ZERO ONE LAWS FOR GRAPHS WITH EDGE PROBABILITIES DECAYING WITH DISTANCE. PART I

SAHARON SHELAH

Abstract. The work prepares the abstract frame for analyzing the following problem. Let G_n be the random graph on $[n] = \{1, \dots, n\}$ with the possible edge $\{i, j\}$ having probability being $p_{|i-j|} = 1/|i-j|^\alpha$, $\alpha \in (0, 1)$ irrational. We prove that the zero one law (for first order logic) holds.

0. Introduction

On 0–1 laws see expository papers e.g., Spencer [Sp]. In Łuczak, Shelah [LuSh 435] the following probabilistic context was investigated. Let $\bar{p} = \langle p_i : i \in \mathbb{N} \rangle$ be a sequence of probabilities, i.e. real numbers in the interval $[0,1]_{\mathbb{R}}$. For each n we draw a graph $G_{n,\bar{p}}$ with set of nodes $[n] \overset{\text{def}}{=} \{1, \dots, n\}$; for this we make the following independent drawing:

- for each (unordered) pair $\{i, j\}$ of numbers from $[n]$ we draw yes/no with probabilities $p_{|i-j|}/\ 1 - p_{|i-j|}$, and let

$$R_n = \{\{i, j\} : i, j \text{ are in } [n] \text{ and we draw yes}\}.$$

We consider R_n a symmetric irreflexive 2-place relation. So we have gotten a random model $\mathcal{M}^0_{n,\bar{p}} = ([n], R_n)$ (i.e. a graph), but we also consider the graph expanded by the successor relation $\mathcal{M}^1_{n,\bar{p}} = ([n], S, R_n)$ where $S = \{(\ell, \ell+1) : \ell \in \mathbb{N}\}$, (more exactly we use $S_n = S \restriction [n]$), and we may also consider the graph expanded by the natural order on the natural numbers $\mathcal{M}^2_{n,\bar{p}} = ([n], <, R_n)$. (Here we will give a little background on this structure below. But the question whether $0 - 1$ law holds is not discussed here). Though we shall start dealing generally with random models, the reader can restrict himself to the case of graphs without losing comprehensibility.

In [LuSh 435] much information was gotten, on when the 0-1 law holds (see Definition 1.1(1)) and when the convergence law holds (see Definition 1.1(2)), depending on conditions such as $\sum_{i \in \mathbb{N}} p_i < \infty$ and $\sum_{i \in \mathbb{N}} i p_i < \infty$.

The sequences $\bar{p}$ considered in [LuSh 435] were allowed to be quite chaotic, and in those circumstances the theorems were shown to be the best possible,

The research partially supported by the United States – Israel Binational Science Foundation; Publication no 467; SAHARON (08.6.27) CHECK THE END..

e.g. counterexamples were gotten by replacing $\bar{p}$ by $\bar{p}'$ where for some fast increasing sequence $\langle i_k : k \in \mathbb{N} \rangle$ we let $p'_j = \begin{cases} p_k & j = i_k \\ 0 & (\forall k) j \neq i_k. \end{cases}$

In [Sh 463] a new version of the 0-1 law was introduced, the very weak zero one law (see 0.1(3), the h variant says that the difference between the probabilities for n and for m_n when $|n - m_n| \leq h(n)$, converges to zero) and it was proved for $\mathcal{M}^2_{n,\bar{p}}$ when $\sum\limits_i p_i < \infty$ (we omit h when $h(n) = 1$, $m_n = n + 1$ and investigate only the very weak 0-1 law). In [Sh 548] the very weak zero one law was proved for models with a random two place function and for ordered graphs; Boppana and Spencer [BoSp] continue this determining the best h for which this holds.

Naturally arise the question what occurs if the p_i's are "well behaved". As in Shelah, Spencer [ShSp 304] this leads to considering $p_i = 1/i^\alpha$ (independently of n). By the results of [LuSh 435], and (essentially) [ShSp 304], the "real" cases are (on the definition of $\mathcal{M}^\ell_{n,\bar{p}}$ see above):

(A) $\mathcal{M}^0_{n,\bar{p}}$ where $p_i = 1/i^\alpha$ for $i > 1$, $\alpha \in (0,1)_\mathbb{R}$ irrational and $p_1 = p_2$
(B) $\mathcal{M}^1_{n,\bar{p}}$ where $p_i = 1/i^\alpha$, $\alpha \in (0,1)_\mathbb{R}$ irrational
(C) $\mathcal{M}^2_{n,\bar{p}}$ where $p_i = 1/i^\alpha$, $\alpha \in (1,2)_\mathbb{R}$

The main aim of this work is to show that in the case **(A)** we have the 0-1 law, also in case **(B)** we prove the convergence law but at present we do not know the answer to problem **(C)** (actually analysis indicates that the problem is whether there is a formula $\varphi(x)$ which holds in $\mathcal{M}^2_n$ for x small enough and fails for $n - x$, x small enough). Here we didn't consider linear order case. For external reasons the work is divided to two parts, the second is [Sh 517]. *Note:* if we let $p_i = 1/i^\alpha$ for $i \geq 1$, surely $\{\ell, \ell+1\}$ is an edge, so it is fine, just case **(A)** becomes similar to case **(B)**. To preserve the distinction between **(A)** and **(B)** we set $p_1 = 1/2^\alpha$ in case **(A)**. This is one of many ways to preserve this distinction; the choice does not matter.

Main and original context

Random graph on $[n]$, with $p_i = 1/i^\alpha$ for $i > 1$ and $p_1 = p_2$; i.e. probability of the edge $\{i,j\}$ is $p_{|i-j|}$ and $\alpha \in (0,1)_\mathbb{R} \setminus \mathbb{Q}$ i.e. is irrational.

But the proofs apply to wider family of cases. We can make a case such that both [ShSp 304] and [LuSh 435] are particular cases: the probability for $\{i,j\}$ being an edge of $\mathcal{M}_n$ for $i,j \in [n]$ is $p^n_{i,j}$. So in [ShSp 304], $p^n_{i,j} = p_n$ and in [LuSh 435], $p^n_{i,j} = p_{|i-j|}$. We can consider $p^n_{i,j} = p^n_{|i-j|}$. We hope to show in another paper that we shall get the same theory as in case **(A)** above in the limit, while simplifying the probabilistic arguments, if we change the context to:

Second context

for $\mathcal{M}_n$ (graph on $\{1,\ldots,n\}$) with probability of $\{i,j\}$ being an edge is $p^n_{i,j} = \frac{1}{n^\alpha} + \frac{1}{2^{|i-j|}}$.

So the probability basically has two parts

1) $(\frac{1}{2^{|i-j|}})$: Depends only on the distance, but decays fast, so the average valency it contributes is bounded.

2) $(\frac{1}{n^\alpha})$: Does not depend on the distance, locally is negligible (i.e. for any particular $\{i,j\}$) but has "large integral". Its contribution for the valency of a node i is on the average "huge" (still $\ll n$).

We can think of this as two kinds of edges. The edges of the sort $n^{-\alpha}$ are as in the paper [ShSp 304]. The other ones still give large probability for some i to have valency with no *a priori* bound (though not compared to n, e.g. $\log n$). In this second context the probability arguments are simpler (getting the same model theory), but we shall not deal with it here.
Note: If we look at all the intervals $[i, i+k)$, and want to get some graph there (i.e. see on H below) and the probability depends only on k (or at least has a lower bound >0 depending only on k), then the chance that for some i we get this graph (by "second kind edges") is ~ 1, essentially this behavior stops where $k \approx (\log(n))^b$ for some appropriate $b>0$ (there is no real need here to calculate it). Now for any graph H on $[k]$ the probability that for a particular $i < [n-k]$ the mapping $\ell \mapsto i+\ell$ embeds H into $\mathcal{M}_n$ is $\geq (\frac{1}{k^\alpha})^{\binom{k}{2}}$ but is $\leq (\frac{1}{(k/3)^\alpha})^{(k/3)^2}$ (exactly

$$\prod_{\{\ell,m\}\in J_1}(\frac{1}{|l-m|^\alpha})\cdot\prod_{\{\ell,m\}\in J_2}(1-\frac{1}{|l-m|^\alpha})\cdot p_1^{|\{\ell:(\ell,\ell+1) \text{ is an edge}\}|}\cdot(1-p_1)^{|\{\ell:(\ell,\ell+1) \text{ is not an edge}\}|}$$

where $\ell, m \leq k$ and $J_1 = \{\ \{\ell,m\}$: (ℓ,m) is an edge and $|\ell-m|>1\}$, $J_2 = \{\ \{\ell,m\}$: (ℓ,m) is not an edge and $|\ell-m|>1\}$). Hence the probability that for no $i < \lfloor n/k \rfloor$ the mapping $\ell \mapsto (k\cdot i+\ell)$ does embed H into $\mathcal{M}_n$ is $\leq \left(1-(\frac{1}{k^\alpha})^{\binom{k}{2}}\right)^{n/k}$. Hence if $\beta k^{\alpha\cdot\binom{k}{2}} = n/k$ that is $\beta = (\frac{n}{k^{\alpha\cdot\binom{k}{2}+1}})$ then this probability is $\leq e^{-\beta}$. This is because $e^{-\beta} \sim \left(1-(\frac{\beta}{n})\right)^n$. We obtain $(\frac{k\beta}{n}) \leq (\frac{1}{k^{\alpha\cdot\binom{k}{2}}})$. So the probability is small, i.e. β large if $k \geq (\frac{2}{\alpha}\log n)^{1/2}$; note that the bound for the other direction has the same order of magnitude. So with parameters, we can interpret, using a sequence of formulas $\bar\varphi$ and parameter $\bar a$, quite long initial segment of the arithmetic (see definition below). This is very unlike [ShSp 304], the irrational case, where first order formula $\varphi(\bar x)$ really says little on $\bar x$: normally it says just that the cl^k–closure of $\bar x$ is $\bar x$ itself or something on the few elements which are in $\mathrm{cl}^k(\bar x)$ (so the first order sentences say not little on the model, but inside a model the first order formula says little). So this sound more like the α rational case of [ShSp 304]. This had seemed like a sure sign of failure of the 0-1 law, but if one goes in this direction one finds it problematic to define $\bar a_0$ such that $\bar\varphi$ with the parameter $\bar a_0$ defines a maximal such initial segment of arithmetic, or at least find $\psi(\bar y)$ such that for random enough $\mathcal{M}_n$, there is $\bar a_0$ satisfying $\psi(\bar y)$ and if $\bar a_0$ satisfies $\psi(\bar y)$ then φ with such parameter define an initial segment of arithmetic of size, say, $> \log\log\log n$. To interpret an initial segment of arithmetic of size k in $\mathcal{M}_n$ for $\bar\varphi$ and $\bar a_0$, mean that $\bar\varphi = \langle \varphi_0(x, \bar y_0), \varphi_1(\bar x^0, \bar y), \varphi_2(\bar x^1, \bar y), \varphi_3(\bar x^2, \bar y)\rangle$ is a sequence of (first order) formulas, and $\bar a_0$ is a sequence of length $\ell g(\bar y)$ such that: the set $\{x : \mathcal{M}_n \models \varphi_0(x, \bar a_0)\}$ has k elements, say $\{b_0, \dots, b_{k-1}\}$, satisfying:

$$\mathcal{M}_n \models \varphi_1(x_0, x_1, \bar a_0) \Leftrightarrow \bigvee_{\ell<m<k} (x_0, x_1) = (b_\ell, b_m),$$

$$\mathcal{M}_n \models \varphi_2(x_0, x_1, x_2, \bar a_0) \Leftrightarrow \bigvee_{\substack{\ell_0,\ell_1,\ell_2<k \\ \ell_2=\ell_0+\ell_1}} (x_0, x_1, x_2) = (b_{\ell_0}, b_{\ell_1}, b_{\ell_2}),$$

$$\mathcal{M}_n \models \varphi_3(x_0, x_1, x_2, \bar a_0) \Leftrightarrow \bigvee_{\substack{\ell_0,\ell_1,\ell_2<k \\ \ell_2=\ell_0\ell_1}} (x_0, x_1, x_2) = (b_{\ell_0}, b_{\ell_1}, b_{\ell_2}).$$

But it is not *a priori* clear whether our first order formulas distinguish between large size and small size in such interpretation.

Note: all this does not show why the $0-1$ law holds, just explain the situation, and show we cannot prove the theory is too nice (as in [ShSp 304]) on the one hand but that this is not sufficient for failure of $0-1$ law on the other hand. Still what we say applies to both contexts, which shows that results are robust. A nice result would be if we can characterize $\langle p_i : i \in \mathbb{N}\rangle$ such that $\mathrm{Prob}\{i, j\} = p_i \Rightarrow 0-1$ holds (see below).

Our idea (to show the $0-1$ law) is that though the "algebraic closure" (suitably defined) is not bounded, it is small and we can show that a first order formula $\varphi(\bar x)$ is equivalent (in the limit case) to one speaking on the algebraic closure of $\bar x$.

Model theoretically we do not get in the limit a first order theory which is stable and generally "low in the stability hierarchy", see Baldwin, Shelah [BlSh 528], for cases with probability $\sim n^{-\alpha}$ (the reason is of course that restricted to "small" formulas in some cases there is a definable linear order (or worse)). However we get a variant of stability over a predicate: on "small" definable sets the theory is complicated, but for types with no small formulas we are in the stable situation. In fact the model theoretic setting is similar to the one in [Sh 463] but we shall not pursue this.

Note that Baldwin, Shelah [BlSh 528] deal with random models with more relations R with probabilities $n^{\alpha(R)}$ (satisfying the parallel to irrationality of α). There, the almost sure theory is stable. In [Sh 550] we define a family of 0-1 contexts where further drawings of relations give us a new context in this family and in all such contexts, elimination of quantifiers to the algebraic closure (as in [ShSp 304], [BlSh 528]) holds, but the context is possibly "almost nice" not nice, i.e. we allow that every $\bar a$ has a nontrivial closure, as in the case in which we have the successor function. Here this is dealt within the general treatment of the elimination, but not used in the main case $\mathcal{M}^0_n$. We could deal with abstract version allowing further drawing as in [Sh 550] also here.

See more [Bl96], [Sh 637].

We have chosen here quite extreme interpretation of "$\bar{p}$ is simple, simply defined". It seems desirable to investigate the problem under more lenient conditions. A natural such family of $\bar{p}$'s is the family of monotonic ones. Can we in this family characterize

$$\{\bar{p} : \bar{p} \text{ monotone sequence, } \mathcal{M}^0_{n,\bar{p}} \text{ satisfies the } 0-1 \text{ law}\}?$$

This will be addressed and solved in [Sh 581].

The two cases considered above are protypes of some families with the 0-1 law, but there are some others, for example with the value of the exponent α "in the appropriate neighbourhood" of a rational (and some degenerate ones of course).

Let us review the paper.

Note: in §1 – §3 we deal with general contexts. In these three sections sufficient conditions are proven for the 0-1 law to hold in 0-1 context; for notational simplicity we restrict ourselves to vocabulary which contains finitely many predicate relations (not only a symmetric irreflexive 2-place relation). The proof is based on elimination of quantifiers by the help of the closure without using probability arguments. Note that in the application we have in mind, the closure has order of magnitude up to $\sim \log|\mathcal{M}_n|$. In [ShSp 304] cl is bounded i.e. $|\text{cl}(A)|$ has a bound depending on $|A|$ (and α of course) only while here is not bounded. In the second part, §4 – §6 deal with $\mathcal{M}^0_n$ and §7 deal with $\mathcal{M}^1_n$.

In §1 we give the basic definitions, including $A <_i B$ (intended to mean: B is the algebraic closure of A but this closure has no *a priori* bound). The restriction to: $\mathcal{M}_n$ has set of elements $[n]$ (rather than some finite set) is not important for the proof. In §1, $A <_i B$ and $A <_s B$ are defined in terms of the number of embeddings of A into $\mathcal{M}_n$ in a sufficiently random model, and from $<_i$ we define $\text{cl}^k(A, M)$.

In §2 a fundamental relation (i.e. given a priori) on structures M is cl^k. From it notions of $A <_i B$ and $A <_s B$ are defined in terms of embeddings $f \subseteq g$ of A, B into a sufficiently random $\mathcal{M}_n$ and the relations between $g(B)$ and $\text{cl}^k_{\mathcal{M}_n}(f(A), \mathcal{M}_n)$. Then these definitions are reconciled with those in §1, when the closure is chosen as in §1. Two axiomatic frameworks for an abstract elimination of quantifiers argument are presented. (This generalizes [BlSh 528].) These frameworks and further conditions on cl^k provide sufficient conditions for $0-1$ laws and convergence laws.

Note: in §2 we retain using "relation free amalgamation" (as in [BlSh 528], but in [Sh 550] we will use more general one). However we waive "random A has no non-trivial closure", hence use "almost nice" rather than nice (and also waive the *a priori* bounds on closure).

In §3 we deal with the case where the natural elimination of quantifiers is to monadic logic. This seems natural, although it is not used later.

We now proceed to describe part II, the main point of §4 is to introduce a notion of weight $\mathbf{w}(A, B, \lambda)$ which depends on an equivalence relation λ

on $B \setminus A$. (Eventually such λ will be defined in terms of the "closeness" of images of points in B under embeddings into $\mathcal{M}_n$.) Relations $A \leq_i^* B$ and $A \leq_s^* B$ are defined in terms of $\mathbf{w}$. The intension is that $\leq_i^*$ is $\leq_i$ etc, thus we will have direct characterization of the later.

§5 contains the major probability estimates. The appropriate λ is defined and thus the interpretations of $<_i^*$ and $<_s^*$ in the first context $(\mathcal{M}_n^0, p_i = \frac{1}{i^\alpha})$. Several proofs are analogous to those in [ShSp 304] and [BlSh 528], so we treat them only briefly. The new point is the dependence on distance, and hence the equivalence relations λ.

In §6 it is shown that the $<_i^*$ and $<_s^*$ of §5 agree with the $<_i$ and $<_s$ of §1. Further, if cl^k is defined from the weight function in §4, these agree with $<_i$, $<_s$ as in §2 and we prove the "simple almost niceness" of Definition 2.12, so the 'elimination of quantifiers modulo quantification on (our) algebraic closure' result applies. This completes the proof of the $0-1$ law for the first context. The model theoretic considerations in the proof of this version of niceness (e.g. the compactness) were less easy than I expect.

§7 deals with the changes needed for $\mathcal{M}^1_{n,\bar{p}}$ where only the convergence law is proved.

Note: our choice "$\mathcal{M}_n$ has set of element $[n]$" is just for simplicity (and tradition), we could have $\mathcal{M}_n$ has set of elements a finite set (not even fixed) and replace n^ε by $\|\mathcal{M}_n\|^\varepsilon$ as long as "for each k for every random enough $\mathcal{M}_n$ we have $\|\mathcal{M}_n\| > k$". Also the choice of n^ε in Definition 1.2 is the most natural but not unique case. The paper is essentially self contained, assuming only basic knowledge of first order logics and probability.

Notation 0.1.
- $\mathbb{N}$ is the set of natural numbers $(\{0, 1, 2, \dots\})$
- $\mathbb{R}$ is the set of reals
- $\mathbb{Q}$ is the set of rationals
- $i, j, k, \ell, m, n, r, s, t$ are natural numbers and
- p, q are probabilities
- α, β, γ, δ are reals
- ε, ζ, ξ are positive reals (usually quite small) and also $\mathbf{c}$ (for constant in inequalities)
- λ is an equivalence relation
- M, N, A, B, C, D are graphs or more generally models (that is structures, finite of fixed finite vocabulary, for notational simplicity with predicates only, if not said otherwise; the reader can restrict himself to graphs)
- $|M|$ is the set of nodes or elements of M, so $\|M\|$ is the number of elements.
- $\mathcal{M}$ denotes a random model,
- μ denotes a distribution (in the probability sense),
- $[n]$ is $\{1, \dots, n\}$
- $A \subseteq B$ means A is a submodel of B i.e. A is B restricted to the set of elements of A (for graphs: induced subgraph)

We shall not always distinguish strictly between a model and its set of elements. If X is a set of elements of M, $M \restriction X$ is M restricted to X.

- a, b, c, d are nodes of graphs / elements of models
- $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are finite sequences of nodes / elements
- x, y, z variables
- $\bar{x}$, $\bar{y}$, $\bar{z}$ are finite sequences of variables
- X, Y, Z are sets of elements
- τ is a vocabulary for simplicity with predicates only (we may restrict a predicate to being symmetric and/or irreflexive (as for graphs)),
- $\mathcal{K}$ is a family of models of fixed vocabulary, usually $\tau = \tau_{\mathcal{K}}$
- the vocabulary of a model M is τ_M,
- $\bar{a}\hat{\ }\bar{b}$ or $\bar{a}\bar{b}$ is the concatenation of the two sequences, $\bar{a}\hat{\ }b$ or $\bar{a}b$ is $\bar{a}\hat{\ }\langle b\rangle$
- the extensions g_1, g_2 of f are disjoint if $x_1 \in \mathrm{dom}(g_\ell) \setminus \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g_{3-\ell}) \Rightarrow x_1 \neq x_2$.

Acknowledgements: We thank John Baldwin and Shmuel Lifsches and Çiğdem Gencer and Alon Siton for helping in various ways and stages to make the paper more user friendly.

1. Weakly nice classes

We interpret here "few" by: "for each ε for every random enough $\mathcal{M}_n$, there are (for each parameter) $< n^{\varepsilon}$". We could use other functions as well.

General Context 1.1. (i) Let τ be fixed vocabulary which for simplicity having only predicates, i.e. symbols for relations.

(ii) $\mathcal{K}$ be a class of finite τ-models closed under isomorphism and submodels. For $n \in \mathbb{N}$, $\mathcal{K}_n$ is a set of τ-models which usually have universe $[n] = \{1, ..., n\}$ (just for notational simplicity).

(iii) Let $\mathcal{M}_n$ be a random model in a fixed vocabulary τ which is an element of $\mathcal{K}_n$, that is we have μ_n a function such that $\mu_n : \mathcal{K}_n \to [0,1]_{\mathbb{R}}$ and $\sum\{\mu_n(\mathcal{M}) : \mathcal{M} \in \mathcal{K}_n\} = 1$, so μ_n is called a distribution and $\mathcal{M}_n$ the random model for μ_n, so we restrict ourselves to finite or countable $\mathcal{K}_n$. We omit μ_n when clear from the context.

(iv) We call $(\mathcal{K}, \langle(\mathcal{K}_n, \mu_n) : n < \omega\rangle)$ a $0-1$ context and denote it by $\mathfrak{K}$ and usually consider it fixed; we may 'forget' to mention $\mathcal{K}$. So

(v) The probability of $\mathcal{M}_n \models \varphi$; $Prob(\mathcal{M}_n \models \varphi)$ is

$$\sum\{\mu_n(\mathcal{M}) : \mathcal{M} \in \mathcal{K}_n, \mathcal{M} \models \varphi\}.$$

(vi) The meaning of "for every random enough $\mathcal{M}_n$ we have Ψ" is

$$\langle \mathrm{Prob}(\mathcal{M}_n \models \Psi) : n < \omega\rangle \text{ converges to } 1;$$

alternatively, we may write "almost surely $\mathcal{M}_n \models \Psi$".

(vii) We call $\mathfrak{K}$ a $0-1$ context if it is as above.

Definition 1.2. (1) The $0-1$ law (for $\mathfrak{K}$) says: whenever φ is a f.o. (=first order) sentence in vocabulary τ,

$$\langle \mathrm{Prob}(M_n \models \varphi) : n < \omega \rangle n < \omega \rangle \text{ converges to 0 or to 1.}$$

(2) The convergence law says: whenever φ is a f.o. sentence in τ,

$$\langle \mathrm{Prob}(\mathcal{M}_n \models \varphi) : n < \omega \rangle \text{ is a convergent sequence.}$$

(3) The very weak $0-1$ law says: whenever φ is a f.o. sentence in τ,

$$\lim_n [\mathrm{Prob}(\mathcal{M}_{n+1} \models \varphi) - \mathrm{Prob}(\mathcal{M}_n \models \varphi)] = 0.$$

(4) The h-very weak $0-1$ law for $h : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ says: whenever φ is a f.o. sentence in τ,

$$0 = \lim_n \max_{\ell,k \in [0,h(n)]} |\mathrm{Prob}(\mathcal{M}_{n+k} \models \varphi) - \mathrm{Prob}(\mathcal{M}_{n+\ell} \models \varphi)|$$

Notation 1.3. $f : A \hookrightarrow B$ means: f is an embedding of A into B (in the model theoretic sense, for graphs: isomorphism onto the induced subgraph).

Definition 1.4. (1) Let

$$\mathcal{K}_\infty = \Big\{ A : \quad A \text{ is a finite } \tau\text{-model} \quad 0 < \lim_n \sup[\mathrm{Prob}((\exists f)(f : A \hookrightarrow \mathcal{M}_n))] \Big\}$$

recall (1.1(v)) that $\mathrm{Prob}((\exists f)(f : A \hookrightarrow \mathcal{M}_n)) = \sum\{\mu_n(\mathcal{M}_n) : \mathcal{M}_n \in \mathcal{K}_n$ and there is an embedding $f : A \hookrightarrow \mathcal{M}_n\}$, $n < \omega$.

Also let $T_\infty =^{\mathrm{df}} \{\varphi : \varphi$ is a f.o. sentence in the vocabulary of $\mathcal{K}$ such that every random enough $\mathcal{M}_n$ satisfies it$\}$.

(2) $A \le B$ means: $A, B \in \mathcal{K}_\infty$ and A is a submodel of B; of course $A < B$ means $A \le B$ and $A \ne B$, similarly for others below.

(3) $A \le_i B$ means: $A \le B$ and for each $\varepsilon \in \mathbb{R}^+$ we have:

$$1 = \lim_n \left[\mathrm{Prob} \left(\begin{array}{l} \text{if } f_0 : A \hookrightarrow \mathcal{M}_n \\ \text{then the number of } f_1 \text{ satisfying} \\ f_0 \subseteq f_1 : B \hookrightarrow \mathcal{M}_n \text{ is } \le n^\varepsilon. \end{array} \right) \right]$$

Also let $\mathrm{ex}(f_0, B, M) = \mathrm{ex}(f_0, A, B, M) =^{\mathrm{df}} \{f : f$ is an embedding of B into M extending $f_0\}$.

(4) $A \le_s B$ means: $A \le B$ and there is no C with $A <_i C \le B$.

(5) $A <_{pr} B$ means: $A <_s B$ and there is no C with $A <_s C <_s B$ (*pr* abbreviates *primitive*).

(6) $A <_a B$ means that $A \le B$ and, for every $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathcal{M}_n$, for no $f : A \hookrightarrow \mathcal{M}_n$ do we have n^ε pairwise disjoint extensions g of f satisfying $g : B \hookrightarrow \mathcal{M}_n$.

(7) $A \le^s_m B$ means $A \subseteq B$ are from $\mathcal{K}$ and for every $X \subseteq B$ with $\le m$ elements, we have $A \restriction (A \cap X) \le_s (B \restriction X)$.

(8) $A \le^i_{k,m} B$ means $A \subseteq B$ are from $\mathcal{K}$ and for every $X \subseteq B$ with $\le k$ elements there is Y, $X \subseteq Y \subseteq B$ with $\le m$ elements such that $A \restriction (A \cap Y) \le_i (B \restriction Y)$.

(9) For $h : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$, we define $A \le_i^h B$ as in part (3) replacing n^ε by $h(n,\varepsilon)$, and similarly $A \le_a^h B$ (in part (6)), hence $A \le_s^h B$, $A \le_{pr}^h B$, $A <_a^h B$, $A \le_m^{s,h} B$, $A \le_{k,m}^{i,h} B$.

Remark 1.5. (1) In these circumstances the original notion of algebraic closure is not well behaved. $A \le_i B$ provides a reasonable substitute for $A \subseteq B \subseteq \mathrm{acl}(A)$.

(2) Note: for $\le_i^h$ to be transitive we need: for every $\varepsilon_1 > 0$ for some $\varepsilon_2 > 0$ for every n large enough $h(n,\varepsilon_2) \times h(n,\varepsilon_2) \le h(n,\varepsilon_1)$.

(3) Why do we restrict ourselves to $\mathcal{K}_\infty$ (in 1.4(1)-(6))? The relations in 1.4(1)-(6) describe situation in the limit. So why in 1.4(7), (8) do we not restrict ourselves to $A, B \in \mathcal{K}_\infty$? As for $A \in \mathcal{K}_\infty$, for quite random $\mathcal{M}_n$, and $f : A \hookrightarrow \mathcal{M}_n$ the set $\mathrm{cl}^k(f(A), \mathcal{M}_n)$ may be quite large, say with $\log(n)$ elements, so it (more exactly the restriction of $\mathcal{M}_n$ to it) is not necessarily in $\mathcal{K}_\infty$; this is a major point here.

Let us expand.
If $A \in \mathcal{K}$ has a copy in a random enough $\mathcal{M}_n$ and we have $0-1$ law then T_∞(see 1.4(1)) says that copies of A occur. But if $\mathcal{M}_n$ is random enough, and for example $A = \{a_1, a_2, a_3\} \le \mathcal{M}_n$, and $B = \mathcal{M}_n \restriction \mathrm{cl}^k(\{a_1,a_2,a_3\}, \mathcal{M}_n)$ has, say, $\log(n)$ elements *then* it does not follow that $T_\infty \models$"a copy of B occurs", as $\mathcal{M}_n$ may not be random enough for B. Still for the statements like

$$(\exists x_1, x_2, x_3)(\mathrm{cl}^k(\{x_1,x_2,x_3\}) \models \varphi)$$

the model $\mathcal{M}_n$ may be random enough. The point is that the size of B could be computed only after we have $\mathcal{M}_n$.
Another way to look at it: models M_∞ of T_∞ are very random in a sense, but $\mathrm{cl}(\{a_1,a_2,a_3\}, M_\infty)$ is infinite, may even be uncountable, so randomness concerning it becomes meaningless.

Definition 1.6. For $A \subseteq M$ and $k < \omega$ define

(a) $\mathrm{cl}^k(A, M) = \bigcup\{B : B \subseteq M,\ B \cap A \le_i B, \text{ and } |B| \le k\}$,
(b) $\mathrm{cl}^{k,0}(A, M) = A$,
(c) $\mathrm{cl}^{k,m+1}(A, M) = \mathrm{cl}^k(\mathrm{cl}^{k,m}(A, M), M)$.

Observation 1.7. *1) For all $\ell, k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$ we have*

$$1 = \lim_n \big[\mathrm{Prob}\big(A \subseteq \mathcal{M}_n, |A| \le \ell \Rightarrow |\mathrm{cl}^k(A, \mathcal{M}_n)| < n^\varepsilon\big)\big].$$

2) Moreover, for every $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$ for some $\zeta \in \mathbb{R}^+$ (actually, any $\zeta < \varepsilon/(k+1)$ will do) we have

$$1 = \lim_n \big[\mathrm{Prob}(A \subseteq \mathcal{M}_n, |A| \le n^\zeta \Rightarrow |\mathrm{cl}^k(A, \mathcal{M}_n)| < n^\varepsilon)\big].$$

Remark 1.8. 1.7 is true for $\mathrm{cl}^{k,m}$ too, but we can use claim 1.16 instead.

Definition 1.9. $\mathfrak{K} = \langle \mathcal{M}_n : n < \omega \rangle$ is *weakly nice* if whenever $A <_s B$ (so $A \neq B$), there is $\varepsilon \in \mathbb{R}^+$ with

$$1 = \lim_n \left[\text{Prob} \begin{pmatrix} \text{if } f_0 : A \hookrightarrow \mathcal{M}_n \text{ then there is } F \text{ with } |F| \geq n^\varepsilon \text{ and} \\ \text{(i) } f \in F \Rightarrow f_0 \subseteq f : B \hookrightarrow \mathcal{M}_n \\ \text{(ii) } f' \neq f'' \in F \Rightarrow \text{Rang}(f') \cap \text{Rang}(f'') = \text{Rang}(f_0) \end{pmatrix} \right].$$

If clause (ii) holds we say the $f \in F$ are pairwise disjoint over f_0 or over A. In such circumstances we say that ε witnesses $A <_s B$.

Remark 1.10. Being weakly nice means there is a gap between being pseudo algebraic and non-pseudo algebraic (both in our sense), so we have a strong dichotomy.

Fact 1.11. *For every A, B, C in $\mathcal{K}_\infty$:*

(1) $A \leq_i A$,
(2) $A \leq_i B,\ B \leq_i C \Rightarrow A \leq_i C$,
(3) $A \leq_s A$,
(4) *if* $A_1 \leq B_1,\ A_2 \leq B_2,\ A_1 \leq A_2,\ B_1 \leq B_2,\ B_1 \setminus A_1 = B_2 \setminus A_2$ *then* $A_2 \leq_s B_2 \Rightarrow A_1 \leq_s B_1$ *and* $A_1 \leq_i B_1 \Rightarrow A_2 \leq_i B_2$,
(5) $A <_i B$ *iff for every* C *we have* $A \leq C < B \ \Rightarrow\ C <_a B$.

PROOF Easy (e.g. 1.11(5) by the Δ-system argument (for fixed size of the sets and many of them); note $|B|$ is constant). $\blacksquare_{1.11}$

Claim 1.12. *If $A <_s B <_s C$ then $A <_s C$*

PROOF *First proof:*

If not, then for some B' we have $A <_i B' \leq C$. If $B' \subseteq B$ we get contradiction to $A <_s B$, so assume $B' \nsubseteq B$. By 1.13(1) below we have $(B' \cap B) <_i B'$ so by 1.11(4) we have $B <_i (B \cup B')$, hence we get contradiction to $B <_s C$.

Second proof: (Assuming $\mathfrak{K}$ is weakly nice i.e. if we define $<_s$ by 1.9.) Let $\varepsilon > 0$ witness $A <_s B$ in Definition 1.9 and let $\zeta > 0$ witness $B <_s C$ in Definition 1.9. Choose $\xi = \min\{\varepsilon/2, \zeta/2\}$; (actually just $\xi < \varepsilon \wedge \xi < \zeta$ suffice). Let n be large enough; in particular $n^\varepsilon > |C|$ and let $f_0 : A \hookrightarrow \mathcal{M}_n$. So we have (almost surely) $\{f_1^i : i < i^*\}$, where $i^* \geq n^\varepsilon$, and $f_0 \subseteq f_1^i$ and $f_1^i : B \hookrightarrow \mathcal{M}_n$ and the f_1^i's are pairwise disjoint over A.

Now, almost surely for every i we have $\{f_2^{i,j} : j < j_i^*\}$ with $f_1^i \subseteq f_2^{i,j}$ and $f_2^{i,j} : C \hookrightarrow \mathcal{M}_n$ and, fixing i, the $f_2^{i,j}$'s are pairwise disjoint over B and $j_i^* \geq n^\zeta$.

Clearly (when the above holds) for $\ell^* = n^\xi$ we can find $\{j_k : k \leq \ell^*\}$ such that $\{f_2^{k,j_k} : k < \ell^*\}$ are pairwise disjoint over A (just choose j_k by induction on k such that: $\text{Rang}(f_2^{k,j_k} \restriction (C \setminus B))$ is disjoint to

$$\bigcup \{\text{Rang}(f_1^i \restriction (B \setminus A)) : i < \ell^*\} \cup \bigcup \{\text{Rang}(f_2^{i,j_i} \restriction (C \setminus B)) : i < k\};$$

at stage k, the number of inappropriate $j < n^\zeta$ is

$$\leq |C \setminus B| \times k + |B \setminus A| \times \ell^* \leq |C| \times \ell^* = |C| \times n^\xi).$$

$\blacksquare_{1.12}$

Fact 1.13. *Suppose $A \leq B \leq C$.*

(1) *If $A \leq_i C$* then *$B \leq_i C$.*
(2) *If $A \leq_s C$* then *$A \leq_s B$.*
(3) *If $A <_{pr} C$ and $A \leq_s B \leq_s C$* then *either $B = A$ or $B = C$.*

Proof Reread the definitions.

Fact 1.14. (1) *If $A \leq_s B$* then *there is some $n < \omega$ and a sequence $\langle A_l : l \leq n \rangle$ such that $A = A_0 <_{pr} A_1 <_{pr} \ldots <_{pr} A_n = B$ (possibly $n = 0$).*
(2) *If $A <_{pr} C$ and $A < B < C$ then $B <_i C$*

Proof For proving (2), choose a maximal B' such that $B \leq_i B' \leq C$, it exists as C is finite (being in $\mathcal{K}_\infty$), and as $B \leq_i B$ (by 1.11(1)). It follows that if $B' < B'' \leq C$ then $\neg B' \leq_i B''$ (by 1.11(2)). Hence $B' \leq_s C$. But $A <_{pr} C$ hence by the Definition 1.4(5) we have $A <_s C$ so by 1.13(2) $A <_s B'$; so by the definition of $<_{pr}$ we have $B' = C$, so $B \leq_i B' = C$ as required. Part (1) is clear as B is finite (being in $\mathcal{K}_\infty$) and the definition of $\leq_{pr}$. $\blacksquare_{1.14}$

Claim 1.15. *$\mathcal{K}$ is weakly nice iff whenever $A <_{pr} B$ there is $\varepsilon \in \mathbb{R}^+$ such that*

$$1 = \lim_n \left[\text{Prob} \begin{pmatrix} \textit{if } f_0 : A \hookrightarrow \mathcal{M}_n \textit{ then there is } F \textit{ with } |F| \geq n^\varepsilon \textit{ and} \\ f_1 \in F \;\Rightarrow\; f_0 \subseteq f_1 : B \hookrightarrow \mathcal{M}_n \end{pmatrix} \right]$$

Proof $\Rightarrow$ is obvious (as $A <_{pr} B$ implies $A <_s B$).

Let us prove $\Leftarrow$: we have $A \leq_s B$ and by fact 1.14(1) there is a sequence $A = A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k = B$. The proof is by induction on k. The induction step for $k > 1$ is by the second proof of 1.12 and $k = 0$ is 1.11(3). So assume $k = 1$, hence $A <_{pr} B$. By fact 1.14(2) if $A < B' \leq B$ then $B' \leq_i B$. Fix $p \in (0,1)_\mathbb{R}$. If n is large enough then the probability of having both

(a) for every $f_0 : A \hookrightarrow \mathcal{M}_n$ there are at least n^ε different extensions f_1^i satisfying $f_0 \subseteq f_1^i : B \hookrightarrow \mathcal{M}_n$ and
(b) for every $a \in B \setminus A$ and $f_0^+ : A \cup \{a\} \hookrightarrow \mathcal{M}_n$ there are at most $n^{\varepsilon/2}$ different extensions f_2^i satisfying $f_0^+ \subseteq f_2^i : B \hookrightarrow \mathcal{M}_n$

is $\geq 1 - p$ (for clause (b) use $A \cup \{a\} <_i B$ for every $a \in B \setminus A$ which holds by 1.14(2)). Let $f_0 : A \hookrightarrow \mathcal{M}_n$, and let $\langle f_1^j : j < j^* \rangle$ be a maximal family of pairwise disjoint extensions of f_0 to an embedding of B into $\mathcal{M}_n$. Let $F = \{f : f$ is an embedding of B into $\mathcal{M}_n$ extending $f_0\}$. By (b) we have

$$n^\varepsilon \leq |F| \leq j^* \times |B \setminus A| \times |B \setminus A| \times n^{\varepsilon/2}.$$

Hence if n is large enough, $j^* > n^{\varepsilon/3}$ (with probability $\geq 1-p$), and this is enough. $\blacksquare_{1.15}$

Claim 1.16. $\mathrm{cl}^{k,m}(A,M) \subseteq \mathrm{cl}^{k^*}(A,M)$ *where* $k^* = k^m$.

Proof For $\ell \leq m$ let $k(\ell) = k^\ell$. For $\ell \leq m$ define $A_\ell = \mathrm{cl}^{k,\ell}(A,M)$. Now if $x \in A_m$ then there is some $\ell < m$ such that $x \in A_{\ell+1} \setminus A_\ell$. Let us prove by induction on $\ell \leq m$ that $x \in A_\ell \Rightarrow x \in \mathrm{cl}^{k(\ell)}(A,M)$. For $\ell = 0$ and $\ell = 1$ this is clear. If $x \in A_{\ell+1} \setminus A_\ell$ then there is C with $|C| \leq k$ such that $x \in C$ and $C \cap A_\ell <_i C$. By the induction hypothesis, for $y \in C \cap A_\ell$ we have $y \in \mathrm{cl}^{k(\ell)}(A,M)$ hence there is C_y with $|C_y| \leq k(\ell)$ such that $y \in C_y$ and $C_y \cap A <_i C_y$. Let $C^0 = \bigcup_{y \in C \cap A_\ell} C_y \cap A$, $C^1 = \bigcup_{y \in C \cap A_\ell} C_y$ and $C^2 = C^1 \cup C$. As $|C| \leq k$, we get

$$|C^2| \leq k(\ell) \cdot |C \cap A_\ell| + |C \setminus A_\ell| \leq k(\ell) \cdot k \leq k(\ell+1),$$

so (as $x \in C^2$) it suffices to show that $C^0 \leq_i C^2$ and by transitivity (i.e. by 1.11(2)) it suffices to show that $C^0 \leq_i C^1$ and that $C^1 \leq_i C^2$. Why $C^1 \leq_i C^2$? Because $C \cap A_\ell \leq_i C$ and $C \cap A_\ell \subseteq C^1 \subseteq A_\ell$ and hence $C^1 \leq_i C^1 \cup C = C^2$ by 1.11(4). Why $C^0 \leq_i C^1$? Let $C \cap A_\ell = \{y_s : s < r\}$. Now $C^0 \leq_i C^0 \cup C_{y_0}$ by 1.11(4) because $A \cap C_{y_0} \leq_i C_{y_0}$ and $A \cap C_{y_0} \subseteq C^0$ and similarly by induction

$$C^0 \leq_i C^0 \cup C_{y_0} \leq_i C^0 \cup C_{y_0} \cup C_{y_1} \leq_i \ldots \leq_i C^0 \cup \bigcup_{s<r} C_{y_s} = C^1.$$

So as $\leq_i$ is transitive (1.11(2)) we are done. $\blacksquare_{1.16}$

Claim 1.17. *For every* $\varepsilon \in \mathbb{R}^+$ *and* ℓ, k, m *we have*

$$1 = \lim_n \left[\mathrm{Prob} \begin{pmatrix} \textit{if } A \in \mathcal{K}_\infty, |A| \leq \ell \textit{ and } f : A \hookrightarrow \mathcal{M}_n \\ \textit{then } |\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)| < n^\varepsilon \end{pmatrix} \right].$$

Proof By the previous claim 1.16, w.l.o.g. $m = 1$. This holds by Definition 1.4(3) and Definition 1.6. $\blacksquare_{1.17}$

Fact 1.18. (1) *For every* A *and* m, k, *for any* $M \in \mathcal{K}$ *if* $f : A \hookrightarrow M$ *then*

(α) $\mathrm{cl}^{k,m}(f(A), M) \leq^i_{1,k} \mathrm{cl}^{k,m+1}(f(A), M)$,

(β) *for some* $m' = m'(k,m)$ *we have*

$$f(A) \leq^i_{k,m'} \mathrm{cl}^{k,m}(f(A), M)$$

(we can get more),

(γ) $f(A) \leq_i \mathrm{cl}^{k,m}(f(A), M_n)$ *or the second is not in* $\mathcal{K}_\infty$.

(2) *For every* m, k, ℓ *for some* r *we have:*
for any $A \in \mathcal{K}_\infty$,

$$1 = \lim_n \left[\mathrm{Prob} \left(\textit{if } f : A \hookrightarrow \mathcal{M}_n \textit{ then } f(A) \leq^i_{\ell,r} \mathrm{cl}^{k,m}(f(A), \mathcal{M}_n) \right) \right].$$

Remark 1.19. In our main case $\mathcal{K} = \mathcal{K}_\infty$.
Recall for 1.18(1)(γ) that $\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)$ is in general not necessarily in $\mathcal{K}_\infty$.

Proof 1) We leave the proof of (α) and (β) to the reader. For proving clause (γ), let $A_0 = f(A)$ and for $\ell \le m$ let $A_\ell = \mathrm{cl}^{k,\ell}(f(A), M)$, and assume $A_m \in \mathcal{K}_\infty$. So for $\ell < m$ we have $A_{\ell+1} = A_\ell \cup \bigcup_{j<m_\ell} C_{\ell,j}$ with $|C_{\ell,j}| \le k$ and $A_{\ell+1} \cap C_{\ell,j} \le_i C_{\ell,j}$. It follows by 1.11(4) that $\langle A_\ell \cup \bigcup_{i<j} C_{\ell,i} : j \le m_\ell \rangle$ is $\le_i$-increasing and $A_\ell \le_i A_{\ell+1}$. By induction we get $A_0 \le_i A_m$ which is the desired conclusion.
2) Read the proofs of 1.18(1) + 1.16. $\blacksquare_{1.18}$

Remark 1.20. In a more general context the previous conclusion is part of the definition of "$\mathcal{K}$ is nice" and also $\biguplus$ of 1.23 below is a basic property (on the later see [Sh 550]).

Fact 1.21. *$\mathcal{K}_\infty$ is closed under isomorphisms and taking submodels.*

Fact 1.22. *For every ℓ, k, m there is a first order formula $\varphi(y, x_0, \ldots, x_{\ell-1})$ such that for every $M \in \mathcal{K}$ and $b, a_0, \ldots, a_{\ell-1}$ in M*

$$M \models \varphi(b, a_0, \cdots, a_{\ell-1}) \quad \text{iff} \quad b \in \mathrm{cl}^{k,m}(\{a_0, \cdots, a_{\ell-1}\}, M).$$

Proof By finiteness of τ (as $\tau_{\mathcal{K}}$ is having no function symbols); or see proof of clause (β) of 2.6.

Definition 1.23. $C_1 \biguplus_B^D C_2$ means: they are all submodels of $D \in \mathcal{K}$, and $C_1 \cap C_2 \subseteq B$ and for every relation symbol R in τ, if $\bar{a} \subseteq C_1 \cup B \cup C_2$ and $R(\bar{a})$ holds then $\bar{a} \subseteq C_1 \cup B$ or $\bar{a} \subseteq C_2 \cup B$ (possibly both).

When D is clear from the context we may omit it.

2. Abstract Closure Context

Here we are inside the 0-1-context but without the $\le_i$ and $\le_s$ as defined in §1, however cl^k is given. The main result is a sufficient condition for having 0-1 law or at least convergence. We have here some amount of freedom, so we give two variants of the main result of this section: 2.16, 2.17, we shall use 2.17. Thus on a reading one may skip Definitions 2.8 ("possible"), 2.9 and 2.10, Remark 2.11 and Lemma 2.16 in favour of the alternative development in Definitions 2.12, 2.13 and 2.17. Lemma 2.15 is needed in both cases and we have made the two independent at the price of some repetition. We want to "eliminate quantifiers" in a restricted sense: in the simple form we quantify only on the closure so each $\varphi(\bar{x})$ is equivalent to some ψ_φ in which quantifiers are over $\mathrm{cl}^{k,m}(\bar{x})$; all this is for a random enough model where $\mathrm{cl}^{k,m}$ is "small", still it is not necessarily "tiny". The closure does not need to be in $\mathcal{K}_\infty$ (though in our application it is). The quantifier elimination result generalizes the result of [BlSh 528]. The chief additional ingredient in the proof here is the use of the addition (= Feferman–Vaught) theorem to analyze a pair of models in stable amalgamation; this is necessary as we do not have *a priori* bound on the size of the closure, whereas there we have. Moreover, the argument in [BlSh 528] is simpler because $<_i$ is defined concretely from a dimension function and moreover it deals with the "nice" rather than almost nice case.

Note that the "simply*" version (2.20 – 2.24) is used in §7.

Note that in this section we can forget about the probability distribution: just deal with elimination of quantifiers. Note that the assumption "cl is f.o. definable" (2.2 clause (d)) is not serious: if it fails all we have to do is to allow "$y \in \mathrm{cl}^k(\bar{x})$" as atomic formulas in ψ_φ.

Context 2.1. In this context in addition to $\mathfrak{K}$ (defined in 1.1) we have an additional basic operation cl which is a closure operation for $\mathcal{K}$ (see 2.2), so cl is in general not defined by Definition 1.6 and $\leq_i$, $\leq_s$, $\leq_a$ are defined by Definition 2.5 and in general are not the ones defined in Definition 1.4. However, we use $\mathcal{K}_\infty$ (from 1.4(1)). Lastly $\biguplus$ is as in 1.23 (can be axiomatized too and moreover generalize to the case of non–uniqueness, as in [Sh 550]). For simplicity assume $\tau_{\mathcal{K}}$ (the vocabulary of $\mathcal{K}$) is finite with no function symbols. In later sections (§4 – §7 but not §3) saying $\mathfrak{K}$ means cl is from §1.

Definition 2.2. 1) We say cl is a *closure operation* for $\mathcal{K}$ if for $M \in \mathcal{K}$ and $k \in \mathbb{N}$ the operation $\mathrm{cl}^k(X, M)$ is defined if and only if $X \subseteq M$ and the operation satisfies:

(a) $X \subseteq \mathrm{cl}^k(X, M) \subseteq M$, and $X \subseteq Y \subseteq M \Rightarrow \mathrm{cl}^k(X, M) \subseteq \mathrm{cl}^k(Y, M) \subseteq M$,

(b) (i) if $\mathrm{cl}^k(X, M) \subseteq N \subseteq M$ then $\mathrm{cl}^k(X, N) = \mathrm{cl}^k(X, M)$,

(ii) If $X \subseteq N \subseteq M$ then $\mathrm{cl}^k(X, N) \subseteq \mathrm{cl}^k(X, M)$

(c) for $k \leq \ell$, $\mathrm{cl}^k(X, M) \subseteq \mathrm{cl}^\ell(X, M)$.

(d) the relation "$b \in \mathrm{cl}^k(A, M)$" is preserved by isomorphism.

2) We say that the closure operation cl is *f.o. definable* if (e) below is true (and we assume this when not said otherwise)

(e) the assertion "$b \in \mathrm{cl}^k(\{a_0, \ldots a_{l-1}\}, M)$" is f.o. definable in $\mathcal{K}$ that is there is a formula $\psi(y, x_0, ..., x_{l-1})$ such that if $M \in \mathcal{K}$ and $b, a_0, ..., a_{l-1} \in M$ then $b \in cl^k(\{a_0, ..., a_{l-1}\}, M)$ iff $M \models \psi[y, x_0, ..., x_{l-1}]$.

3) We say cl is *transitive* if for every k for some m, for every $X \subseteq M \in \mathcal{K}$ we have $\mathrm{cl}^k(\mathrm{cl}^k(X, M), M) \subseteq \mathrm{cl}^m(X, M)$.

Definition 2.3. (1) For $X \subseteq M$ and $k, m \in \mathbb{N}$ we define $\mathrm{cl}^{k,m}(X, M)$ by induction on m:

$$\mathrm{cl}^{k,0}(X, M) = X$$
$$\mathrm{cl}^{k,1}(X, M) = \mathrm{cl}^k(X, M)$$

$$cl^{k,m+1}(X,M) = cl^{k,1}(cl^{k,m}(X,M),M)$$

(if we write $\mathrm{cl}^{k,m-1}(X,M)$ and $m=0$ we mean $\mathrm{cl}^{k,0}(X,M) = X$).

(2) We say the closure operation cl^k is (ℓ,r)-local when:
for $M \in \mathcal{K}$, $X \subseteq M$ and $Z \subseteq M$ if $Z \subseteq \mathrm{cl}^k(X,M)$, $|Z| \le \ell$ then for some Y we have $Z \subseteq Y$, $|Y| \le r$ and $\mathrm{cl}^k(Y \cap X, M \restriction Y) = Y$.

(3) We say the closure operation cl is local *if* for every k, for some r, cl^k is $(1,r)$-local. We say that cl is simply local if cl^k is $(1,k)$-local for every k.

Remark 2.4. (1) Concerning "possible in $\mathfrak{K}$"(from Definition 2.8 below), in the main case $\mathcal{M}^0_{n,\bar{p}}$, it is degenerate, i.e. if $\bar{a} \subseteq N \in \mathcal{K}_\infty$, $B \subseteq N$ then $(N,B,\bar{a},k,m)$ is possible. But for the case with the successor relation it has a real role.

(2) Note: *if* cl^k is $(1,r)$-local and "$y \in \mathrm{cl}^k(\{x_1,\dots,x_r\},M)$" is f.o. definable *then* for every m, s we have "$y \in \mathrm{cl}^{k,m}(\{x_1,\dots,x_s\},M)$" is f.o. definable.

(3) Clearly $\mathrm{cl}^{k,m_1}(\mathrm{cl}^{k,m_2}(X,M)) = \mathrm{cl}^{k,m_1+m_2}(X,M)$ and $k_1 \le k_2 \wedge m_1 \le m_2 \Rightarrow \mathrm{cl}^{k_1,m_1}(A,M) \subseteq \mathrm{cl}^{k_2,m_2}(A,M)$.

(4) Note that if cl^k is (ℓ_1,r_1)-local and $r_2 \ge mr_1$ and $\ell_2 \le m\ell_1$ then cl^k is (ℓ_2,r_2)–local.

Definition 2.5 (For our 0-1 context $(\mathcal{K}, cl)$ with cl as a basic operation)**.**

(1) $A \le_i B$ if and only if $A \subseteq B \in \mathcal{K}_\infty$ and for some $k,m \in \mathbb{N}$ and every random enough $\mathcal{M}_n$ and for every embedding $g : B \hookrightarrow \mathcal{M}_n$ we have $g(B) \subseteq \mathrm{cl}^{k,m}(g(A),\mathcal{M}_n)$.

(2) $A <_s B$ if and only if $A \subseteq B \in \mathcal{K}_\infty$ and for every $k,m \in \mathbb{N}$ and random enough $\mathcal{M}_n$ and $f : A \hookrightarrow M_n$ there is g such that $f \subseteq g$, and $g : B \hookrightarrow \mathcal{M}_n$ with $g(B) \cap \mathrm{cl}^{k,m}(f(A),\mathcal{M}_n) = f(A)$. We define $\le_{pr}$, $\le^s_m$,$\le^i_{k,m}$ as in 1.4(5), (7), (8) respectively and $A <_a B$ means $A < B$ and $\neg(A <_s B)$.

(3) $(\mathfrak{K},\mathrm{cl})$ is *weakly nice* if for every $A \subseteq C \in \mathcal{K}_\infty$, for some B we have $A \le_i B \le_s C$.

(4) We say $\mathfrak{K}$ (more exactly $(\mathfrak{K},\mathrm{cl})$) is *smooth*[1] when:

if $A \subseteq B \subseteq N \in \mathcal{K}_\infty$, $A \subseteq C \subseteq N$, $B \underset{A}{\overset{N}{\biguplus}} C$,

then $B <_i B \cup C \Leftrightarrow A <_i C$

(note that $\Leftarrow$ is always true).

(5) We say that cl^k is r-transparent if

$$A \le_i B \;\&\; |B| \le r \qquad \Rightarrow \qquad \mathrm{cl}^k(A,B) = B.$$

We say that cl is transparent if for every r for some k we have: cl^k is r-transparent. We say that cl is simply transparent if for every k, cl^k is k-transparent.

[1]Smoothness is not used in [Sh 550], but the closure there has *a priori* bound, so the definitions there will be problematic here. See more in [Sh:F192].

Fact 2.6. *Assume $\mathfrak{K}$ is a $0-1$ context (see 1.1) and* cl *is defined in 1.6* then

(α) cl *is a closure operation for $\mathcal{K}_\infty$ (see Def.2.2(1)),*
(β) cl *is f.o. definable (for $\mathcal{K}$),*
(γ) $\mathrm{cl}^{k,m}$ *as defined in 1.6(c) and as defined 2.3 are equal,*
(δ) cl *is transitive,*
(ε) cl *is simply local (see Def.2.3(2),(3)),*
(ζ) cl *is transparent, in fact simply transparent,*
(η) $\leq_i$ *as defined in 2.5(1) and in 1.4 are equal,*
(θ) *If in §1, $\mathfrak{K}$ is weakly nice (see Def.1.9) then* $(\mathcal{K}_\infty, \mathrm{cl})$ *is weakly nice by Def.2.5(3); if so then $\leq_s$ as defined in 2.5(2) and 1.4(4) are the same and $<_a$ as defined in 2.5(2) and in 1.4(6) are equal.*

PROOF. (α) We have to show that $(\mathcal{K}_\infty, \mathrm{cl})$ from §1 satisfies clauses $(a), (b), (c), (d)$ from Def. 2.2(1).
(a) By the Def. 1.6 of $(\mathcal{K}_\infty, \mathrm{cl})$ the following holds: trivially $X \subseteq \mathrm{cl}^k(X, M) \subseteq M$. Assume $X \subseteq Y \subseteq M$. If $b \in \mathrm{cl}^k(X, M)$ then for some B, $|B| \leq k$ and $b \in B$, $X \cap B \leq_i B$ by Def.1.6. As $X \subseteq Y$ and $X \cap B \leq_i B$ we obtain $Y \cap B \leq_i B$ by Fact 1.11(4). So $B \subseteq \mathrm{cl}^k(Y, M)$ witnessing that $b \in \mathrm{cl}^k(Y, M)$. Hence $cl^k(X, M) \subseteq cl^k(Y, M)$.

(b) (i) First, let's show $\mathrm{cl}^k(X, N) \subseteq \mathrm{cl}^k(X, M)$. If $b \in \mathrm{cl}^k(X, N)$ then let B witness it and we have $b \in B$, $B \subseteq N$, $B \cap X \leq_i B$, $|B| \leq k$. As $N \subseteq M$ the witness B is in M, $B \cap X \leq_i B$ so $b \in \mathrm{cl}^k(X, M)$. Second we will show that $\mathrm{cl}^k(X, M) \subseteq \mathrm{cl}^k(X, N)$: if $b \in \mathrm{cl}^k(X, M)$ then there is B witnessing it such that $b \in B \subseteq M$,$B \cap X \leq_i B$, $|B| \leq k$. Now clearly $B \subseteq \mathrm{cl}^k(X, M)$ hence by assumption $B \subseteq N$ so $b \in B \subseteq N$, $B \cap X \leq_i B$, $|B| \leq k$ and so B witnesses $b \in \mathrm{cl}^k(X, N)$. So we get the result.

(ii) Included in the proof of clause (i).
(c) It follows immediately that $(\mathcal{K}, \mathrm{cl})$ holds by Def.1.6.
(d) Easy.
(β) We show that $(\mathcal{K}, \mathrm{cl})$ is f.o. definable. By Def.2.2(d) this means that for each ℓ, there is a formula $\psi(y, x_0, ..., x_{\ell-1})$ such that if $M \in \mathcal{K}$ and $b, a_0, ...a_{\ell-1} \in M$ then: $b \in \mathrm{cl}^k(\{a_0, ..., a_{\ell-1}\}, M)$ iff $M \models \psi(b, ..., a_{\ell-1})$. It suffice to restrict ourselves to the case $\langle b_0, ..., b_{\ell-1}\rangle$ is with no repetition.

Let $\mathfrak{B} = \{(B, \bar{b}) : B \in \mathcal{K}_\infty$ has $\leq \ell + 1$ elements, $\bar{b}$ is a sequence of length $\leq \ell + 1$ listing the elements of B without repetitions$\}$. On $\mathfrak{B}$ the relation $\cong$ (isom.) is defined. We say $(B', \bar{b}') \cong (B'', \bar{b}'')$ if there is an isomorphism h from B' onto B'' mapping $\bar{b}'$ onto $\bar{b}''$. Now $\cong$ is an equivalence relation on $\mathfrak{B}$. $\mathfrak{B}/\cong$ is finite. So let $\{(B_i, \bar{b}_i) : i < i^*\}$ be a set of representatives. Now i^* is finite as τ is finite (actually locally finite suffice). Let when

$k_i = |B_i| = lg(\bar{b}_i) - 1$

$\varphi_i(x_0,...,x_k) =$

$\bigwedge\{\theta(x_0,...,x_k) : \theta$ is a basic formula (possibly with dummy variables) and $B_i \models \theta[b_0,...,b_k]\}$.

Lastly

$$\psi(y,x_0,...,x_{\ell-1}) = \bigvee_{m<\ell} y = x_m \vee$$
$$\bigvee\{(\exists z_0,\dots,z_{k-1})(\bigwedge_{m<\ell}\bigvee_{t<k} x_m = z_t \wedge \varphi_i(z_0,...,z_{k-1},y)) :$$
$$B_i \text{ has exactly } k+1 \text{ members and } B_i \restriction \{b^i_t : t<\ell\} \leq_i B_i\}$$

(γ) Trivial.
(δ) By 1.16.
(ε) Now, we will show that $(\mathcal{K}_\infty, \mathrm{cl})$ is simply local. For this we have to show that cl^k is $(1,k)$-local for every k. Let $X \subseteq M \in \mathcal{K}_\infty$ be given and $Z \subseteq \mathrm{cl}^k(X,M)$ such that $|Z| \leq 1$. If $Z = \emptyset$ let $Y = \emptyset$. So assume $Z = \{y\}$. As $y \in Z \subseteq \mathrm{cl}^k(X,M)$ there is a witness set Y for $y \in \mathrm{cl}^k(X,M)$ so $Y \cap X \leq_i Y$, $|Y| \leq k$. As $Y \cap X \leq_i Y$, clearly $\mathrm{cl}^k(X\cap Y, Y) = Y$ and $Z = \{y\} \subseteq Y$ and $|Y| \leq k$ so we are done.
(ζ) Trivial by the definition of cl (Def.1.6) and of transparency (Def.2.5(5)).
(η) First assume $A \leq_i B$ by Def. 2.5 and we shall prove that $A \leq_i B$ by Def. 1.4. So for some k, m we have:

- $(*)$ for every random enough $\mathcal{M}_n$ and embedding $g : B \hookrightarrow \mathcal{M}_n$ we have $g(B) \subseteq \mathrm{cl}^{k,m}(g(A), \mathcal{M}_n)$.

Let $\varepsilon > 0$. Let $\mathcal{M}_n$ be random enough and $f : A \hookrightarrow \mathcal{M}_n$. By (*) and 1.16 if g is an embedding of B into $\mathcal{M}_n$ extending f then we have $g(B) \subseteq \mathrm{cl}^{k^m}(g(A), \mathcal{M}_n)$, hence $|\mathrm{ex}(f,B,\mathcal{M}_n)| \leq |\mathrm{cl}^{k^m}(g(A),\mathcal{M}_n)|^{|B\setminus A|}$. Let $\zeta = \varepsilon/(|B\setminus A|+1)$, now if $\mathcal{M}_n$ is random enough, then by 1.17 for every $g : B \hookrightarrow \mathcal{M}_n$ we have $|\mathrm{cl}^{k^m}(g(A),\mathcal{M}_n)|^{|B\setminus A|} \leq n^\zeta$, hence $\mathrm{ex}(f,B,\mathcal{M}_n)| \leq |n^\zeta|^{|B\setminus A|} \leq n^\varepsilon$. As $\varepsilon > 0$ was arbitrary, we have proved that $A \leq_i B$ by Def.1.4.

Next assume $A \leq_i B$ by Def. 1.4 and we shall prove that $A \leq_i B$ by Def.2.5. Choose $k = |B|$ and $m = 1$, so $\mathrm{cl}^{k,m} = \mathrm{cl}^k$. So let $\mathcal{M}_n$ be random enough, and $g : B \hookrightarrow \mathcal{M}_n$. Recall that $\mathrm{cl}^k(g(A),\mathcal{M}_n) = \cup\{C : C \subseteq \mathcal{M}_n, |C| \leq k$ and $C \cap g(A) \leq_i C\}$, so $g(B)$ can serve such C, hence $g(B) \subseteq \mathrm{cl}^k(g(A),\mathcal{M}_n)$.
(θ) We shall use clause (η) freely. First assume that $\mathcal{K}$ is weakly nice by Def.1.9 and we shall prove that $(\mathcal{K},\mathrm{cl})$ is weakly nice by Def.2.5(3). So assume $A \leq B$. We can find C such that $A \leq_i C \leq B$ and for no C', $A \leq_i C' \leq B$, $C \subset C'$; C' exist as $A \leq_i A \leq B$ and B is finite. By 1.11(2) for no C', do we have $C <_i C' \leq B$ hence $C \leq_s B$ by Def.1.4, so it is enough to prove that $C \leq_s B$ by Def.2.5(2), and w.l.o.g. $C \neq B$ so $C <_s B$. Let k, m be given. As we are assuming that $\mathcal{K}$ is weakly nice by Def.1.9 and

$C <_s B$ by Def.1.4(4) we have that there is an $\varepsilon \in \mathbb{R}^+$ such that

$$1 = \lim_n \left[\text{Prob} \begin{pmatrix} \text{if } f_0 : A \hookrightarrow \mathcal{M}_n \text{ then there is } F \text{ with } |F| \geq n^\varepsilon \text{ and} \\ \text{(i) } f \in F \Rightarrow f_0 \subseteq f : B \hookrightarrow \mathcal{M}_n \\ \text{(ii) } f' \neq f'' \in F \Rightarrow \text{Rang}(f') \cap \text{Rang}(f'') = \text{Rang}(f_0) \end{pmatrix} \right].$$

As $\mathcal{M}_n$ is random enough and $f : A \hookrightarrow \mathcal{M}_n$, there is F as above for B with $|F| \geq n^\varepsilon$; but by 1.16 also

$$|\text{cl}^{k,m}(f(A), \mathcal{M}_n)| \leq |\text{cl}^\ell(f(A), \mathcal{M}_n)|$$

for $l = k^m$ and by 1.7 we have

$$|\text{cl}^{k^m}(f(A), \mathcal{M}_n)| < n^\varepsilon$$

so $|\text{cl}^{k,m}(f(A), \mathcal{M}_n)| < n^\varepsilon$.

As the sequence $\langle \text{Rang}(g) \setminus \text{Rang}(f) : g \in F \rangle$ list a family of $\geq n^\varepsilon > |\text{cl}^{k,m}(f(A), \mathcal{M}_n)|$ pairwise disjoint subsets of $\mathcal{M}_n$, for some $g \in F$, we have: $\text{Rang}(g) \cap \text{Rang}(f)$ is disjoint to $\text{cl}^{k,m}(f(A), \mathcal{M}_n)$. So g is as required in Def.2.5(2); so we have finished by proving $C \leq_s B$ by Def.2.5, hence we have finished proving $(\mathcal{K}_\infty, \text{cl})$ is weakly nice according to Def.2.5(3).

So we have proved the implication between the two version of weakly nice. Second, assuming $\mathfrak{K}$ is weakly nice by Def.1.9, we still have to say why the two version of $\leq_s$ (by Def.1.4(4) and by 2.5(2)) are equivalent. Now if $C \leq_s B$ by Def.1.4(4) then $C \leq_s B$ by Def.2.5(2) has been proved inside the proof above that $\mathcal{K}$ weakly nice; by Def.1.4(3) implies $(\mathcal{K}, \text{cl})$ is weakly nice by Def.2.5(3). Lastly assume $A \leq_s B$ by Def.2.5(2), now if $A <_i C \leq B$ we get a contradiction directly from Def.2.5(2): but this confirm $A \leq_s B$ according to Def.1.4(4).

Lastly we leave the statement on $<_a$ to the reader. $\blacksquare_{2.6}$

Remark 2.7.
(1) Note that the assumption "$\mathfrak{K}$ is weakly nice" is very natural in the applications we have in mind.
(2) Why have we not prove the equivalence of the two versions of weakly nice in 2.6(θ)? We can define the following $0-1$ context $\mathfrak{K}$: let $\mathcal{M}_n$ be $\mathcal{M}^0_{n,\bar{p}}$ if n is even with $p_n = 1/n^\alpha$, $\alpha \in (0,1)_\mathbb{R}$ irrational (except $p_1 = 1/2^\alpha$) and $\mathcal{M}_n$ is the random graph with probability 1/2 if n is odd. Now in §1, $\mathcal{K}_\infty$ is the family of finite graphs, and $A \leq_i B$ iff $A = B$ (using the odd n-s). Hence $\text{cl}^k(A, M) = A$ so clearly $A < B \Rightarrow A <_s B$ according to 1.4 hence weakly niceness by 2.5(3) holds trivially but weakly niceness by Def.1.9 fails.
(3) Note that in Definitions 2.8, 2.9, 2.12 below, the "universal" demand speak on a given situation in random enough $\mathcal{M}_n$ whereas the "existential demand" implicit in goodness deal with extensions of an embedding into $\mathcal{M}_n$.
(4) We would like to show that for every formula $\varphi(\bar{x})$ (f.o. in the vocabulary $\tau_\mathcal{K}$) there are (f.o.) $\psi_\varphi(\bar{x})$ and $k = k_\varphi$, $m = m_\varphi$ such that $(*)_\varphi$ for every random enough $\mathcal{M}_n$ and $\bar{a} \in {}^{lg(\bar{x})}\mathcal{M}_n$ we have

$\mathcal{K} \models \varphi[\bar{a}] \Leftrightarrow \mathcal{M}_n \restriction cl^{k,m}(\bar{a}, \mathcal{M}_n) \models \psi_\varphi(\bar{x})$.
Naturally enough we shall do it by induction on the quantifier depth of φ and the non-trivial case is $\varphi(\bar{x}) = (\exists y)\varphi_1(\bar{x}, y)$, and we assume $\psi_{\varphi_1}(\bar{x}, y)$, k_{φ_1}, m_{φ_1} are well defined. So we should analyze the situation: $\mathcal{M}_n$ is random enough, $\bar{a} \in {}^{lg(\bar{x})}(\mathcal{M}_n)$, $\mathcal{M}_n \models \varphi[\bar{a}]$ so there is $b \in \mathcal{M}_n$ such that $\mathcal{M}_n \models \varphi_1[\bar{a}, b]$, and we split it to two cases according to the satisfaction of a suitable statement on a suitable neighbourhood of $\bar{a}$ i.e., $cl^{k',m'}(\bar{a}, \mathcal{M}_n)$. If b belongs to a small enough neighbourhood of $\bar{a}$ this should be clear. If not we would like to find a suitable situation (really a set of possible situation, with a bound on their number depending just on φ) to guarantee the existence of an element b with $\mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n)$ satisfying $\psi_{\varphi_1}(\bar{a}, b)$. Now in general the $\mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}$ can be of large cardinality (for φ, i.e. depending on $\mathcal{M}_n$). In the nice case we are analyzing, to find such a witness b outside a small neighbourhood of $\bar{a}$ it will suffice to look at $\mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}b, \mathcal{M}_n)$ essentially with small cardinality. Why only essentially? As may be $\mathrm{cl}^{k_{\varphi_1}, m_{\varphi_2}}(\bar{a}, \mathcal{M}_n)$ is already large, so what we should have is something is like: $\mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^{k_{\varphi_1}, m_2}(\bar{a}, \mathcal{M}_n)$ can be replaced by a set of small cardinality. For this we need $\bigcup\!\!\!\!\bigcup$ (the relation free amalgamation) to hold, possibly replacing $\mathrm{cl}^{k_{\varphi_1}, m_2}(\bar{a}, \mathcal{M}_n)$ by a subset (in §3 we can make it arbitrary, here quite definable) and the amalgamation base has an a priori bound. By the addition theorem we may replace $(B^*, b)_{b \in B}$ by similar enough $(B', b)_{b \in B}$ (in particular when $B^* \in \mathcal{K}_\infty$ so we need to express in such situation something like B^* exists over B (we can say such B exists by clause (b) of 2.8(4) using quantifiers on $\mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n)$). Well, $B \leq_s B^*$ is a good approximation. But this does not say that $\mathrm{cl}(\bar{a}b, \mathcal{M}_n)$ is suitable. So we need to say first that the closure of $\bar{a}b$ in essentially $B^* \cup B_2$ where $B_2 = cl^{k_{\varphi_1}, m_2}(\bar{a}, \mathcal{M}_n)$, obeys a version of the addition theorem, and secondly that B^* sit in $\mathcal{M}_n$ in a way where the closure is right. All this is carried out in Def.2.8(4) (of good saying: we have a tuple in a situation which exist whenever a copy of B as above exist) and 2.9 (when there are B etc. as above). The proof is carried out in 2.16.

(5) Defining good, by demanding the existence of the embedding $g : B^* \hookrightarrow \mathcal{M}_n$ extending $f : B \hookrightarrow \mathcal{M}_n$, we demand on f only little: it is an embedding. We may impose requirements of the form $cl^{k_i, m_i}(f(B_i), \mathcal{M}_n) \subseteq f(B)$ or $cl^{k_i, m_i}(f(B_i), \mathcal{M}_n) \cap f(B) = f(C_i)$ for some B_i,$C_i \subseteq B$. This make it easier for a tuple to be good. Thus giving a version of almost nice covering more cases. In other possible strengthening we do not replace B^* by $B' \in \mathcal{K}_\infty$ of bounded cardinality but look at it as a family of possible ones all similar in

the relevant sense. On the other hand we may like simpler version which are pursued in 2.13, 2.17.

(6) Note that if cl^k is r-transparent and $A \subseteq M \in \mathcal{K}$ then $\mathrm{cl}^k(A, M) \supseteq \cup\{C \subseteq M : C \cap A \leq_i C$ and $|C| \leq r\}$. [Why? if $C \subseteq M$, $C \cap A \leq_i C$ and $|C| \leq r$ then : first $\mathrm{cl}^k(C \cap A, C) = C$ as cl^k is r-transparent; second $\mathrm{cl}^k(C \cap A, C) \subseteq \mathrm{cl}^k(C \cap A, M)$ by $(b)(ii)$ of Def. 2.2 (1), third $\mathrm{cl}^k(C \cap A, M) \subseteq \mathrm{cl}^k(A, M)$ as $C \cap A \subseteq M$ by clause (a) of Def.2.2(1); together we are done]. Note that if cl^k is $(1, r)$-local we can prove the other inclusion. So obviously if $(\mathfrak{K}, \mathrm{cl})$ is simply local and simply transparent (and $\tau_{\mathcal{K}}$ is finite or at least locally finite of course), then cl is f.o. definable. If we omit the simple we can eliminate the assumption cl is f.o. definable in 2.16, 2.17.

Definition 2.8. (1) We say $(N, B, \bar{B}, k)$ is possible for $(\mathfrak{K}, \mathrm{cl})$ if:

(a) $\bar{B} = \langle B_i : i < lg(\bar{B})\rangle$, $B_i \subseteq N \in \mathcal{K}_\infty$, $B \subseteq N$ and $\mathrm{cl}^k(B_i, N) \subseteq B_{i+1}$ for $i < \ell g(\bar{B}) - 1$

(b) it is not true that:
for every random enough $\mathcal{M}_n$, for no embedding $f : N \hookrightarrow \mathcal{M}_n$, do we have:

$$\text{for } i < \ell g(\bar{B}) - 1,\ \mathrm{cl}^k(f(B_i), \mathcal{M}_n) \subseteq f(\mathrm{cl}^k(B_i), N) \cup \mathrm{cl}^k(f(B), \mathcal{M}_n).$$

(2) If we write (N, C, B, k) we mean $(N, C, \langle B, \mathrm{cl}^k(B, N)\rangle, k)$.

(3) We say $(N, B, \bar{a}, k, m)$ is possible for $\mathfrak{K}$ if $(N, B, \bar{B}, k)$ is possible for $\mathfrak{K}$ where $\bar{B} = \langle \mathrm{cl}^{k,i}(\bar{a}, N) : i \leq m\rangle$.

(4) We say that the tuple $(B^*, B, B_0, B_1, k, m_1, m_2)$ is good for $(\mathcal{K}, \mathrm{cl})$ if

(a) $B \leq B^* \in \mathcal{K}_\infty$ and, $B_0 \leq B_1 \leq B^* \in \mathcal{K}_\infty$

(b) for every random enough $\mathcal{M}_n$ we have: if $f : B \hookrightarrow \mathcal{M}_n$ then there is an extension g of f satisfying $g : B^* \hookrightarrow \mathcal{M}_n$ and

(α) $g(B^*) \cap \mathrm{cl}^{k,m_2}(f(B), \mathcal{M}_n) = f(B)$,

(β) $\mathrm{cl}^{k,m_1}(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^{k,m_2}(g(B), \mathcal{M}_n)))$

(γ) $\mathcal{M}_n \restriction g(B^*) \underset{\mathcal{M}_n \restriction f(B)}{\overset{\mathcal{M}_n}{\bigcup}} \mathcal{M}_n \restriction \mathrm{cl}^{k,m_2}(f(B), \mathcal{M}_n)$.

Definition 2.9. The 0-1 context $\mathfrak{K}$ with closure cl (or the pair $(\mathfrak{K}, \mathrm{cl})$ or $\mathfrak{K}$ when cl is understood) is *almost nice* if it is weakly nice and

(A) *the universal demand:*
for every k, m_0 and ℓ, ℓ' there are

$$m^* = m^*(k, m_0, \ell, \ell') > m_0,\ k^* = k^*(k, m_0, \ell, \ell') \geq k \text{ and } t = t(k, m_0, \ell, \ell')$$

such that, for every random enough $\mathcal{M}_n$ we have:
if $\bar{a} \in {}^\ell|\mathcal{M}_n|$ and $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$
then there are $m_2 \in [m_0, m^*]$ and $m_1 \leq m^* - m_2$ and $B \subseteq \mathrm{cl}^{k,m_1}(\bar{a}, \mathcal{M}_n)$ and $B^* \subseteq \mathcal{M}_n$ such that:

(α) $|B| \leq t$ and $\bar{a} \subseteq B$,

(β) $B^* = [\mathrm{cl}^{k,m_0}(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^{k,m_2}(B, \mathcal{M}_n)] \cup B$ so necessarily $b \in B^*$ and $\bar{a} \subseteq B^*$, (see 2.11 below)

(γ) $B <_s B^*$ or at least:
for every first order formula $\varphi = \varphi(\ldots, x_a, \ldots)_{a \in B}$ of quantifier depth $\leq \ell'$ there is B' such that $B <_s B'$ (so $B' \in \mathcal{K}_\infty$) and

$$B^* \models \varphi(\ldots, a, \ldots)_{a \in B} \quad \text{iff} \quad B' \models \varphi(\ldots, a, \ldots)_{a \in B},$$

(δ) $\mathcal{M}_n \restriction B^* \underset{\mathcal{M}_n \restriction B}{\overset{\mathcal{M}_n}{\bigcup\!\!\!\bigcup}} \mathcal{M}_n \restriction \mathrm{cl}^{k,m_2}(B, \mathcal{M}_n)$,

(ε) $(B^*, B, \bar{a}b, B^* \cap \mathrm{cl}^{k,m_0}(\bar{a}b, \mathcal{M}_n)), k, m_0, m_2)$ is good for $(\mathcal{K}, cl)$ or at least for some B', B'' we have[2]

(i) $(B', B, \bar{a}b, B'', k, m_1, m_2)$ is good for $(\mathcal{K}, cl)$

(ii) $(B^*, \mathrm{cl}^{k,m_0}(\bar{a}b, \mathcal{M}_n) \cap B^*), b, c)_{c \in B} \equiv_{\ell'} (B', B'', b, c)_{c \in B}$,

(ζ) for $m \leq m_0$ we have

$$\mathrm{cl}^{k,m}(\bar{a}b, B^*) = B^* \cap \mathrm{cl}^{k,m}(\bar{a}b, \mathcal{M}_n).$$

Definition 2.10. If in Def 2.9 above, $k^* = k$ in clause (A) *then* we add "k–preserving".

Remark 2.11. (1) Note that if $\mathcal{K} = \mathcal{K}_\infty$ and cl is local (or just cl^k is (l_k, r_k)-local for each k) (which holds in the cases we are interested in) then in clauses (γ), (ε) of (A) in Def.2.9 above the two possibilities are close.

(2) Why in 2.9(A)(β) we have "necessarily $b \in B^*$"? Because

$$b \in \mathrm{Rang}(\bar{a}b) \subseteq \mathrm{cl}^{k,m_0}(\bar{a}b, \mathcal{M}_n) \quad \text{and}$$

$$\begin{aligned} \mathrm{cl}^{k,m_2}(B, \mathcal{M}_n) &\subseteq \mathrm{cl}^{k,m_2}(\mathrm{cl}^{k,m_1}(\bar{a}, \mathcal{M}_n), \mathcal{M}_n) \subseteq \mathrm{cl}^{k,m_1+m_2}(\bar{a}, \mathcal{M}_n) \\ &\subseteq \mathrm{cl}^{k,m^*}(\bar{a}, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n) \end{aligned}$$

and b does not belong to the later.

(3) Why do we use $\mathrm{cl}^{k,m_2}(B, \mathcal{M}_n)$? Part of our needs is that this set is definable from B without b.

(4) In clause (γ), Definition 2.9 clause (A), there is one B' for all such φ (Why? As the set of f.o. formulas of quantifier depth ℓ is closed under Boolean combinations) so for some $B' \in \mathcal{K}_\infty$ we have $B \leq_s B'$, and $(B', c)_{c \in B} \equiv_\ell (B^*, c)_{c \in B}$. So we could have phased clause (ii) of $(A)(\varepsilon)$ in the same way as clause (γ).

In our main case, also the following variant of the property applies (see 2.18 below).

Definition 2.12. 1) We say that the quadruple $(N, B, \langle B_0, B_1 \rangle, k)$ is *simply good* for $(\mathfrak{K}, \mathrm{cl})$ if $(B, B_0, B_1 \leq N \in \mathcal{K}_\infty$ and) for every random enough $\mathcal{M}_n$, for every embedding $f : B \hookrightarrow \mathcal{M}_n$ there is an extension g of f satisfying $g : N \hookrightarrow \mathcal{M}_n$ such that:

[2] $M_1 \equiv_{\ell'} M_2$ means; M_1, M_2 satisfy the same f.o. sentences of quantifier depth $\leq \ell'$

(i) $g(N) \cap \mathrm{cl}^k(f(B), \mathcal{M}_n) = f(B)$,

(ii) $g(N) \underset{f(B)}{\biguplus} \mathrm{cl}^k(f(B), \mathcal{M}_n)$,

(iii) $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^k(g(B), \mathcal{M}_n)$

(natural but not used is $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \cap g(N) = g(\mathrm{cl}^k(B_0, N))$). If we write B_0 instead of $\langle B_0, B_1\rangle$, we mean $B_1 = N$.

2) We say that $(N, B, \langle B_0, B_1\rangle, k, k')$ is *simply good* if part (1) holds replacing (iii) by

(iii)′ $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^{k'}(g(B), \mathcal{M}_n)$.

Definition 2.13. 1) The 0-1 context with closure $(\mathfrak{K}, \mathrm{cl})$ is *simply almost nice* if it is weakly nice and

(A) *the universal demand:*
for every k and ℓ, ℓ' there are

$$m^* = m^*(k, \ell, \ell'), \quad k^* = k^*(k, \ell, \ell') \geq k \ \text{ and } t = t(k, \ell, \ell')$$

such that for every random enough $\mathcal{M}_n$ we have:
if $\bar{a} \in {}^{\ell}|\mathcal{M}_n|$ and $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$
then there are $B \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ and $B^* \subseteq \mathcal{M}_n$ such that:

(α) $|B| \leq t$ and $\bar{a} \subseteq B$ and $\mathrm{cl}^k(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$,

(β) $B^* = [\mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^k(B, \mathcal{M}_n)] \cup B$
(or at least $B^* \supseteq [\mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^k(B, \mathcal{M}_n)] \cup B$),

(γ) $B <_s B^*$ (so $B^* \in \mathcal{K}_\infty$) or at least for every first order formula $\varphi = \varphi(x_b, \ldots, x_a, \ldots)_{a \in B}$ of quantifier depth $\leq \ell'$ there is B' such that $B <_s B'$ (so $B' \in \mathcal{K}_\infty$) and:

$$B^* \models \varphi(b, \ldots, c, \ldots)_{c \in B} \quad \text{iff} \quad B' \models \varphi(b, \ldots, c, \ldots)_{c \in B}$$

(or even , but actually equivalently, $(B^*, b, \ldots, c, \ldots)_{c \in B} \equiv_{\ell'} (B', b, \ldots, c, \ldots)_{c \in B}$),

(δ) $\mathcal{M}_n \restriction B^* \underset{\mathcal{M}_n \restriction B}{\overset{\mathcal{M}_n}{\biguplus}} \mathcal{M}_n \restriction \mathrm{cl}^k(B, \mathcal{M}_n)$

(ε) $B^* \in \mathcal{K}_\infty$ and $(\mathcal{M}_n \restriction B^*, B, \bar{a}b, k)$ is simply good for $(\mathfrak{K}, \mathrm{cl})$ or at least for some B', b' we have:
(i) $(B', B, \bar{a}b', k)$ is simply good for $(\mathfrak{K}, \mathrm{cl})$ and
(ii) $(B^*, b, \ldots, c, \ldots)_{c \in B} \equiv_{\ell'} (B', b', \ldots, c, \ldots)_{c \in B}$.

2) If above always $k^* = k$ we say: $\mathfrak{K}$ is simply almost nice depth preserving.

3) We say that $(\mathfrak{K}, \mathrm{cl})$ is *simply nice* (i.e. omitting the almost) if 2.13(1) holds but we omit clause (ε) and add

(B) if $B <_s B^*$ and $k \in \mathbb{N}$ then (B^*, B, B^*, k) is simply good.

(C) $\mathcal{K}_\infty = \mathcal{K}$ (or at least if $A \in \mathcal{K}_\infty$ and $k, m \in \mathbb{N}$ then for any random enough $\mathcal{M}_n$ for any $f : A \hookrightarrow \mathcal{M}_n$, $\mathrm{cl}^{k,m}(A, \mathcal{M}_n) \in \mathcal{K}_\infty$.

Similarly in Definition 2.9 for "nice".

Remark 2.14. 1) In 2.13(1) we can weaken the demands (and call $(\mathcal{K}, \mathrm{cl})$ simply$^\otimes$ almost nice): get also $k^\otimes = k^\otimes(k, \ell, \ell') \in \mathbb{N}$ replace in clause (β) $\mathrm{cl}^k(B, \mathcal{M}_n)$ by $\mathrm{cl}^{k^\otimes}(B, \mathcal{M}_n)$ and replace (ε) by

- (ε') $(B', B, \bar{a}b, k, k^\otimes)$ is simply good for $(\mathfrak{K}, \mathrm{cl})$ (see 2.12(2)) *or* at least for some B', b' we have:
  - (i) $(B', B, \bar{a}b', k, k^\otimes)$ is simply$^\otimes$ good
  - (ii) $(B^*, b, \ldots, c, \ldots)_{c \in B} \equiv_{\ell'} (B', b', \ldots, c, \ldots)_{c \in B}$

The parallel change in 2.13(3) (that is defining simply$^\otimes$ nice) is

- $(\mathrm{B})'$ for every $k, \ell \in \mathbb{N}$ for some $k^\otimes = k^\otimes(k, \ell) \in \mathbb{N}$ we have: if $B <_s B^*$ and $|B| \leq \ell$, then $(B^*, B, B^*, k, k^\otimes)$ is simply good.

This does not change the conclusions i.e (2.13, 2.17, 2.18, 2.19).
2) We can change Definition 2.9 as we have changed Definition 2.13(1) in 2.13(3) and/or in 2.14(1).
3) If $c\ell$ is transparent we can without loss of generality demand in 2.13(1)(A) that $m^*(k, \ell, \ell') = 1$ at the expense of increasing k^*, as if $\mathrm{cl}^{k^{**}}(\bar{a}, M) \supseteq \mathrm{cl}^{k^*, m^*}(\bar{a}, M)$ whenever $\bar{a} \in {}^\ell|M|$, $M \in \mathcal{K}$ then k^{**} will do.
4) We can omit clause (γ) in Def.2.13(1), but it is natural. Similarly in Def.2.9 (i.e. those omitting do not change the later claims).

Lemma 2.15 below (the addition theorem, see [**?**] or [Gu] and see more [Sh 463]) is an immediate a corollary of the well known addition theorem; this is the point where $\biguplus$ is used.

Lemma 2.15. *For finite vocabulary τ and f.o. formula (in τ) $\psi(\bar{z}, \bar{z}^1, \bar{z}^2)$, $\bar{z} = \langle z_1, \ldots, z_s \rangle$, there are $i^* \in \mathbb{N}$ and τ- formulas $\theta^1_i(\bar{z}, \bar{z}^1) = \theta^1_{i,\psi}(\bar{z}, \bar{z}^1)$, $\theta^2_i(\bar{z}, \bar{z}^2) = \theta^2_{i,\psi}(\bar{z}, \bar{z}^2)$ for $i < i^*$, each of quantifier depth at most that of ψ such that:*

> if N *is τ-model,* $N_1 \biguplus_{N_0}^{N} N_2$, $N_1 \cap N_2 = N_0$, $N_1 \cup N_2 = N$ *and the set of elements of* N_0 *is* $\{c_1, \ldots, c_s\}$, $\bar{c} = \langle c_1, \ldots, c_s \rangle$ *and* $\bar{c}^1 \in {}^{\ell g \bar{z}^1}(N_1)$ *and* $\bar{c}^2 \in {}^{\ell g \bar{z}^2}(N_2)$
> then:

$$N \models \psi[\bar{c}, \bar{c}^1, \bar{c}^2] \quad \text{iff} \quad \text{for some } i < i^*, N_1 \models \theta^1_i[\bar{c}, \bar{c}^1] \text{ and } N_2 \models \theta^2_i[\bar{c}, \bar{c}^2].$$

Main Lemma 2.16 (Context as above)**.** *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is almost nice and* cl *is f.o. definable.*
1) Let $\varphi(\bar{x})$ be a f.o. formula in the vocabulary $\tau_{\mathcal{K}}$. Then *for some* $m_\varphi \in \mathbb{N}$ *and* $k = k_\varphi \geq \ell g(\bar{x}) + q.d.(\varphi(\bar{x}))$ *and for some f.o.* $\psi_\varphi(\bar{x})$ *we have:*

- $(*)_\varphi$ *for every random enough $\mathcal{M}_n$ and $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have*
- $(**)$ $\mathcal{M}_n \models \varphi(\bar{a})$ *if and only if* $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) \models \psi_\varphi(\bar{a})$.

2) Moreover, if for simplicity we will consider "$y \in \mathrm{cl}^{k,m}(\bar{x}, M)$" as an atomic formula when computing the q.d.[3] *of* ψ_φ then *we can demand: the*

[3] q.d stand for quantifier depth

number of alternation of quantifiers of ψ_φ is $\leq$ those of φ, more fully if φ is a Π_n (or Σ_n) then so is ψ_φ.

PROOF We shall ignore (2), (which is not used and is obvious if we understand the proof below). We prove the statement in part (1) by induction on $r = q.d.(\varphi(\bar{x}))$ and first note (by clause (e) of Def.2.2 as "$y \in cl^{k,m}(\bar{x})$" is f.o. definable in $\mathcal{K}$) that $(*)_\varphi$ implies

$(*)^+_\varphi$ in $(*)_\varphi$, possibly changing ψ_φ one can replace $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n)$ by any N with $\mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$.

Case 1 φ atomic. Trivial [Proof of Case 1: If $\varphi(\bar{x})$ is an atomic formula we let $m_\varphi = 0$, $k_\varphi = 0$ or whatever. So $\mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) = \bar{a}$ for our k_φ, m_φ. Assume $\mathcal{M}_n \models \varphi(\bar{a})$ and we let $\psi_\varphi = \varphi$. Now as $\bar{a} \subseteq \mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) \subseteq \mathcal{M}_n$ we have $\mathcal{M}_n \models \varphi(\bar{a})$ iff $cl^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) \models \psi_\varphi(\bar{a})$ as required].

Case 2: φ is a Boolean combination of atomic formulas and the formulas of the form $\exists x \varphi'(x, \bar{y})$ with q.d.$(\varphi') < r$. Clearly follows by case 3 and case 1.
Case 3: $r > 0$ and $\varphi(\bar{x}) = (\exists y)\varphi_1(\bar{x}, y)$. Let

$(*)_1$ $m^* = m^*(k_{\varphi_1}, m_{\varphi_1}, \ell g(\bar{x}), \ell')$, $k_\varphi = k^*(k_{\varphi_1}, m_{\varphi_1}, \ell g(\bar{x}), \ell')$, $t = t(k_{\varphi_1}, m_{\varphi_1}, \ell g(\bar{x}), \ell')$

be as guaranteed in Def.2.9 with ℓ' suitable (see its use below) and let $m_\varphi := m^* + m_{\varphi_1}$. Let $\psi^1_{\varphi_1}$ be such that it witness $(*)_{\varphi_1}$, and let $\psi^2_{\varphi_1}$ be such that it witness $(*)^+_{\varphi_1}$.

So it is enough to prove the following two statements:
Statement 1: There is $\psi^1_\varphi(\bar{x})$ (f.o) such that:

$(\boxtimes)_1$ for every random enough $\mathcal{M}_n$, for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:
- $(\alpha)_1$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar{a}, \mathcal{M}_n) \models \psi^1_\varphi(\bar{a})$
- $(\beta)_1$ $\mathcal{M}_n \models$ "there is $b \in \mathrm{cl}^{k_\varphi, m^*}(\bar{a}, \mathcal{M}_n)$ such that $\varphi_1(\bar{a}, b)$ holds" (i.e. b belongs to a small enough neighbourhood of $\bar{a}$).

Statement 2: There is $\psi^2_\varphi(\bar{x})$ (f.o) such that:

$(\boxtimes)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:
- $(\alpha)_2$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m^*}(\bar{a}, \mathcal{M}_n) \models \psi^2_\varphi(\bar{a})$
- $(\beta)_2$ $\mathcal{M}_n \models$ "there is $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k_\varphi, m^*}(\bar{a}, \mathcal{M}_n)$ such that $\varphi_1(\bar{a}, b)$ holds" (i.e. b is far from $\bar{a}$)

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one of them holds).
Note that as "$y \in \mathrm{cl}^{k_\varphi, m^*}(\bar{x})$" is f.o definable and $m_\varphi = m^* + m_{\varphi_1} \geq m^*$, by 2.2 and clause (e), we can in $(\alpha)_2$ replace m^* by m_φ, changing ψ^2_φ to $\psi^{2.5}_\varphi$.
Clearly these two statements are enough and $\psi^1_\varphi(\bar{x}) \vee \psi^{2.5}_\varphi(\bar{x})$ is as required.

Proof of statement 1:
Easy, recalling that $k_\varphi = k^* \geq k_{\varphi_1}$ by clause (A) of Def.2.9, by the induction hypothesis as (assuming $b \in c\ell^{k_{\varphi_1},m^*}(\bar{a}, M_n)$)

$$\mathrm{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \subseteq \mathrm{cl}^{k_\varphi,m^*+m_{\varphi_1}}(\bar{a}, \mathcal{M}_n) = \mathrm{cl}^{k_\varphi,m_\varphi}(\bar{a}, \mathcal{M}_n)$$

and by the fact that the closure is sufficiently definable.

Proof of statement 2:
We will use a series of equivalent statements $\otimes_\ell$.

$\otimes_1$ is $(\beta)_2$

$\otimes_2$ there are $m_2 \in [m_{\varphi_1}, m^*]$, $m_1 \leq m^* - m_2$, b, B, B^* and B' such that:

(α) $b \in \mathcal{M}_n$, $b \notin \mathrm{cl}^{k_\varphi,m^*}(\bar{a}, \mathcal{M}_n)$, $\bar{a} \subseteq B \subseteq \mathrm{cl}^{k_{\varphi_1},m_1}(\bar{a}, \mathcal{M}_n)$, $|B| \leq t$,

(β) $B^* = B \cup [\mathrm{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n)]$ [hence $B = B^* \cap \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n)$] and

(γ) $B \leq_s B' \in \mathcal{K}_\infty$ and $B' = B^*$ or at least $(B^*, b, c)_{c \in B} \equiv_{\ell'} (B', b, c)_{c \in B}$ (see 2.11(4)) and

(δ)

$$B^* \bigcup_{B}^{\mathcal{M}_n} \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n)$$

(ε) $(B', B, \bar{a}b, B^* \cap \mathrm{cl}^{k_{\varphi_1},m_0}(\bar{a}b, \mathcal{M}_n), k_{\varphi_1}, m_0, m_2)$ is good,

(ζ) for $m \leq m_{\varphi_1}$ we have $\mathrm{cl}^{k_{\varphi_1},m}(\bar{a}b, B^*) = B^* \cap \mathrm{cl}^{k_{\varphi_1},m}(\bar{a}b, \mathcal{M}_n)$

and

$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar{a}, b)$

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial as $\oplus_2$ is included in $\otimes_2$, the implication $\Rightarrow$ holds by clause (A) in the definition of almost nice 2.9, except $b \notin cl^{k_\varphi,m^*}(\bar{a}, \mathcal{M}_n)$ which is explicitly demanded in $(\beta)_2$.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by

$\oplus_3$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \models \psi^1_{\varphi_1}(\bar{a}, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by

$\oplus_4$ $\mathcal{M}_n \restriction [B^* \cup \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar{a}, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+_{\varphi_1}$ in the beginning of the proof, the definition of B^* and the choice of $\psi^2_{\varphi_1}$ (Let $\otimes_3$ be true. As by the choice of B^*, B above, $\mathrm{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \cup \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n) \subseteq B^* \cup \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n) \subseteq \mathcal{M}_n$ we have $\mathcal{M}_n \models \varphi_1(\bar{a}, b)$ iff
$B^* \cup \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n) \models \psi^2_{\varphi_1}(\bar{a}b)$ by $(*)^+_{\varphi_1}$. So $(*)_4$ holds.)

For notational simplicity we assume $B \neq \emptyset$, and similarly assume $\bar{a}$ is with no repetition and we shall apply the lemma 2.15 several times.

First, for $m \le m_{\varphi_1}$ we apply 2.15 to the case $s = t$, $\bar{z} = \langle z_1, \dots, z_t \rangle$, $\bar{z}^1 = \langle z^1_1, z^1_2 \rangle$, $\bar{z}^2$ empty and the formula "$z^1_2 \in \mathrm{cl}^{k_{\varphi_1}, m}(\bar{z}, z^1_1)$" and get $i^*_{1,m} \in \mathbb{N}$ and formulas $\theta^1_{1,m,i}(\bar{z}, z^1_1, z^1_2)$ and $\theta^2_{1,m,i}(\bar{z})$ for $i < i^*_{1,m}$. Let

$$u^*_1 = \{(m, i) : m \le m_{\varphi_1}, i < i^*_{1,m}\}.$$

Second for $m \le m_{\varphi_1}$ we apply 2.15 to the case $s = t$, $\bar{z}^2 = \langle z^2_1 \rangle$, $\bar{z}^1 = \langle z^1_1 \rangle$, $\bar{z} = \langle z_1, \dots, z_t \rangle$ and the formula "$z^2_1 \in \mathrm{cl}^{k_{\varphi_1}, m}(\bar{z}, z^1_1)$" and get $i^*_{2,m} \in \mathbb{N}$ and formulas $\theta^1_{2,m,i}(\bar{z}, \bar{z}^1)$ and $\theta^2_{2,m,i}(\bar{z}, z^2_1)$, for $i < i^*_{2,m}$.

Let $\tau' = \tau_{\mathcal{K}} \cup \{P_1, P_2\}$, with P_1, P_2 new unary predicates: for $\theta \in \mathcal{L}[\tau']$ let $\theta^{[P_\ell]}$ be θ restricting the quantifiers to P_ℓ. Let $\psi^* = \psi^*_1 \wedge \psi^*_2 \wedge \psi^*_3$ where

$$\begin{aligned} \psi^*_1 =: &\ \psi^2_{\varphi_1}(z_1, \dots, z_{\ell g(\bar{x})}, z^1_1) \\ \psi^*_2 =: &\ \bigwedge_{m \le m_{\varphi_1}} (\forall y) \Big[y \in \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\{z_1, \dots, z_{\ell g(\bar{x})}, z^1_1\}) \\ &\qquad \equiv \big(\psi^{*,1}_{2,m}(z_1, \dots, z_t, z^1_1, y) \vee \psi^{*,2}_{2,m}(z_1, \dots, z_t, z^1_1, y) \big) \Big], \end{aligned}$$

where

$$\psi^{*,1}_{2,m}(z_1, \dots, z_t, z^1_1, y) =: \bigvee_{i < i^*_{1,m}} (\theta^1_{1,m,i}(z_1, \dots, z_t, z^1_1, y)^{[P_1]} \wedge \theta^2_{1,m,i}(z_1, \dots, z_t)^{[P_2]})$$

$$\psi^{*,2}_{2,m}(z_1, \dots, z_t, z^1_1, y) =: \bigvee_{i < i^*_{2,m}} (\theta^2_{1,m,i}(z_1, \dots, z_t, z^1_1)^{[P_1]} \wedge \theta^2_{2,m,i}(z_1, \dots, z_t, y)^{[P_2]})$$

and let

$$\begin{aligned} \psi^*_3 =: (\forall y) \Big(P_1(y) \equiv [\bigvee_{\ell=1}^{t} y = z_\ell \vee & \\ (y \in \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\{z_1, \dots, z_{\ell g(\bar{x})}, z^1_1\}) & \\ \wedge\, y \notin \mathrm{cl}^{k_{\varphi_1}, m_2}(\{z_1, \dots, z_{lg(\bar{x})}\}))] \Big). & \end{aligned}$$

So we have defined ψ^*. Now we apply 2.15 the third time, with the vocabulary $\tau_{\mathcal{K}} \cup \{P_1, P_2\}$ to the case $s = t$, $\bar{z}^2$ empty, $\bar{z}^1 = \langle z^1_1 \rangle$, $\bar{z} = \langle z_1, \dots, z_\ell \rangle$, and $\psi(\bar{z}, \bar{z}^1, \bar{z}^2) = \psi(\bar{z}, z^1_1) = \psi^*(\langle z_1, \dots, z_{\ell g(\bar{x})} \rangle, z^1_1)$ and get i^*, $\theta^1_{3,i}(\bar{z}, \bar{z}^1)$ and $\theta^2_{3,i}(\bar{z}, \bar{z}^2)$ as there. Let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_4$ by

$\oplus_5$ letting $c_1, \dots, c_t$ list B possibly with repetitions but such that $\langle c_1, \dots, c_{\ell g(\bar{x})} \rangle = \bar{a}$ and letting

$$P^*_1 = B^* \text{ and } P^*_2 = \mathrm{cl}^{k_{\varphi_1}, m_2}(\{c_1, \dots, c_t\}, \mathcal{M}_n)$$

we have

$(*)$ $(\mathcal{M}_n \restriction (P^*_1 \cup P^*_2), P^*_1, P^*_2) \models \psi^*[c_1, \dots, c_t, b]$ (the model is a τ'-model).

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$.

Why? Look at what the statements mean recalling $\mathcal{M}_n \restriction P_1^* \underset{B}{\overset{\mathcal{M}_n}{\bigcup}} \mathcal{M}_n \restriction P_2^*$.

Next let

$\otimes_6$ like $\otimes_5$ but replacing $\oplus_5$ by

$\oplus_6$ letting $c_1, \dots, c_t$ list B possibly with repetitions but such that $\langle c_1, \dots, c_{\ell g(\bar{x})} \rangle = \bar{a}$ and letting

$$P_1^* = B^* \text{ and } P_2^* = \mathrm{cl}^{k_{\varphi_2}, m_1}(\{c_1, \dots, c_t\}, \mathcal{M}_n)$$

there is $i < i^*$ such that:

(i) $(\mathcal{M}_n \restriction P_1^*, P_1^*, P_2^* \cap P_1^*) \models \theta^1_{3,i}[\langle c_1, \dots, c_t \rangle, b]$,

(ii) $(\mathcal{M}_n \restriction P_2^*, P_1^* \cap P_2^*, P_2^*) \models \theta^2_{3,i}[\langle c_1, \dots, c_t \rangle]$.

Now

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$

Why? By the choice of $\theta^1_{3,i}$, $\theta^2_{3,i}$ $(i < i^*)$.

However in the two τ'-models appearing in $\oplus_6$, the predicates P_1, P_2 are interpreted in a trivial way: as the whole universe of the model or as $\{c_1, \dots, c_t\}$.

So let:

(a) $\theta^1_{4,i}(z_1, \dots, z_t, y)$ be $\theta^1_{3,i}(z_1, \dots, z_t, y)$ with each atomic formula of the form $P_1(\sigma)$ or $P_2(\sigma)$ being replaced by $\sigma = \sigma$ or $\bigvee_{r=1}^t \sigma = z_r$ respectively,

(b) $\theta^2_{4,i}(z_1, \dots, z_t)$ be $\theta^2_{3,i}(z_1, \dots, z_t)$ with each atomic formula of the form $P_1(\sigma)$ or $P_2(\sigma)$ being replaced by $\bigvee_{r=1}^t \sigma = z_r$ or $\sigma = \sigma$ respectively.

So let (recall B' is mentioned in $\otimes_2$, a "replacement" to B^*)

$\otimes_7$ like $\otimes_6$ but replacing $\oplus_6$ by

$\oplus_7$ letting $c_1, \dots, c_t$ list B possibly with repetitions but such that $\langle c_1, \dots, c_{\ell g(\bar{x})} \rangle = \bar{a}$, *there is* $i < i^*$ such that

(i) $\mathcal{M}_n \restriction B' \models \theta^1_{4,i}[\langle c_1, \dots, c_t \rangle, b]$ and

(ii) $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}, m_2}(\langle c_1, \dots, c_t \rangle, \mathcal{M}_n) \models \theta^2_{4,i}(\langle c_1, \dots, c_t \rangle)$.

$(*)_7$ $\otimes_6 \Leftrightarrow \otimes_7$

Why? By the choice of the $\theta^1_{4,i}$, $\theta^2_{4,i}$ and the property of B' (stated in $\otimes_2$).

Let $\mathcal{P} = \{(N, c_1, \dots, c_t) : N \in \mathcal{K}_\infty$, with the set of elements $\{c_1, \dots, c_t\}\}$. Let $\{(N_j, c^j_1, \dots, c^j_t) : j < j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j < j^*$ and $i < i^*$ choose if possible $(N_{j,i}, c^j_1, \dots, c^j_t, b^j_i)$ such that:

(i) $N_j \leq_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i \in N_{j,i} \setminus N_j$,

(iii) $N_{j,i} \models \theta^1_{4,i}(\langle c^j_1, \dots, c^j_t \rangle, b^j_i)$ and

(iv) $(N_{j,i}, B, \{c^j_i : i = 1, \dots, \ell g(\bar{x})\} \cup \{b^j_i\}, k, m_0, m_2)$ is good for $\mathfrak{K}$.

Let

$$w = \{(i,j) : i < i^*, j < j^* \text{ and } (N_{j,i}, c^j_1, \dots, c^j_t, b^j_i) \text{ is well defined}\}.$$

Let

$\otimes_8$ there are $m_2 \le m^*$, $m_1 \le m^* - m_2$, such that $m_2 \ge m_{\varphi_1}$ and, there are b, B such that:
$\bar a \subseteq B \subseteq \mathrm{cl}^{k^*,m_2}(\bar a, \mathcal{M}_n)$, $|B| \le t(k_{\varphi_1}, m_{\varphi_1}, \ell g(\bar x))$, $b \notin \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, $b \in \mathcal{M}_n$, and

$\oplus_8$ for some $c_1, \dots, c_t$ listing B such that $\bar a = \langle c_1, \dots, c_{\ell g \bar x}\rangle$ *there are* $i < i^*$, $j < j^*$ such that $(i,j) \in w$ and:

(i) $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embed N_j into $\mathcal{M}_n$,

(ii) $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1},m_2}(B, \mathcal{M}_n) \models \theta^2_{4,i}(\langle c_1, \dots, c_t\rangle)$

$(*)_8$ $\otimes_7 \Leftrightarrow \otimes_8$

Why? For proving $\otimes_7 \Rightarrow \otimes_8$ let $c_1, \dots, c_t$ as well as $i < i^*$ be as in $\oplus_7$, let $j < j^*$ be such that $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$. The main point is that B' exemplifies that $(i,j) \in w$.
For proving $\otimes_8 \Rightarrow \otimes_7$ use the definition of goodness in clause (ε) (see $\otimes_2$ and Def. in 2.8(4).

We now have finished as $\otimes_8$ can be expressed as a f.o formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof. $\blacksquare_{2.16}$

Lemma 2.17. *1) Assume* $(\mathfrak{K}, \mathrm{cl})$ *is simply almost nice and* cl *is f.o. definable. Let* $\varphi(\bar x)$ *be a f.o. formula in the vocabulary* $\tau_{\mathcal{K}}$. Then *for some* $k = k_\varphi$ *and f.o. formula* $\psi_\varphi(\bar x)$ *we have:*

$(*)$ *for every random enough* $\mathcal{M}_n$ *and* $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$

$(**)$ $\mathcal{M}_n \models \varphi(\bar a)$ *if and only if* $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar a, \mathcal{M}_n) \models \psi_\varphi(\bar a)$

2) The number of alternation of quantifiers of ψ_φ *in* (1) *is* $\le$ *the number of alternation of quantifiers of* φ *if we consider "*$y \in \mathrm{cl}^{k,m}(\bar x, M)$*" as atomic. More fully, if* φ *is* Π_n *(or* Σ_n*) then* ψ_φ *is.*

Remark 2.18.

(1) Of course we do not need to assume that closure operation is definable, it is enough if there is a variant cl_* which is definable and for every k, m there are k^1, m^1, k^2, m^2 such that always $\mathrm{cl}^{k,m}(A, M) \subseteq \mathrm{cl}^{k^1,m^1}_*(A, M) \subseteq \mathrm{cl}^{k^2,m^2}(A, M)$.
(2) Similarly in 2.16 (using Def.2.10).
(3) We can weaken "simply almost nice" as in Remark 2.14(1) and still part (1) is true, with essentially the same proof.
(4) The proof of 2.17 is somewhat simpler than the proof of 2.16.

Proof 1) We prove the statement by induction on $r = q.d.(\varphi(\bar x))$. First note (by clause (e) of 2.2)

$(*)^+_\varphi$ in $(*)$ (of 2.17, possibly changing ψ_φ) one can replace $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar a, \mathcal{M}_n)$ by any N with $\mathrm{cl}^{k_\varphi}(\bar a, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$.

Case 1: Let φ be atomic. Trivial.

Case 2: φ a Boolean combination of atomic formulas and formulas of the form $\exists x \varphi'(x\bar y)$ with q.d.$(\ell') < r$. Clearly follows by case 3 and case 1. Trivial.

Case 3: $r > 0$ and $\varphi(\bar x) = (\exists y)\varphi_1(\bar x, y)$. Let (the functions are from 2.13(1))

$$m^* = m^*(k_{\varphi_1}, \ell g(\bar x), \ell'), \quad k^* = k^*(k_{\varphi_1}, \ell g(\bar x), \ell'), \quad t = t(k_{\varphi_1}, \ell g(\bar x), \ell')$$

with ℓ' suitable (just the quantifier depth of $\psi^2_{\varphi_1}$ defined below) and let k_φ be[4] such that:

$(*)_1$ $|A| \le \ell g(\bar x)+1 \;\&\; A \subseteq N \in \mathcal{K} \Rightarrow \mathrm{cl}^{k_{\varphi_1}}(\mathrm{cl}^{k^*,m^*}(A,N),N) \subseteq \mathrm{cl}^{k_\varphi}(A,N)$.

Let $\psi^1_{\varphi_1}(\bar x, y)$ be such that it witness $(*)_{\varphi_1}$ holds, and let $\psi^2_{\varphi_1}(\bar x, y)$ be such that it witness $(*)^+_{\varphi_1}$.

It is enough to prove the following two statements (see below):

Statement 1: There is $\psi^1_\varphi(\bar x)$ (f.o.) such that:

- $(\boxtimes)_1$ for every random enough $\mathcal{M}_n$, for every $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:
  - $(\alpha)_1$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar a, \mathcal{M}_n) \models \psi^1_\varphi(\bar a)$
  - $(\beta)_1$ $\mathcal{M}_n \models$ "there is $b \in \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$ such that $\varphi_1(\bar a, b)$ holds."

Statement 2: There is $\psi^2_\varphi(\bar x)$ (f.o) such that:

- $(\boxtimes)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:
  - $(\alpha)_2$ $\mathcal{M}_n \restriction \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n) \models \psi^2_\varphi(\bar a)$
  - $(\beta)_2$ $\mathcal{M}_n \models$ "there is $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$ such that $\varphi_1(\bar a, b)$ holds"

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one holds).

Note that as "$y \in \mathrm{cl}^{k^*,m^*}(\bar x)$" is f.o. definable, by 2.2, clause (e) and the choice of k_φ we can in $(\alpha)_2$ replace cl^{k^*,m^*} by cl^{k_φ}, changing ψ^2_φ to $\psi^{2.5}_\varphi$; (just as from $(*)$ we have deduced $(*)^+_\varphi$).

Clearly these two statements are enough as $\psi^1_\varphi(\bar x) \vee \psi^{2.5}_\varphi(\bar x)$ is as required.

Proof of statement 1:

---

[4]if we change clause (A) of 2.13(1) a little, $k_\varphi = k^*$ will be O.K.: instead of assuming $b \notin \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$ assume just $\mathrm{cl}^k(\bar a b, \mathcal{M}_n) \not\subseteq \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$. Allowing to increase m^*, the two versions are equivalent. $m^{**} = m^{**}(k,\ell,\ell') = m^*(k,\ell,\ell') + k$. Now by 2.4(3) we have $b \in \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$ and $c \in \mathrm{cl}^k(\bar a b, \mathcal{M}_n) \Rightarrow c \in \mathrm{cl}^{m^*+k}(\bar a, \mathcal{M}_n) = \mathrm{cl}^{m^{**}}(\bar a, \mathcal{M}_n)$ hence $b \in \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n) \Rightarrow \mathrm{cl}^k(\bar a b, \mathcal{M}_n) \subseteq \mathrm{cl}^{k,m^{**}}(\bar a, \mathcal{M}_n)$ hence $\mathrm{cl}^k(\bar a b, \mathcal{M}_n) \not\subseteq \mathrm{cl}^{k,m^{**}}(\bar a, \mathcal{M}_n) \Rightarrow b \notin \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, so our new assumption for m^{**} implies our old for m^*. Of course our new assumption for m^* implies the old for m^*. See section 3 where this is done.

Easily, by the induction hypothesis as

$$\mathrm{cl}^{k_{\varphi_1}}(\bar a b, M_n) \subseteq \mathrm{cl}^{k_{\varphi_1}}(\mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n), \mathcal{M}_n) \subseteq \mathrm{cl}^{k_\varphi}(\bar a, \mathcal{M}_n)$$

and by the fact that the closure is sufficiently definable. So in this case $\psi_\varphi(\bar a)$ can be chosen as $(\exists y)\psi^2_{\varphi_1}(\bar a, y)$.

Proof of statement 2:
We will use a series of equivalent statements $\otimes_\ell$.

$\otimes_1$ is $(\beta)_2$
$\otimes_2$ there are b, B and B^*, B' such that:
(α) $b \in \mathcal{M}_n$, $b \notin \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$,
(β) $\bar a \subseteq B \subseteq \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, moreover $\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, and $|B| \le t$,
(γ) $B^* \supseteq B \cup [\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \setminus \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)]$ and
(δ) $B \le_s B' \in \mathcal{K}_\infty$ and: $B' = B^*$ or just $(B^*, b, c)_{c\in B} \equiv_{\ell'} (B', b, c)_{c \in B}$ (see 2.11(4)) and
(ε) $B^* \underset{B}{\overset{\mathcal{M}_n}{\biguplus}} \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$ (and so $B = B^* \cap \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$) and
(ζ) $(B', B, \bar a b, k_{\varphi_1})$ is simply good
(η) $\mathrm{cl}^{k_{\varphi_1}}(\bar a b, B^*) \setminus B = B^* \cap \mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \setminus \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$, actually this follows from clauses $(\epsilon), (\beta)$, and
$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar a, b)$

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial as $\oplus_2$ is included in $\otimes_2$, the implication $\Rightarrow$ holds by clause (A) in the definition 2.13 of simply almost nice.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by
$\oplus_3$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \models \psi^1_{\varphi_1}(\bar a, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis and our choices.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by
$\oplus_4$ $\mathcal{M}_n \restriction [B^* \cup \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar a, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+_{\varphi_1}$ in the beginning of the proof, the requirements on B^* and the choice of $\psi^2_{\varphi_1}$.

For notational simplicity we assume $B \ne \emptyset$, and similarly assume $\bar a$ has no repetitions and apply the lemma 2.15 with the vocabulary $\tau_{\mathfrak{K}}$ to the case $s = t$, $\bar z^2$ empty, $\bar z^1 = \langle z^1_1 \rangle$, $\bar z = \langle z_1, \dots, z_t \rangle$, and $\psi(\bar z, \bar z^1, \bar z^2) = \psi(\bar z, z^1_1) = \psi^2_{\varphi_1}(\langle z_1, \dots, z_{\ell g(\bar x)}\rangle, z^1_1)$ and get i^*, $\theta^1_i(\bar z, \bar z^1)$ and $\theta^2_i(\bar z)$ for $i < i^*$ as there; in particular the quantifier depth of θ^1_i, θ^2_i for $i < i^*$ is at most the quantifier depth of $\psi^2_{\varphi_1}$.
Next let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_4$ by

$\oplus_5$ letting $c_1, \dots, c_t$ list B possibly with repetitions but such that $\langle c_1, \dots, c_{\ell g(\bar{x})} \rangle = \bar{a}$, $i < i^*$ such that:

(i) $B^* \models \theta^1_i[\langle c_1, \dots, c_t \rangle, b]$

(ii) $\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \models \theta^2_i[\langle c_1, \dots, c_t \rangle]$

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$

Why? By the choice of θ^1_i, θ^2_i for $i < i^*$, so by lemma 2.15.

Let $\mathcal{P} = \{(N, c_1, \dots, c_t) : N \in \mathcal{K}_\infty$, with the set of elements $\{c_1, \dots, c_t\}\}$. Let $\{(N_j, c^j_1, \dots, c^j_t) : j < j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j < j^*$ and $i < i^*$ choose if possible $(N_{j,i}, c^j_1, \dots, c^j_t, b^j_i)$ such that:

(i) $N_j \leq_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i \in N_{j,i} \setminus N_j$,

(iii) $N_{j,i} \models \theta^1_i(\langle c^j_1, \dots, c^j_t \rangle, b^j_i)$ and

(iv) $(N_{j,i}, \{c^j_1, \dots, c^j_t\}, \{c^j_1, \dots, c^j_{\ell g(\bar{x})}, b^j_i\}, k_{\varphi_1})$ is simply good for $\mathfrak{K}$.

$$w = \{(i,j) : i < i^*, j < j^* \text{ and } (N_{j,i}, c^j_1, \dots, c^j_t, b^j_i) \text{ is well defined}\}.$$

Let

$\otimes_6$ like $\otimes_5$ replacing $\oplus_5$ by

$\oplus_6$ like $\oplus_5$ adding

(iii) for some j, $(i,j) \in w$ and $(B, c_1, ..., c_t) \cong N_{j,i}$

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$

Why? By the definition of w.

Let

$\otimes_7$ there is B such that: $b \in \mathcal{M}_n$, $\bar{a} \subseteq B \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $|B| \leq t$, and

$\oplus_7$ for some $c_1, \dots, c_t$ listing B such that $\bar{a} = \langle c_1, \dots, c_{\ell g(\bar{x})} \rangle$ *there are* $i < i^*$, $j < j^*$ such that $(i,j) \in w$ and:

(i) $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embeds N_j into $\mathcal{M}_n$,

(ii) $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \models \theta^2_i(\langle c_1, \dots, c_t \rangle)$

$(*)_7$ $\otimes_6 \Leftrightarrow \otimes_7$

Why? For proving $\otimes_6 \Rightarrow \otimes_7$ let $c_1, \dots, c_t$ as well as $i < i^*$, $j < j^*$ be as in $\oplus_6$, let $j < j^*$ be such that $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$. The main point is that B' exemplifies that $(i,j) \in w$ (remember: B' is from $\otimes_2$, and if $B^* \in \mathcal{K}_\infty$, we normally could have chosen $B' = B^*$).

For proving $\otimes_7 \Rightarrow \otimes_6$ use definition of simply good tuples in Definition 2.12(1).

We now have finished as $\otimes_7$ can be expressed as a f.o. formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof.

2) Similar $\blacksquare_{2.17}$

Conclusion 2.19. (1) *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is almost nice or simply almost nice and* cl *is f.o. definable.*

Then*:* $\mathfrak{K}$ *satisfies the 0-1 law* iff *for any* k, m *we have*

$(*)_{k,m}$ $\langle \mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n) : n < \omega\rangle$ *satisfies the 0-1 law.*

(2) *Similarly with convergence and the very weak* $0-1$ *law.*

Proof 1) We first prove the "only if". There is a f.o. formula $\theta(x)$ such that for every random enough $\mathcal{M}_n$, $\theta(x)$ define $\mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n)$. Hence for every f.o. sentence φ there is a f.o. sentence ψ_φ which is the relativization of φ to $\theta(x)$; hence, for every model $M \in \mathcal{K}$, $M \models \psi_\varphi \Leftrightarrow M \restriction \{a : M \models \theta[a]\} \models \varphi$. Now for every random enough $\mathcal{M}_n$ we have $a \in \mathcal{M}_n \Rightarrow \mathcal{M}_n \models \theta[a] \Leftrightarrow a \in \mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n)$, hence together

$$\mathcal{M}_n \models \psi_\varphi \Leftrightarrow \mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n) \models \varphi.$$

As we are assuming that $\mathfrak{K}$ satisfies the 0-1 law, for some truth value $\mathfrak{t}$ for every random enough $\mathcal{M}_n$

$$\mathcal{M}_n \models \text{“}\psi_\varphi \equiv \mathfrak{t}\text{”}$$

hence (as required)

$$\mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n) \models \text{“}\varphi = \mathfrak{t}\text{”}.$$

The other direction is similar by the main lemma 2.16 when $(\mathcal{K}, \mathrm{cl})$ is almost nice, 2.17 when $(\mathcal{K}, \mathrm{cl})$ is simply almost nice.
2) Similar, so left to the reader. $\blacksquare_{2.19}$

Definition 2.20. (1) The tuple $(N, \bar b, \psi(\bar x), \langle B_0, B_1\rangle, k, k_1)$ is *simply* good* for $(\mathfrak{K}, \mathrm{cl})$ if: B_0, $B_1 \le N \in \mathcal{K}_\infty$, $\mathrm{cl}^k(B_0, N) \subseteq B_1$, $\bar b \in {}^{\ell g(\bar x)}N$, $\psi(\bar x)$ a f.o. formula and k, $k_1 \in \mathbb{N}$ and for every random enough $\mathcal{M}_n$, for every $\bar b' \in {}^{\ell g(\bar x)}(\mathcal{M}_n)$ such that $\mathcal{M}_n \restriction \mathrm{cl}^{k_1}(\bar b', \mathcal{M}_n) \models \psi(\bar b')$, letting $B' = \mathcal{M}_n \restriction \mathrm{Rang}(\bar b')$, there is an embedding g of N into $\mathcal{M}_n$ such that

(i) $g(\bar b) = \bar b'$

(ii) $g(N) \cap \mathrm{cl}^{k_1}(\bar b', \mathcal{M}_n) = B'$

(iii) $g(N) \underset{B'}{\coprod} \mathrm{cl}^{k_1}(\bar b', \mathcal{M}_n)$

(iv) $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^{k_1}(B', \mathcal{M}_n)$.

(2) We may write B_0 instead $\langle B_0, B_1\rangle$ if $B_1 = N$.

(3) We say "normally simply* good" if (iv) is replaced by

(iv)' $\mathrm{cl}^{k_1}(B', \mathcal{M}_n) = g(\mathrm{cl}^k(B_0, N)) \setminus B$.

Definition 2.21. The 0-1 context with closure $(\mathfrak{K}, \mathrm{cl})$ is *(normally) simply* almost nice* if:

(A) for every k, ℓ, ℓ' there are $m^* = m^*(k, \ell, \ell')$, $k^* = k^*(k, \ell, \ell')$, $t = t(k, \ell, \ell')$, $k_0 = k_0(k, \ell, \ell')$, $k_1 = k_1(k, \ell, \ell')$ such that for every random enough $\mathcal{M}_n$ we have *if* $\bar{a} \in {}^{\ell}|\mathcal{M}_n|$ and $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ *then* there are $B \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ and $B^* \subseteq \mathcal{M}_n$ such that

(α) $|B| \le t$, $\bar{a} \subseteq B$, $\mathrm{cl}^{k_1}(B,\mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ and

(β) $B^* \supseteq B \cup [\mathrm{cl}^k(\bar{a}b,\mathcal{M}_n) \setminus \mathrm{cl}^{k_1}(B,\mathcal{M}_n)]$

(γ) $B <_s B^*$ (so $B^* \in \mathcal{K}_\infty$) *or* at least there is B' such that $B <_s B'$, $(B',b,\bar{c}) \equiv_{\ell'} (B^*,b,\bar{c})$

(δ) $\mathcal{M}_n \restriction B^* \underset{B^*}{\overset{\mathcal{M}_n}{\bigcup\!\!\!\!\!\;\bigcup}} \mathcal{M}_n \restriction \mathrm{cl}^{k_1}(B,\mathcal{M}_n)$

(ε) letting $\bar{c}$ list the element of B and

$$\psi(\bar{x}) = \bigwedge\{\varphi(\bar{x}) : \mathcal{M}_n \restriction \mathrm{cl}^{k_1}(\bar{c},\mathcal{M}_n) \models \varphi(\bar{x}) \text{ and } q.d.(\varphi(\bar{x})) \le k_0\}$$

we have $(\mathcal{M}_n \restriction B^*, \bar{c}, \psi(\bar{x}), \bar{a}b, k, k_1)$ is (normally) simply* good or at least for some B', b' we have

(i) $(B', \bar{c}, \psi(\bar{x}), \bar{a}b, k, k_1)$ is (normally) simply* good

(ii) $(B^*, b, \bar{c}) \equiv_{\ell'} (B', b, \bar{c})$

Remark 2.22. We may restrict ψ e.g. demand that it is in Π_1 (most natural in the cases we have.

Claim 2.23. *In 2.17 we can replace simply by simply***, i.e.*
1) Assume $(\mathfrak{K}, \mathrm{cl})$ *is simply** *almost nice. Let* $\varphi(\bar{x})$ *be a f.o. formula.* Then *for some* $k = k_\varphi$ *and first order formula* $\psi_\varphi(\bar{x})$ *we have:*

for every random enough $\mathcal{M}_n$ *and* $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$

$(*)$ $\mathcal{M}_n \models \varphi(\bar{a})$ *if and only if* $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar{a},\mathcal{M}_n) \models \psi_\varphi(\bar{a})$.

2) We have $[\varphi \in \Pi_n \Rightarrow \psi_\varphi \in \Pi_n]$, $[\varphi \in \Sigma_n \Rightarrow \psi_\varphi \in \Sigma_n]$.

Conclusion 2.24. (1) *Assume that the 0-1 context with closure* $(\mathfrak{K}, \mathrm{cl})$ *is (normally) simply** *almost nice. Then* $\mathfrak{K}$ *satisfies the 0-1 law* iff *for any* k, m *we have* $\langle \mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\emptyset,\mathcal{M}_n) : n < \omega\rangle$ *satisfies the 0-1 law.*

(2) *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is simply** *almost nice. Then* $\mathfrak{K}$ *has convergence (respectively very weak* $0{-}1$*) low iff for every* k, m $\langle \mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\emptyset,\mathcal{M}_n) : n < \omega\rangle$ *satisfies convergence (resp. very weak* $0-1$*) low.*

3. Further abstract closure context

The context below is not used later so it can be skipped but it seems natural. In this section we are lead to deal with the $0-1$ law holding for monadic second order logic (i.e. we quantify over the sets). For this aim we will use similar tools to those of §2. Looking again at Definition 2.9 or 2.12(2), clause (A), we note that there is an asymmetry: we try to represent $\mathrm{cl}^{k,m}(\bar{a}b,\mathcal{M}_n)$ and some $C \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ as free amalgamation over some B, small enough (with *a priori* bound depending on $\ell g(\bar{a})$ and k only, there $C = \mathrm{cl}^k(B,\mathcal{M}_n)$). Now this basis, B, of free amalgamation is included in $\mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ so it is without elements from $\mathrm{cl}^{k,m}(\bar{a}b,\mathcal{M}_n) \setminus \mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$.

Suppose we allow this and first we deal with the case $\mathcal{M}_n$ is a graph. Hence a member d of $\mathrm{cl}^{k,m}(\bar{a}b,\mathcal{M}_n)$ may code a subset of $\mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$: the set

$$\{c \in cl^{k^*,m^*}(\bar{a},\mathcal{M}_n) : \text{ the pair } \{c,d\} \text{ is an edge}\}.$$

So though we are interested in f.o. formulas $\varphi(\bar{x})$ speaking on $\mathcal{M}_n$, we are drawn into having $\psi_\varphi(\bar{x})$, the formula speaking on $\mathrm{cl}^{k_\varphi,m_\varphi}(\bar{x})$, being a monadic formula. Once we allow also three place relations and more, we have to use second order logic (still we can say which quantifiers we need because the witnesses for the elimination will come from the extensions of the $\mathrm{cl}^{k,m}(\bar{a},\mathcal{M}_n)$). For this elimination, thinking of an $\mathcal{M}_n$, we need that any possible kind of extension of $\mathrm{cl}^{k,m}(\bar{a},\mathcal{M}_n)$ occurs; so in the most natural cases, $|\mathrm{cl}^{k,m+1}(\bar{a},\mathcal{M}_n)|$ may be with $2^{|\mathrm{cl}^{k,m}(\bar{a},\mathcal{M}_n)|}$ elements, so in the natural case which we expect to be able to understand the situation is when there $\mathrm{cl}^{k,m}(\bar{a},\mathcal{M}_n) < \log_*(|\mathcal{M}_n|)$. Still possibly $\mathrm{cl}^{k,m+1}(\bar{a},\mathcal{M}_n)$ is not larger than $\mathrm{cl}^{k,m}(\bar{a},\mathcal{M}_n)$.

However there is a big difference between the monadic (e.g. graph where the relations coded on $\mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$ by members of $\mathrm{cl}^k(\bar{a}b,\mathcal{M}_n)$ are monadic) case and the more general case. For monadic logic addition theorems like 2.15 are known, but those are false for second order logic.

So we have good enough reason to separate the two cases. For readability we choose here to generalize the "simply almost nice with $\mathcal{K} = \mathcal{K}_\infty$" case only.

Context 3.1. As in §2 for $(\mathfrak{K},\mathrm{cl})$.

Definition 3.2. 1) The 0-1 context with a closure operation, $(\mathfrak{K},\mathrm{cl})$ is *s.m.a. (simply monadically almost) nice* if it is weakly nice, $\mathcal{K} = \mathcal{K}_\infty$, cl is transitive smooth local transparent (see Definitions 2.3(3),2.5(2),(3) and 2.9(4),(5)) and

(A) for every k and ℓ, there are $r = r(k,\ell)$, $k^* = k^*(k,\ell)$ and $t_1 = t_1(k,\ell)$, $t_2 = t_2(k,\ell)$ such that:

for every $\mathcal{M}_n$ random enough we have:
if $\bar{a} \in {}^\ell(\mathcal{M}_n)$, $b \in \mathcal{M}_n$, $\mathrm{cl}^k(\bar{a}b,\mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*}(\bar{a},\mathcal{M}_n)$
then there are B^*, B^1, B^2 such that:

(α) $\bar{a} \subseteq B^1$ and $\mathrm{cl}^r(B^1,\mathcal{M}_n) \subseteq \mathrm{cl}^{k^*}(\bar{a},\mathcal{M}_n)$ and $|B^1| \leq t_1$,

(β) $B^1 \subseteq B^2$, $B^2 \cap \mathrm{cl}^r(B^1,\mathcal{M}_n) = B^1$, $|B^2| \leq t_2$, $b \in B^2$,

(γ) $B^* \supseteq [\mathrm{cl}^k(\bar{a}b,\mathcal{M}_n) \setminus \mathrm{cl}^r(B^1,\mathcal{M}_n)] \cup B^2$, and $B_1 \leq_S B^*$ and $\mathrm{cl}^k(\bar{a}b,\mathcal{M}_n) \subseteq B^*$ (hence $\mathrm{cl}^k(\bar{a}b,B^*) = \mathrm{cl}^k(\bar{a}b,\mathcal{M}_n)$),

(δ) $\mathcal{M}_n \restriction B^* \underset{\mathcal{M}_n \restriction B^2}{\overset{\mathcal{M}_n}{\biguplus}} \mathcal{M}_n \restriction (B^2 \cup \mathrm{cl}^r(B^1,\mathcal{M}_n))$ (also here $\biguplus$ is the relation of being in free amalgamation),

(ε) if Q is a predicate from $\tau_\mathcal{K}$ and $\mathcal{M}_n \models Q(\bar{c})$, $\mathrm{Rang}(\bar{c}) \subseteq \mathrm{cl}^r(B^1,\mathcal{M}_n) \cup B^2$ *then:* $\mathrm{Rang}(\bar{c}) \cap B^2 \subseteq B^1$ or $\mathrm{Rang}(\bar{c}) \setminus B^2$ has at most one member; if this holds we say B^2 is monadic over $\mathrm{cl}^r(B^1,\mathcal{M}_n)$ inside $\mathcal{M}_n$,

(ζ) $(B^*, B^1, B^2, \bar{a}, b, k, r)$ is m.good (see below, m stands for monadically), so clearly $B \in \mathcal{K}_\infty$.

2) We say $(B^*, B^1, B^2, \bar{a}, b, k, r)$ is *m.good* when: B^*, B^1, $B^2 \in \mathcal{K}_\infty$ and $B^1 \le B^2 \le B^*$, $\bar{a} \subseteq B^1, b \in B^2$ and *for every* random enough $\mathcal{M}_n$, and $f : B^1 \hookrightarrow \mathcal{M}_n$, and $C^1 \in \mathcal{K}_\infty$ such that $\mathcal{M}_n \restriction \mathrm{cl}^r(f(B^1), \mathcal{M}_n) \subseteq C^1$, and $f^+ : B^2 \hookrightarrow C^1$ extending f such that $C^1 = f^+(B^2) \cup \mathrm{cl}^r(f(B^1), \mathcal{M}_n)$ and $f^+(B^2)$ is monadic over $\mathrm{cl}^r(f(B^1), \mathcal{M}_n)$ inside C^1 (see above, but not necessarily $C^1 \subseteq \mathcal{M}_n$) *there are* $g^+ : C^1 \hookrightarrow \mathcal{M}_n$ and $g : B^* \hookrightarrow \mathcal{M}_n$ such that $g \restriction B^2 = (g^+ \circ f^+) \restriction B^2$ and

$$g(B^*) \underset{g(B^2)}{\biguplus} g^+(C^1) \quad \text{and} \quad \mathrm{cl}^k(g(\bar{a}b), \mathcal{M}_n) \subseteq g(B^*) \cup \mathrm{cl}^r(g(B^1), \mathcal{M}_n).$$

3) Assume $\mathbf{E} \subseteq \{(C, B^1, B^2) : B^1 \le B^2 \le C \in \mathcal{K}\}$ is closed under isomorphism. We say B^2 is $\mathbf{E}$-over D inside N if $B^2 \le N \in \mathcal{K}$, $D \le N$ and $(N \restriction (B^2 \cup D), B^2 \cap D, B^2) \in \mathbf{E}$.

4) We say $(B^*, B^1, B^2, \bar{a}, b, k, r)$ is $\mathbf{E}$-good when $B^*, B^1, B^2 \in \mathcal{K}_\infty$ and $B^1 \le B^2 \le B^*$, $\bar{a} \subseteq B^1$, $b \in B^2$ and for every random enough $\mathcal{M}_n$ and $f : B^1 \hookrightarrow \mathcal{M}_n$ and $C^1 \in \mathcal{K}_\infty$ such that $\mathcal{M}_n \restriction c\ell^r(f(B^1), \mathcal{M}_n) \subseteq C^1$ and $f^+ : B^2 \hookrightarrow C^1$ extending f such that $C^1 = f^+(B^2) \cup \mathrm{cl}^r(f(B^1), \mathcal{M}_n)$ and $f^+(B^2)$ is $\mathbf{E}$-over $cl^r(f(B^1), \mathcal{M}_n)$ inside C^1(see above but not necessarily $C^1 \subseteq \mathcal{M}_n$) there are $g^+ : C^1 \hookrightarrow \mathcal{M}_n$ and $g : B^* \hookrightarrow \mathcal{M}_n$ such that $g \restriction B^2 = (g^+ \circ f^+) \restriction B^2$ and $g(B^*) \underset{g(B^2)}{\biguplus} g^+(C^1)$ and $\mathrm{cl}^k(g(\bar{a}b), \mathcal{M}_n) \subseteq g(B^*) \cup \mathrm{cl}^r(g(B^1), \mathcal{M}_n)$.

5) We say $\mathfrak{K}$ is s.$\mathbf{E}$.a nice if in 3.2(1) we replace clauses (ε), (ζ) by

(ε)$'$ B^2 is $\mathbf{E}$-over $\mathrm{cl}^r(B^1, \mathcal{M}_n)$ inside $\mathcal{M}_n$
(ζ)$'$ $(B^*, B^1, B^2, \bar{a}, b, k, r)$ is $\mathbf{E}$-good.

6) We say $\mathbf{E}$ is monadic if it is as in part (3) and $(C, B^1, B^2) \in \mathbf{E}$ implies

$$(\bar{a} \in Q^C \Rightarrow \mathrm{Rang}(\bar{a}) \cap B^2 \subseteq B^1) \ \vee \ (|\mathrm{Rang}(\bar{a}) \setminus B^2| \le 1).$$

7) We say $\mathbf{E}$ as in 3.2(3) is simply monadic *if* it is monadic and for any $B^1 \le B^2 \in \mathcal{K}$, letting

$$\Gamma_{B^2} = \Big\{\theta(y, \bar{b}) : \ \bar{b} \subseteq B^2 \text{ is with no repetition, } \theta(y, \bar{x}) \text{ is an atomic formula, each variable actually appearing}\Big\}$$

we have: the class

$$\Big\{(D, R_{\theta(y,\bar{b})}, c)_{\theta(y,\bar{b}) \in \Gamma_{B^2}, c \in B^1} : \ \begin{array}{l} D \in \mathcal{K}, \\ B^1 \le D, \ R_{\theta(y,\bar{b})} \text{ is a subset of } D \setminus B^1 \text{ and} \\ \text{there are } C^1, f \text{ such that:} (C^1, B^1, B^2) \in \mathbf{E} \\ D \le C^1 \in \mathcal{K}, \ f : B^2 \hookrightarrow C^1, \ f(B^2) \cap D = B^1, \\ f \restriction B^1 = \mathrm{id}_{B^1}, \text{ and for } \theta(y, \bar{b}) \in \Gamma \text{ we have} \\ R_{\theta(y,\bar{b})} = \{d \in D \setminus B^1 : C^1 \models \theta[d, f(\bar{b})]\}\Big\} \end{array}$$

is definable by a monadic formula[5].
8) We say that cl is monadically definable for $\mathcal{K}$ *if* for each k, letting $\bar{x} = \langle x_\ell : \ell < k\rangle$ some monadic formula $\Theta_k(y,x)$ we have $y \in \mathrm{cl}^k(\bar{x},\mathcal{M}_n) \iff \mathcal{M}_n \restriction \mathrm{cl}^k(\bar{x},\mathcal{M}_n) \models \Theta_k(\bar{x},y)$ holds for every random enough $\mathcal{M}_n$.
9) We say that $\mathbf{E}$ is trivial if it is $\{(C,B^1,B^2) : C \bigcup_{B^1} B^2,\ B^1 \leq B^2 \leq C\}$.

Lemma 3.3. *Assume* $(\mathfrak{K},\mathrm{cl})$ *is s.*$\mathbf{E}$*.a. nice and* $\mathbf{E}$ *is simply monadic and* cl *is f.o. definable or at least monadically definable (see 3.2(8)).* Then *for every f.o. formula* $\varphi(\bar{x})$ *there are* k *and a monadic formula* $\psi_\varphi(\bar{x})$ *such that:*

$(*)_{\varphi(\bar{x})}$ *for every random enough* $\mathcal{M}_n$*, for every* $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ *we have*

$$\mathcal{M}_n \models \varphi(\bar{a}) \quad \Leftrightarrow \quad \mathcal{M}_n \restriction \mathrm{cl}^k(\bar{a},\mathcal{M}_n) \models \psi_\varphi(\bar{a}).$$

Discussion 3.4. Some of the assumptions of 3.3 are open to manipulations; others are essential.
1) As said above, the "monadic" is needed in order to use an addition theorem (see 3.5), the price of removing it is high: essentially above we need that after finding the copy $g(B^2)$ realizing the required type over $\mathrm{cl}^k(B^2,\mathcal{M}_n)$, we need to find $g(B^*)$, or a replacement like B' in the proofs in §2 but only the holding of some formula $\varphi(\ldots,b,\ldots)_{b\in B^2}$ in B^* is important. Now what if the requirements on the type of $g(B^2)$ over $\mathrm{cl}^r(B^1,\mathcal{M}_n)$ are not coded by some subsets of $\mathrm{cl}^r(B^1,\mathcal{M}_n)$ but e.g. by two place relations on $\mathrm{cl}^r(B^1,\mathcal{M}_n)$? So naturally we allow quantification over two place relations in the formulas $\psi_\varphi(\bar{x})$. But then even though

$$B^* \bigcup_{B^2} \mathrm{cl}^r(B^1,\mathcal{M}_n) \cup B^2$$

not only the small formulas satisfied by $(B^2,b)_{b\in B^1}$ are important but also e.g. the answer to $B^* \cong \mathrm{cl}^r(B^1,\mathcal{M}_n)$.

It is natural to demand that all possibilities for the set of small formulas in second order logic satisfied by $B^* \cup \mathrm{cl}^r(B^1,\mathcal{M}_n)$ occur so this may include cases where B^* has to be of cardinality much larger than $\mathrm{cl}^k(\bar{a},\mathcal{M}_n)$. So we do not formulate such lemma. Of course some specific information may help to control the situation. We may however consider adding (in 3.2), the demand:

$\boxtimes_s$ if $Y \subseteq B^* \cup \mathrm{cl}^r(B^1,\mathcal{M}_n)$ and $Y \cap B^* \nsubseteq B^2$, $Y \cap \mathrm{cl}^r(B^1,\mathcal{M}_n) \nsubseteq B^1$, then Y is not s-connected, that is for some Y_1, Y_2, we have $Y = Y_1 \cup Y_2$, $|Y_1 \cap Y_2| \leq s$ and $Y_1 \bigcup_{Y_1\cap Y_2} Y_2$ (i.e. $\mathcal{M}_n \restriction Y_1 \bigcup_{\mathcal{M}_n \restriction Y_1 \cap Y_2} \mathcal{M}_n \restriction Y_2$).

In this case we can allow e.g. quantification on 2-place relations R such that $\mathcal{M}_n \restriction \mathrm{Dom}(R)$ is s-connected.

[5]We can restrict ourselves to the cases $C = \mathrm{cl}^k(B,C)$.

2) If $\mathbf{E}$ is monadic but not simply monadic, not much is changed: we should allow new quantifiers in ψ_φ. Let $C^1 <^{\mathbf{E}}_B C^2$ if $B \le C^1 \le C^2$ and $(C^2, B, B \cup (C^2 \setminus C^1)) \in \mathbf{E}$. We want the quantifier to say for $(C^1, R_{\theta(y,\bar b)}, c)_{\theta(\bar y,\bar b)\in\Gamma, c\in B}$ that it codes C^2 with $C^1 \le^{\mathbf{E}}_B C^2$ where $\Gamma = \Gamma_{B\cup(C^2\setminus C^1)}$, but then the logic should be defined such that we would be able to iterate.

The situation is similar to the case that in §2, we have: cl is definable or at least monadically definable.

3) In 3.3 we essentially demand

(∗) for each t, for random enough $\mathcal{M}_n$, for every $B \subseteq \mathcal{M}_n$, $|B| \le t$, if $\mathcal{M}_n \restriction \mathrm{cl}^k(\bar a, \mathcal{M}_n) <^{\mathbf{E}}_{\bar a} C$ then C is embeddable into $\mathcal{M}_n$ over $\mathrm{cl}^k(\bar a, \mathcal{M}_n)$.

Of course we need this just for a dense set of such C's, dense in the sense that a monadic sentence is satisfied, just like the use of B' in 2.12. That is we may replace clause (ζ) of Definition 3.2(1)(A) by

$(\zeta)'$ there is B' such that $(B', c, b)_{c\in B_2} \equiv'_\ell (B^*, c, b)_{c\in B_2}$ and $(B', B^1, B^2, \bar a, k)$ is m. good (and ℓ' large enough e.g. quantifier depth of ψ_{φ_1} in main case).

4) As we have done in 2.16(2), 2.17(2), we can add that the number of alternation of quantifiers of φ and the number of (possibly) alternation of monadic quantifier of ψ_φ are equal as long as the depth of the formulas from "simply monadic" is not counted (Always we can trivially increase the q.d. so we may ask about ψ_φ with minimal number. But for a specific $\langle \mathcal{M}_n : n < \omega\rangle$ we may get better. We can though look at minimal q.d. on all cases then it should be trivial.

5) Can we find a reasonable context where the situation from 3.3 and 3.4(1) above holds? Suppose we draw edges as here in $\mathcal{M}^0_n$ and redraw in the neighborhood of each edge. Let us describe drawing fully, this for a model on $[n]$. For each $i < j$ from $[n]$ we flip a coin $\mathcal{E}_{i,j}$ on whether we have (i, j) as a pre-edge, with probability $p^n_{i,j}$. If we succeed for $\mathcal{E}_{i,j}$ then for any pair (i', j') from $[n]$ we flip a coin $\mathcal{E}_{i,j,i',j'}$ with probability $p^n_{i,j,i',j'}$. The flippings are independent and finally for $i' < j'$, (i', j') is an edge if and only if for some $i < j$, (i, j) is a pre-edge, that is we succeed in $\mathcal{E}_{i,j}$ and we also succeed in $\mathcal{E}_{i,j,i',j'}$. For our case let ($\alpha \in (0, 1)_{\mathbb{R}}$ is irrational):

Distribution 1

$$p^n_{i,j} = p_{|i-j|} = \begin{cases} 1/|i-j|^\alpha & \text{when } |i-j| > 1 \\ 1/2^\alpha & \text{if } \ |i-j| = 1 \end{cases}$$

and $p^n_{i,j,i',j'} = \frac{1}{2^{|i-i'|+|j-j'|}}$;

Distribution 2

$p^n_{i,j}$ is as above and

$$p^n_{i,j,i',j'} = \begin{cases} \frac{1}{2^{|i-i'|+|j-j'|}} & \text{if } i = i' \ \vee \ j = j' \\ 0 & \text{if otherwise.} \end{cases}$$

Now distribution 2 seems to give us an example as in Lemma 3.3, distribution 1 fits the non-monadic case. Distribution 1 will give us, for some pre-edges (i, j), a lot of edges in the neighbourhood of it; of course for the average pre-edge there will be few. This give us a lot of $\leq_i$ extensions in that neighbourhood. We may wonder whether actually the 0–1 law holds. It is intuitively clear that for distribution 2 the answer is “yes”, for distribution 1 the answer is “no”.

6) Why in distribution 1 from (5) the 0–1 law should fail (in fact fails badly)? It seems to me that for distribution 1 we can find $A \subseteq B$ such that for every random enough $\mathcal{M}_n$, for some $f : A \hookrightarrow \mathcal{M}_n$, the number of $g : B \hookrightarrow \mathcal{M}_n$ extending f is quite large, and on the set of such g we can interpret an initial segment N_f of arithmetic even with $f(A)$ a segment, N_f in its neighbourhood. The problem is to compare such N_{f_1}, N_{f_2} with possibly distinct parameters, which can be done using a path of pre-edges from $f_1(A)$ to $f_2(A)$. But this requires further thoughts.

The case of distribution 2 should be similar to this paper.

We intend to return to this.

7) If $\mathbf{E}$ is trivial, then the claim above becomes (a variant) of the main claims in section 2 (the variant fulfill promises there).

Proof of 3.3:

This proof is similar to that of Lemma 2.16 and 2.17. We say in the claim that $\psi_\varphi(\bar{x})$ or $\psi_\varphi(\bar{x})$, k_φ witness $(*)_{\varphi(\bar{x})}$. We prove the statement by induction on q.d.$(\varphi(\bar{x}))$ and first note (by clause (d) of Definition 2.2) that $(*)_{\varphi(\bar{x})} \Longrightarrow (*)^+_{\varphi(\bar{x})}$ where $\psi_\varphi(\bar{x})$ will be monadic logic.

$(*)^+_{\varphi(\bar{x})}$ for every random enough $\mathcal{M}_n$, for every $\bar{a} \in {}^{lg(\bar{x})}(\mathcal{M}_n)$ and N if $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$ then $\mathcal{M}_n \models \varphi[\bar{a}] \Longleftrightarrow N \models \psi_\varphi[\bar{a}]$.

Case 1: Let $\varphi(\bar{x})$ be an atomic formula. Trivial.

Case 2: $\varphi(\bar{x})$ a Boolean combination of atomic formulas and formulas $\varphi(\bar{x})$ of the form $\exists y \varphi'(\bar{x}, y)$, φ' of quantifier depth$< r$, such that $(*)_{\exists y\varphi'(\bar{x},y)}$ holds. Clearly follows by case 3 and case 1.

Case 3: $\varphi(\bar{x}) = (\exists y)\varphi_1(\bar{x}, y)$. Let k_{φ_1}, ψ_{φ_1} be a witness for $(*)_{\varphi(\bar{x})}$ of 3.3 and let k_{φ_1} $\psi^2_{\varphi_1}$ be witness for $(*)^+_{\varphi_1(\bar{x})}$ holds for it (for φ_1). Let $r = r(k_{\varphi_1}, \ell g(\bar{x}))$, $k^* = k^*(k_{\varphi_1}, \ell g(\bar{x}))$, $t_1 = t_1(k_{\varphi_1}, \ell g(\bar{x}))$ and $t_2 = t_2(k_{\varphi_1}, \ell g \bar{x})$ be as in Definition 3.2(1)(A), more exactly its 3.2(4) variant. Let k_φ be k^*.

It is enough to prove the following two statements:

Statement 1: There is $\psi^1_\varphi(\bar{x})$ a monadic formula such that:

$(*)_1$ for every random enough $\mathcal{M}_n$, for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:
- $(\alpha)_1$ $\mathcal{M}_n \restriction \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n) \models \psi^1_\varphi(\bar{a})$
- $(\beta)_1$ $\mathcal{M}_n \models$“there is b satisfying $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n)$ such that $\varphi_1(\bar{a}, b)$ holds.”

Statement 2: There is $\psi^2_\varphi(\bar{x})$ a monadic formula such that:

$(*)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:

$(\alpha)_2$ $\mathcal{M}_n \restriction \mathrm{cl}^{k^*}(\bar a, \mathcal{M}_n) \models \psi^2_\varphi(\bar a)$

$(\beta)_2$ $\mathcal{M}_n \models$ "there is b satisfying $\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*}(\bar a, \mathcal{M}_n)$ such that $\varphi_1(\bar a, b)$ holds"

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one holds).

Note that as "$y \in \mathrm{cl}^{k^*}(\bar x)$" is monadically definable, by 3.2(8) and by the choice of k_φ we can in $(\alpha)_2$ replace cl^{k^*} by cl^{k_φ}, changing ψ^2_φ to $\psi^{2.5}_\varphi$, and similarly in $(\alpha)_1$ replace cl^{k^*} by cl^{k_φ} changing ψ^1_φ to $\psi^{1.5}_\varphi$.

Clearly these two statements are enough and $\psi^{1.5}_\varphi(\bar x) \vee \psi^{2.5}_\varphi(\bar x)$ is as required.

Proof of statement 1:

Easily, by the induction hypothesis and by the fact that the closure is sufficiently definable.

Proof of statement 2:

We will use a series of equivalent statements $\otimes_\ell$.

$\otimes_1$ is $(\beta)_2$,

$\otimes_2$ there are b and B^*, B_1, B_2 such that:

$b \in \mathcal{M}_n$, $\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*}(\bar a, \mathcal{M}_n)$, $\bar a \subseteq B_1 \subseteq \mathrm{cl}^{k^*}(\bar a, \mathcal{M}_n)$, $\mathrm{cl}^r(B_1, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*}(\bar a, \mathcal{M}_n)$, $|B_1| \le t_1$, $|B_2| \le t_2$, $B_1 \le B_2 \le B^*$, $b \in B^*$, $B^* \setminus B_2$ disjoint to $\mathrm{cl}^r(B_1, \mathcal{M}_n)$ and [6] $B_1 \le_s B^* \in \mathcal{K}_\infty$ and $B^* \underset{B_2}{\overset{\mathcal{M}_n}{\biguplus}} \mathrm{cl}^r(B_1, \mathcal{M}_n) \cup B_2)$ and $\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \subseteq B^*$ (hence $\mathrm{cl}^{k_{\varphi_1}}(\bar a b, B^*) = \mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n)$) and

$(B^*, B_1, B_2, \bar a, b, k, r)$ is $\mathbf{E}$–good and

$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar a, b)$.

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial, the implication $\Rightarrow$ holds by clause (A) in the definition 3.2.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by

$\oplus_3$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}}(\bar a b, M_n) \models \psi_{\varphi_1}(\bar a, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis i.e. choice of k_{φ_1} ψ_{φ_1}.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by

$\oplus_4$ $\mathcal{M}_n \restriction [B^* \cup \mathrm{cl}^{k_{\varphi_1}}(B_1, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar a, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+_{\varphi_1}$ being witnessend by $\psi^2_{\varphi_1}$, k_{φ_1} see the beginning of the proof, the definition of B^* and the choice of $\psi^2_{\varphi_1}$.

[6] the B' does not appear for simplicity only

For notational simplicity we assume $B \neq \emptyset$, and similarly assume $\bar{a}$ is with no repetition and apply Lemma 3.5 below with the vocabulary $\tau_{\mathcal{K}}$ to the case $s = \ell$, $\bar{z}^2$ empty, $\bar{z}^1 = \langle z^1_1 \rangle$, $\bar{z} = \langle z_1, \dots, z_\ell \rangle$, and $\psi(\bar{z}, \bar{z}^1, \bar{z}^2) = \psi(\bar{z}, z^1_1) = \psi^2_\varphi(\langle z_1, \dots, z_{\ell g(\bar{x})} \rangle, z^1_1)$ and get i^*, $\theta^1_i(\bar{z}, \bar{z}^1)$ and $\theta^2_i(\bar{z})$ for $i < i^*$ as there.
Next let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_5$ by

$\oplus_5$ letting $c_1, \dots, c_{t_2}$ list B_2 possibly with repetitions but such that $\{c_1, \dots, c_{t_1}\} = B_1$ and $\langle c_1, \dots, c_{\ell g(\bar{x})} \rangle = \bar{a}$ and *there is* $i < i^*$ such that:

(i) $B^* \models \theta^1_i[\langle c_1, \dots, c_{t_2} \rangle, b]$

(ii) $\mathcal{M}_n \restriction (B_2 \cup \mathrm{cl}^k(B_1, \mathcal{M}_n)) \models \theta^2_i[\langle c_1, \dots, c_{t_2} \rangle]$.

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$

Why? by the choice of θ^1_i, θ^2_i $(i < i^*)$.

Let $\mathcal{P} = \{(N, c_1, \dots, c_{t_2}) : N \in \mathcal{K}_\infty$, with the set of elements $\{c_1, \dots, c_{t_2}\}\}$. Let $\{(N_j, c^j_1, \dots, c^j_{t_2}) : j < j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j < j^*$ and $i < i^*$ choose if possible $(N_{j,i}, c^j_1, \dots, c^j_{t_2}, b^j_i)$ such that:

(i) $N_j \leq_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i \in N_{j,i} \setminus N_j$,

(iii) $N_{j,i} \models \theta^1_i(\langle c^j_1, \dots, c^j_{t_2} \rangle, b^j_i)$ and

(iv) $(N_{j,i}, \{c^j_1, \dots, c^j_{t_1}\}, \{c^j_1, \dots, c^j_{t_2}\}, \{c^j_1, \dots, c^j_{\ell g \bar{x}}, b^j_i\}, k)$ is **E**–good.

Let

$$w = \{(i,j) : i < i^*, j < j^* \text{ and } (N_{j,i}, c^j_1, \dots, c^j_t, b^j_i) \text{ is well defined}\}.$$

Let $\Gamma = \{\theta(y, \bar{x}) : \theta$ is a basic formula, $\bar{x} \subseteq \{x_1, \dots, x_{t_2}\}\}$.

As **E** is simply monadic (see Definition 3.2(4)) we have: for some monadic formula θ^3_i such that

(*) if $\{d_1, \dots, d_{t_1}\} \leq C \in \mathcal{K}$ letting $\Gamma =^{\mathrm{df}} \{\theta(y, \dots, x_{i(\ell)}, \dots)_{\ell < \ell(*)} : \theta$ an atomic formula for $\tau_{\mathcal{K}}$, every variable actually appear and $i(\ell) \in \{1, \dots, t_2\}\}$;

the following are *equivalent*:

(a) there are subsets R_θ of C for $\theta \in \Gamma$ and there are $C_1, d_t (t = t_1 + 1, \dots, t_2)$ satisfying $R_{\varphi(y,\bar{x})} \subseteq C$ and $C \leq C_1 \in \mathcal{K}$, $C_1 \setminus C = \{d_{t_1+1}, \dots, d_{t_2}\}$, and

$$R_{\theta(y,\dots,x_i,\dots)} = \{e \in C : C_1 \models \theta[e, \dots, d_i, \dots]\} \quad \text{for } \theta(y, \dots, x_i, \dots) \in \Gamma$$

and $C_1 \models \theta^2_i[d_1, \dots, d_{t_2}]$

(b) $C \models \theta^3_i[d_1, \dots, d_{t_1}]$.

Let

$\otimes_6$ there are b, B_1 such that: $b \in \mathcal{M}_n$, $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n)$, $\bar{a} \subseteq B_1 \subseteq \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n)$, $\mathrm{cl}^r(B_1, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n)$, $|B_1| \le t_1(k_{\varphi_1}, \ell\mathrm{g}(\bar{x}))$, and

$\oplus_6$ for some $c_1, \ldots, c_{t_1}$ listing B_1 such that $\bar{a} = \langle c_1, \ldots, c_{\ell\mathrm{g}(\bar{x})} \rangle$ *there are* $i < i^*$, $j < j^*$ such that:

(i) $(\mathcal{M}_n \restriction B_1, c_1, \ldots, c_{t_1}) \cong (N_j, c^j_1, \ldots, c^j_{t_1})$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embeds N_j into $\mathcal{M}_n$,

(ii) $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}}(B_1, \mathcal{M}_n) \models \theta^3_i(\langle c_1, \ldots, c_{t_1} \rangle)$.

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$.

Why? For proving $\otimes_5 \Rightarrow \otimes_6$ let $c_1, \ldots, c_t$ as well as $i < i^*$ be as in $\oplus_5$, let $j < j^*$ be such that $(\mathcal{M}_n \restriction B_1, c_1, \ldots, c_{t_1}) \cong (N_j, c^j_1, \ldots, c^j_{t_1})$. A main point is that B^* exemplifies that $(i, j) \in w$.
For proving $\otimes_6 \Rightarrow \otimes_5$ use part (B) of Definition 2.9. [Mor: irrelevant reference, Saharon check]

Now we have finished as $\otimes_6$ can be expressed as a monadic formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof. $\blacksquare_{3.3}$

The following is the parallel of 2.15 for monadic logic (see Gurevich [Gu], more [Sh 463]).

Lemma 3.5. *For finite vocabulary τ and monadic formula (in the vocabulary τ) $\psi(\bar{z}, \bar{z}^1, \bar{z}^2)$, $\bar{z} = \langle z_1, \ldots, z_s \rangle$, there are $i^* \in \mathbb{N}$ and monadic τ-formulas $\theta^1_i(\bar{z}, \bar{z}^1) = \theta^1_{i,\psi}(\bar{z}, \bar{z}^1)$, $\theta^2_i(\bar{x}, \bar{z}) = \theta^2_{i,\psi}(\bar{z}, \bar{z}^2)$ for $i < i^*$ each of quantifier depth at most that of ψ such that:*

if $N_1 \coprod_{N_0}^{N} N_2$, *$N_1 \cap N_2 = N_0$, $N_1 \cup N_2 = N$ and the set of elements of N_0 is $\{c_1, \ldots, c_s\}$, $\bar{c} = \langle c_1, \ldots, c_s \rangle$ and $\bar{c}^1 \in {}^{\ell\mathrm{g}(\bar{z}^1)}(N_1)$ and $\bar{c}^2 \in {}^{\ell\mathrm{g}(\bar{z}^2)}(N_2)$*
then

$$N \models \psi[\bar{c}, \bar{c}^1, \bar{c}^2] \quad \textit{iff} \quad \textit{for some } i < i^*, N_1 \models \theta^1_i[\bar{c}, \bar{c}^1] \textit{ and } N_2 \models \theta^2_i[\bar{c}, \bar{c}^2].$$

References

[Bl96] John T. Baldwin. Near model completeness and 0–1 laws. *preprint*, 1996.

[BlSh 528] John T. Baldwin and Saharon Shelah. Randomness and Semigenericity. *Transactions of the American Mathematical Society*, **349**:1359–1376, 1997. arxiv:math.LO/9607226.

[BoSp] Ravi B. Boppana and Joel Spencer. Smoothness laws for random ordered graphs. In *Logic and random structures (New Brunswick, NJ, 1995)*, volume 33 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 15–32. American Mathematical Society, Providence, Rhode Island, 1997.

[Gu] Yuri Gurevich. Monadic Second–Order Theories. In J. Barwise and S. Feferman, editors, *Model Theoretic Logics*, Perspectives in Mathematical Logic, chapter XIII, pages 479–506. Springer-Verlag, New York Berlin Heidelberg Tokyo, 1985.

[LuSh 435] Tomasz Łuczak and Saharon Shelah. Convergence in homogeneous random graphs. *Random Structures & Algorithms*, **6**:371–391, 1995. arxiv:math.LO/9501221.

[Sh 550] Saharon Shelah. 0–1 laws. *Preprint*. arxiv:math.LO/9804154.

[Sh 637] Saharon Shelah. 0.1 Laws: Putting together two contexts randomly . *in preparation*.

[Sh:F192] Saharon Shelah. Lecture notes on 0–1 laws, October'95, Rutgers.

[Sh 581] Saharon Shelah. When 0–1 law hold for $G_{n,\bar{p}}$, $\bar{p}$ monotonic. *in preparation*.

[Sh 463] Saharon Shelah. On the very weak $0-1$ law for random graphs with orders. *Journal of Logic and Computation*, **6**:137–159, 1996. arxiv:math.LO/9507221.

[Sh 548] Saharon Shelah. Very weak zero one law for random graphs with order and random binary functions. *Random Structures & Algorithms*, **9**:351–358, 1996. arxiv:math.LO/9606230.

[Sh 517] Saharon Shelah. Zero-one laws for graphs with edge probabilities decaying with distance. Part II. *Fundamenta Mathematicae*, **185**:211–245, 2005. arxiv:math.LO/0404239.

[ShSp 304] Saharon Shelah and Joel Spencer. Zero-one laws for sparse random graphs. *Journal of the American Mathematical Society*, **1**:97–115, 1988.

[Sp] Joel Spencer. Survey/expository paper: zero one laws with variable probabilities. *Journal of Symbolic Logic*, **58**:1–14, 1993.

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

E-mail address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah