_1 are finite subsets of $\mathrm{succ}(\omega_1)$ and $\beta = \sup(W_0 \cap W_1)$, there exists $\delta > \beta$ and $y_{\delta,0}$, $y_{\delta,1} \in A_{\delta+1}$ such that $0 \neq py_{\delta,1} - y_{\delta,0} \in A_{\beta+1}$, and $t_\nu(y_{\delta,\ell}) = 0$ for all $\nu \in W_\ell$ ($\ell = 0, 1$).

Suppose for a moment that we can carry out the construction. Then A is $\aleph_1$-separable since $\{t_\nu : \nu \in \mathrm{succ}(\omega_1)\}$ is an unbounded system of projections. Also, (2) implies that $\Gamma(A) = \tilde{S}$ and A has quotient type $\mathbb{Q}^{(p)}$.

We claim that there is no coherent unbounded system of projections. Suppose, to the contrary that $\{\pi_i : i \in I\}$ is a coherent unbounded system of projections where $\mathrm{rge}(\pi_i) = H_i$. Then by (I), for each π_i there is a finite set W_i such that for all $a \in A$, if $t_\nu(a) = 0$ for all $\nu \in W_i$, then $\pi_i(a) = 0$. Now apply the Δ-system Lemma [5, p. 225]: there is a finite set $\Delta \subseteq \omega_1$ and an uncountable subset Z of I such that for all $i \neq i'$ in Z, $W_i \cap W_{i'} = \Delta$. Let $\beta = \sup(\Delta)$. Choose i_0, $i_1 \in Z$ such that $A_{\beta+1} \subseteq H_{i_0}$ and $H_{i_0} \subseteq H_{i_1}$. Let δ and $y_{\delta,0}$ and $y_{\delta,1}$ be as in (II) for W_{i_0} and W_{i_1}. Then by (I) and (II) we have $\pi_{i_\ell}(y_{\delta,\ell}) = 0$ for $\ell = 0, 1$. By coherence we then have $\pi_{i_0}(y_{\delta,1}) = \pi_{i_0}(\pi_{i_1}(y_{\delta,1})) = 0$, so $\pi_{i_0}(py_{\delta,1} - y_{\delta,0}) = 0$, which is a contradiction because $py_{\delta,1} - y_{\delta,0}$ is non-zero and belongs to $A_{\beta+1} \subseteq H_{i_0}$.

So it remains to do the construction. First let us write S as the disjoint union

$$S = S_0 \amalg S_1$$

of (stationary) sets such that $\diamondsuit_{\omega_1}(S_i)$ holds for $i = 0, 1$. Also, choose a surjection ψ from S_0 onto the set of all pairs (W_0, W_1) of finite subsets of $\mathrm{succ}(\omega_1)$ such that for each $\delta \in S_0$, if $\psi(\delta) = (W_0, W_1)$, then $\delta > \sup(W_0 \cap W_1) + \omega$.

Suppose now that we have constructed A_α and $t_{\alpha\nu}$ for all $\alpha < \gamma$. There are four cases to consider.

In the first case, γ is a limit ordinal. In this case, we let $A_\gamma = \cup_{\alpha<\gamma} A_\alpha$ and $t_{\gamma\nu} = \cup_{\nu<\alpha<\gamma} t_{\alpha\nu}$ for all successor ordinals $\nu < \gamma$. Clearly (1) and (2) are satisfied. So now we can assume that $\gamma = \delta + 1$ for some δ.

In the second case, $\delta \notin S$. In this case we let $A_\gamma = A_\delta \oplus \bigoplus_{n\in\omega} \mathbb{Z}x_{\delta,n}$ and for each successor $\nu \leq \delta$ we define $t_{\gamma\nu}$ to be an extension of $t_{\delta\nu} : A_\delta \to A_\nu$ (where $t_{\delta\delta}$ is the identity map if $\delta \notin \mathrm{succ}(\omega_1)$) such that the $t_{\gamma\nu}$ ($\nu \in \mathrm{succ}(\omega_1) \cap \gamma$) satisfy:

(3) $t_{\gamma\nu}(x_{\delta,0}) = 0$ and for every finite subset F of $\mathrm{succ}(\omega_1) \cap \gamma$ and function $\theta : F \to A_\delta$, there exists $k \geq 1$ such that $t_{\gamma\nu}(x_{\delta,k}) = \theta(\nu)$ for all $\nu \in F$, and $t_{\gamma\nu}(x_{\delta,k}) = 0$ for $\nu \notin F$.

Since the number of pairs (F, θ) is countable, this is easy to arrange.

In the third case, $\delta \in S_0$. Here we will do the construction to insure that (II) holds. Let $\psi(\delta) = (W_0, W_1)$ and let $\beta = \sup(W_0 \cap W_1)$. Choose a ladder η on δ such that $\eta(0) = \beta$ and $\eta(n)$ is a successor ordinal greater than $\sup(W_0 \cup W_1)$ for all $n \geq 1$. By (3) there exists k_1 such that

$$t_{\delta\nu}(x_{\eta(1),k_1}) = -t_{\delta\nu}(x_{\eta(0),0})$$

for all $\nu \in W_0 \setminus W_1$ and

$$t_{\delta\nu}(x_{\eta(1),k_1}) = 0$$

for all other successor $\nu \leq \eta(1)$ (hence for all $\nu \in W_1$).

Now let $a_0 = px_{\eta(0),0}$, $a_1 = x_{\eta(1),k_1}$ and $a_j = x_{\eta(j),0}$ for $j \geq 2$. Let

$$y_{\delta,n} = (y_\delta + \Sigma_{j<n} p^j a_j)/p^n \in D_{\delta+1}$$

(so $y_{\delta,0} = y_\delta$). Let $A_{\delta+1} = A_\gamma$ be the subgroup of $D_{\delta+1}$ generated by

$$A_\delta \cup \{y_{\delta,n} : n \in \omega\}.$$

For all successor $\nu < \delta$ let

$$t_{\gamma\nu}(y_{\delta,n}) = -\Sigma_{j=n}^{\infty} p^{j-n} t_{\delta\nu}(a_j)$$

for all $n \in \omega$. This is easily seen to be a finite sum, by our choice of the a_j, and the projections are well-defined. Moreover, for $\nu \in W_1 \setminus W_0$,

$$t_{\gamma\nu}(y_{\delta,0}) = -p \cdot t_{\delta\nu}(x_{\eta(0),0})$$

and $t_{\gamma\nu}(y_{\delta,n}) = 0$ for $n \geq 1$. For $\nu \in W_0 \setminus W_1$,

$$t_{\gamma\nu}(y_{\delta,1}) = t_{\delta\nu}(x_{\eta(0),0})$$

and $t_{\gamma\nu}(y_{\delta,n}) = 0$ for $n \neq 1$. For $\nu \in W_0 \cap W_1$, since $\nu \leq \beta = \eta(0)$, $t_{\gamma\nu}(x_{\eta(n),0}) = 0$ by definition; hence $t_{\gamma\nu}(y_{\delta,n}) = 0$ for all n. Note also that

$$py_{\delta,1} - y_{\delta,0} = a_0 = px_{\eta(0),0} = px_{\beta,0} \in A_{\beta+1}.$$

Hence, (II) is satisfied.

In the fourth and last case, $\delta \in S_1$. Then $\diamondsuit(S_1)$ gives us a prediction of a function $\pi_\delta : A_\delta \to A_\delta$. If π_δ is not a projection, or if there is a finite subset W of $\mathrm{succ}(\omega_1) \cap \delta$ such that for all $a \in A_\delta$, $t_{\delta\nu}(a) = 0$ for all $\nu \in W$ implies $\pi_\delta(a) = 0$, then define A_γ and $t_{\gamma\nu}$ in any way that satisfies (1) and (2). Otherwise, we want to define A_γ so that, in addition, π_δ does not extend to A_γ. Now π_δ is a projection: $A_\delta \to H$ (for some countable $H = \mathrm{rge}(\pi_\delta)$) and if we write $\mathrm{succ}(\omega_1) \cap \delta$ as the increasing union, $\cup_{n\in\omega} W_n$, of finite sets, then for each $n \in \omega$ there exists $a_n \in A_\delta$ such that $\pi_\delta(a_n) \neq 0$ but $t_{\delta\nu}(a_n) = 0$ for all $\nu \in W_n$. By the Lemma following, there is a choice of $c_n \in \mathbb{Z}$ such that the sequence $\langle \Sigma_{j=0}^{n} p^j c_j \pi_\delta(a_j) : n \in \omega\rangle$ does not have a limit in H (in the p-adic topology). Define

$$y_{\delta,n} = (y_\delta + \Sigma_{j<n} p^j c_j a_j)/p^n \in D_{\delta+1}$$

and let $A_{\delta+1} = A_\gamma$ be the subgroup of $D_{\delta+1}$ generated by $A_\delta \cup \{y_{\delta,n} : n \in \omega\}$. Define

$$t_{\gamma\nu}(y_{\delta,n}) = -\Sigma_{j\geq n} p^{j-n} t_{\delta\nu}(a_j)$$

which is well defined since almost all the $t_{\delta\nu}(a_j)$ are 0. Then π_δ does not extend to a homomorphism $h: A_\gamma \to H$ since if it did, $h(y_\delta)$ would be a limit of $\langle \Sigma_{j=0}^{n} p^j c_j \pi_\delta(a_j) : n \in \omega\rangle$.

This completes the inductive construction. It remains to check that (I) holds. Given any projection $\pi: A \to H$, by the diamond property, there is a stationary subset S' of S_1 such that for $\delta \in S'$, $\pi \restriction A_\delta = \pi_\delta$. Hence, since π_δ does extend to $A_{\delta+1}$, there is a finite subset W_δ of $\mathrm{succ}(\omega_1) \cap \delta$ such that for all $a \in A_\delta$, $t_{\delta\nu}(a) = 0$ for all $\nu \in W_\delta$ implies $\pi(a) = 0$. Then by Fodor's Lemma (cf. [2, II.4.11]) and a coding argument, there is a finite set W_π such that for a stationary subset S'' of S', $\delta \in S''$ implies $W_\delta = W_\pi$. Since S'' is unbounded in ω_1, we are done. $\square$

Lemma 2 *Let H be a countable free group and $\hat{H}$ its closure in the p-adic topology. If $\langle b_n : n \in \omega\rangle$ is a sequence of non-zero elements of H, then*

$$\{\Sigma_{j\in\omega} p^j c_j b_j : \langle c_j : j \in \omega\rangle \in \mathbb{Z}^\omega\}$$

is a subset of $\hat{H}$ of cardinality $2^{\aleph_0}$.

Proof. By induction choose an increasing sequence (m_n), so that p^{m_n+n} does not divide any element of $\{p^{m_k+k} b_k : k < n\}$. For any $\xi \in {}^\omega 2$ let $c_{\xi n} = \xi(n) p^{m_n}$. It remains to check that if $\xi_0 \neq \xi_1$ then $\sum_{k=0}^{\infty} p^k c_{\xi_0 k} b_k \neq \sum_{k=0}^{\infty} p^k c_{\xi_1 k} b_k$. Let n be minimal so that $\xi_0(n) \neq \xi_1(n)$, then

$$\sum_{k=0}^{n} p^k c_{\xi_0 k} b_k - \sum_{k=0}^{n} p^k c_{\xi_1 k} b_k = \pm p^{m_n+n} b_n \not\equiv 0 \bmod p^{m_{n+1}+n+1} H.$$

However, $p^{m_{n+1}+n+1}$ divides $\sum_{k=n+1}^{\infty} p^k c_{\xi_0 k} b_k - \sum_{k=n+1}^{\infty} p^k c_{\xi_1 k} b_k$. $\square$

Corollary 3 *It is consistent with ZFC that there are filtration-equivalent $\aleph_1$-separable groups A and B such that B has a coherent system of projections with respect to a filtration but A does not have a coherent unbounded system of projections.*

PROOF. Let A be as constructed in the Theorem. Associated with each $\delta \in S$ there is a ladder η_δ on δ such that p^{n+1} divides $y_{\delta,0}$ mod A_ν if and only if $\nu \geq \eta_\delta(n)$. If we construct B as in [2, VIII.1.1] (with $p_\delta = p$ for all $\delta \in S$), then by [2, VII.1.10] B has a coherent system of projections with respect to a filtration and by [1, Thm. 1.4], A and B are filtration-equivalent. □

The following should be compared with [2, XIV.3.1]. (See also the introductory remarks concerning dual groups.)

COROLLARY 4 *It is consistent with ZFC that there is an $\aleph_1$-separable group A such that $\Gamma(A) \neq 1$ and A does not have a coherent system of complementary summands.* □

4 Counterexamples where CH fails

Theorem 1 requires $\diamondsuit(S)$ which implies CH. We know that it is consistent with $\neg$CH that every $\aleph_1$-separable group of cardinality $\aleph_1$ has a coherent unbounded system of projections (cf. [7]). So the question naturally arises whether it is consistent with $\neg$CH that there is an $\aleph_1$-separable group of cardinality $\aleph_1$ which does not have a coherent unbounded system of projections. Here we shall prove that the answer to the question is "yes". In fact the forcing used is just the simplest possible, namely $\mathrm{Fn}(\kappa, 2, \omega)$, the forcing for adding κ Cohen reals, where $\kappa \geq \aleph_2$, to make CH fail. ($\mathrm{Fn}(\kappa, 2, \omega)$ is the poset consisting of all partial functions from κ to 2 whose domains have cardinality less than ω.)

THEOREM 5 *It is consistent with $\neg$CH that for every stationary subset S of* $\lim(\omega_1)$ *there is an $\aleph_1$-separable group A of cardinality $\aleph_1$ with $\Gamma(A) = \tilde{S}$ which does not have a coherent unbounded system of projections.*

PROOF. We shall prove the following lemma.

LEMMA 6 *Suppose $\mathbb{P} = Fn(\aleph_1, 2, \omega)$ and suppose $S \in V$ is a stationary subset of* $\lim(\omega_1)$. *If G is generic for $\mathbb{P}$, then in $V[G]$ there is an $\aleph_1$-separable group A of cardinality $\aleph_1$ with $\Gamma(A) = \tilde{S}$ which does not have a coherent unbounded system of projections.*

Assume for the moment that the lemma is correct. Let $\mathbb{P}'$ be $\mathrm{Fn}(\kappa, 2, \omega)$ where $\kappa \geq \aleph_2$, and let G' be generic for $\mathbb{P}'$. Given any stationary set, S, in the generic extension, $V[G']$, we recast the forcing as a two-step iteration, say $\mathbb{P}_0 \times \mathbb{P}_1$ with generic set $G' = G_0 \times G_1$, where $\mathbb{P}_0$ adds some number of Cohen reals, $\mathbb{P}_1$ adds $\aleph_1$ Cohen reals and $S \in V[G_0]$. By Lemma 6 there is an $\aleph_1$-separable group A of cardinality $\aleph_1$ in $V[G_0][G_1] = V[G']$ with $\Gamma(A) = \tilde{S}$ and with no coherent unbounded system of projections.

Thus it remains to prove Lemma 6. We will describe an iterated forcing which forces the existence of the desired A. The forcing will be an iteration of length ω_1. Afterward we will note that the forcing is equivalent to adding $\aleph_1$ Cohen reals. We follow the usual notation where at each step α the iterate is Q_α and the result of the iteration up to α is $\mathbb{P}_\alpha$. We will let G_α denote an arbitrary $\mathbb{P}_\alpha$-generic set and talk of members of $V[G_\alpha]$ where more correctly we should talk of $\mathbb{P}_\alpha$-names.

As well as constructing the sequence Q_α we will define a sequence of groups A_α and projections $t_{\alpha\nu}$ where $A_\alpha, t_{\alpha\nu} \in V[G_\alpha]$ and the A_α's, $t_{\alpha\nu}$ are as in Theorem 1 (except for properties (I) and (II) which we will have to verify). The groups A_α will be constructed to be subgroups of $D_\alpha \subseteq D$, as in Theorem 1. By coding we can assume that the set underlying D is ω_1. As in the proof of Theorem 1, partition S into two disjoint stationary subsets S_0 and S_1.

The construction goes by cases. If $\alpha \notin S_1$ then define Q_α to be trivial (the one element poset). The construction of $A_{\alpha+1}$ and $\{t_{\alpha+1\nu}: \nu \leq \alpha, \nu \in \mathrm{succ}(\omega_1)\}$ is as in Theorem 1 (i.e., as in the second or third case). Of course the construction of A_δ and $t_{\delta,\nu}$ is determined when δ is a limit ordinal.

Suppose now that $\delta \in S_1$. We will work in $V[G_\delta]$ and define $\mathbb{Q}_\delta$. Then Q_δ will be the obvious $\mathbb{P}_\delta$-name. List as $(\alpha_n: n < \omega)$ the ordinals in $\delta \cap \mathrm{succ}(\omega_1)$. The forcing $\mathbb{Q}_\delta$ is defined to be the set of sequences of the form $(c_0, a_0, \ldots, c_{n-1}, a_{n-1})$ where for all $m < n$, $c_m \in \{0,1\}$, $a_m \in A_\delta$ and if $j < m$ then $t_{\delta,\alpha_j}(a_m) = 0$. $\mathbb{Q}_\delta$ is ordered by extension. A generic set for $\mathbb{Q}_\delta$ can be identified with a sequence of length ω. Given a generic set $G_{\delta+1}$ for $\mathbb{P}_{\delta+1}$ and so a generic sequence $(c_j, a_j: j < \omega)$ for $\mathbb{Q}_\delta$, let

$$y_{\delta,n} = (y_\delta + \Sigma_{m<n} p^m c_m a_m)/p^n \in D_{\delta+1}$$

Let $A_{\delta+1} = A_\gamma$ be the subgroup of $D_{\delta+1}$ generated by $A_\delta \cup \{y_{\delta,n}: n \in \omega\}$. The definition of the projections is as in Theorem 1; they are well-defined because for all $j \in \omega$, for all $m > j$, $t_{\delta,\alpha_j}(a_m) = 0$.

In $V[G_{\omega_1}]$, we let $A = \bigcup_{\alpha<\omega_1} A$ and for every successor ordinal ν, we let $t_\nu = \bigcup_{\beta>\nu} t_{\beta\nu}$. We will observe that $\mathbb{P}_{\omega_1}$ is equivalent to adding $\aleph_1$ Cohen reals. In particular, the forcing is c.c.c. and so ω_1 is preserved and A is an $\aleph_1$-separable group of cardinality $\aleph_1$. To see that A is the desired group we have to check that property (I) from Theorem 1 holds. (The construction guarantees that property (II) holds for exactly the same reasons as in the proof of Theorem 1). The proof that A satisfies property (I) is contained in the following two lemmas.

Lemma 7 *Use the notation above. Suppose $\delta \in S_1$. Furthermore suppose $\pi \in V[G_\delta]$ and π is a projection from A_δ to H so that for every finite set $w \subseteq \{\alpha < \delta: \alpha \in succ(\omega_1)\}$ there is $a \in A_\delta$ such that $t_{\delta\alpha}(a) = 0$ for all $\alpha \in w$ and $\pi(a) \neq 0$. Then π does not extend to a projection from $A_{\delta+1}$ to H.*

PROOF. We will work in $V[G_\delta]$. Fix some such π. It suffices to show for all $a \in A_\delta$ that

$$D_a \stackrel{\text{def}}{=} \{q \in \mathbb{Q}_\delta : q \Vdash \text{ "if } \hat{\pi} \text{ is an extension of } \pi \text{ to } A_{\delta+1} \text{ then } \hat{\pi}(y_\delta) \neq a\text{"}\}$$

is dense. Fix $a \in A_\delta$ and consider any condition $(c_0, a_0, \ldots, c_{n-1}, a_{n-1})$. Choose a_n so that $\pi(a_n) \neq 0$ and $t_{\delta\alpha_m}(a_n) = 0$ for all $m < n$. For some choice of $c_n \in \{0, 1\}$, $\sum_{m=0}^{n} p^m c_m \pi(a_m) \neq a$. Since A_δ is free, there is $k > n$ so that $\sum_{m=0}^{n} p^m c_m \pi(a_m) \not\equiv a \bmod p^k A_\delta$. For m so that $n < m < k$ let $c_m = 0$ and let $a_m = 0$. Notice that if b_i $(i \geq k)$ are any elements of A_δ we have

$$\sum_{m=0}^{k-1} p^m c_m \pi(a_m) + \sum_{m=k}^{\infty} p^m b_m \equiv \sum_{m=0}^{n} p^m c_m \pi(a_m) \not\equiv a \bmod p^k A_\delta.$$

Hence $(c_0, a_0, \ldots c_{k-1}, a_{k-1})$ belongs to D_a. □

(We could have replaced Lemma 2 by an argument like that in the preceding proof.)

LEMMA 8 *Suppose $\pi \in V[G_{\omega_1}]$ is a projection of A to a subgroup H. Then there is a closed unbounded set C so that for all $\alpha \in C$, $\pi \restriction A_\alpha \in V[G_\alpha]$. (We assume here, as we have done tacitly above, that G_α is the restriction of G_{ω_1} to $\mathbb{P}_\alpha$.)*

PROOF. This is a standard fact for finite support iterations of c.c.c. forcing, so we will just sketch the argument. Take $\tilde{\pi}$ a name for π. For each $\alpha \in A$, take X_α a maximal antichain of conditions so that for all $q \in X_\alpha$, there is $a_{q\alpha}$ so that $q \Vdash \tilde{\pi}(\alpha) = a_{q\alpha}$. (Recall that the underlying set of A is contained in ω_1). Since $\mathbb{P}$ is c.c.c., each X_α is countable. Our cub C consists of $\{\alpha < \omega_1 : \text{for all } \beta \in A_\alpha, X_\beta \subseteq \mathbb{P}_\alpha \text{ and for all } q \in X_\beta, a_{q\beta} \in A_\alpha\}$. □

It remains to observe that $\mathbb{P}_{\omega_1}$ is equivalent to adding $\aleph_1$ Cohen reals. The proof uses two pieces of folklore. The first one that any countable poset with the property that any element has two incompatible extensions is equivalent to the forcing for adding a Cohen real. The second, which uses the first, is that an iteration of length ω_1 such that each iterate is forced to be a countable poset with the property that any element has two incompatible extensions is equivalent to adding $\aleph_1$ Cohen reals. A somewhat fuller explanation can be found in the proof of Lemma 1.5 of [8]. If we view $\mathbb{P}_{\omega_1}$ as the iteration of $\{Q_\delta : \delta \in S_1\}$, then the second piece of folklore applies. □

5 Questions

One question that we do not know the answer to is whether or not the existence of an $\aleph_1$-separable group of cardinality $\aleph_1$ without a coherent unbounded system of projections follows from CH alone. (Presumably one would use weak diamond in such a proof.) To put the question a different way, is it consistent with CH that every $\aleph_1$-separable group of cardinality $\aleph_1$ has a coherent unbounded system of projections?

Another question along the same lines is whether MA + $\neg$CH implies that every $\aleph_1$-separable group of cardinality $\aleph_1$ has a coherent unbounded system of projections. Since PFA implies MA + $\neg$CH, we know that it is consistent with MA + $\neg$CH that every $\aleph_1$-separable group of cardinality $\aleph_1$ has a coherent system of projections with respect to a filtration. Our methods cannot be immediately translated over to a model of MA + $\neg$CH, since we have built a group which is filtration equivalent to a group with a coherent system of projections, while under MA + $\neg$CH any two filtration equivalent $\aleph_1$-separable groups of cardinality $\aleph_1$ are isomorphic ([1])

Finally, there is the question of whether the existence of a coherent unbounded system of projections for an $\aleph_1$-separable group A of cardinality $\aleph_1$ implies the existence of a coherent system of projections with respect to a filtration of A. (It clearly implies the existence of a filtration $\{A_\nu : \nu \in \omega_1\}$ of A and a coherent family of projections $\{\pi_\nu : A \to A_\nu : \nu \in \mathrm{succ}(\omega_1)\}$; the problem is to define coherently projections π_ν when ν is a limit ordinal not in E.)

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