

# All meager filters may be null

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October 6, 2003

## Abstract

We show that it is consistent with ZFC that all filters which have the Baire property are Lebesgue measurable. We also show that the existence of a Sierpinski set implies that there exists a nonmeasurable filter which has the Baire property.

The goal of this paper is to show yet another example of nonduality between measure and category.

Suppose that  $\mathcal{F}$  is a nonprincipal filter on  $\omega$ . Identify  $\mathcal{F}$  with the set of characteristic functions of its elements. Under this convention  $\mathcal{F}$  becomes a subset of  $2^\omega$  and a question about its topological or measure-theoretical properties makes sense.

It has been proved by Sierpinski that every non-principal filter has either Lebesgue measure zero or is nonmeasurable. Similarly it is either meager or does not have the Baire property.

In [T] Talagrand proved that

**Theorem 0.1** *There exists a measurable filter which does not have the Baire property. ■*

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\*The author thanks the Lady Davis Fellowship Trust for full support

<sup>†</sup>Supported by the Israel Academy of Sciences (Basic Research Fund)

<sup>‡</sup>Publication 434

In fact we have an even stronger result. In [Ba] it is proved that

**Theorem 0.2** *Every measurable filter can be extended to a measurable filter which does not have the Baire property. ■*

We show that the dual result is false.

## 1 A model where all meager filters are null

In this section we prove the following theorem:

**Theorem 1.1** *It is consistent with ZFC that every filter which has the Baire property is measurable.*

PROOF We will use the following more general result:

**Theorem 1.2** *Let  $\mathbf{V} \models GCH$  and suppose that  $\mathbf{V}[G]$  is a generic extension of  $\mathbf{V}$  obtained adding  $\omega_2$  Cohen reals. Then in  $\mathbf{V}[G]$  for any two sets  $A, B \subset 2^\omega$  if  $A + B = \{a + b : a \in A, b \in B\}$  is a meager set then either  $A$  or  $B$  has measure zero.*

PROOF Note that we apply this lemma only for the case  $A = B$ . Therefore to simplify the notation we assume that  $A = B$ . The proof of the general case is almost the same. We follow [Bu].

We will use the following notation. Let

$$Fn(X, 2) = \{s : \text{dom}(s) \in [X]^{<\omega} \text{ and } \text{range}(s) \subset \{0, 1\}\}$$

be the notion of forcing adding  $|X|$ -many Cohen reals. For  $s \in Fn(X, 2)$  let  $[s] = \{f \in 2^X : s \subset f\}$ .

Let  $\mathbf{V} \models GCH$  be a model of ZFC and let  $G_{\omega_2}$  be a  $Fn(\omega_2, 2)$ -generic filter over  $\mathbf{V}$ . Clearly  $c = \bigcup G_{\omega_2}$  is a generic sequence of  $\omega_2$  Cohen reals and  $\mathbf{V}[c] = \mathbf{V}[G_{\omega_2}]$ .

Let  $\{F_n : n \in \omega\}$  be a sequence of closed, nowhere dense sets such that  $A + A \subseteq \bigcup_{n \in \omega} F_n$ . Without loss of generality we can assume that  $\{F_n : n \in \omega\} \in \mathbf{V}$ .

Let  $\{a_\xi : \xi < \omega_2\}$  be an enumeration of all elements of  $A$ . For every  $\xi < \omega_2$  let  $\dot{a}_\xi$  be a name for  $a_\xi$ . In other words for every  $\xi < \omega_2$  we have a countable set  $I_\xi \subset \omega_2$  such that  $\dot{a}_\xi$  is a Borel function from  $2^{I_\xi}$  into  $2^\omega$ . Moreover  $a_\xi$  is the value of of the function  $\dot{a}_\xi$  on Cohen real i.e.  $\dot{a}_\xi(c|I_\xi) = a_\xi$ . In addition we can find a dense  $G_\delta$  set  $H_\xi \subseteq 2^{I_\xi}$  such that  $\dot{a}_\xi|H_\xi$  is a continuous function.

For  $\alpha, \xi, \eta < \omega_2$  define  $\xi \simeq_\alpha \eta$  if

1.  $I_\xi$  and  $I_\eta$  are order isomorphic,
2. the order-isomorphism between  $I_\xi$  and  $I_\eta$  transfers  $\dot{a}_\xi$  onto  $\dot{a}_\eta$  and  $H_\xi$  onto  $H_\eta$ ,

$$3. I_\xi \cap \alpha = I_\eta \cap \alpha.$$

Notice that for every  $\alpha < \omega_2$  the relation  $\simeq_\alpha$  is an equivalence relation with  $\omega_1$  many equivalence classes.

**Lemma 1.3** *There exists  $\alpha^* < \omega_2$  such that*

$$\forall \xi, \beta \exists \eta (\xi \simeq_{\alpha^*} \eta \ \& \ I_\eta \cap (\beta - \alpha^*) = \emptyset) .$$

PROOF For every  $\alpha < \omega_2$  let  $\mathcal{E}_\alpha$  be the set  $\{[\xi]_\alpha : \xi < \omega_2\}$  of  $\simeq_\alpha$ -equivalence classes. Let

$$\begin{aligned} \mathcal{E}_\alpha^0 &= \{E \in \mathcal{E}_\alpha : \sup_{\eta \in E} (\min(I_\eta - \alpha)) < \omega_2\} \text{ and} \\ \mathcal{E}_\alpha^1 &= \mathcal{E}_\alpha - \mathcal{E}_\alpha^0 . \end{aligned}$$

Let

$$\gamma(\alpha) = \sup_{E \in \mathcal{E}_\alpha^0} (\sup_{\eta \in E} (\min(I_\eta - \alpha))) .$$

Note that  $\gamma(\alpha) < \omega_2$  since  $|\mathcal{E}_\alpha| < \aleph_1$ .

Find  $\alpha^* < \omega_2$  such that  $\gamma(\alpha) < \alpha^*$  for all  $\alpha < \alpha^*$  and  $\text{cf}(\alpha^*) = \omega_1$ . We claim that  $\alpha^*$  satisfies the statement of the lemma.

Take any  $\xi < \omega_2$  and any  $\beta$ . If  $\beta < \alpha^*$  or  $I_\xi \subseteq \alpha^*$ , then we can choose  $\eta = \xi$ . So assume  $\beta > \alpha^*$  and  $I_\xi - \alpha^* \neq \emptyset$ . There is  $\alpha < \alpha^*$  such that  $I_\xi \cap \alpha = I_\xi \cap \alpha^*$ . Let  $E = [\xi]_\alpha$ .

CASE 1  $E \in \mathcal{E}_\alpha^0$ . Then

$$\sup_{\eta \in E} (\min(I_\eta - \alpha)) \leq \gamma(\alpha) < \alpha^*$$

which is a contradiction since  $\min(I_\xi - \alpha) \geq \alpha^*$  and  $\xi \in E$ .

CASE 2  $E \notin \mathcal{E}_\alpha^0$ . So

$$\sup_{\eta \in E} (\min(I_\eta - \alpha)) = \omega_2$$

hence there is  $\eta \in E$  with  $\min(I_\eta - \alpha) \geq \beta$  i.e.  $I_\eta \cap (\beta - \alpha) = \emptyset$ .

So  $I_\xi \cap \alpha^* = I_\xi \cap \alpha = I_\eta \cap \alpha = I_\eta \cap \alpha^*$ , where the last equality holds because  $I_\eta \cap (\alpha^* - \alpha) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset$ . Also  $I_\eta \cap (\beta - \alpha^*) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset$ . ■

Let  $\alpha^*$  be the ordinal from the above lemma. Work in  $\mathbf{V}' = \mathbf{V}[c|\alpha^*]$ .

For every  $\xi < \omega_2$  define

$$D_\xi = \{s \in Fn(\omega_2 - \alpha^*, 2) : \text{cl}(\dot{a}_\xi([s])) \text{ has measure zero} \} .$$

**Lemma 1.4**  $D_\xi$  is dense in  $Fn(\omega_2 - \alpha^*, 2)$  for every  $\xi < \omega_2$ .

PROOF Notice that it is enough to show that  $D_\xi \cap Fn(I_\xi - \alpha^*, 2)$  is dense in  $Fn(I_\xi - \alpha^*, 2)$  for  $\xi < \omega_2$ .

Suppose that this fails. Find  $\xi < \omega_2$  and  $s_0 \in Fn(I_\xi - \alpha^*, 2)$  such that for all  $s \supseteq s_0$  the set  $cl(\dot{a}_\xi([s]))$  has positive measure.

Using the lemma with  $\beta > \sup(I_\xi)$  we can find  $\eta < \omega_2$  such that  $\xi \simeq_{\alpha^*} \eta$  and  $(I_\xi - \alpha^*) \cap (I_\eta - \alpha^*) = \emptyset$ . Notice that there exists  $t_0 \in Fn(I_\eta - \alpha^*, 2)$  (the image of  $s_0$  under the isomorphism between  $I_\xi$  and  $I_\eta$ ) such that for every  $t \supseteq t_0$  the set  $cl(\dot{a}_\eta([t]))$  has positive measure.

Since  $s_0$  and  $t_0$  have disjoint domains,  $s_0 \cup t_0 \in Fn(\omega_2 - \alpha^*, 2)$ . Find  $n \in \omega$  and a condition  $u \in Fn(\omega_2 - \alpha^*, 2)$  extending  $s_0 \cup t_0$  such that  $u \Vdash \dot{a}_\xi(\dot{c}) + \dot{a}_\eta(\dot{c}) \in F_n$ .  $u$  can be written as  $u_1 \cup u_2 \cup u_3$  where  $s_0 \subseteq u_1 \in Fn(I_\xi - \alpha^*, 2)$ ,  $t_0 \subseteq u_2 \in Fn(I_\eta - \alpha^*, 2)$  and  $u_3 \in Fn(\omega_2 - (I_\xi \cup I_\eta \cup \alpha^*), 2)$ . By the assumption the sets  $cl(\dot{a}_\xi([u_1]))$ ,  $cl(\dot{a}_\eta([u_2]))$  have positive measure. By well-known theorem of Steinhaus the set  $cl(\dot{a}_\xi([u_1])) + cl(\dot{a}_\eta([u_2]))$  contains an open set (hence also  $(cl(\dot{a}_\xi([u_1])) + cl(\dot{a}_\eta([u_2]))) - F_n$  contains an open set). Using the fact that  $\dot{a}_\xi$  and  $\dot{a}_\eta$  are continuous functions we can find  $u_1 \subseteq s_1 \in Fn(I_\xi - \alpha^*, 2)$  and  $u_2 \subseteq t_1 \in Fn(I_\eta - \alpha^*, 2)$  such that  $(cl(\dot{a}_\xi([s_1])) + cl(\dot{a}_\eta([t_1]))) \cap F_n = \emptyset$ . But this is a contradiction since

$$s_1 \cup t_1 \cup u_3 \Vdash \dot{a}_\xi(\dot{c}) + \dot{a}_\eta(\dot{c}) \notin F_n \quad \blacksquare$$

Notice that for  $\xi < \omega_2$

$$D_\xi = \{s \in Fn(I_\xi) : \text{there exists a closed measure zero set } F \in V' \text{ such that } s \Vdash \dot{a}_\xi(\dot{c}) \in F\} .$$

Therefore by the above lemma

$$A \subseteq \bigcup \{F : F \text{ is a closed measure zero set coded in } V'\} .$$

Since  $\mathbf{V}$  contains Cohen reals over  $V'$ , the union of all closed measure zero sets coded in  $V'$  has measure zero in  $\mathbf{V}$ . We conclude that  $A$  has measure zero.  $\blacksquare$

Let  $\mathcal{F}$  be a non-principal filter. Denote by  $\mathcal{F}^c = \{X \subseteq \omega : \omega - X \in \mathcal{F}\}$ .  $\mathcal{F}^c$  is an ideal and it is very easy to see that  $\mathcal{F}$  is measurable (has the Baire property) iff  $\mathcal{F}^c$  is measurable (has the Baire property).

**Lemma 1.5**  $\mathcal{F} + \mathcal{F} = \mathcal{F}^c$ .

PROOF Suppose that  $X, Y \in \mathcal{F}$ . Then  $\{n : X(n) + Y(n) = 0\} \supseteq X^{-1}(1) \cap Y^{-1}(1) \in \mathcal{F}$ . In general  $\mathcal{F} + \dots + \mathcal{F}$  is equal to  $\mathcal{F}$  or  $\mathcal{F}^c$  depending whether there is an even or odd number of  $\mathcal{F}$ 's.

Let  $\mathbf{V} \models GCH$  and suppose that  $\mathbf{V}[G]$  is a generic extension of  $\mathbf{V}$  obtained by adding  $\omega_2$  Cohen reals. By the above lemma if  $\mathcal{F}$  is a meager filter then  $\mathcal{F}^c = \mathcal{F} + \mathcal{F}$  is meager. So by 1.2  $\mathcal{F}$  has measure zero.  $\blacksquare$

## 2 Filters which are meager and nonmeasurable

Theorem 1.1 shows that in order to construct a filter which is meager and nonmeasurable we need some extra assumptions.

In [T] Talagrand showed that

**Theorem 2.1** *Suppose that the real line is not the union of  $< 2^{\aleph_0}$  many measure zero sets. Then there exists a nonmeasurable filter which is meager. ■*

Let  $\kappa$  be a regular uncountable cardinal. Recall that  $S$  is a generalized Sierpinski set of size  $\kappa$  if  $|S \cap H| < \kappa$  for every null set  $H$ . It is clear that all  $S' \subseteq S$  of size  $\kappa$  are also nonmeasurable.

**Theorem 2.2** *Assume that there exists a generalized Sierpinski set. Then there exists a nonmeasurable meager filter.*

PROOF Let  $S$  be a generalized Sierpinski set of size  $\kappa$ . Build a sequence  $\{x_\xi : \xi < \kappa\} \subset S$  and an elementary chain of models  $\{M_\xi : \xi < \kappa\}$  of size  $\kappa$  such that

1.  $\{x_\xi : \xi < \alpha\} \subset M_\alpha$  for  $\alpha < \kappa$ ,
2.  $x_\beta$  is a random real over  $M_\alpha$  for  $\beta > \alpha$ .

Suppose that  $M_\beta, x_\beta$  are already constructed for  $\beta < \alpha$ . Since  $S$  is a Sierpinski set

$$\bigcup \{S \cap H : H \text{ is a null set coded in } M_\beta \text{ for } \beta < \alpha\}$$

has size  $< \kappa$ . Let  $x_\alpha$  be any element of  $S$  avoiding this set.

Let  $X_\xi = x_\xi^{-1}(1)$  for  $\xi < \kappa$ . Let  $\mathcal{F}$  be the filter generated by the family  $\{X_\xi : \xi < \kappa\}$ . We will show that  $\mathcal{F}$  has the required properties.

For  $X \subset \omega$  let

$$d(X) = \lim_{n \rightarrow \infty} \frac{|X \cap n|}{n}$$

if the above limit exists.

By easy induction we show that for  $\xi_1, \dots, \xi_n < \kappa$  we have  $d(X_{\xi_1} \cap \dots \cap X_{\xi_n}) = 2^{-n}$ . This shows that

$$\mathcal{F} \subseteq \{X \subset \omega : \liminf_{n \rightarrow \infty} \frac{|X \cap n|}{n} > 0\}$$

which is a meager set. To check that  $\mathcal{F}$  is nonmeasurable notice that  $\mathcal{F}$  contains the nonmeasurable set  $\{x_\xi : \xi < \kappa\}$ . ■

It is an open problem whether one can construct a meager nonmeasurable filter assuming the existence of a nonmeasurable set of size  $\aleph_1$ . We only have some partial results.

Let  $\mathbf{b}$  be the size of the smallest unbounded family in  $\omega^\omega$  and let  $\mathbf{unif}$  be the size of the smallest nonmeasurable set.

For  $X \subseteq \omega$  let  $f_X \in \omega^\omega$  be an increasing function enumerating  $X$ . For a filter  $\mathcal{F}$  let  $\mathcal{F}^* = \{f_X : X \in \mathcal{F}\}$ . In [J] it is proved that

**Theorem 2.3** For every filter  $\mathcal{F}$ ,  
 $\mathcal{F}$  has the Baire property iff  $\mathcal{F}^*$  is bounded. ■

**Theorem 2.4** Suppose that  $\mathbf{unif} < \mathbf{b}$ . Then there exists a nonmeasurable filter which is meager.

PROOF Let  $X \subseteq 2^\omega$  be a nonmeasurable set of size  $\mathbf{unif}$ . Let  $M$  be a model of the same size containing  $X$  as a subset. Then  $M \cap 2^\omega$  does not have measure zero, so it is nonmeasurable. Consider any filter  $\mathcal{F}$  such that  $M \models \mathcal{F}$  is an ultrafilter.  $\mathcal{F}$  generates a filter in  $\mathbf{V}$  and this filter is meager by 2.3 and the fact that it is generated by  $\mathbf{unif} < \mathbf{b}$  many elements. On the other hand  $M \models 2^\omega = \mathcal{F} \cup \mathcal{F}^c$  and we know that  $M \cap 2^\omega$  is a nonmeasurable set. Hence  $\mathcal{F}$  is nonmeasurable. ■

The previous theorem depended on the implication:

If  $\mathcal{F}$  has measure zero then  $M \cap 2^\omega$  has measure zero.

This implication is not true in general for any set  $X \in M$  having outer measure 1 in  $M$  as is showed by the following example.

EXAMPLE It is consistent with ZFC that there are models  $M \subset \mathbf{V}$  such that only *some* sets which have outer measure 1 in  $M$  have measure 0 in  $V$ .

Let  $\mathbf{V} = \mathbf{L}[c][\langle r_\xi : \xi < \omega_1 \rangle]$  where  $c$  is a Cohen real over  $\mathbf{L}$  and  $\langle r_\xi : \xi < \omega_1 \rangle$  is a sequence of random reals over  $\mathbf{L}[c]$  (added side by side). Let  $M = \mathbf{L}[\langle r_\xi : \xi < \omega_1 \rangle]$ . Consider the set  $X = \mathbf{L} \cap 2^\omega$ . It is known that  $X$  is a nonmeasurable set in  $M$  but  $X$  has measure 0 in  $\mathbf{V}$ . On the other hand the set  $\{r_\xi : \xi < \omega_1\}$  is nonmeasurable in  $\mathbf{V}$ . ■

We conclude the paper with a canonical example of a filter which does not generate an ultrafilter. In other words we have the following:

**Theorem 2.5** Let  $M$  be a model for ZFC and let  $r$  be a real which does not belong to  $M$ . Then there exists a filter  $\mathcal{F}$  such that  $M \models \mathcal{F}$  is an ultrafilter but

$$M[r] \models \{X \subseteq \omega : \exists Y \in \mathcal{F} Y \subseteq X\} \text{ is not an ultrafilter .}$$

PROOF Let  $\{k_n : n \in \omega\}$  be a fast increasing sequence of natural numbers. Let  $T$  be a tree on  $2^{<\omega}$  such that:

1. For  $s \in T$  we have  $|s| = k_n$  iff  $s \frown 0 \in T$  and  $s \frown 1 \in T$ ,
2. let  $\{s_1, \dots, s_{2^n}\}$  be the list of  $T \cap 2^{k_n}$  in lexicographical order. Then for every  $w \subseteq \mathcal{P}(2^n) - \{\emptyset, 2^n\}$  there exists  $m \in [k_n + 1, k_{n+1})$  such that  $s_l(m) = 0$  iff  $l \in w$ ,
3. there is no  $m \in \omega$  such that for all  $s \in T \cap 2^{m+1}$  we have  $s(m) = 0$  or for all  $s \in T \cap 2^{m+1}$  we have  $s(m) = 1$ .

Let  $S \subseteq T$  be a subtree of  $T$ . Define

$$A_S^0 = \{m : \forall s \in S \cap 2^{m+1} \ s(m) = 0\} \text{ and}$$

$$A_S^1 = \{m : \forall s \in S \cap 2^{m+1} \ s(m) = 1\} .$$

Let  $\mathcal{J}$  be the ideal generated by sets  $\{A_S^0, A_S^1 : S \text{ is a perfect subtree of } T\}$ .

One can easily verify that all finite subsets of  $\omega$  belong to  $\mathcal{J}$ .

**Lemma 2.6**  *$\mathcal{J}$  is a proper ideal.*

PROOF Let  $S_1, \dots, S_m$  be perfect subtrees of  $T$ . Find  $n$  sufficiently big so that  $|S_j \cap 2^{k_n}| > m$  for  $j \leq m$ . Let  $s_1, \dots, s_{2^n}$  be the list of  $T \cap 2^{k_n}$  in lexicographical ordering. Let  $w_1, \dots, w_m$  be such that  $S_j \cap 2^{k_n} = \{s_i : i \in w_j\}$  for  $j \leq m$ . Let  $w = \{\min(w_1), \dots, \min(w_m)\}$ . Then for all  $j$ ,  $w_j \not\subseteq w$  and  $w_j \cap w \neq \emptyset$ . By the definition of  $T$  there is  $k < k_n$  such that  $w = \{l : s_l(k) = 0\}$ . By the property of  $w$  for every  $j \leq m$  there exist  $s^0, s^1 \in S_j \cap 2^{k_n}$  such that  $s^0(k) = 0$  and  $s^1(k) = 1$ . Therefore  $k \notin A_{S_1}^0 \cup A_{S_1}^1 \cup \dots \cup A_{S_m}^0 \cup A_{S_m}^1$ .

Let  $\mathcal{F}$  be any ultrafilter in  $M$  extending the filter  $\{\omega - X : X \in \mathcal{J}\}$ . Let  $r$  be a real which does not belong to  $M$ . Without loss of generality we can assume that  $r$  is a branch through  $T$ .

Assume that  $\mathcal{F}$  generates an ultrafilter and let  $X_r = \{n : r(n) = 1\}$ . We can assume that there exists an element  $X \in \mathcal{F}$  such that  $X \subseteq X_r$ . Let  $S = \{s \in T : \forall k \in X \ (|s| > k \rightarrow s(k) = 1)\}$ . Clearly  $r$  is a branch through  $S$ . But in that case  $S$  contains a perfect subtree  $S_1 \subseteq S$  (since it contains a new branch). Therefore  $X \subseteq A_{S_1}^1 \in \mathcal{J}$ . Contradiction. ■

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