

Linear liftings for non-complete probability spaces

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Abstract: We show that it is consistent with ZFC that $L^\infty(Y, \mathcal{B}, \nu)$ has no linear lifting for many non-complete probability spaces (Y, \mathcal{B}, ν) , in particular for $Y = [0, 1]^A$, $\mathcal{B} =$ Borel subsets of Y , $\nu =$ usual Radon measure on \mathcal{B} .

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§1 Introduction

In [S 83] the second author showed that it is consistent that Lebesgue measure on $[0, 1]$ has no Borel lifting. This argument was generalized in [J 89] and [BJ 89] to produce a model where there is no lifting ρ for the usual product measure on $[0, 1]^A$ such that for each measurable set E , $\rho(E) = E' \times [0, 1]^{A-B}$ where $B \subseteq A$ is countable and $E' \subseteq [0, 1]^B$ is projective. In particular $[0, 1]^A$ has no Baire lifting. The approach taken there did not shed any light on the question of whether one can produce in ZFC a Borel lifting for $[0, 1]^A$ when A is uncountable. In this paper we show that this is not possible. D.H. Fremlin suggested the use of linear liftings for this purpose. The technique is a modification of the one used in [S 83]. We assume that the reader is familiar with [S 83]. Most definitions which we need are given below. See [IT 69] for more details concerning liftings.

1.1 Definitions

1. If (Y, \mathcal{B}, ν) is any probability space (not necessarily complete) then as usual we say that $f: Y \rightarrow \mathbf{R}$ is *measurable* if $f^{-1}(a, b) \in \mathcal{B}$ for every rational interval $(a, b) \subseteq \mathbf{R}$.
2. $L^\infty(Y, \mathcal{B}, \nu) = \{f \in \mathbf{R}^Y : f \text{ is bounded and measurable}\}$.
3. $\rho: L^\infty(Y, \mathcal{B}, \nu) \rightarrow L^\infty(Y, \mathcal{B}, \nu)$ is a *linear lifting* if for all $f, g \in L^\infty(Y, \mathcal{B}, \nu)$ and all $x, y \in \mathbf{R}$,
 - (a) $f = g$ a.e. implies $\rho(f) = \rho(g)$ (everywhere).
 - (b) $\rho(f) = f$ a.e.
 - (c) $\rho(xf + yg) = x\rho(f) + y\rho(g)$.
 - (d) $\rho(1) = 1$ where 1 is the constant function with value 1.
 - (e) $f \geq 0$ a.e. implies $\rho(f) \geq 0$.
4. $\rho: L^\infty(Y, \mathcal{B}, \nu) \rightarrow L^\infty(Y, \mathcal{B}, \nu)$ is a *lifting* if ρ is a linear lifting and $\rho(fg) = \rho(f)\rho(g)$ for all $f, g \in L^\infty(Y, \mathcal{B}, \nu)$. In this case ρ corresponds in a canonical way to a lifting for the measure algebra of (Y, \mathcal{B}, ν) . See [IT 69].
5. When ρ is a linear lifting for $L^\infty(Y, \mathcal{B}, \nu)$ and $E \in \mathcal{B}$, we will write $\rho(E)$ instead of $\rho(\chi_E)$, where χ_E is the characteristic function of the set E .
6. For sequences of real numbers, we will use the expressions *increasing* and *decreasing* to mean *strictly increasing* and *strictly decreasing*, respectively.
7. For real numbers $a \neq b$, (a, b) will denote $\{x \in \mathbf{R} : a < x < b\}$ if $a < b$, and $\{x \in \mathbf{R} : b < x < a\}$ if $b < a$.

We will prove the following theorem:

1.2 Theorem *The following is consistent with ZFC:*

*Let $\Sigma =$ the σ -algebra of Borel subsets of $[0, 1]$, $\mu =$ Lebesgue measure on Σ .
Then $L^\infty([0, 1], \Sigma, \mu)$ has no linear lifting.*

1.3 Corollary *The following is consistent with ZFC:*

Suppose that

1. (Y, \mathcal{B}, ν) is a probability space (not necessarily complete),
2. There is a measurable inverse-measure-preserving function $\varphi: Y \rightarrow [0, 1]$,
3. There is a Borel disintegration of ν , i.e., there is a family $\langle \nu_x : x \in [0, 1] \rangle$ of probability measures on \mathcal{B} such that for each $g \in L^\infty(Y, \mathcal{B}, \nu)$, the function $x \mapsto \int g d\nu_x$ is Borel measurable and equal a.e. to $E(g|\varphi^{-1}(\Sigma))$. (Here $E(\cdot)$ is the conditional expectation operator.)

Then $L^\infty(Y, \mathcal{B}, \nu)$ has no linear lifting. (In particular (Y, \mathcal{B}, ν) has no lifting.)

[Proof: If ρ is a linear lifting for $L^\infty(Y, \mathcal{B}, \nu)$, then $\bar{\rho}$ is a linear lifting for $L^\infty([0, 1], \Sigma, \mu)$ where $\bar{\rho}(f)(x) = \int \rho(f \circ \varphi) d\nu_x$. [For a.a. x we have $\bar{\rho}(f)(x) = E(f \circ \varphi | \varphi^{-1}(\Sigma))(x) = f(x)$.]]

1.4 Examples

1. $Y = [0, 1]^A$, \mathcal{B} = Borel subsets of $[0, 1]^A$, ν = usual Radon product measure on \mathcal{B} .
2. $Y = \{0, 1\}^A$, \mathcal{B} = Borel subsets of $\{0, 1\}^A$, ν = usual Haar measure on \mathcal{B} .
3. $(Z, \mathcal{C}, \lambda)$ is any probability space, $Y = [0, 1] \times Z$, \mathcal{B} = the σ -algebra generated by the rectangles $E \times F$, $E \in \Sigma$, $F \in \mathcal{C}$, ν = the usual product measure on \mathcal{B} .

Note that the third hypothesis of the corollary is needed. To see this, consider the hyperstonian space (Y, \mathcal{B}, ν) of $[0, 1]$ and the canonical projection $\varphi: Y \rightarrow [0, 1]$. We know that (Y, \mathcal{B}, ν) has a lifting (even a continuous lifting). (See [F 89].) However in the model which we will construct, none of the disintegrations of ν will be Borel, so there is no contradiction.

1.5 Problem Is it consistent with ZFC that there is a translation-invariant linear lifting for $L^\infty([0, 1], \Sigma, \mu)$? (ρ is translation invariant if $\rho(f_a)(x) = \rho(f)(a+x)$, where $f_a(y) = f(a+y)$ (all additions are mod 1), for $a, x, y \in [0, 1]$, $f \in L^\infty([0, 1], \Sigma, \mu)$.)

§2 Proof of theorem 1.2

Let L^∞ stand for $L^\infty([0, 1], \Sigma, \mu)$.

Assume $V = L$. As in [S 83] (the technique is explained in [S 82]), we use an oracle-cc iteration of length \aleph_2 , and it will suffice to prove the following lemma.

2.1 Main Lemma *Let \bar{M} be an \aleph_1 -oracle and let ρ be a linear lifting of L^∞ . Then there is a forcing notion P satisfying the \bar{M} -cc and a P -name \dot{X} of an open set such that for every $G \subseteq P \times Q$ generic over V (where Q is Cohen forcing), there is no Borel function h in $V[G]$ such that*

- (a) $h = \chi_{\dot{X}[G]}$ a.e.
- (b) for every $g \in (L^\infty)^V$, if $g \leq \chi_{\dot{X}[G]}$ a.e. then $\rho(g) \leq h$.
- (c) for every $g \in (L^\infty)^V$, if $\chi_{\dot{X}[G]} \leq g$ a.e. then $h \leq \rho(g)$.

2.2 Proof of the main lemma

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Let \mathcal{S} denote the set of triples

$$\bar{a} = (\langle a_{0i} : i < \omega \rangle, \langle a_{1i} : i < \omega \rangle, a_\omega)$$

such that the a_{ji} are rational numbers in $(0, 1)$ ($j < 2, i < \omega$), a_ω is irrational, $\langle a_{0i} : i < \omega \rangle$ is an increasing sequence converging to a_ω and $\langle a_{1i} : i < \omega \rangle$ is a decreasing sequence converging to a_ω .

Define a partial order $P = P(\langle \bar{a}^\alpha : \alpha < \beta \rangle)$ where $\beta \leq \omega_1$, $\bar{a}^\alpha \in \mathcal{S}$, and the numbers a_ω^α are pairwise distinct, as follows: $p \in P$ iff the following conditions hold:

- (a) $p = (U_p, f_p)$, where U_p is an open subset of $(0, 1)$, $\text{cl}(U_p)$ has measure $< 1/2$, and $f_p: U_p \rightarrow \{0, 1\}$.
- (b) There is a finite sequence of rational numbers $0 = b_0 < b_1 < \dots < b_n = 1$ such that $U_p = \bigcup_{\ell=0}^{n-1} I_\ell$, $\text{cl}(I_\ell) \subseteq (b_\ell, b_{\ell+1})$.
- (c) I_ℓ is either a rational interval, in which case $f_p|_{I_\ell}$ is constant, or there are $\alpha < \beta$ and $n(\ell) < \omega$ such that

$$I_\ell = \bigcup_{j < 2} \bigcup_{n(\ell) \leq m < \omega} (a_{j,2m}^\alpha, a_{j,2m+1}^\alpha)$$

and $f_p|(a_{j,4m+2k}^\alpha, a_{j,4m+2k+1}^\alpha)$ is identically equal to k , ($j < 2, n(\ell) \leq 2m + k, m < \omega, k < 2$).

The order on P is: $p \leq q$ if and only if $U_p \subseteq U_q$, $f_p \subseteq f_q$, and $\text{cl}(U_p) \cap U_q = U_p$.

Let \dot{X} be a P -name for $\bigcup\{(a, b) : (a, b) \text{ is a rational interval } \subseteq (0, 1) \text{ and for some } p \in G_P, (a, b) \subseteq U_p \text{ and } f_p|(a, b) \text{ is identically zero}\}$.

As in [S 83], the main lemma will follow if we prove the following claim.

2.3 Main Claim *Let $P_\delta = P(\langle \bar{a}^\alpha : \alpha < \delta \rangle)$, $\delta < \omega_1$ be given, as well as a countable M_δ , $P_\delta \in M_\delta$, a condition $(p^*, r^*) \in P_\delta \times Q$ and a $P_\delta \times Q$ -name τ for a code for a member of L^∞ . (We shall identify Borel functions and their codes. This should not cause any confusion.) Then we can find $\bar{a}^\delta \in \mathcal{S}$ such that, letting $P_{\delta+1} = P(\langle \bar{a}^\alpha : \alpha \leq \delta \rangle)$, the following conditions hold:*

(A) *Every predense subset of P_δ which belongs to M_δ is a predense subset of $P_{\delta+1}$.*

(B) *There is a condition $(p', r') \in P_{\delta+1} \times Q$ such that $(p^*, r^*) \leq (p', r')$ and one of the following two conditions holds for some n :*

$$\begin{aligned} \text{(B1)} \quad (p', r') \Vdash_{P_{\delta+1} \times Q} \text{ “ } \tau(a_\omega^\delta) \geq 1/2 \\ \text{and } \rho(\bigcup_{j < 2} \bigcup_{n \leq m < \omega} (a_{j,4m+2}^\delta, a_{j,4m+3}^\delta))(a_\omega^\delta) \geq 3/4 \\ \text{and } \dot{X} \cap \bigcup_{j < 2} \bigcup_{n \leq m < \omega} (a_{j,4m+2}^\delta, a_{j,4m+3}^\delta) = \emptyset. \text{”} \end{aligned}$$

or

$$\begin{aligned} \text{(B2)} \quad (p', r') \Vdash_{P_{\delta+1} \times Q} \text{ “ } \tau(a_\omega^\delta) \leq 1/2 \\ \text{and } \rho(\bigcup_{j < 2} \bigcup_{n \leq m < \omega} (a_{j,4m}^\delta, a_{j,4m+1}^\delta))(a_\omega^\delta) \geq 3/4 \\ \text{and } \bigcup_{j < 2} \bigcup_{n \leq m < \omega} (a_{j,4m}^\delta, a_{j,4m+1}^\delta) \subseteq \dot{X}. \text{”} \end{aligned}$$

[The proof of the Main Lemma is a bookkeeping argument using the Main Claim. P is obtained, in the notation of the Main Claim, as $P = \bigcup_{\delta < \omega_1} P_\delta$, and the bookkeeping

is needed to ensure that all triples (p^*, r^*, τ) are considered in the construction, where $(p^*, r^*) \in P \times Q$ and τ is a $P \times Q$ -name for a code of a Borel function. If there were an h contradicting the Main Lemma, then there would be a $P \times Q$ -name τ for h and a condition $(p^*, r^*) \in P \times Q$ forcing that τ satisfies (a), (b), (c) of the Main Lemma. But then condition (B) of the Main Claim gives a contradiction. For more details of such oracle-cc arguments see pp. 114ff of [S 82].]

2.4 Proof of main claim 2.3

Choose a sufficiently large regular λ and choose a countable $N \prec H_\lambda$ such that $\rho, P_\delta, \langle \bar{a}^\alpha : \alpha < \delta \rangle, \tau, M_\delta \in N$. Choose a random real over N , $a_\omega^\delta \in (0, 1) - \text{cl}(U_{P^*})$. Note that for any rational interval $(a, b) \subseteq (0, 1)$ we have $\rho((a, b))(a_\omega^\delta) = \chi_{(a,b)}(a_\omega^\delta)$.

Let $u_0 = \rho((0, a_\omega^\delta))(a_\omega^\delta)$, $u_1 = \rho((a_\omega^\delta, 1))(a_\omega^\delta)$. Then $u_0 + u_1 = 1$.

Note that for any number x , if $0 \leq x < a_\omega^\delta$, then $\rho((x, a_\omega^\delta))(a_\omega^\delta) = u_0$. [Otherwise, for any rational number b such that $x < b < a_\omega^\delta$, we have $\rho((0, b))(a_\omega^\delta) > 0$, contradicting the choice of a_ω^δ .] A similar statement holds for u_1 . Putting these together we see that $\rho((x, y))(a_\omega^\delta) = 1$ for any numbers x and y such that $0 \leq x < a_\omega^\delta < y \leq 1$.

Choose an increasing sequence of rational numbers $\langle b_{0n} : n < \omega \rangle \in N[a_\omega^\delta]$ converging to a_ω^δ , and choose a decreasing sequence of rational numbers $\langle b_{1n} : n < \omega \rangle \in N[a_\omega^\delta]$ also converging to a_ω^δ . In $N[a_\omega^\delta]$ define the partial order R for adding a Mathias real as follows:

$$R = \{(s, A) : s \text{ is a finite subset of } \omega, A \subseteq \omega, \max(s) < \min(A)\},$$

ordered by $(s, A) \geq (t, B)$ iff t is an initial segment of s , $A \subseteq B$, $s - t \subseteq B$.

For sets $A \subseteq \omega$, let us identify A with its enumerating function, so that we may write $A = \{A(i) : i < |A|\}$. We need the following special case of the known fact that an infinite subset of a Mathias real is a Mathias real. (See [M 77: Theorem 2.0]; the special case which we need here is a fairly routine exercise.)

2.5 Fact *If $X \subseteq \omega$ is R -generic over $N[a_\omega^\delta]$, and $g \in \omega^\omega \cap N[a_\omega^\delta]$ is increasing, then $Y = \{X(g(n)) : n < \omega\}$ is also R -generic over $N[a_\omega^\delta]$. ■*

Let f^* be the enumerating function of a set which is R -generic over $N[a_\omega^\delta]$.

In $N[a_\omega^\delta][f^*]$, define for increasing functions $f \in \omega^\omega$,

$$A_m^k(f) = \bigcup_{j < 2} \bigcup_{k \leq \ell < \omega} (b_{j, f(4\ell+m)}, b_{j, f(4\ell+m+1)}).$$

Define $f_3^*(\ell) = f^*(3\ell)$ for $\ell < \omega$.

Then $\{A_m^0(f_3^*) : m < 4\}$ is a partition of $(b_{0, f^*(0)}, b_{1, f^*(0)})$. For some $m < 4$ we have

$$(*) \quad \rho(A_m^0(f_3^*))(a_\omega^\delta) \leq 1/4.$$

2.6 Claim *For any $\bar{m} < 4$ and $k < \omega$, we can find an increasing function $g \in N[a_\omega^\delta] \cap \omega^\omega$ such that $g(i) = i$ for all $i < k$ and $\rho(A_{\bar{m}}^0(f^* \circ g))(a_\omega^\delta) \geq 3/4$.*

Proof of claim Let $g(i) = i$ for $i < 4k + \bar{m} + 1$ and define $g(4\ell + \bar{m} + 1 + j) = 12\ell + 3m + j$ for $\ell \geq k$ and $j < 4$. We leave it for the reader to check, using (*), that g has the desired property. (The reader might find it helpful, for seeing the role of g , to mark off the first few elements of its range on a line.) ■

Let us provisionally let $\bar{a}^\delta = (\langle b_{0,f^*(\ell)} : \ell < \omega \rangle, \langle b_{1,f^*(\ell)} : \ell < \omega \rangle, a_\omega^\delta)$.

2.7 Proof of condition (A) of main claim 2.3

Let $J \subseteq P_\delta$ be predense, $J \in M_\delta$. We must show that J is predense in $P_{\delta+1}$. Let $p \in P_{\delta+1}$, $p \notin P_\delta$. By the definition of $P_{\delta+1}$, there are $q \in P_\delta$ and rational numbers c_0, c_1 and $\ell(0) \in \omega$ such that

$$0 < b_{0,f^*(4\ell(0)-1)} < c_0 < b_{0,f^*(4\ell(0))} < a_\omega^\delta < b_{1,f^*(4\ell(0))} < c_1 < b_{1,f^*(4\ell(0)-1)} < 1,$$

$\text{cl}(U_q) \cap [c_0, c_1] = \emptyset$, $U_p = U_q \cup A_0^{\ell(0)}(f^*) \cup A_2^{\ell(0)}(f^*)$, $f_p = f_q \cup 0_{A_0^{\ell(0)}(f^*)} \cup 1_{A_2^{\ell(0)}(f^*)}$. (For $i = 0, 1$, i_A denotes the function with domain A and constant value i .)

The proof of the following fact is exactly as in [S 83].

2.8 Fact *If $r \in P_\delta$, $J \subseteq P_\delta$ is dense, $(c_0, c_1) \subseteq (0, 1)$ and $(c_0, c_1) \cap U_r = \emptyset$, then*

$$\mu((c_0, c_1) \cap \bigcap \{\text{cl}(U_{r_1}) : r_1 \in J, r_1 \geq r\}) = 0. \quad \blacksquare$$

Let $J_1 = \{r \in P_\delta : \exists q_1 \in J \ q_1 \leq r\}$. For every $k > f^*(4\ell(0))$ let

$T_k = \{t \in P_\delta : U_t \text{ is the union of finitely many intervals whose endpoints are from } \{b_{j,\ell} : j < 2, f^*(4\ell(0)) \leq \ell \leq k\} \text{ and } \mu(U_q \cup U_t) < 1/2\}$.

So T_k is finite and for each $t \in T_k$, $q \leq q \cup t \in P_\delta$ and $a_\omega^\delta \notin \text{cl}(U_t)$. In N , define for each $k > f^*(4\ell(0))$ and $t \in T_k$,

$$J_t = (b_{0,k}, b_{1,k}) \cap \bigcap \{\text{cl}(U_{r_1}) : r_1 \in J_1, r_1 \geq q \cup t\}.$$

By fact 2.8, J_t has measure zero, and hence $a_\omega^\delta \notin J_t$. Thus there is an $r_t \in J_1$, such that $r_t \geq q \cup t$ and $a_\omega^\delta \notin \text{cl}(U_{r_t})$. Define $g: \bigcup \{T_k : k > f^*(4\ell(0))\} \rightarrow \omega$ and $G: \omega \rightarrow \omega$ such that $[b_{0,g(t)}, b_{1,g(t)}] \cap \text{cl}(U_{r_t}) = \emptyset$, $b_{1,g(t)} - b_{0,g(t)} < (1/2) - \mu(U_{r_t})$, $G(k) = \max\{g(t) : t \in T_k\}$.

Since f^* is R -generic over $N[a_\omega^\delta]$, for all but finitely many $\ell < \omega$ we have

$$f^*(4\ell + 2) \geq G(f^*(4\ell + 1)).$$

Choose such an $\ell \geq \ell(0)$. Let $k = f^*(4\ell + 1)$, $t = (U_t, f_t)$, where

$$U_t = U_p \cap ([b_{0,f^*(4\ell(0))}, b_{0,k}] \cup [b_{1,k}, b_{1,f^*(4\ell(0))}]),$$

$f_t = f_p|_{U_t}$. Then $t \in T_k$ and we have $r_t \in J_1$, $r_t \geq q \cup t$. Also, $[b_{0,G(k)}, b_{1,G(k)}] \cap \text{cl}(U_{r_t}) = \emptyset$ and hence $[b_{0,f^*(4\ell+2)}, b_{1,f^*(4\ell+2)}] \cap \text{cl}(U_{r_t}) = \emptyset$. Thus p and r_t are compatible, and this proves part (A) of main claim 2.3.

2.9 Proof of condition (B) of main claim 2.3

Let

$$p_1^* = (U_p \cup A_0^k(f^*) \cup A_2^k(f^*), f_{p^*} \cup 0_{A_0^k(f^*)} \cup 1_{A_2^k(f^*)})$$

where k is large enough so that $p_1^* \in P_{\delta+1}$. So $p_1^* \in N[a_\omega^\delta][f^*]$ and $(p_1^*, r^*) \geq (p^*, r^*)$. In $N[a_\omega^\delta][f^*]$, choose $(p', r') \geq (p_1^*, r^*)$ deciding whether $\tau(a_\omega^\delta) \geq 1/2$ or $\tau(a_\omega^\delta) \leq 1/2$, say the first. We will get (p', r') so that condition (B1) of main claim 2.3 is satisfied. The other case is handled similarly. For some $(t, B) \in R \cap N[a_\omega^\delta]$ we have $f^*(n) = t(n)$ for all $n < |t|$, $f^*(n) \in B$ for all $n \geq |t|$, and

$$N[a_\omega^\delta] \models (t, B) \Vdash_R \text{“}(p', r') \Vdash_{P_{\delta+1} \times Q} \tau(a_\omega^\delta) \geq 1/2\text{”}.$$

By claim 2.6 and fact 2.5 above, we can replace f^* by another R -generic real, maintaining $f^*(n) = t(n)$ for $n < |t|$ and $f^*(n) \in B$ for $n \geq |t|$, so that $\rho(A_2^0(f^*))(a_\omega^\delta) \geq 3/4$. (B1) is now satisfied. This completes the proof of main claim 2.3 and of theorem 1.2. ■

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