A saturated model of an unsuperstable theory of cardinality greater than its theory has the small index property

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Abstract

A model M of cardinality λ is said to have the small index property if for every $G \subseteq Aut(M)$ such that $[Aut(M) : G] \leq \lambda$ there is an $A \subseteq M$ with $\big|A\big| < \lambda$ such that $Aut_A(M) \subseteq G$. We show that if M^* is a saturated model of an unsuperstable theory of cardinality $> Th(M)$, then M^* has the small index property.

1 Introduction

Throughout the paper we work in $\mathfrak{C}^{eq}$, and we assume that M^* is a saturated model of T of cardinality λ. We denote the set of automorphisms of M^* by $Aut(M^*)$ and the set of automorphisms of M^* fixing A pointwise by $Aut_A(M^*)$. M^* is said to have the small index property if whenever G is a subgroup of $Aut(M^*)$ with index not larger than λ then for some $A \subset M^*$ with $|A| < \lambda$, $Aut_A(M^*) \subseteq G$. The main theorem of this paper is the following result of Shelah: If M^* is a saturated model of cardinality $\lambda > |T|$ and there is a tree of height some uncountable regular cardinal

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$\kappa \geq \kappa_r(T)$ with $\mu > \lambda$ many branches but at most λ nodes, then M^* has the small index property, in fact

$$[Aut(M^*) : G] \geq \mu$$

for any subgroup G of $Aut(M^*)$ such that for no $A \subseteq M^*$ with $|A| < \lambda$ is $Aut_A(M^*) \subseteq G$. By a result of Shelah on cardinal arithmetic this implies that if $Aut(M^*)$ does not have the small index property, then for some strong limit μ such that $cf\,\mu = \aleph_0$,

$$\mu < \lambda < 2^\mu$$

So in particular, if T is unsuperstable, M^* has the small index property.

In the paper "Uncountable Saturated Structures have the Small Index Property" by Lascar and Shelah, the following result was obtained:

Theorem 1.1 *Let M^* be a saturated model of cardinality λ with $\lambda > |T|$ and $\lambda^{<\lambda} = \lambda$. Then if G is a subgroup of $Aut(M^*)$ such that for no $A \subseteq M^*$ with $|A| < \lambda$ is $Aut_A(M^*) \subseteq G$ then $[Aut(M^*) : G] = \lambda^\lambda$.*

PROOF See [L Sh].

Corollary 1.2 *Let M^* be a saturated model of cardinality λ with $\lambda > |T|$ and $\lambda^{<\lambda} = \lambda$. Then M^* has the small index property.*

Theorem 1.3 *T has a saturated model of cardinality λ iff $\lambda = \lambda^{<\lambda} + D(T)$ or T is stable in λ.*

PROOF See [Sh c] chp. VIII.

So we can assume in the rest of this paper that T is stable in λ.

Theorem 1.4 *T is stable in μ iff $\mu = \mu_0 + \mu^{<\kappa(T)}$ where μ_0 is the first cardinal in which T is stable.*

PROOF See [Sh c] chp. III.

Since T is stable in λ, we must have $\lambda = \lambda^{<\kappa(T)}$, so $cf\,\lambda \geq \kappa(T)$. Since the first cardinal κ, such that $\lambda^\kappa > \lambda$ is regular, we also know that $cf\,\lambda \geq \kappa_r(T)$.

Definition 1.5 *Let* Tr *be a tree. If* $\eta, \nu \in Tr$, *then* $\gamma[\eta, \nu] =$ *the least* γ *such that* $\eta(\gamma) \neq \nu(\gamma)$ *or else it is* $min(height(\eta), height(\nu))$.

Notation 1.6 *Let* Tr *be a tree. If* $h \in Aut(M^*)$ *and* $\alpha < height(Tr)$, η, $\nu \in Tr$, *then*

$$h^{\eta(\alpha) < \nu(\alpha)} = h$$

if $\eta(\alpha) < \nu(\alpha)$ *and* id_{M^*} *otherwise.*

Lemma 1.7 *Let* $\Big\{C_i \mid i \in I\Big\}$ *be independent over* A *and let* $\Big\{D_i \mid i \in I\Big\}$ *be independent over* B. *Suppose that for each* $i \in I$, $tp(C_i/A)$ *is stationary. Let* f *be an elementary map from* A *onto* B, *and let for each* $i \in I$, f_i *be an elementary map extending* f *which sends* C_i *onto* D_i. *Then*

$$\bigcup_{i \in I} f_i$$

is an elementary map from $\bigcup_{i \in I} C_i$ *onto* $\bigcup_{i \in I} D_i$.

PROOF Left to the reader.

Lemma 1.8 *Let* $|T| < \lambda$. *Let* Tr *be a tree of height* ω *with* κ_n *nodes of height* n *for some* $\kappa_n < \lambda$. *Let* $n < \omega$ *and let* $\langle M_i \mid i \leq n \rangle$ *be an increasing chain of models. Let* $M_n \subseteq N_0 \subseteq N_1 \subseteq M^*$ *with* $|N_1| < \lambda$. *Suppose* $\langle h_i \mid i \leq n \rangle$ *are automorphisms of* M^* *such that*

1. $h_i = id_{M_i}$
2. $h_i[N_j] = N_j$ *for* $j \leq 1$
3. $h_i[M_k] = M_k$ *for* $k \leq n$

For each $\nu \in Tr \restriction level(n+1)$ *let* m_ν, l_ν *be automorphisms of* N_0. *Let* $\eta \in Tr \restriction level(n+1)$. *Suppose* $g_\eta \in Aut(N_0)$ *such that for all* $\nu \in Tr \restriction level(n+1)$,

$$g_\eta m_\eta (m_\nu)^{-1} (g_\eta)^{-1} = l_\eta (l_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}$$

Let m_ν^+, l_ν^+ *be extensions of* m_ν *and* l_ν *to automorphisms of* N_1 *for all* $\nu \in Tr \restriction level(n+1)$. *Then there exists a model* $N_2 \subseteq M^*$ *containing* N_1 *such that* $|N_2| \leq |N_1| + |T| + \kappa_{n+1}$ *and* $h_i[N_2] = N_2$ *for* $i \leq n$

and a $g'_\eta \in Aut(N_2)$ *extending* g_η *and for all* $\nu \in Tr \restriction level(\alpha+1)$ *automorphisms of* N_2, m'_ν *and* l'_ν *extending* m^+_ν *and* l^+_ν *respectively such that*

$$g'_\eta m'_\eta (m'_\nu)^{-1}(g'_\eta)^{-1} = l'_\eta (l'_\nu)^{-1} h^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}_{\gamma[\eta,\nu]}$$

PROOF Let g^+_η be a map with domain N_1 such that $g^+_\eta(N_1) \underset{N_0}{\bigcup} N_1$, $g^+_\eta(N_1) \subseteq M^*$ and g^+_η extends g_η. Let g^{++}_η be a map extending g_η such that the domain of $(g^{++}_\eta)^{-1}$ is N_1, $(g^{++}_\eta)^{-1}(N_1) \subseteq M^*$ and $(g^{++}_\eta)^{-1}(N_1) \underset{N_0}{\bigcup} N_1$. So $g^+_\eta \cup g^{++}_\eta$ is an elementary map. Let l''_η and m''_η be an extensions of l^+_η and m^+_η to an automorphisms of M^*. Let

$$m^{++}_\nu = (g^{++}_\eta)^{-1}(h_{\eta,\nu})^{-1} l_\nu (l_\eta)^{-1} g^{++}_\eta m''_\eta \restriction \ (m'_\eta)^{-1}[(g^{++})^{-1}[N_1]]$$

Note that $m^+_\nu \cup m^{++}_\nu$ is an elementary map. Let

$$l^{++}_\nu = (l''_\eta)^{-1} g^+_\eta m^+_\eta (m^+_\nu)^{-1}(g^+_\eta)^{-1}(h_{\eta,\nu})^{-1} \restriction \ h_{\eta,\nu}[g^+_\eta[N_1]]$$

Note that $l^+_\nu \cup l^{++}_\nu$ is an elementary map. Let g''_η, m''_ν, l''_ν be elementary extensions to M^* of $g^+_\eta \cup g^{++}_\eta$, $m^+_\nu \cup m^{++}_\nu$, and $l^+_\nu \cup l^{++}_\nu$. Let N_2 be a model of size $|N_1| + |T| + \kappa_{n+1}$ containing N_1 such that N_2 is closed under m''_η, g''_η, l''_η all the $h_{\eta,\nu}$ and m''_ν, l''_ν. Let m'_ν, l'_ν, g'_η, $h'_{\eta,\nu}$, m'_η, l'_η be the restrictions to N_2 of the m''_ν, l''_ν, g''_η, $h''_{\eta,\nu}$, m''_η, l''_η.

Theorem 1.9 *If* $\lambda > |T|$, $cf\,\lambda = \omega$, M^* *is a saturated model of cardinality* λ *and if* G *is a subgroup of* $Aut(M^*)$ *such that for no* $A \subseteq M^*$ *with* $|A| < \lambda$ *is* $Aut_A(M^*) \subseteq G$ *then* $[Aut(M^*) : G] = \lambda^\omega$.

PROOF Suppose not. Let $\{\kappa_i \mid i < \omega\}$ be an increasing sequence of cardinals each greater than $|T|$ with $sup = \lambda$. Let $Tr = \{\eta \in {}^{<\omega}\lambda \mid \eta(i) < \kappa_i\}$. Let $M^* = \bigcup_{i<\omega} B_i$ with $|B_i| \le \kappa_i$. By induction on $n < \omega$ for every $\eta \in Tr \restriction level\, n$ we define models $N_n \subset M^*$ and $h_n \in Aut_{N_n}(M^*) - G$ such that $B_n \subseteq N_n$ and $|N_n| \le \kappa_n$, and automorphisms g_η, m_η, l_η of N_n such that if $\rho \neq \nu$ then $l_\rho \neq l_\nu$ and

$$g_\rho m_\rho (m_\nu)^{-1}(g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

Suppose we have defined the g_η, m_η, l_η for $height(\eta) \le m$, and N_j for $j \le m$. If $n = m+1$, for each $i < \kappa_n$ we define models $N_{n,i}$ such that

$B_n \subseteq N_{n,i}$, $N_m \subseteq N_{n,i}$, $\langle N_{n,i} \mid i < \kappa_n \rangle$ is increasing continuous, and for some $\eta_i \in Tr \restriction level\, n$, $g_{\eta_i} \in Aut(N_{n,i})$ such that for each $\eta \in Tr \restriction level\, n$, $\eta = \eta_i$ cofinally many times in κ_n, and for every $\nu \in Tr \restriction level\, n$, $m^i_\nu \neq l^i_\nu \in Aut(N_{n,i})$ such that

$$g_{\eta_i} m^i_{\eta_i} (m^i_\nu)^{-1} (g_{\eta_i})^{-1} = l^i_{\eta_i} (l^i_\nu)^{-1} h^{\eta_i(\gamma[\eta_i,\nu]) < \nu(\gamma[\eta_i,\nu])}_{\gamma[\eta_i,\nu]}$$

The g_{η_i}, m^i_ν, l^i_ν are easily defined by induction on $i < \kappa_n$ using lemma 1.8 so that if $i_1 < i_2$ then $m^{i_1}_\nu \subseteq m^{i_2}_\nu$, $l^{i_1}_\nu \subseteq l^{i_2}_\nu$, and if $\eta_{i_1} = \eta_{i_2}$ then $g_{\eta_{i_1}} \subseteq g_{\eta_{i_2}}$. Then if we let $g_\eta = \bigcup \{ g_{\eta_i} \mid \eta_i = \eta \}$, $m_\eta = \bigcup\limits_{i<\kappa_n} m^i_\eta$, $l_\eta = \bigcup\limits_{i<\kappa_n} l^i_\eta$, $N_n = \bigcup\limits_{i<\kappa_n} N_{n,i}$ and $h_n \in Aut_{N_n}(M^*) - G$ we have finished. Let Br be the set of branches of Tr of height ω. For $\rho \in Br$ let $g_\rho = \bigcup \{ g_\eta \mid \eta < \rho \}$, $m_\rho = \bigcup \{ m_\eta \mid \eta < \rho \}$, and $l_\rho = \bigcup \{ l_\eta \mid \eta < \rho \}$. If $\rho \neq \nu$, $g_\rho \neq g_\nu$ since without loss of generality $\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])$ and

$$g_\rho m_\rho (m_\nu)^{-1} (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

and

$$g_\nu m_\nu (m_\rho)^{-1} (g_\nu)^{-1} = l_\nu (l_\rho)^{-1}$$

implies

$$g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

So if $g_\rho = g_\nu$ this would imply $h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} = id_{M^*}$ a contradiction. If

$$[Aut(M^*) : G] < \lambda^\omega$$

then for some $\rho, \nu \in Br$ we must have $l_\rho (l_\nu)^{-1} \in G$ and $g_\rho (g_\nu)^{-1} \in G$, but then we get a contradiction as $g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G$ and $l_\rho (l_\nu)^{-1} \in G$, but $h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} \notin G$.

Corollary 1.10 *If* $\lambda > |T|$, $cf\, \lambda = \omega$ *and* M^* *is a saturated model of cardinality* λ *then* M^* *has the small index property.*

So we will assume in the remainder of the paper that in addition to T being stable, $cf\, \lambda \geq \kappa_r(T) + \aleph_1$ and T, M^*, and λ are constant.

2 Constructing M^* as a chain from K_δ

Definition 2.1 *Let* $\delta < \lambda^+$, $cf\,\delta \geq \kappa_r(T)$.

$$K^s_\delta = \Big\{ \bar{N} \mid \bar{N} = \langle N_i \mid i \leq \delta \rangle,\ N_i \textit{ is increasing continuous, } |N_i| = \lambda,$$

$$N_0 \textit{ is saturated, } N_\delta = M^*, \textit{ and } (N_{i+1}, c)_{c \in N_i} \textit{ is saturated} \Big\}$$

For $\mu > \aleph_0$,

$$K^\mu_\delta = \Big\{ \bar{A} \mid \bar{A} = \langle A_i \mid i \leq \delta \rangle,$$

$$A_i \textit{ is increasing continuous, } |A_\delta| < \mu,\ acl\, A_i = A_i \Big\}$$

If $\bar{A} \in K^{\lambda^+}_\delta$, *then* $f \in Aut(\bar{A})$ *if* f *is an elementary permutation of* A_δ *and if* $i \leq \delta$, *then* $f \restriction A_i$ *is a permutation of* A_i.

Definition 2.2 *Let* $\bar{A}^0, \bar{A}^1 \in K^\mu_\delta$. *Then* $\bar{A}^0 \leq \bar{A}^1$ *iff* $\bigwedge_{i \leq \delta} A^0_i \subseteq A^1_i$ *and* $i < j \leq \delta \ \Rightarrow\ A^1_i \biguplus_{A^0_i} A^0_j$.

Lemma 2.3 1. $(K^\mu_\delta, \leq)$ *is a partial order*

2. *Let* $\bar{A}^\zeta \in K^\mu_\delta$ *for* $\zeta < \zeta(*)$ *and let* $\xi < \zeta \ \Rightarrow\ \bar{A}^\xi \leq \bar{A}^\zeta$. *If we let* $A_i = \bigcup_{\zeta < \zeta(*)} A^\zeta_i$, *and* $\left| \bigcup_{\zeta < \zeta(*)} A^\delta_i \right| < \mu$, *then*

$$\bar{A} = \langle A_i \mid i \leq \delta \rangle \in K^\mu_\delta$$

and for every $\zeta < \zeta(*)$, $\bar{A}_\zeta \leq \bar{A}$.

3. *If* $\bar{A}^\zeta \leq \bar{A}^*$ *for* $\zeta < \zeta(*)$, *and* $\bar{A}$ *is as above, then* $\bar{A} \leq \bar{A}^*$

PROOF

1. By the transitivity of nonforking.
2. By the finite character of forking.
3. By the finite character of forking.

Definition 2.4 *Let* $A \subseteq M$, *with* $|A| < \kappa_r(T)$ *and let* $p \in S(acl\, A)$. *Then* $dim(p, M) =$ *the minimal cardinality of an maximal independent set of realizations of* p *inside* M. *If* M *is* $\kappa_r^\epsilon(T)$ *-saturated (* κ_r^ϵ *-saturated means* $\aleph_\epsilon$ *-saturated if* $\kappa_r(T) = \aleph_0$ *and* $\kappa_r(T)$ *saturated otherwise) then by [Sh c] III 3.9.* $dim(p, M) =$ *the cardinality of any maximal independent set of realizations of* p *inside* M.

Lemma 2.5 *Let* $|M| = \lambda$ *and assume that* M *is* $\kappa_r^\epsilon(T)$ *-saturated. Then* M *is saturated if and only if for every* $A \subseteq M$, *with* $|A| < \kappa_r(T)$ *and* $p \in S(acl\, A)$, $dim(p, M) = \lambda$.

PROOF See [Sh c] III 3.10.

Lemma 2.6 *Let* $\langle \bar{A}^\alpha \mid \alpha < \lambda \rangle$ *be an increasing continuous sequence of elements of* $K_\delta^{\lambda^+}$ *such that* $\forall\, \gamma < \delta$, $\forall\, A \subseteq \bigcup_{\alpha<\lambda} A_\gamma^\alpha$ *if* $|A| < \kappa_r(T)$ *and* $p \in S(acl\, A)$ *then for* λ *many* $\alpha < \lambda$,

1. $A_\zeta^\alpha = A_\zeta^{\alpha+1}\ \forall\, \zeta \leq \gamma$
2. *There exists* $a \in A_{\gamma+1}^{\alpha+1}$ *such that the type of* $a / A_{\gamma+1}^\alpha$ *is the stationarization of* p

then

$$\langle N_\gamma \mid \gamma < \delta \rangle \in K_\delta^s$$

where $N_\gamma = \bigcup_{\alpha<\lambda} A_\gamma^\alpha$.

PROOF It is enough to show $\forall\, \gamma < \delta$ that $(N_{\gamma+1},\ c)_{c \in N_\gamma}$ is saturated. For this by lemma 2.5 it is enough to show $\forall\, A \subseteq N_{\gamma+1}$ such that $|A| < \kappa_r(T)$ and for every type $p \in S(acl\, A \cup N_\gamma)$,

$$dim(p, N_{\gamma+1}) = \lambda$$

By the assumption of the lemma, there exists $\{a_i \mid i < \lambda\}$ realizations of $p \restriction acl\, A$ and $\langle A_{\gamma+1}^{\alpha_i} \mid i < \lambda \rangle$ such that for each $i < \lambda$, $a_i \in A_{\gamma+1}^{\alpha_i+1}$, $A_\gamma^{\alpha_i+1} = A_\gamma^{\alpha_i}$, and

$$a_i \mathop{\bigcup\!\!\!\!\!\!|\;\;}_{A} A_{\gamma+1}^{\alpha_i} \quad \text{and} \quad a_i A_{\gamma+1}^{\alpha_i} \mathop{\bigcup\!\!\!\!\!\!|\;\;}_{A_\gamma^{\alpha_i}} N_\gamma$$

which implies

$$a_i \underset{A^{\alpha_i}_{\gamma+1}}{\bigcup\!\!\!\!\!\mid} N_\gamma \quad \text{and} \quad a_i \underset{A}{\bigcup\!\!\!\!\!\mid} N_\gamma$$

Since $cf\,\lambda \geq \kappa_r(T)$ without loss of generality $A \subseteq A^{\alpha_0}_{\gamma+1}$. We must show the $\langle a_i \mid i < \lambda\rangle$ are independent over $N_\gamma \cup A$. By induction on $i < \lambda$, we show that

$$\langle a_j \mid j \leq i\rangle$$

are independent over $A \cup \{A^{\alpha_j}_\gamma \mid j \leq i\}$. This is enough as

$$\{a_j \mid j \leq i\} \underset{A \cup \{A^{\alpha_j}_\gamma \mid j \leq i\}}{\bigcup\!\!\!\!\!\mid} N_\gamma$$

Since $\langle a_j \mid j < i\rangle$ are independent over $A \cup \{A^{\alpha_i}_\gamma \mid j < i\}$, and

$$\{a_j \mid j < i\} \underset{A \cup \{A^{\alpha_j}_\gamma \mid j < i\}}{\bigcup\!\!\!\!\!\mid} A^{\alpha_i}_\gamma$$

$\langle a_j \mid j < i\rangle$ are independent over $A \cup A^{\alpha_i}_\gamma$. Since $a_i \underset{A \cup A^{\alpha_i}_\gamma}{\bigcup\!\!\!\!\!\mid} A^{\alpha_i}_{\gamma+1}$ we have

$$a_i \underset{A \cup A^{\alpha_i}_\gamma}{\bigcup\!\!\!\!\!\mid} \{a_j \mid j < i\}$$

Lemma 2.7 *Let* $\langle \bar{N}^\alpha \mid \alpha < \delta\rangle$ *be an increasing continuous sequence of elements of* $K^{\mu^+}_\delta$ *such that* $\bigcup_{\alpha<\delta} N^\alpha_\delta = M^*$ *and for every* $\gamma < \delta$, *and* $\alpha < \delta$,

$$(N^{\alpha+1}_{\gamma+1}, c)_{c \in N^\alpha_{\gamma+1} \cup N^{\alpha+1}_\gamma}$$

and

$$(N^{\alpha+1}_0, c)_{c \in N^\alpha_0}$$

are saturated of cardinality λ. *Then*

$$\langle N_\gamma \mid \alpha < \delta\rangle \in K^s_\delta$$

where $N_\gamma = \bigcup_{\alpha<\delta} N^\alpha_\gamma$.

PROOF Similar to the proof of the previous lemma.

Lemma 2.8 *Let* $cf\,\delta \geq \kappa_r(T)+\aleph_1$. *Let* $\bar{M} \in K^s_\delta$. *Let* $A_\delta \subseteq M^*$ *such that* $|A_\delta| < \lambda$ *and* $A_\delta = \bigcup_{i<\delta} A_i$ *where* $\langle A_i \mid i < \delta \rangle$ *is an increasing continuous chain. Suppose* $\forall \beta < \delta$, *and* $\forall i < \delta$,

$$M_\beta \underset{A_i \cap M_\beta}{\bigcup} A_i$$

Let $a \subseteq M_{\beta^*}$ *such that* $|a| < \kappa_r(T)$. *Then there exists a continuous increasing sequence* $\langle A'_i \mid i < \delta \rangle$ *and a set* B *such that* $|B| < \kappa_r(T)$, $A_i \subset A'_i$, $a \subset \bigcup A'_i = A'_\delta$, $|A'_\delta| < \lambda$, *for some non-limit* $i^* < \delta$, $A'_i = A_i$ *if* $i < i^*$, *and* $A'_i = A_i \cup B$ *if* $i^* \leq i$ *and* $\forall\ i, \beta < \delta$,

$$M_\beta \underset{A'_i \cap M_\beta}{\bigcup} A'_i$$

and $\forall\ i, \beta < \delta$,

$$M_\beta \cup (M_{\beta+1} \cap A_\delta) \underset{M_\beta \cup (M_{\beta+1} \cap A_i)}{\bigcup} A'_i \cap M_{\beta+1}$$

and

$$A_\delta \underset{A_i}{\bigcup} A'_i$$

PROOF First by induction on $n \in \omega$, we define $\langle B_n \mid n < \omega \rangle$ such that $B_0 = a$, $|B_n| < \kappa_r(T)$ and $\forall\ i < \delta$, $\forall \beta < \delta$,

$$B_n \underset{(M_\beta \cap (A_i \cup B_{n+1})) \cup A_i}{\bigcup} M_\beta \cup A_i$$

So suppose B_n has been defined. By induction on $m < \omega$ we define subsets C_1 and C_2 of δ such that $0 \in C_i$, $|C_i| \leq \kappa_r(T)$ and such that if $(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_2)$ are four neighboring points in $C_1 \times C_2$ with $a_1 < a_2$ and $b_1 < b_2$, then for all i, j such that $a_1 \leq i < a_2$ and $b_1 \leq j < b_2$

$$B_n \underset{M_{a_1} \cup A_{b_1}}{\bigcup} M_{a_1+i} \cup A_{b_1+j}$$

So it is enough to find $|B_{n+1}| < \kappa_r(T)$ such that for every $(a, b) \in C_1 \times C_2$,

$$B_n \underset{(M_a \cap (A_b \cup B_{n+1})) \cup A_b}{\bigcup} M_a \cup A_b$$

As $|C_1 \times C_2| < \kappa_r(T)$ this is possible. Let $B = \bigcup\limits_{n\in\omega} B_n$. (If $\kappa_r(T) = \aleph_0$ then without loss of generality we can define the B_n such that for some $k < \omega$, $\bigcup\limits_{n\in\omega} B_n = \bigcup\limits_{n\in k} B_n$.) It is enough to prove the following statement.

There exists a non-limit $i^* < \delta$ such that if $A'_i = A_i$ for $i < i^*$, and $A'_i = A_i \cup B$ for $i \geq i^*$ then the conditions of the theorem hold.

PROOF $\forall\beta < \delta$, $\forall i < \delta$, if $A'_i = A_i \cup B$, then since

$$B \underset{(M_\beta \cap (A_i \cup B)) \cup A_i}{\bigcup\!\!\!\!\!\!\;\smile} M_\beta \cup A_i$$

we have

$$A'_i \underset{A'_i \cap M_\beta}{\bigcup\!\!\!\!\!\!\;\smile} M_\beta$$

Let $i^{**} < \delta$ such that for all $i \geq i^{**}$,

$$A_\delta \underset{A_i}{\bigcup\!\!\!\!\!\!\;\smile} A'_i$$

It is enough to find $i^{**} \leq i^* < \delta$ such that $\forall\beta < \delta$,

$$B \underset{M_\beta \cup (M_{\beta+1} \cap A_{i^*})}{\bigcup\!\!\!\!\!\!\;\smile} M_\beta \cup (M_{\beta+1} \cap A_\delta)$$

Let $\langle \beta_\alpha \mid \alpha \in \gamma \rangle$ where $\gamma < \kappa_r(T)$ be the set of all places such that

$$B \underset{M_{\beta_\alpha - 1} \cup (M_{\beta_\alpha} \cap A_\delta)}{\not\bigcup\!\!\!\!\!\!\;\smile} M_{\beta_\alpha} \cup (M_{\beta_\alpha+1} \cap A_\delta)$$

For each $\beta \in \langle \beta_\alpha \mid \alpha \in \gamma \rangle$ let i_α be such that

$$B \underset{M_{\beta_\alpha} \cup (M_{\beta_\alpha+1} \cap A_{i_\alpha})}{\bigcup\!\!\!\!\!\!\;\smile} M_{\beta-1} \cup (M_{\beta_\alpha+1} \cap A_\delta)$$

Let i_γ be such that

$$B \underset{M_0 \cup (M_1 \cap A_{i_\gamma})}{\bigcup\!\!\!\!\!\!\;\smile} M_0 \cup (M_1 \cap A_\delta)$$

Let $i^* = sup\{i_\alpha \mid \alpha \in \gamma + 1\} + 1 + i^{**}$. As $|B| < \kappa_r(T)$ and $cf\,\delta \geq \kappa_r(T)$, $i^* < \delta$, so there is no problem.

Lemma 2.9 *Let $\bar{M} \in K^s_\delta$. Let $A \subseteq M^*$ such that $|A| < \lambda$ and $A = \bigcup\limits_{i<\delta} A_i$ where $\langle A_i \mid i < \delta \rangle$ is increasing continuous, each A_i is algebraically closed and $\forall\, i < \delta$, $\forall\, \beta < \delta$,*

$$M_\beta \underset{M_\beta \cap A_i}{\bigcup\!\!\!\!\!\!\mid}\; A_i$$

Let i^* be a successor $< \delta$, $\beta^* < \delta$, β^* a successor, and let $p \in S(A_{i^*} \cap M_{\beta^*})$. (Or even a $< \lambda$ type over $A_i \cap M_{\beta^*}$.) Let $p' \in S((A_{i^*} \cap M_{\beta^*}) \cup M_{\beta^*-1})$ such that p' does not fork over p. Then there exists an $a \in M_{\beta^*}$ such that a realizes p',

$$A \underset{M_{\beta^*} \cap A_{i^*}}{\bigcup\!\!\!\!\!\!\mid}\; a$$

and if $A'_i = A_i \cup \{a\}$ for $i \geq i^*$ and $A'_i = A_i$ for $i < i^*$, then $\forall\, \beta < \delta$, $\forall\, i < \delta$,

$$M_\beta \underset{M_\beta \cap A'_i}{\bigcup\!\!\!\!\!\!\mid}\; A'_i$$

PROOF Let $B \subseteq M_{\beta^*}$ such that $|B| < \lambda$, $A^*_i \cap M_{\beta^*} \subseteq B$, and

$$M_{\beta^*} \underset{M_{\beta^*-1}B}{\bigcup\!\!\!\!\!\!\mid}\; A$$

Let $a \in M_{\beta^*}$ such that a realizes p and

$$a \underset{A^*_i \cap M_{\beta^*}}{\bigcup\!\!\!\!\!\!\mid}\; B \cup M_{\beta^*-1}$$

Since

$$M_{\beta^*} \underset{M_{\beta^*-1} \cup B}{\bigcup\!\!\!\!\!\!\mid}\; A$$

we have

$$a \underset{M_{\beta^*-1} \cup B}{\bigcup\!\!\!\!\!\!\mid}\; A$$

which implies

$$a \underset{A^*_i \cap M_{\beta^*}}{\bigcup\!\!\!\!\!\!\mid}\; M_{\beta^*-1} \cup A$$

Since for all $i \geq i^*$,

$$a \underset{A_i}{\bigcup} M_{\beta^*-1} \cup A$$

we have for all $\gamma < \beta^*$,

$$a \underset{A_i}{\bigcup} M_\gamma \cup A$$

which implies

$$a \cup A_i \underset{A_i \cap M_\gamma}{\bigcup} M_\gamma$$

Since $a \subseteq M_{\beta^*}$ we also have $\forall \gamma \geq \beta^*$,

$$a \cup A_i \underset{(a \cup A_i) \cap M_\gamma}{\bigcup} M_\gamma$$

Lemma 2.10 *Let $\bar{M} \in K^s_\delta$. Let $A \subseteq M^*$ such that $|A| < \lambda$ and $A = \bigcup_{i<\delta} A_i$ where $\langle A_i \mid i < \delta \rangle$ is increasing continuous, each A_i is algebraically closed and $\forall\, i < \delta$, $\forall\, \beta < \delta$,*

$$M_\beta \underset{M_\beta \cap A_i}{\bigcup} A_i$$

Let $i^* < \delta$, $\beta^* < \delta$, β^*, i^* successors, and let $p \in S(A_i \cap M_\beta)$. Let $p' \in S((A_i \cap M_{\beta^*}) \cup M_{\beta^*-1})$ such that p' does not fork over p. Let $f \in Aut(A)$ such that $\forall\, i < \delta$, $f[A_i] = A_i$. Then there exists $\{a_i \mid i \in \mathbb{Z}\} \subseteq M^*$ and an extension f' of f with domain $A \cup \{a_i \mid i \in \mathbb{Z}\}$ such that a_0 realizes p', $a_0 \in M_{\beta^*}$, and $\forall\, i \in \mathbb{Z}$ $\mho'(\partial\sqsupset) = \partial_{\sqsupset+\nVdash}$ and if $A'_i = A_i \cup \{a_i \mid i \in \mathbb{Z}\}$ for $i \geq i^*$ and $A'_i = A_i$ for $i < i^*$, then for all $\beta < \delta$,

$$M_\beta \underset{M_\beta \cap A'_i}{\bigcup} A'_i$$

$$A_\delta \underset{A_i}{\bigcup} A'_i$$

and

$$M_{\beta-1} \cup (M_\beta \cap A) \underset{M_{\beta-1} \cup (M_\beta \cap A_i)}{\bigcup} M_\beta \cap A'_i$$

PROOF We define $\{a_i \mid i \in -n, \ldots, 0, \ldots, n\}$ by induction on n such that if $A'_i = acl(A_i \cup \{a_i \mid i \in -n, \ldots, 0, \ldots, n\})$ if $i \geq i^*$ and $A'_i = A_i$ if $i < i^*$, then $\forall i < \delta$, $\forall \beta < \delta$,

$$M_\beta \underset{M_\beta \cap A'_i}{\bigcup\!\!\!\!|} A'_i$$

$$A_\delta \underset{A_i}{\bigcup\!\!\!\!|} A'_i$$

and

$$M_{\beta-1} \cup (M_\beta \cap A) \underset{M_{\beta-1} \cup (M_\beta \cap A)}{\bigcup\!\!\!\!|} M_\beta \cap A'_i$$

and $f_n = f \cup \{(a_i, a_{i+1}) \mid -n \leq i < n\}$ is an elementary map. In addition we define a sequence of successor ordinals $\langle \beta_i \mid i \in \mathbb{Z} \rangle$ such that $\beta_i < \beta_j$ if $|i| < |j|$, and $\beta_n < \beta_{-n}$ such that

$$a_{n+1} \underset{M_{\beta_{n+1}} \cap A_{i^*}}{\bigcup\!\!\!\!|} M_{\beta_{n+1}-1} \cup A \cup \{a_{-n} \ldots, a_0, \ldots, a_n\}$$

and

$$a_{-(n+1)} \underset{M_{\beta_{-(n+1)}} \cap A_{i^*}}{\bigcup\!\!\!\!|} M_{\beta_{-(n+1)}-1} \cup A \cup \{a_{-n}, \ldots, a_0, \ldots, a_n, a_{n+1}\}$$

Define a_0 as in the previous lemma. Suppose that $\{a_{-n}, \ldots, a_0, \ldots, a_n\}$ and β_i for $-n \leq i \leq n$ have been defined satisfying the conditions. Let $C = acl\, C$ such that for some $B \subseteq C$ with $|B| < \kappa_r(T)$, $aclB = C$, $C \subseteq M_{\beta_{-n}} \cap A_{i^*}$ and

$$a_n \underset{C}{\bigcup\!\!\!\!|} A \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\}$$

Let $\beta_{n+1} > \beta_{-n}$ be a successor such that $f(C) \subseteq M_{\beta_{n+1}} \cap A_{i^*}$. Let $a_{n+1} \in M_{\beta_{n+1}}$ realize

$$f_n\Big(tp(a_n / A \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\})\Big)$$

and in addition

$$a_{n+1} \underset{M_{\beta_{n+1}} \cap A_i^*}{\bigcup\!\!\!\!|} A \cup M_{\beta_{n+1}-1}$$

Similarly for $a_{-(n+1)}$. Now as in the proof of the previous lemma, all the conditions of the induction hold.

Lemma 2.11 *Let* δ *be an ordinal less than* λ^+ *such that* $cf\,\delta \geq \aleph_1 + \kappa_r(T)$. *Let* $f \in Aut_E(M^*)$ *with* $|E| < \lambda$. *Let* $\bar{M} \in K^s_\delta$. *Then there exists* $\bar{N}^1, \bar{N}^2 \in K^s_\delta$, $f_1 \in Aut_E(\bar{N}^1)$, $f_2 \in Aut_E(\bar{N}^2)$ *with* $E \subseteq N^1_0$, $E \subseteq N^2_0$ *such that*

1. $f = f_2 f_1$
2. $\forall\, i, \beta < \delta,\ \forall\, l \in \{0,1\}$,

$$M_\beta \bigcup_{M_\beta \cap N^l_i} N^l_i$$

3. $\forall\, i, \beta < \delta,\ \forall\, l \in \{0,1\}$,

$$(N^l_{i+1} \cap M_{\beta+1}, c)_{c \in (N^l_{i+1} \cap M_\beta) \cup (N^l_i \cap M_{\beta+1})}$$

   is saturated of cardinality λ

4. $(N^l_{i+1} \cap M_0, c)_{c \in N^l_i \cap M_0}$ *is saturated of cardinality* λ

PROOF Without loss of generality $E = \emptyset$. By induction on $\alpha < \lambda$ we build increasing continuous sequences $\langle A^\alpha_i \mid i \leq \delta \rangle$, $\langle B^\alpha_i \mid i \leq \delta \rangle$, $\langle f^\alpha_1 \mid \alpha < \lambda \rangle$, $\langle f^\alpha_2 \mid \alpha < \lambda \rangle$ such that

1. $M^* = \bigcup_{\alpha < \lambda} A^\alpha_\delta = \bigcup_{\alpha < \lambda} B^\alpha_\delta$
2. $N^1_i = \bigcup_{\alpha < \lambda} A^\alpha_i \quad N^2_i = \bigcup_{\alpha < \lambda} B^\alpha_i$
3. $f^\alpha_1 \in Aut(A^\alpha_\delta)$ such that $f^\alpha_1[A^\alpha_i] = A^\alpha_i$
4. $f^\alpha_2 \in Aut(B^\alpha_\delta)$ such that $f^\alpha_2[B^\alpha_i] = B^\alpha_i$
5. $f[A^\alpha_i] = A^\alpha_i, \quad f[B^\alpha_i] = B^\alpha_i$
6. $|A^\alpha_\delta| < |\alpha|^+ + \kappa_r(T) + \aleph_1$
7. $|B^\alpha_\delta| < |\alpha|^+ + \kappa_r(T) + \aleph_1$
8. $A^\alpha_\delta = B^\alpha_\delta$

9. $f^2_\alpha f^1_\alpha = f \restriction A^\alpha_\delta$

10. $\forall\, \beta < \delta,\ \forall\, i < \delta,\ \forall\, \alpha < \lambda,$

$$M_\beta \underset{M_\beta \cap A^\alpha_i}{\bigcup\!\!\!\!\!\mid}\ A^\alpha_i$$

11. $\forall\, \beta < \delta,\ \forall\, i < \delta,\ \forall\, \alpha < \lambda,$

$$M_\beta \underset{M_\beta \cap B^\alpha_i}{\bigcup\!\!\!\!\!\mid}\ B^\alpha_i$$

12. $\forall\, i, \beta < \delta,\ \forall\, l \in \{0, 1\},$

$$(N^l_{i+1} \cap M_{\beta+1}, c)_{c \in (N^l_{i+1} \cap M_\beta) \cup (N^l_i \cap M_{\beta+1})}$$

is saturated of cardinality λ

13. $(N^l_{i+1} \cap M_0, c)_{c \in N^l_i \cap M_0}$ is saturated of cardinality λ

14. $\forall i < \delta,\ \forall \alpha < \lambda,$

$$A^\alpha_\delta \underset{A^\alpha_i}{\bigcup\!\!\!\!\!\mid}\ A^{\alpha+1}_i$$

15. $\forall i < \delta,\ \forall \alpha < \lambda,$

$$B^\alpha_\delta \underset{B^\alpha_i}{\bigcup\!\!\!\!\!\mid}\ B^{\alpha+1}_i$$

16. $\forall \beta < \delta,\ \forall i < \delta, \forall \alpha < \lambda,$

$$M_\beta \cup (M_{\beta+1} \cap A^\alpha_\delta) \underset{M_\beta \cup (M_{\beta+1} \cap A^\alpha_i)}{\bigcup\!\!\!\!\!\mid}\ M_{\beta+1} \cap A^{\alpha+1}_i$$

17. $\forall \beta < \delta,\ \forall i < \delta, \forall \alpha < \lambda,$

$$M_\beta \cup (M_{\beta+1} \cap B^\alpha_\delta) \underset{M_\beta \cup (M_{\beta+1} \cap B^\alpha_i)}{\bigcup\!\!\!\!\!\mid}\ M_{\beta+1} \cap B^{\alpha+1}_i$$

At limit stages we take unions. Let α be even. Let $M^* = \langle m_\alpha \mid \alpha < \lambda\rangle$. In the induction we define $\langle p_\alpha \mid \alpha$ is even and $\alpha < \lambda\rangle$ such that each $p_\alpha \in S((M_{\beta+1} \cap A^\alpha_{i+1}) \cup M_\beta)$ for some $i, \beta < \delta$ and such that $\forall\, i < \delta$, $\forall\, \beta < \delta$, $\forall\, A \subseteq M^*$ such that $|A| < \kappa_r(T)$, $\forall\, p \in S(acl\, A)$ there exists λ many $p_\alpha \in \langle p_\alpha \mid \alpha < \lambda\rangle$ such that $p_\alpha \in S((M_{\beta+1} \cap A^\alpha_{i+1}) \cup M_\beta)$, p_α is a nonforking extension of p, p_α is realized in $A^{\alpha+1}_{i+1} \cap M_{\beta+1}$, and $\forall j \le i, A^\alpha_j = A^{\alpha+1}_j$. By the proof of lemma 2.6 this insures 12. and 13. holds for $l = 1$ when we finish our construction. So let $i^*, \beta^* < \delta$ such that $p_\alpha \in S((M_{\beta^*+1} \cap A^\alpha_{i+1^*}) \cup M_{\beta^*})$. By lemma 2.10 we can find an extensions $(A^\alpha_i)'$ of A^α_i with $(A^\alpha_i)' = A^\alpha_i$ for $i \le i^*$ and extension f_1' of f_1 such that $f_1'[(A^\alpha_i)'] = (A^\alpha_i)'$, p_α is realized in $M_{\beta^*+1} \cap (A^\alpha_{i^*+1})'$ and $\forall\, \beta < \delta$, $\forall\, i < \delta$,

$$M_{\beta-1} \cup (M_\beta \cap A^\alpha_\delta) \underset{M_{\beta-1} \cup (M_\beta \cap A^\alpha_i)}{\bigcup} M_\beta \cap (A^\alpha_i)'$$

$$A^\alpha_\delta \underset{A^\alpha_i}{\bigcup} (A^\alpha_i)'$$

and

$$M_\beta \underset{M_\beta \cap (A^\alpha_i)'}{\bigcup} (A^\alpha_i)'$$

Let F_1' be an extension of f_1' to an automorphism of M^*. By iterating ω times the procedure in the proof of lemma 2.8 we can find $D \subset M^*$ such that $|D| < \kappa_r(T) + \omega_1$, if m is the least element of $\langle m_\alpha \mid \alpha < \lambda\rangle$ then $m \in D$, D is closed under $f, f^{-1}, F_1', (F_1')^{-1}$ and for some $i^{**}, i^{***} < \delta$ if $A^{\alpha+1}_i = (A^\alpha_i)' \cup D$, for $i \ge i^{**}$ and $(A^\alpha_i)'$ for $i < i^{**}$ and if $B^{\alpha+1}_i = B^\alpha_i \cup D$, for $i \ge i^{***}$ and B^α_i for $i < i^{***}$ then

$$M_\beta \cup (M_\beta \cap A^\alpha_\delta) \underset{M_\beta \cup (M_{\beta+1} \cap A^\alpha_i)}{\bigcup} M_{\beta+1} \cap A^{\alpha+1}_i$$

$$A^\alpha_\delta \underset{A^\alpha_i}{\bigcup} A^{\alpha+1}_i$$

and

$$M_\beta \underset{M_\beta \cap A^{\alpha+1}_i}{\bigcup} A^{\alpha+1}_i$$

and

$$M_\beta \cup (M_{\beta+1} \cap B^\alpha_\delta) \underset{M_\beta \cup (M_{\beta+1} \cap B^\alpha_i)}{\bigcup} M_{\beta+1} \cap B^{\alpha+1}_i$$

$$B^\alpha_\delta \underset{B^\alpha_i}{\bigcup} B^{\alpha+1}_i$$

and

$$M_\beta \underset{M_\beta \cap B^{\alpha+1}_i}{\bigcup} B^{\alpha+1}_i$$

Similarily for α odd. Let $f^{\alpha+1}_1 = F_1 \restriction A^{\alpha+1}_i$ and $f^{\alpha+1}_2 = f(f^{\alpha+1}_1)^{-1}$.

3 The proof of the small index property

Definition 3.1 *Let* δ *be a limit ordinal and let* $\bar{N} \in K^s_\delta$. *Then* $f \in Aut^*(\bar{N})$ *if and only if* $f \in Aut(M^*)$ *and for some* $n \in \omega$, $f[N_\alpha] = N_\alpha$ *for every* α *such that* $n \leq \alpha \leq \delta$. $Aut^*_A(\bar{N}) = \{f \in Aut^*(\bar{N}) \mid f \restriction A = id_A\}$.

Definition 3.2 *Let* δ *be a limit ordinal and let* $\bar{N} \in K^s_\delta$. *Let* $B \subseteq N_0$ *as in the above definition. If for every* $f \in Aut(M^*)$

$$(\, f \in Aut^*(\bar{N}) \;\wedge\; f \restriction B = id_B \,) \;\Rightarrow\; f \in G$$

then we define

$$E = \Big\{ C \subseteq B \mid f \in Aut^*(\bar{N}) \;\wedge\; f \restriction C = id_C \Rightarrow f \in G \Big\}$$

Lemma 3.3 *Let* δ *be a limit ordinal and let* $\bar{N} \in K^s_\delta$. *Let* $B \subseteq N_0$ *such that* $(N_0, c)_{c \in B}$ *is saturated. Let* $C = acl\, C$, $C \subseteq B$, *and* g *an elementary map with* $dom\, g = B$, $g \restriction C = id_C$, $(N_0, c)_{c \in B \cup g[B]}$ *is saturated, and*

$$B \underset{C}{\bigcup} g(B)$$

Then the following are equivalent.

1. $C \in E$
2. *All extensions of* g *in* $Aut^*(\bar{N})$ *are in* G

3. Some extension of g in $Aut^*(\bar{N})$ is in G

PROOF 1. $\Rightarrow$ 2. is trivial.
2. $\Rightarrow$ 3. We just need to prove g has some extension in $Aut^*(\bar{N})$. But this follows easily by the saturation for every $j < \delta$ of $(N_{j+1}, c)_{c \in N_j}$.
3. $\Rightarrow$ 1. Let $f \in Aut^*(\bar{N})$ such that $f \restriction C = id_C$. Let $n \in \omega$ and $g^* \in Aut^*(\bar{N})$ such that $g^* \supseteq g$, $f, g^* \in Aut(\bar{N} \restriction [n, \delta))$, and $g^* \in G$. Let $B' \subseteq N_{n+1}$ such that $B' \bigcup\limits_{C} N_n$ and $tp(B'/C) = tp(B/C)$. Let $g_1 \in Aut(\bar{N} \restriction [n+2, \delta))$ such that g_1 maps $g(B)$ onto B' and $g_1 \restriction B = id_B$. Since $g_1 \restriction B = id_B$, $g_1 \in G$. Let $g_2 = g_1 g^* (g_1)^{-1}$. Again $g_2 \in G$, $g_2 \restriction C = id_C$, and $g_2[B] = B'$. As

$$B' \bigcup_{C} N_n$$

$f \in Aut(\bar{N} \restriction [n, \delta))$ and $f \restriction C = id_C$, clearly

$$f(B') \bigcup_{C} N_n$$

Therefore there exists $g_3 \in Aut(\bar{N} \restriction [n+2, \delta))$ such that $g_3 \restriction B' = f \restriction B'$ and $g_3 \restriction N_n = id_{N_n}$, hence $g_3 \in G$. $(g_3)^{-1} f \restriction B' = id_{B'}$ so $(g_2)^{-1}(g_3)^{-1} f g_2 = id_B$ hence $(g_2)^{-1}(g_3)^{-1} f g_2 \in G$. But this implies $f \in G$.

Theorem 3.4 *Let $|T| < \lambda$. Let $\bar{M} \in K^s_\delta$. Let $G \subseteq Aut^*(M)$. If*

$$f \in Aut^*_{M_0}(\bar{M}) \;\Rightarrow\; f \in G$$

but for no $C \subseteq M_0$ with $|C| < \lambda$ does

$$f \in Aut^*_C(\bar{M}) \;\Rightarrow\; f \in G$$

then

$$[Aut(M^*) : G] > \lambda$$

PROOF Suppose not. Let $\langle h_i \mid i < \lambda \rangle$ be a list of the representatives of the left G cosets of $Aut(\bar{M} \restriction [1, \delta))$ possibly with repetition. Let $\lambda = \bigcup\limits_{\zeta < cf\,\lambda} \lambda_\zeta$ with $\langle \lambda_\zeta \mid \zeta < cf\,\lambda \rangle$ increasing continuous and $|T| \leq |\lambda_0| \leq |\lambda_\zeta| < \lambda$. Let

$M_0 = \bigcup_{\zeta < cf\,\lambda} M^0_\zeta$ and $M_1 = \bigcup_{\zeta < cf\,\lambda} M^1_\zeta$ with each being a continuous chain such that $|M^i_\zeta| \leq |\lambda_\zeta|$.

Now we define by induction on $\zeta < cf\,\lambda$, $N_{0,\zeta}$, $N_{1,\zeta}$, f_ζ, B_ζ, and $h_{j,\zeta}$ for $j < \lambda_\zeta$ such that

1. f_ζ is an automorphism of $N_{1,\zeta}$
2. $\langle f_\zeta \mid \zeta < cf\,\lambda \rangle$ is increasing continuous
3. If $j < \lambda_\zeta$ and there is an $h \in Aut(\bar{M} \restriction [1,\delta))$ such that
   (a) h extends f_ζ
   (b) $hG = h_j G$

   then $h_{j,\zeta}$ satisfies a. and b.
4. B_ζ is a subset of $N_{1,\zeta}$ of cardinality $\leq |\lambda_\zeta|$
5. $M^1_\zeta \subseteq B_\zeta$
6. $N_{0,\zeta} \subseteq B_{\zeta+1}$ and $B_{\zeta+1}$ is closed under $h_{j,\epsilon}$ and $h^{-1}_{j,\epsilon}$ for $j < \lambda_\epsilon$ and $\epsilon \leq \zeta$
7. $f^{-1}_{\zeta+1}(B_{\zeta+1}) \underset{N_{0,\zeta}}{\Cup} N_{0,\zeta+1}$
8. $N_{1,\zeta} \underset{N_{0,\zeta}}{\Cup} M_0$
9. $M_1 = \bigcup_{\zeta < cf\,\lambda} N_{1,\zeta} \quad M_0 = \bigcup_{\zeta < cf\,\lambda} N_{0,\zeta}$
10. $|N_{0,\zeta}| \leq |\lambda_\zeta|$
11. $(N_{1,\zeta+1}, c)_{c \in N_{1,\zeta}}$ is saturted of cardinality λ
12. $(M_1, c)_{c \in M_0 \cup N_{1,\zeta}}$ is saturated of cardinality λ

For $\zeta = 0$ let B_0 be empty, let $N_{0,0}$ be a submodel of M_0 of cardinality $|\lambda_0|$, let $N_{1,0}$ be a saturated submodel of M_1 of cardinality λ such that $N_{1,0} \underset{N_{0,0}}{\Cup} M_0$ and let $f_\zeta = id_{N_{1,0}}$. At limit stages take unions. If $\zeta = \epsilon + 1$, let B_ζ be as in 4,5,6. Let $N_{0,\zeta} \subseteq M_0$ such that

$B_\zeta \underset{N_{0,\zeta}}{\bigcup} M_0,\ N_{0,\epsilon} \subseteq N_{0,\zeta},\ M^0_\zeta \subseteq N_{0,\zeta},\ |N_{0,\zeta}| \leq \lambda_\zeta$. Let $N_{1,\zeta} \subseteq M_1$ such that $B_\zeta \subseteq N_{1,\zeta},\ N_{1,\zeta} \underset{N_{0,\zeta}}{\bigcup} M_0,\ (N_{1,\zeta}, c)_{c \in N_{1,\epsilon}}$ is saturated of cardinality λ, and $(M_1, c)_{c \in M_0 \cup N_{1,\zeta}}$ is saturated of cardinality λ. Let f_ζ be an extension of $f_\epsilon \restriction N_{1,\epsilon}$ to an automorphism of $N_{1,\zeta}$ so that

$$f_\zeta^{-1}(B_\zeta) \underset{N_{1,\epsilon}}{\bigcup} N_{0,\zeta}$$

Since

$$N_{0,\zeta} \underset{N_{0,\epsilon}}{\bigcup} N_{1,\epsilon}$$

we have

$$f_\zeta^{-1}(B_\zeta) \underset{N_{0,\epsilon}}{\bigcup} N_{0,\zeta}$$

Let f be an extension of $\bigcup_{\zeta < cf\,\lambda} f_\zeta$ to an element of $Aut(\bar{M} \restriction [1, \delta))$. We have defined f so that

1. (By nonforking calculus) $\forall \zeta < cf\,\lambda,\ \forall j < \lambda_\zeta$,

$$f^{-1} h_{j,\zeta}(M_0) \underset{N_{0,\zeta}}{\bigcup} M_0$$

2. $f^{-1} h_{j,\zeta} \restriction N_{0,\zeta} = id$

By lemma 3.3 none of the $f^{-1} h_{j,\zeta}$ are in G, a contradiction as for some $j < \lambda$, $fG = h_j G$ so for some $\zeta,\ j < \lambda_\zeta,\ \ h_j G = h_{j,\zeta} G = fG$.

Lemma 3.5 *Let $|T| < \lambda$. Let $cf\,\delta \geq \kappa_r(T) + \aleph_1$. Suppose $[Aut(M^*) : G] \leq \lambda$ and assume that for no $A \subseteq M^*$ with $|A| < \lambda$ is $Aut_A(M^*) \subseteq G$. Then for some $\bar{N} \in K^s_\delta$,*

$$\bigwedge_{\alpha < \delta} Aut^*_{N_\alpha}(\bar{N}) \not\subseteq G$$

PROOF Suppose not. Let $\bar{M} \in K^s_\delta$. Then there exists an $\alpha < \delta$ such that $Aut^*_{M_\alpha}(\bar{M}) \subseteq G$. Without loss of generality $\alpha = 0$. By lemma 3.4 there exists $E \subseteq M_0$ such that $|E| < \lambda$ and $Aut_E(\bar{M}) \subseteq G$. Let $f \in Aut_E(M^*) \backslash G$. By lemma 2.11 we can find $\bar{N}^1, \bar{N}^2 \in K^s_\delta$ and automorphisms $f_1 \in Aut_E(\bar{N}^1)$ and $f_2 \in Aut_E(\bar{N}^2)$ such that

1. $E \subset N^1_0,\ E \subset N^2_0$
2. $f = f_2 f_1$
3. $f_1 \restriction E = f_2 \restriction E = id_E$
4. $\forall\, \alpha, \beta < \delta,$
   (a) $N^1_\alpha \underset{N^1_\alpha \cap M_\beta}{\bigcup} M_\beta$
   (b) $N^2_\alpha \underset{N^2_\alpha \cap M_\beta}{\bigcup} M_\beta$
   (c) $(N^1_{\alpha+1} \cap M_{\beta+1},\, c)_{c \in (N^1_{\alpha+1} \cap M_\beta) \cup (N^1_\alpha \cap M_{\beta+1})}$ is saturated of cardinality λ
   (d) $(N^2_{\alpha+1} \cap M_{\beta+1},\, c)_{c \in (N^2_{\alpha+1} \cap M_\beta) \cup (N^2_\alpha \cap M_{\beta+1})}$ is saturated of cardinality λ
   (e) $(N^1_{\alpha+1} \cap M_0)_{c \in N^1_\alpha \cap M_0}$ is saturated of cardinality λ
   (f) $(N^2_{\alpha+1} \cap M_0)_{c \in N^2_\alpha \cap M_0}$ is saturated of cardinality λ

Since $f \notin G$ we can assume without loss of generality that $f_1 \notin G$. Also, by the hypothesis of suppose not we can assume there is a $F \subseteq N^1_0$ such that $(N^1_0, c)_{c \in F}$ is saturated and $Aut_F(\bar{N}^1) \subseteq G$. By lemma 3.4 we can assume that $|F| < \lambda$ and without loss of generality $E \subseteq F$. Let for $\alpha < \delta$,

$$F_\alpha = F \cap M_\alpha$$

By the lemma 3.6 we can find a sequence $\langle F'_\alpha \mid \alpha < \delta \rangle$ such that for each α, $F_\alpha \subseteq F'_\alpha$ with $|F'_\alpha| < \lambda$ and for each $\beta < \alpha$ $F'_\alpha \cap M_\beta = F'_\beta$ and if $F' = \bigcup\limits_{\alpha < \delta} F'_\alpha$ then

$$M_\alpha \cap N^1_0 \underset{F'_\alpha}{\bigcup} F'$$

We define by induction on $\alpha < \delta$ a map g_α an automorphism of $M_\alpha \cap N^1_0$ such that

1. $\forall\, \beta, \alpha < \delta,\quad \beta < \alpha \ \Rightarrow\ g_\beta \subseteq g_\alpha$
2. If α is a limit then $g_\alpha = \bigcup\limits_{\beta < \alpha} g_\beta$

3. $g_\alpha(F'_\alpha) \underset{E}{\bigcup\!\!\!\!|}\; F'_\alpha$

4. $g_\alpha \restriction E = id_E$

Let $\alpha = \beta + 1$ and suppose g_β has been defined. Let $X \subseteq M_\alpha \cap N^1_0$ such that $X \frown g_\beta(F'_\beta) \equiv F'_\alpha \frown F'_\beta$ by h_β an extension of $g_\beta \restriction F'_\beta$ and

$$X \underset{g_\beta(F'_\beta)}{\bigcup\!\!\!\!|}\; F'_\alpha \cup (M_\beta \cap N^1_0)$$

Let $g'_\alpha = g_\beta \cup h_\beta$. Since $X \underset{g_\beta(F'_\beta)}{\bigcup\!\!\!\!|}\; g_\beta(M_\beta \cap N^1_0)$ and $F'_\alpha \underset{F'_\beta}{\bigcup\!\!\!\!|}\; M_\beta \cap N^1_0$, g'_α is an elementary map. Now let g_α be an extension of g'_α to an automorphism of $M_\alpha \cap N^1_0$. Let $g' = \bigcup\limits_{\alpha<\delta} g_\alpha$. g' is an automorphism of N^1_0 such that for every $\alpha < \delta$,

$$g'[M_\alpha \cap N^1_0] = [M_\alpha \cap N^1_0]$$

By the saturation and independence of the N^1_α, M_β we can find an extension g of g' such that $g \in Aut(\bar{N}_1)$ and $g \in Aut(\bar{M})$. This gives a contradiction since $g(F) \underset{E}{\bigcup\!\!\!\!|}\; F$ and $g \in Aut(\bar{N}_1)$ implies $g \notin G$, but $g \in Aut(\bar{M})$ and $g \restriction E = id_E$ implies $g \in G$.

Lemma 3.6 *Let $\bar{M} = \langle M_\beta \mid \beta \leq \delta \rangle \in K^s_\delta$. Let $F \subseteq M^*$ with $|F| < \lambda$. Then there exits a set F' such that $|F'| < \lambda$, $F \subseteq F'$, and $\forall\, \beta < \delta$,*

$$* \;\; M_\beta \underset{F' \cap M_\beta}{\bigcup\!\!\!\!|}\; F'$$

PROOF Let $w \subseteq F$ be finite. There are less than $\kappa_r(T)$ many $\alpha < \delta$ such that

$$w \underset{M_\alpha}{\not\bigcup\!\!\!\!|}\; M_{\alpha+1}$$

Let a_w be the set of such α. For each $\alpha \in a_w$ let $w_\alpha \subseteq M_\alpha$ such that $|w_\alpha| < \kappa_r(T)$, and

$$w \underset{w_\alpha}{\bigcup\!\!\!\!|}\; M_\alpha$$

Let $w^1 = \bigcup_{\alpha \in a_w} w_\alpha$. Let $F^1 = \bigcup_{w \underset{finite}{\subset} F} w^1$ and repeat this procedure ω times with F^n relating to F^{n+1} as F is related to F^1. Let $F' = \bigcup_{n \in \omega} F^n$. F' satisfies $*$.

Lemma 3.7 *Let Tr be a tree of infinite height. Let $\alpha < height(Tr)$ and let $\eta \in Tr \restriction level(\alpha+1)$. Let $\langle M_\beta \mid \beta \leq \alpha \rangle$ be an increasing chain of models such that for all $\beta < \alpha$, $(M_{\beta+1},\ c)_{c \in M_\beta}$ is saturated. Let $M_\alpha \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3$ with $(N_{i+1},\ c)_{c \in N_i}$ saturated for $i \leq 2$. Suppose $\langle h_\beta \mid \beta \leq \alpha \rangle$ are such that*

1. $h_\beta = id_{M_\beta}$
2. $h_\beta[N_i] = N_i$ *for* $i \leq 3$
3. $h_\beta[M_\gamma] = M_\gamma$ *for* $\gamma \leq \alpha$

For each $\nu \in Tr \restriction level(\alpha+1)$ let m_ν, l_ν be automorphisms of N_0. Suppose $g_\eta \in Aut(N_0)$ such that for all $\nu \in Tr \restriction level(\alpha+1)$,

$$g_\eta m_\eta (m_\nu)^{-1} (g_\eta)^{-1} = l_\eta (l_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}$$

Let m_ν^+, l_ν^+ be extensions of m_ν and l_ν to automorphisms of N_1 for all $\nu \in Tr \restriction level(\alpha+1)$. Then there exists a $g'_\eta \in Aut(N_3)$ extending g_η and for all $\nu \in Tr \restriction level(\alpha+1)$ automorphisms of N_3, m'_ν and l'_ν extending m_ν^+ and l_ν^+ respectively such that

$$g'_\eta m'_\eta (m'_\nu)^{-1} (g'_\eta)^{-1} = l'_\eta (l'_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}$$

PROOF Similar to the proof of lemma 1.8.

Theorem 3.8 *Let $|T| < \lambda$. Let M^* be a saturated model of cardinality λ, and let $G \subseteq Aut(M^*)$. Suppose that for no $A \subseteq M$ with $|A| < \lambda$ is $Aut_A(M^*) \subseteq G$. Suppose Tr is a tree of height κ, where κ is a regular cardinal $\geq \kappa_r(T) + \aleph_1$ such that each level of Tr is of size at most λ, but Tr having more than λ branches. Then*

$$[Aut(M^*) : G] > \lambda$$

PROOF Suppose not. Then by lemma 3.5 there is a $\bar{N} \in K^s_{\lambda\times\kappa}$, such that

$$\bigwedge_{\alpha<\lambda\times\kappa} Aut^*_{N_\alpha}(\bar{N}) \not\subseteq G$$

By thinning $\bar{N}$ if necessary we can assume for each $\alpha < \kappa$ there exists an automorphism $h_\alpha \in Aut_{N_{\lambda\times\alpha}}(\bar{N})$ such that $h_\alpha \notin G$. By induction on $\alpha < \kappa$ for every $\eta \in Tr \restriction level\,\alpha$ we define automorphisms g_η, m_η, l_η of $N_{\lambda\times\alpha}$ such that if $\rho \neq \nu$ then $l_\rho \neq l_\nu$ and

$$g_\rho m_\rho (m_\nu)^{-1}(g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu])\,<\,\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

At limit steps we take unions. If $\alpha = \beta + 1$, for each $i < \lambda$ we define for some $\eta_i \in Tr \restriction level\,\alpha$, $g_{\eta_i} \in Aut(N_{\lambda\times\beta+3i})$ such that for each $\eta \in Tr \restriction level\,\alpha$, $\eta = \eta_i$ cofinally many times in λ, and for every $\nu \in Tr \restriction level\,\alpha$, $m^i_\nu \neq l^i_\nu \in Aut(N_{\lambda\times\beta+3i})$ such that

$$g_{\eta_i} m^i_{\eta_i} (m^i_\nu)^{-1}(g_{\eta_i})^{-1} = l^i_{\eta_i} (l^i_\nu)^{-1} h^{\eta_i(\gamma[\eta_i,\nu])\,<\,\nu(\gamma[\eta_i,\nu])}_{\gamma[\eta_i,\nu]}$$

The g_{η_i}, m^i_ν, l^i_ν are easily defined by induction on $i < \lambda$ using lemma 3.7. Then if we let $g_\eta = \bigcup\{g_{\eta_i} \mid \eta_i = \eta\}$, $m_\eta = \bigcup_{i<\lambda} m^i_\eta$ and $l_\eta = \bigcup_{i<\lambda} l^i_\eta$ we have finished. Let Br the set of branches of Tr of height κ. For $\rho \in Br$ let $g_\rho = \bigcup\{g_\eta \mid \eta < \rho\}$, $m_\rho = \bigcup\{m_\eta \mid \eta < \rho\}$, and $l_\rho = \bigcup\{l_\eta \mid \eta < \rho\}$. If $\rho \neq \nu$, $g_\rho \neq g_\nu$ since without loss of generality $\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])$ and

$$g_\rho m_\rho (m_\nu)^{-1}(g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu])\,<\,\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

and

$$g_\nu m_\nu (m_\rho)^{-1}(g_\nu)^{-1} = l_\nu (l_\rho)^{-1}$$

implies

$$g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu])\,<\,\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

So if $g_\rho = g_\nu$ this would imply $h^{\rho(\gamma[\rho,\nu])\,<\,\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} = id_{M^*}$ a contradiction. If

$$[Aut(M^*) : G] \leq \lambda$$

then for some $\rho, \nu \in Br$ we must have $l_\rho(l_\nu)^{-1} \in G$ and $g_\rho(g_\nu)^{-1} \in G$, but then we get a contradiction as $g_\rho(g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G$ and $l_\rho(l_\nu)^{-1} \in G$, but $h^{\rho(\gamma[\rho,\nu])\,<\,\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} \notin G$.

Corollary 3.9 *Let* $G \subseteq Aut(M^*)$. *Suppose that for no* $A \subseteq M$ *with* $|A| < \lambda$ *is* $Aut_A(M^*) \subseteq G$. *Suppose* $|T| < \lambda$ *and* M^* *does not have the small index property. Then*

1. *There is no tree of height an uncountable regular cardinal* κ *with at most* λ *nodes, but more than* λ *branches.*
2. *For some strong limit cardinal* μ, $cf\,\mu = \aleph_0$ *and* $\mu < \lambda < 2^\mu$.
3. T *is superstable.*

PROOF

1. By the previous theorem
2. By 1. and [Sh 430, 6.3]
3. If T is stable in λ, then $\lambda = \lambda^{<\kappa_r(T)}$, so if $\kappa_r(T) > \aleph_0$ we can let κ from the previous theorem be the least κ such that $\lambda < \lambda^\kappa$.

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