Explicitly Non-Standard Uniserial Modules

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Abstract

A new construction is given of non-standard uniserial modules over certain valuation domains; the construction resembles that of a special Aronszajn tree in set theory. A consequence is the proof of a sufficient condition for the existence of non-standard uniserial modules; this is a theorem of ZFC which complements an earlier independence result.

Introduction

This paper is a sequel to [ESh]. Both papers deal with the existence of non-standard uniserial modules over valuation domains; we refer to [ESh] for history and motivation. While the main result of the previous paper was an independence result, the main results of this one are theorems of ZFC, which complement and extend the results of [ESh].

We are interested in necessary and sufficient conditions for a valuation domain R to have the property that there is a non-standard uniserial R-module of a given type J/R. (Precise definitions are given below.) The question is interesting only when R is uncountable, and since additional complications arise for higher cardinals, we confine ourselves to rings of cardinality $\aleph_1$. Associated to any type J/R is an invariant, denoted $\Gamma(J/R)$, which is a member of a Boolean algebra $D(\omega_1)$ (equal to $\mathcal{P}(\omega_1)$ modulo the filter of closed unbounded sets). For example, if R is an almost maximal valuation domain, then $\Gamma(J/R) = 0$ for all types J/R; but there are natural and easily defined examples where $\Gamma(J/R) = 1$.

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It is a fact that

if $\Gamma(J/R) = 0$*, then there is no non-standard uniserial* R*-module of type* J/R

(cf. [ESh, Lemma 5]). In [ESh] we showed that the converse is independent of ZFC + GCH; the consistency proof that the converse fails involved the construction of a valuation domain R associated with a stationary and co-stationary subset of ω_1 — that is, $0 < \Gamma(J/R) < 1$. The existence of such sets requires a use of the Axiom of Choice; no such set can be explicitly given. Thus — without attempting to give a mathematical definition of "natural" — we could say that for *natural* valuation domains, R, it is the case that for every type J/R, $\Gamma(J/R)$ is either 0 or 1. For natural valuation domains, it turns out that the converse *is* true: if there is no non-standard uniserial R-module of type J/R, then $\Gamma(J/R) = 0$. This is a consequence of the following result which is proved below (for *all* valuation domains of cardinality $\aleph_1$):

if $\Gamma(J/R) = 1$*, then there is a non-standard uniserial* R*-module of type* J/R*.*

(Theorem 12.) This vindicates a conjecture made by Barbara Osofsky in [O1, (9), p. 164]. (See also the Remark following Theorem 12.)

The proof of Theorem 12 divides into several cases; the key new result which is used is a construction of a non-standard uniserial module in the *essentially countable* case; this construction is done in ZFC and is motivated by the construction of a special Aronszajn tree. (See Theorem 7.) Moreover, the uniserial constructed is "explicitly non-standard" in that there is an associated "special function" which demonstrates that it is non-standard. This special function continues to serve the same purpose in any extension of the universe, V, of set theory, so the module is "absolutely" non-standard. In contrast, this may not be the case with non-standard uniserials constructed using a prediction (diamond) principle. (See the last section.)

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Preliminaries

For any ring R, we will use R^* to denote the group of units of R. If $r \in R$ we will write $x \equiv y \pmod{r}$ to mean $x - y \in rR$.

A module is called uniserial if its submodules are linearly ordered by inclusion. An integral domain R is called a valuation domain if it is a uniserial R-module. If R is a valuation domain, let Q denote its quotient field; we assume $Q \neq R$. The residue field of R is R/P, where P is the maximal ideal of R. [FS] is a general reference for modules over valuation domains.

If J and A are R-submodules of Q with $A \subseteq J$, then J/A is a uniserial R-module, which is said to be *standard*. A uniserial R-module U is said to be *non-standard* if it is not isomorphic to a standard uniserial.

Given a uniserial module U, and a non-zero element, a, of U, let $\mathrm{Ann}(a) = \{r \in R\colon ra = 0\}$ and let $D(a) = \cup\{r^{-1}R\colon r \text{ divides } a \text{ in } U\}$. We say U is of *type* J/A if $J/A \cong D(a)/\mathrm{Ann}(a)$. This is well-defined in that if b is another non-zero element of U, then $D(a)/\mathrm{Ann}(a) \cong D(b)/\mathrm{Ann}(b)$. For example, U has type Q/R if and only if U is divisible torsion and the annihilator ideal of every non-zero element of U is principal. (But notice that there is no $a \in U$ with $\mathrm{Ann}(a) = R$.) It is not hard to see that if U has type J/A, then U is standard if and only if it is isomorphic to J/A. We will only consider types of the form J/R; it is a consequence of results of [BFS] that the question of the existence of a non-standard uniserial R-module of type J_1/A can always be reduced to the question of the existence of a non-standard uniserial of type J/R for an appropriate J.

¿From now on we will assume that R has cardinality $\aleph_1$. We always have

$$(*) \qquad J = \cup_{\sigma<\omega_1} r_\sigma^{-1} R$$

for some sequence of elements $\{r_\sigma\colon \sigma < \omega_1\}$ such that for all $\tau < \sigma$, $r_\tau | r_\sigma$. If J is countably generated, then U is standard, so generally we will be assuming that J is not countably generated; then it has a set of generators as in $(*)$, where, furthermore, r_σ does not divide r_τ if $\tau < \sigma$.

If $\delta \in \lim(\omega_1)$, let

$$(**) \qquad J_\delta \stackrel{\text{def}}{=} \cup_{\sigma<\delta} r_\sigma^{-1} R.$$

By results in [BS] every uniserial module U, of type J/R, is described up to isomorphism by a family of units, $\{e_\sigma^\tau\colon \sigma < \tau < \omega_1\}$ such that

$$(\dagger) \qquad e_\tau^\delta e_\sigma^\tau \equiv e_\sigma^\delta \pmod{r_\sigma}$$

for all $\sigma < \tau < \delta < \omega_1$. Indeed, U is a direct limit of submodules $a_\sigma R$ where $\mathrm{Ann}(a_\sigma) = r_\sigma R$; then $a_\sigma R \cong r_\sigma^{-1}R/R$ and U is isomorphic to a direct limit of the $r_\sigma^{-1}R/R$, where the morphism from $r_\sigma^{-1}R/R$ to $r_\tau^{-1}R/R$ takes r_σ^{-1} to $e_\sigma^\tau r_\sigma^{-1}$ if $a_\sigma = e_\sigma^\tau r_\sigma^{-1} r_\tau a_\tau$.

If U is given by $(\dagger)$, then U is standard if and only if there exists a family $\{c_\sigma\colon \sigma < \omega_1\}$ of units of R such that

$$(\dagger\dagger) \qquad c_\tau \equiv e_\sigma^\tau c_\sigma \pmod{r_\sigma}$$

for all $\sigma < \tau < \omega_1$. Indeed, if the family $\{c_\sigma : \sigma < \omega_1\}$ satisfying (††) exists, then multiplication by the c_σ gives rise to isomorphisms from $r_\sigma^{-1}R/R$ to $a_\sigma R$, which induce an isomorphism of J/R with U.

Essentially Countable Types

Definition. Suppose $J = \cup_{\sigma<\omega_1} r_\sigma^{-1}R$ as in (*). Call the type J/R *essentially uncountable* if for every $\sigma < \omega_1$ there exists $\tau > \sigma$ such that $r_\sigma R/r_\tau R$ is uncountable. Otherwise, J/R is *essentially countable*; this is equivalent to saying that there is a $\gamma < \omega_1$ such that for all $\gamma < \sigma < \omega_1$, $r_\gamma R/r_\sigma R$ is countable. Say that J/R is *strongly countable* if for all $\sigma < \omega_1$, $R/r_\sigma R$ is countable; clearly, a strongly countable type is essentially countable.

It is easily seen that the notions of being essentially or strongly countable are well-defined, that is, independent of the choice of the representation (*). If the residue field of R is uncountable, then, except in trivial cases, the types J/R have to be essentially uncountable; but if the residue field is countable, the question is more delicate.

Proposition 1 *If the residue field of R is uncountable, then every type J/R such that J is not countably generated is essentially uncountable.*

Proof. Let $J = \cup_{\sigma<\omega_1} r_\sigma^{-1}R$ as in (*). It suffices to prove that if $\sigma < \tau$, then $r_\sigma R/r_\tau R$ is uncountable. But $r_\sigma R/r_\tau R \cong R/tR$ where $t = r_\tau r_\sigma^{-1} \in P$. So we have $tR \subseteq P \subseteq R$, and hence $(R/tR)/(P/tR) \cong R/P$, the residue field of R. Since R/P is uncountable, so is R/tR. □

Theorem 2 *For any countable field K there are valuation domains R_1 and R_2, both of cardinality $\aleph_1$ with the same residue field K and the same value group, whose quotient fields, Q_1 and Q_2, respectively, are generated by $\aleph_1$ but not countably many elements, and such that Q_1/R_1 is essentially uncountable and Q_2/R_2 is strongly countable.*

Proof. Let G be the ordered abelian group which is the direct sum $\oplus_{\alpha<\omega_1}\mathbb{Z}\alpha$ ordered anti-lexicographically; that is, $\Sigma_\alpha n_\alpha \alpha > 0$ if and only if $n_\beta > 0$, where β is maximal such that $n_\beta \neq 0$. In particular, the basis elements have their natural order and if $\alpha < \beta$, then $k\alpha < \beta$ in G for any $k \in \mathbb{Z}$. Let $G^+ = \{g \in G : g \geq 0\}$.

Let $\hat{R} = K[[G]]$, that is, $\hat{R} = \{\sum_{g\in\Delta} k_g X^g : k_g \in K, \Delta \text{ a well-ordered subset of } G^+\}$, with the obvious addition and multiplication (cf. [O1, p. 156]). Given an element $y = \sum_{g\in\Delta} k_g X^g$ of $\hat{R}$, let $\text{supp}(y) = \{g \in \Delta : k_g \neq 0\}$; let $\text{p-supp}(y) = \{\alpha \in \omega_1 : \exists g \in \text{supp}(y)$ whose projection on $\mathbb{Z}\alpha$ is non-zero$\}$. Define $v(y) =$ the least element of $\text{supp}(y)$. If $X \subseteq G$, then $y|X$ is defined to be $\sum_{g\in X\cap\Delta} k_g X^g$. Let $y|\nu = y|\{g \in G^+ : g < \nu\}$.

Let $R_1 = \{y \in \hat{R}: p\text{-supp}(y) \text{ is finite}\}$. Then R_1 is a valuation domain since p-supp$(xy^{-1}) \subseteq p\text{-supp}(x) \cup p\text{-supp}(y)$. Let R_2 be the valuation subring of R_1 generated by $\{X^g : g \in G\}$. We have $Q_j = \cup_{\alpha<\omega_1} X^{-\alpha} R_j$ for $j = 1, 2$. Now Q_1/R_1 is essentially uncountable since for all $\beta > \alpha$, $X^\alpha R_1/X^\beta R_1$ contains the $2^{\aleph_0}$ elements of the form

$$\sum_{n\in\omega} \zeta(n) X^{(n+1)\alpha}$$

(with p-supp $= \{\alpha\}$) where ζ is any function: $\omega \to 2$. R_1 has cardinality $2^{\aleph_0}$; if $2^{\aleph_0} > \aleph_1$, to get an example of cardinality $\aleph_1$, choose a valuation subring of R_1 which contains all the monomials X^g ($g \in G$) and $\aleph_1$ of the elements $\sum_{n\in\omega} \zeta(n) X^{(n+1)\alpha}$ for each α.

We claim that Q_2/R_2 is essentially countable. Let $K[G]$ be the subring of $\hat{R}$ generated by $\{X^g: g \in G^+\}$; thus $K[G]$ consists of the elements of $\hat{R}$ with finite support; we shall refer to them as *polynomials*. R_2 consists of all elements of the form xy^{-1} where x and y are polynomials and $v(x) \geq v(y)$. We claim that $R_2/X^\beta R_2$ is countable for any $\beta < \omega_1$. There are uncountably many polynomials, but we have to show that there are only countably many truncations $xy^{-1}|\beta$.

Given polynomials x and y with $v(x) \geq v(y)$, there is a finitely generated subgroup $\oplus_{1\leq i\leq d} \mathbb{Z}\sigma_i$ of G (with $\sigma_1 < \sigma_2 < \ldots < \sigma_d$) such that x and y are linear combinations of monomials X^g with $g \in \oplus_{1\leq i\leq d} \mathbb{Z}\sigma_i$. More precisely, there exist $k, r \in \omega$ and a $(k+r)$-tuple $(a_1, \ldots, a_{k+r})$ of elements of K and $k + r$ linear terms t_j of the form

$$t_j = \sum_{i=1}^{d} n_{ij} v_i$$

($n_{ij} \in \mathbb{Z}$, v_i variables) such that if we let $t_j(\sigma)$ denote $\sum_{i=1}^d n_{ij}\sigma_i$, then

$$(\natural) \qquad x = \textstyle\sum_{j=1}^k a_j X^{t_j(\sigma)} \quad \text{and} \quad y = \textstyle\sum_{j=k+1}^r a_j X^{t_j(\sigma)}.$$

Finally, there is $q \leq d$ such that σ_q is maximal with $\sigma_q < \beta$.

Now, consideration of the algorithm for computing xy^{-1} shows that, for fixed $(a_1, \ldots, a_{k+r})$ and t_j, there are linear terms

$$s_\ell = \sum_{i=1}^{d} m_{i\ell} v_i$$

($m_{i\ell} \in \mathbb{Z}$, $\ell \in \omega$) and elements $c_\ell \in K$ such that for *any* strictly increasing sequence $\sigma = \langle \sigma_1, \ldots, \sigma_d \rangle$, if x and y are as in ($\natural$), then

$$xy^{-1} = \sum_{\ell\in\omega} c_\ell X^{s_\ell(\sigma)}.$$

For any $q \le d$, only certain of the s_ℓ involve only variables v_i with $i \le q$ (i.e. $m_{i\ell} = 0$ if $i > q$); say these are the s_ℓ with $\ell \in T$ ($T \subseteq \omega$). If σ is such that $\sigma_i < \beta$ iff $i \le q$, then $xy^{-1}|\beta = \sum_{\ell \in T} c_\ell X^{s_\ell(\sigma)}$.

There are only countably many choices for q, d, k and r in ω, $(a_1, \ldots, a_{k+r}) \in K^{k+r}$, and for $\sigma_1 < \ldots < \sigma_q < \beta$. Therefore, there are only countably many possibilities for the truncations $xy^{-1}|\beta$. □

By the first part of the following, the type Q_2/R_2 of the previous theorem *must* be strongly countable; on the other hand, there are types which are essentially countable but not strongly countable.

PROPOSITION 3 *(i) If Q/R is essentially countable, then it is strongly countable.*

(ii) For any countable field K, there is a valuation domain R with residue field K which has a type J/R which is essentially countable but not strongly countable.

PROOF. (i) Since Q/R is essentially countable, we can write $Q = \cup_{\sigma<\omega_1} r_\sigma^{-1}R$ such that for all $\sigma < \tau$, $r_\sigma R/r_\tau R$ is countable. We claim that R/r_0R is countable, which clearly is equivalent to $R/r_\tau R$ countable for all $\tau < \omega_1$. Suppose not. There is a $\sigma < \omega_1$ such that $r_\sigma = r_0^2 t$ for some $t \in R$ (since $r_0^{-2} \in Q$). But then $r_0R/r_\sigma R \cong R/tr_0R$, which is uncountable since R/r_0R is uncountable, and this contradicts the choice of the r_σ.

(ii) Let $G = \oplus_{\alpha \le \omega_1} \mathbb{Z}\alpha$, ordered anti-lexicographically. Let $\hat{R} = K[[G]]$ (cf. proof of Theorem 2), and let R be the smallest valuation subring of $\hat{R}$ containing all the monomials X^g ($g \in G$). Let $J = \cup_{\alpha<\omega_1} r_\alpha^{-1}R$ where $r_\alpha = X^{\alpha+\omega_1}$. Then the proof that $r_\alpha R/r_\beta R \cong R/X^{\beta-\alpha}R$ is countable for all $\alpha < \beta$ is the same as in Theorem 2. But $R/r_0R = R/X^{\omega_1}R$ is clearly uncountable. □

Remark. More generally, referring to a dichotomy in [O1, Prop. 7, p. 155], if the type J/R is essentially countable and falls into Case (A), then J/R is strongly countable; if it falls into case (B), then it is not strongly countable.

Gamma Invariants

A subset C of ω_1 is called a *cub* — short for closed unbounded set — if $\sup C = \omega_1$ and for all $Y \subseteq C$, $\sup Y \in \omega_1$ implies $\sup Y \in C$. Call two subsets, S_1 and S_2, of ω_1 equivalent iff there is a cub C such that $S_1 \cap C = S_2 \cap C$. Let $\tilde{S}$ denote the equivalence class of S. The inclusion relation induces a partial order on the set, $D(\omega_1)$, of equivalence classes, i.e., $\tilde{S}_1 \le \tilde{S}_2$ if and only if there is a cub C such that $S_1 \cap C \subseteq S_2 \cap C$. In fact, this induces a Boolean algebra structure on $D(\omega_1)$, with least element, 0, the equivalence class of sets disjoint from a cub; and greatest element, 1, the equivalence class of sets containing a cub. We say S is *stationary* if $\tilde{S} \neq 0$, i.e., for every cub C, $C \cap S \neq \emptyset$. We say S is *co-stationary* if $\omega_1 \setminus S$ is stationary.

Given R and a type J/R, where J is as in (*), define $\Gamma(J/R)$ to be $\tilde{S}$, where

$$S = \{\delta \in \lim(\omega_1) : R/\cap_{\sigma<\delta} r_\sigma R \text{ is not complete}\}$$

where the topology on $R/\cap_{\sigma<\delta} r_\sigma R$ is the metrizable linear topology with a basis of neighborhoods of 0 given by the submodules $r_\sigma R$ ($\sigma < \delta$). This definition is independent of the choice of the representation of J as in (*) — see [ESh].

For any limit ordinal $\delta < \omega_1$, let

$$\mathcal{T}^\delta_{J/R} = \{\langle u_\sigma : \sigma < \delta \rangle : \forall \sigma < \tau < \delta (u_\sigma \in R^*, \text{ and } u_\tau - u_\sigma \in r_\sigma R)\};$$

that is, $\mathcal{T}^\delta_{J/R}$ consists of sequences of units which are Cauchy in the metrizable topology on $R/\cap_{\sigma<\delta} r_\sigma R$. Let $\mathcal{L}^\delta_{J/R}$ consist of those members of $\mathcal{T}^\delta_{J/R}$ which have limits in R, i.e.

$$\mathcal{L}^\delta_{J/R} = \{\langle u_\sigma : \sigma < \delta \rangle \in \mathcal{T}^\delta_{J/R} : \exists u_\delta \in R^* \text{ s.t. } \forall \sigma < \delta (u_\delta - u_\sigma \in r_\sigma R)\}.$$

Note that $\Gamma(J/R) = \tilde{S}$ where

$$S = \{\delta \in \lim(\omega_1) : \mathcal{T}^\delta_{J/R} \neq \mathcal{L}^\delta_{J/R}\}.$$

If J is not countably generated, then $\neg$CH implies that $\Gamma(J/R) = 1$, since the completion of $R/\cap_{\sigma<\delta} r_\sigma R$ has cardinality $2^{\aleph_0} > \aleph_1$. An *ω_1-filtration of R by subrings* is an increasing chain $\{N_\alpha : \alpha \in \omega_1\}$ of countable subrings of R such that $R = \cup_{\alpha\in\omega_1} N_\alpha$, and for limit α, $N_\alpha = \cup_{\beta<\alpha} N_\beta$.

Define $\Gamma'(J/R) = \tilde{E}'$ where

$$E' = \{\delta \in \lim(\omega_1) : \exists \langle u_\sigma : \sigma < \delta \rangle \in \mathcal{T}^\delta_{J/R} \text{ s.t. } \forall f \in R^* \exists \sigma < \delta \text{ s.t. } u_\sigma f \notin N_\delta \pmod{r_\sigma}\}.$$

Again, it can be shown that the definition does not depend on the choice of $\{r_\nu : \nu < \omega_1\}$ or of $\{N_\alpha : \alpha < \omega_1\}$. Notice that $\Gamma'(J/R) \leq \Gamma(J/R)$ since if $\mathcal{T}^\delta_{J/R} = \mathcal{L}^\delta_{J/R}$, then we can let f be a limit of $\langle u_\sigma^{-1} : \sigma < \delta \rangle$.

In [ESh, Theorem 7] it is proved that if $\Gamma'(J/R) \neq 0$, then there is a non-standard uniserial R-module.

THEOREM 4 *Suppose J/R is essentially countable. Then*
(i) $\Gamma'(J/R) = 0$;
(ii) $\Gamma(J/R) = 1$.

Proof. Without loss of generality we can assume that $J = \cup_{\sigma<\omega_1} r_\sigma^{-1}R$ where $r_0R/r_\sigma R$ is countable for all $\sigma < \omega_1$.

(i) We can also assume that the ω_1-filtration of R by subrings, $R = \cup_{\alpha<\omega_1} N_\alpha$, has the property that for all α, N_α contains a complete set of representatives of $r_0R/r_\sigma R$ for each $\sigma < \alpha$. For any $\delta \in \lim(\omega_1)$, and any $\langle u_\sigma : \sigma < \delta\rangle$ in $\mathcal{T}^\delta_{J/R}$, let $f = u_0^{-1}$. To show that $\delta \notin E'$, it suffices to show that $u_\sigma f \in N_\delta \pmod{r_\sigma}$ for all $\sigma < \delta$. Now $u_\sigma f = u_\sigma u_0^{-1} \equiv 1 \pmod{r_0}$, since $u_\sigma \equiv u_0 \pmod{r_0}$, by definition of $\mathcal{T}^\delta_{J/R}$. Say $u_\sigma u_0^{-1} - 1 = y \in r_0R$. By the assumption on N_δ, there exists $a \in N_\delta$ such that $y \equiv a \pmod{r_\sigma}$. Then $u_\sigma f = u_\sigma u_0^{-1} = 1 + y \equiv 1 + a \pmod{r_\sigma}$, and $1 + a \in N_\delta$ since N_δ is a subring of R.

(ii) To show that $\Gamma(J/R) = 1$, it suffices to show that for all limit ordinals $\delta < \omega_1$, $R/\cap_{\nu<\delta} r_\nu R$ is not complete. Assuming that it is complete, we shall obtain a contradiction by showing that $r_0R/r_\delta R$ is uncountable. Fix a ladder on δ, i.e., a strictly increasing sequence $\langle \nu_n : n \in \omega\rangle$ whose sup is δ. For each function $\zeta : \omega \to 2$, define $u^\zeta = \langle u^\zeta_\sigma : \sigma < \delta\rangle \in \mathcal{T}^\delta_{J/R}$ as follows: if $\nu_m < \sigma \le \nu_{m+1}$, then

$$u_\sigma = \sum_{i\le m} \zeta(i) r_{\nu_i}.$$

Clearly $u_\sigma \in r_0R$, and if $\tau > \sigma$, where $\nu_k < \tau \le \nu_{k+1}$, then $m \le k$ and

$$u_\tau - u_\sigma = \sum_{i=m+1}^{k} \zeta(i) r_{\nu_i} \in r_{\nu_{m+1}}R \subseteq r_\sigma R.$$

Since $R/\cap_{\nu<\delta} r_\nu R$ is assumed to be complete, for each ζ there is an element $u^\zeta_* \in R$ which represents the limit of $\langle u^\zeta_\sigma : \sigma < \delta\rangle$ in $R/\cap_{\nu<\delta} r_\nu R$. To obtain a contradiction, we need only show that if $\eta \neq \zeta$, then $u^\zeta_* - u^\eta_* \notin r_\delta R$. Without loss of generality there exists m such that $\zeta|m = \eta|m$ and $\zeta(m) = 0$, $\eta(m) = 1$, then

$$u^\eta_* - u^\eta_{\nu_{m+1}} \in r_{\nu_{m+1}}R; \text{ and}$$

$$u^\zeta_* - u^\zeta_{\nu_{m+1}} \in r_{\nu_{m+1}}R;$$

but $u^\eta_{\nu_{m+1}} - u^\zeta_{\nu_{m+1}} = r_{\nu_m} \notin r_{\nu_{m+1}}R$, so $u^\zeta_* - u^\eta_* \notin r_{\nu_{m+1}}R \supseteq r_\delta R$. □

Special Aronszajn trees

This section contains standard material from set theory. (See, for example, [J, §22] or [Dr, Ch. 7, §3].) It is included simply to provide motivation for the notation and proof in the next section.

A *tree* is a partially ordered set $(T, <)$ such that the predecessors of any element are well ordered. An element x of T is said to have *height* α, denoted $\mathrm{ht}(x) = \alpha$, if the order-type of $\{y \in T\colon y < x\}$ is α. The *height* of T is defined to be $\sup\{\mathrm{ht}(x)+1\colon x \in T\}$. If T is a tree, a *branch* of T is a maximal linearly ordered initial subset of T; the *length* of a branch is its order type. If T is a tree, let $T_\alpha = \{y \in T : ht(y) = \alpha\}$. We say that a tree T is a κ-*Aronszajn tree* if T is of height κ, $|T_\alpha| < \kappa$ for every $\alpha < \kappa$, and T has no branch of length κ.

A tree T of height ω_1 is a *special Aronszajn tree* if T_α is countable for all $\alpha < \omega_1$ and for each $\alpha < \omega_1$ there is a function $f_\alpha\colon T_\alpha \to \mathbb{Q}$ such that

($) whenever $x \in T_\alpha$ and $y \in T_\beta$ and $x < y$, then $f_\alpha(x) < f_\beta(y)$.

Notice that a special Aronszajn tree is an ω_1-Aronszajn tree, since an uncountable branch would give rise to an uncountable increasing sequence of rationals.

König's Lemma implies that there is no ω-Aronszajn tree. However, there is an ω_1-Aronszajn tree:

Theorem 5 *There is a special Aronszajn tree.*

Proof. Let ${}^{<\alpha}\omega$ denote the set of all functions from $\{\beta \in \omega_1\colon \beta < \alpha\}$ to ω. We shall construct T_α and f_α by induction on $\alpha < \omega_1$ such that T_α is a countable subset of ${}^{<\alpha}\omega$ and the partial ordering is inclusion, i.e., if $x \in T_\alpha$ and $y \in T_\beta$ then $x < y$ if and only if $\alpha < \beta$ and $y|\alpha = x$. Finally, T will be defined to be $\bigcup_{\alpha<\omega_1} T_\alpha$.

Let $T_0 = \{\emptyset\}$, $f_0(\emptyset) = 0$, $T_1 = {}^{\{0\}}\omega$, and $f_1 : T_1 \to \mathbb{Q}$ be onto $(0, \infty)$. Suppose now that T_α and f_α have been defined for all $\alpha < \delta$ such that for all $\sigma < \rho < \delta$:

($\star$) for any $\epsilon > 0$, and $x \in T_\sigma$ there is $y \in T_\rho$ such that $x < y$ and $f_\rho(y) < f_\sigma(x) + \epsilon$.

There are two cases. In the first case, if δ is a successor ordinal, $\delta = \tau + 1$, let

$$T_\delta = \{x \cup \{(\tau, n)\} : n \in \omega,\ x \in T_\tau\}.$$

Define f_δ so that for every $x \in T_\tau$,

$$\{f_\delta(x \cup \{(\tau, n)\}) : n \in \omega\} = \{r \in \mathbb{Q} : r > f_\tau(x)\}.$$

Clearly ($\star$) continues to hold.

In the second case, δ is a limit ordinal. Choose a ladder $\langle \nu_n : n \in \omega\rangle$ on δ. For each $\sigma < \delta$, $x \in T_\sigma$ and $k > 0$, by inductive hypothesis ($\star$) there exists a sequence $\langle y_n : n \in \omega$ s.t. $\nu_n > \sigma\rangle$ such that $y_n \in T_{\nu_n}$, $x < y_n < y_m$ for $n < m$ and $f_{\nu_n}(y_n) < f_\sigma(x)+(1/k-1/n)$. Let $y[\sigma, x, k] = \bigcup_{n\in\omega} y_n \in {}^{<\delta}\omega$. Let T_δ consist of one such $y[\sigma, x, k]$ for each σ, x, k. Define $f_\delta(y[\sigma, x, k]) = f_\sigma(x) + 1/k$. Then it is clear that ($\star$) still holds. □

Special Uniserial Modules

Definition. Suppose U is a uniserial module of type J/R where $J = \cup_{\sigma<\omega_1} r_\sigma^{-1}R$ as in (*). For each $\sigma > \omega_1$, fix an element a_σ of U such that $\mathrm{Ann}(a_\sigma) = r_\sigma R$ (so that the submodule $a_\sigma R$ of U is isomorphic to $R/r_\sigma R \cong r_\sigma^{-1}R/R$). Let I_σ be the set of all R-module isomorphisms $\varphi: a_\sigma R \to r_\sigma^{-1}R/R$. We say that $\{f_\sigma: \sigma \in \omega_1\}$ is a *special family of functions* for U if for each $\sigma < \omega_1$, $f_\sigma : I_\sigma \to \mathbb{Q}$ such that whenever $\sigma < \rho$ and $\varphi \in I_\rho$ extends $\psi \in I_\sigma$, then $f_\sigma(\psi) < f_\rho(\varphi)$.

Lemma 6 *If U has a special family of functions, then U is non-standard.*

Proof. Suppose there is an isomorphism $\theta: U \to J/R$. Then for every $\sigma < \omega_1$, θ restricts to an isomorphism φ_σ of $a_\sigma R$ onto $r_\sigma^{-1}R/R$. But then $\langle f_\sigma(\varphi_\sigma): \sigma < \omega_1\rangle$ is an uncountable strictly increasing sequence of rationals, a contradiction. □

With this lemma as justification, we will say that U is *explicitly non-standard* if U has a special family of functions.

If the uniserial module uniserial module U, of type J/R, is described up to isomorphism by a family of units, $\{e_\sigma^\rho: \sigma < \rho < \omega_1\}$ as in (†), then it is clear that U is explicitly non-standard if and only if for every $\sigma < \omega_1$, there is a function

$$f_\sigma: (R/r_\sigma R)^* \longrightarrow \mathbb{Q}$$

such that

($$) whenever $\sigma < \rho$ and $c_\sigma, c_\rho \in R^*$ satisfy $c_\rho \equiv c_\sigma e_\sigma^\rho \pmod{r_\sigma}$, then $f_\sigma(c_\sigma) < f_\rho(c_\rho)$.

(Here, and hereafter, we abuse notation and regard f_ρ and f_σ as functions on R^*.)

Note that we have a tree, T, such that $T_\sigma = (R/r_\sigma R)^*$ and the partial ordering is given by:

$$c_\sigma + r_\sigma R < c_\rho + r_\rho R \iff \sigma < \rho \text{ and } c_\rho \equiv c_\sigma e_\sigma^\rho \pmod{r_\sigma}.$$

Assume $\sigma < \rho$. Each c_σ has at least one successor of height ρ, namely $c_\sigma e_\sigma^\rho$, and if $r_\sigma R/r_\rho R$ is countable, then c_σ has only countably many successors of height ρ. For each $c_\rho \in T_\rho$, its unique predecessor in T_σ is $c_\rho (e_\sigma^\rho)^{-1}$. (Here again we abuse notation and write, for example, c_ρ for an element of T_ρ instead of $c_\rho + r_\rho R$.)

Without loss of generality we can assume that the r_σ are such that for all $\sigma < \omega_1$, $r_\sigma R/r_{\sigma+1}R$ is infinite. (Just choose a subsequence of the original r_σ's if necessary.) Thus for all $\sigma < \omega_1$, there is an infinite subset W_σ of R^* such that for all $u \neq v \in W_\sigma$, $u \equiv 1 \pmod{r_\sigma}$ and $u \not\equiv v \pmod{r_{\sigma+1}}$.

THEOREM 7 *If J/R is an essentially countable type, then there is an explicitly non-standard uniserial R-module of type J/R.*

PROOF. We will first give the construction in the case when J/R is strongly countable, and afterward indicate the modifications needed for the general case. Thus T_σ is assumed countable for all $\sigma < \omega_1$.

We will define, by induction on δ, e^τ_σ for $\sigma < \tau < \delta$ as in (†) and, at the same time, the maps $f_\sigma \colon (R/r_\sigma R)^* \to \mathbb{Q}$ for $\sigma < \delta$. We will do this so that ($$) holds and the following condition is satisfied for all $\sigma < \rho < \omega_1$:

($\star\star_{\sigma,\rho}$) for any $\epsilon > 0$, $m \in \omega$, $c^j_\sigma \in T_\sigma$, and $c^j_\rho \in T_\rho$ $(j = 1, \ldots, m)$ such that $c^j_\sigma < c^j_\rho$, there exists $u \in R^*$ such that $u \equiv 1 \pmod{r_\sigma}$ and $f_\rho(uc^j_\rho) < f_\sigma(c^j_\sigma) + \epsilon$ for all $j = 1, \ldots, m$.

Note that the c^j_ρ determine the c^j_σ — $c^j_\sigma \equiv c^j_\rho(e^\rho_\sigma)^{-1} \pmod{r_\sigma}$ — and uc^j_ρ is another successor of c^j_σ of height ρ. For any given $\epsilon > 0$, $\sigma < \rho < \delta$, $m \in \omega$, and $c^j_\rho \in R^*$ $(j = 1, \ldots, m)$, there exist infinitely many u as in ($\star\star_{\sigma,\rho}$), since we can decrease ϵ as much as we like.

Suppose we have defined e^ρ_σ and f_σ for all $\sigma < \rho < \delta$ satisfying the inductive hypotheses. Let $\langle u_n : n \in \omega \rangle$ enumerate representatives of all the elements of $(R/r_\delta R)^*$. Also, let $\langle \theta_q : q \in \omega \rangle$ enumerate all instances of ($\star\star_{\sigma,\delta}$), for all $\sigma < \delta$, with each instance repeated infinitely often. More precisely, we enumerate (with infinite repetition) all tuples of the form

$$\langle \epsilon = \frac{1}{n},\ \sigma,\ c^j_\delta + r_\delta R \colon j = 1, \ldots, m \rangle$$

with $n \in \omega \setminus \{0\}$, $\sigma < \delta$, and $c^j_\delta \in R^*$.

We will define f_δ as the union of a chain of functions $f_{\delta,k}$ into $\mathbb{Q}$, each with a finite domain. When k is even we will concentrate on insuring that the domain of f_δ will be T_δ; and when k is odd, we will work at satisfying the conditions ($\star\star_{\sigma,\delta}$).

Suppose first that $\delta = \tau + 1$ and define $e^\delta_\tau = 1$ and $e^\delta_\sigma = e^\tau_\sigma$ for $\sigma < \tau$. Suppose that $f_{\delta,i}$ has been defined for $i < k$, and assume first that k is even. Let n be minimal such that $u_n \notin \mathrm{dom}(f_{\delta,k-1})$. Let $\mathrm{dom}(f_{\delta,k}) = \mathrm{dom}(f_{\delta,k-1}) \cup \{u_n\}$ and let $f_{\delta,k}(u_n)$ be any rational greater than $f_\tau(u_n(e^\delta_\tau)^{-1})$ $(= f_\tau(u_n))$.

Now suppose k is odd; say $k = 2q+1$. It's easy to see that it's enough to construct f_δ to satisfy ($\star\star_{\tau,\delta}$). So if θ_q is an instance of ($\star\star_{\sigma,\delta}$) for $\sigma < \tau$, let $f_{\delta,k} = f_{\delta,k-1}$. Otherwise, suppose θ_q is the instance of ($\star\star_{\tau,\delta}$) given by

$$\frac{1}{n},\ c^j_\delta + r_\delta R \colon j = 1, \ldots, m.$$

Since W_τ is infinite (see above), there is a unit u such that $u \equiv 1 \pmod{r_\tau}$ and $uc^j_\delta \notin \mathrm{dom}(f_{\delta,k-1})$ for $j = 1, \ldots, m$. Then define $f_{\delta,k}$ to be the extension of $f_{\delta,k-1}$ with domain $= \mathrm{dom}(f_{\delta,k-1}) \cup \{uc^j_\delta : j = 1, ..., m\}$ such that

$$f_{\delta,k}(uc^j_\delta) = f_\tau(c^j_\delta(e^\delta_\tau)^{-1}) + \frac{1}{2n}.$$

Now we consider the case where δ is a limit ordinal. Fix a ladder $\langle \nu_n : n \in \omega \rangle$ on δ. We are going to define units $e^\delta_{\nu_n}$ by induction such that $e^\delta_{\nu_n} \equiv e^\delta_{\nu_m} e^{\nu_m}_{\nu_n} \pmod{r_{\nu_n}}$ whenever $n < m < \omega$. This will easily determine the sequence $\langle e^\delta_\sigma : \sigma < \delta \rangle$ such that for all $\sigma < \tau < \delta$, $e^\delta_\sigma \equiv e^\delta_\tau e^\tau_\sigma \pmod{r_\sigma}$; then (†) will be satisfied for $\langle e^\tau_\sigma : \sigma < \tau \leq \delta \rangle$.

For simplicity of notation, let e_n denote $e^\delta_{\nu_n}$. Suppose we've already defined $f_{\delta,k-1}$ and e_k such that for all $x \in \mathrm{dom}(f_{\delta,k-1})$,

$$f_{\nu_k}(xe_k^{-1}) < f_{\delta,k-1}(x).$$

(Recall that if $x \in T_\delta$, then xe_n^{-1} is the unique predecessor of x in T_{ν_n}.) If k is even, we proceed as in the even case above (when δ is a successor). If $k = 2q + 1$ and θ_q is

$$\langle \frac{1}{n},\ \sigma,\ c^j_\delta + r_\delta R : j = 1, \ldots, m \rangle$$

we can assume — since each instance is repeated infinitely often — that $\sigma < \nu_k$. Thus $e^\delta_\sigma = e^\delta_{\nu_k} e^{\nu_k}_\sigma$ is defined. Note that $c^j_\delta (e^\delta_\sigma)^{-1} \equiv c^j_\delta e_k^{-1} (e^{\nu_k}_\sigma)^{-1} \pmod{r_\sigma}$, so we can apply $(\star\star_{\sigma,\nu_k})$ [with $c^j_{\nu_k} = c^j_\delta e_k^{-1}$] and obtain a unit $w \equiv 1 \pmod{r_\sigma}$ such that for all $j = 1, \ldots, m$

$$f_{\nu_k}(wc^j_\delta e_k^{-1}) < f_\sigma(c^j_\delta(e^\delta_\sigma)^{-1}) + \frac{1}{2n}.$$

Moreover, since there are infinitely many such w, we can choose one so that the elements wc^j_δ ($j = 1, \ldots, m$) do not belong to $\mathrm{dom}(f_{\delta,k-1})$. Let these be the new elements of the domain of $f_{\delta,k}$ and define

$$f_{\delta,k}(wc^j_\delta) = f_\sigma(c^j_\delta(e^\delta_\sigma)^{-1}) + \frac{1}{2n}.$$

Now we will define e_{k+1} (for k odd or even). For each $x \in \mathrm{dom}(f_{\delta,k})$ we have committed ourselves to $f_\delta(x)$ $(= f_{\delta,k}(x))$ and to the predecessor of x in T_{ν_k} $(= xe_k^{-1})$; we need to choose e_{k+1} so that x and its predecessor, xe_{k+1}^{-1}, in $T_{\nu_{k+1}}$ satisfy ($$).

Let $e' = e_k(e^{\nu_{k+1}}_{\nu_k})^{-1}$. The desired element e_{k+1} will have the form ue' for some unit $u \equiv 1 \pmod{r_{\nu_k}}$. Choose $\epsilon' < f_{\delta,k}(x) - f_{\nu_k}(xe_k^{-1})$ for each $x \in \mathrm{dom}(f_{\delta,k})$. Apply $(\star\star_{\nu_k,\nu_{k+1}})$ to this ϵ' and $xe_k^{-1} \in T_{\nu_k}$, $xe'^{-1} \in T_{\nu_{k+1}}$ ($x \in \mathrm{dom}(f_{\delta,k})$). (Note that $xe_k^{-1} < xe'^{-1}$ by choice of e'.) This gives us $v \equiv 1 \pmod{r_{\nu_k}}$ such that for all x

$$f_{\nu_{k+1}}(vxe'^{-1}) < f_{\nu_k}(xe_k^{-1}) + \epsilon' < f_{\delta,k}(x).$$

Then we let $e_{k+1} = v^{-1}e'$, and we have completed the inductive step.

This completes the proof in the strongly countable case. We turn now to the general (essentially countable) case. In this case, R/r_0R may be uncountable; let Z be a complete set of representatives of $(R/r_0R)^*$. Fix $z_0 \in Z$. We first define, by induction on σ, $f_\sigma(c_\sigma)$ — or, more precisely, $f_\sigma(c_\sigma + r_\sigma R)$ — for all $c_\sigma \in R^*$ such that $c_\sigma \equiv z_0 e_0^\sigma \pmod{r_0}$. We do the construction exactly as in the previous strongly countable case; this will work since there are only countably many cosets $c + r_\sigma R$ such that $c \equiv z_0 e_0^\sigma \pmod{r_0}$ since $r_0R/r_\sigma R$ is countable.

Having done this, the e_σ^τ are determined. We claim that there is no family $\{c_\sigma : \sigma < \omega_1\}$ satisfying (††). Indeed, suppose we had such a family. Let $z \in Z$ be such that $c_0 \equiv z \pmod{r_0}$. Then for all $\sigma < \omega_1$, $c_\sigma \equiv z e_0^\sigma \pmod{r_0}$. Hence the family $\{z_0 z^{-1} c_\sigma : \sigma < \omega_1\}$ satisfies (††) and also satisfies $z_0 z^{-1} c_\sigma \equiv z_0 e_0^\sigma \pmod{r_0}$; but this is impossible by construction. □

Consequences

Now we consider some of the general consequences, for the question of the existence of non-standard uniserials, of the results of the previous sections. First of all, we can construct non-standard uniserial modules associated to any residue field of cardinality $\leq \aleph_1$.

Proposition 8 *(i) For any countable field K, there exists a valuation domain R of cardinality $\aleph_1$ with residue field K such that there is an explicitly non-standard uniserial module of type Q/R.*

(ii) For any field K of cardinality $\leq \aleph_1$, there exists a valuation domain R of cardinality $\aleph_1$ with residue field K such that there is a non-standard uniserial module of type Q/R.

Proof. Part (i) is an immediate consequence of Theorem 2 and Theorem 7. Part (ii) follows from (i) in the case of a countable K and from the Osofsky construction in the case of an uncountable K (cf. [O1]; see also [ESh, Theorem 11]). □

The following improves [ESh, Corollary 15], in that it is a theorem of ZFC rather than a consistency result. It shows that the condition $\Gamma'(J/R) > 0$ is not necessary for the existence of a non-standard uniserial of type J/R.

Proposition 9 *There is a valuation domain R of cardinality $\aleph_1$ such that $\Gamma'(Q/R) = 0$ and there is a non-standard uniserial R-module of type Q/R.*

Proof. Let R be such that Q/R is essentially countable (cf. Theorem 2). By Theorem 4(i), $\Gamma'(Q/R) = 0$, but there is a non-standard uniserial R-module of type Q/R by Theorem 7. □

The following sums up some old results which we want to combine with results proved here.

Theorem 10 *Suppose that R is a valuation domain of cardinality $\aleph_1$.*

(i) If CH does not hold and J/R is an essentially uncountable type, then there is a non-standard uniserial R-module of type J/R.

(ii) If CH holds and $\Gamma(J/R) = 1$, then there is a non-standard uniserial R-module of type J/R.

Proof. Part (i) is Theorem 8 of [ESh]. Part (ii) is because the weak diamond principle, $\Phi_{\omega_1}(\omega_1)$, is a consequence of CH (see [DSh]) and this implies that there exists a non-standard uniserial of type J/R when $\Gamma(J/R) = 1$ (see [ESh, Proposition 3] or [FrG]). □

Now we can completely handle the cases when either CH fails, or $\Gamma = 1$.

Theorem 11 *If CH does not hold, then for every valuation domain R of cardinality $\aleph_1$, and every type J/R such that $\Gamma(J/R) \neq 0$, there is a non-standard uniserial R-module of type J/R.*

Proof. Use Theorem 7 for the essentially countable case, and Theorem 10(i) otherwise. □

Remark. This result shows that CH is needed for the independence result in [ESh, Thm. 14].

Theorem 12 *For every valuation domain R of cardinality $\aleph_1$ and every type J/R, if $\Gamma(J/R) = 1$, then there is a non-standard uniserial R-module of type J/R.*

Proof. If CH fails, use the previous theorem. If CH holds, use Theorem 10(ii). □

Remark. Osofsky's original conjecture ([O1, (9), p. 164], restricted to valuation domains of cardinality $\aleph_1$, said — in our notation — that there is a non-standard uniserial R-module of type J/R if and only if $\Gamma(J/R) = 1$. This is now seen to be true assuming $\neg$CH. On the other hand, it cannot be true in this form assuming CH, since CH implies the weak diamond principle for some co-stationary subsets of ω_1 (cf. [EM, VI.1.10]) Indeed, as in the proof of [ESh, Prop. 3] it is possible to construct R with a type J/R where $\Gamma(J/R) = \tilde{S}$ and $\Phi_{\omega_1}(S)$ holds; so there is a non-standard uniserial R-module of type J/R (cf. [ESh, proof of Prop. 3]). On the other hand, to construct such an R

one has to begin with the stationary and co-stationary set S, so such rings will not be "natural", i.e. will not be ones ordinarily met in algebraic contexts.

Recall, from [ESh], that it is in the case when the hypotheses of the previous theorems fail — i.e., when CH holds and $\Gamma(J/R) < 1$ (and non-zero) — that the independence phenomena occur.

Absoluteness

Finally, let us briefly discuss absoluteness. Consider Theorem 11; if CH fails and $\Gamma(J/R) \neq 0$, we always have a non-standard uniserial module of type J/R, but there are two separate constructions involved. In one case, when J/R is essentially countable, we construct an explicitly non-standard uniserial. If the universe of set theory is extended to a larger universe (with the same $\aleph_1$) this module remains non-standard because the special family of functions remains a special family for U in the extension of the universe. In the essentially uncountable case we use the fact that $\Gamma'(J/R) = 1$ ([ESh, Theorem 8]) and construct our non-standard uniserial U as in [ESh, Theorem 7]. In this case too U remains non-standard in an extension of the universe (preserving $\aleph_1$). The reason here is more subtle; relative to a fixed ω_1-filtration of R by subrings, N_α, the e^ρ_σ we construct satisfy the following property for every $\delta \in \lim(\omega_1)$ and every $c \in R^*$:

$$(\#_{c,\delta}) \qquad \forall \langle c_\sigma \colon \sigma < \delta\rangle \in {}^\delta N^*_\delta[\exists \sigma < \delta(\forall t \in R(c - c_\sigma e^\delta_\delta \neq r_\sigma t))]$$

It is a theorem of ZFC that if $(\#_{c,\delta})$ holds for U (defined by the e^ρ_σ) for all c, δ, then U is non-standard. Now $(\#_{c,\delta})$ is, by a coding argument, a Π^1_1 statement (with parameters in the ground model) about ω. Hence, by a theorem of Mostowski (cf. [Dr, Thm. 7.13, p. 160]), it remains true in an extension of the universe, so U remains non-standard.

On the other hand, in the proof of Theorem 12, there is one additional case: when J/R is essentially uncountable and $\Gamma'(J/R) = 0$ (so CH holds). In this case the existence of a non-standard uniserial is proved using the weak diamond principle, which is a consequence of CH. Here the U we construct may not remain non-standard in an extension of the universe. Consider for example that R is constructed as in [ESh, Theorem 14], but with $\Gamma(J/R) = 1$. If $\mathbb{P}$ is the forcing defined in the proof there, then $\mathbb{P}$ is proper, so it preserves $\aleph_1$ and, moreover, in the $\mathbb{P}$-generic extension U is standard. (Of course, in the generic extension we can construct another non-standard module.)

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