

WAS SIERPINSKI RIGHT III?
CAN CONTINUUM-C.C. TIMES
C.C.C. BE CONTINUUM-C.C.?
SH481

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ABSTRACT. We prove the consistency of: if B_1, B_2 are Boolean algebras satisfying the c.c.c. and the 2^{\aleph_0} -c.c. respectively then $B_1 \times B_2$ satisfies the 2^{\aleph_0} -c.c.

We start with a universe with a Ramsey cardinal (less suffice).

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§0 INTRODUCTION

We heard the problem from Velickovic who got it from Todorćević, it says “are there P , a c.c.c. forcing notion, and Q is a 2^{\aleph_0} -c.c. forcing such that $P \times Q$ is not 2^{\aleph_0} -c.c.?” We can phrase it as a problem of cellularity of Boolean algebras or topological spaces.

We give a negative answer even for 2^{\aleph_0} regular, this by proving the consistency of the negation. The proof is close to [Sh 288],§3 which continues [Sh 276],§2 and is close to [Sh 289]. A recent use is [Sh 473].

We start with $V \models “\lambda$ is a Ramsey cardinal” , use c.c.c. forcing blowing the continuum to λ . Originally the paper contained the consistency of e.g. $2^{\aleph_0} \rightarrow [\aleph_2]_3^2$, 2^{\aleph_0} the first k_2^2 -Mahlo, (weakly inaccessible)(remember $k_2^2 < \omega$) but the theorem presented arrive here to satisfactory state (for me) earlier. See more [Sh 546]. I thank Mariusz Rabus for corrections.

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What problems do [Sh 276], [Sh 288], [Sh 289], [Sh 473] and [Sh 481] raise? The most important are (we state the simplest uncovered case for each point):

- A Question.** 1) Can we get e.g. $CON(2^{\aleph_0} \rightarrow [\aleph_2]_3^2)$; more generally raise μ^+ to higher.
 2) Can we get $CON(\aleph_\omega > 2^{\aleph_0} \rightarrow [\aleph_1]_3^2)$; generally lower 2^μ , the exact \aleph_n seems to me less exciting.
 3) Can get e.g. $CON(2^\mu > \lambda \rightarrow [\mu^+]_3^2)$?

Also concerning [Sh 473].

- B Question.** 1) Can we get the continuity on a non-meagre set for functions $f : {}^\kappa 2 \rightarrow {}^\kappa 2$?
 2) what can we say on continuity of 2-place functions?
 3) What about n -place functions (after [Sh 288]).

- C Question.** 1) Can we get e.g. for $\mu = \mu^{<\mu} > \aleph_0, CON(\text{if } P \text{ is } 2^\mu\text{-c.c., } Q \text{ is } \mu^+\text{-c.c. then } P \times Q \text{ is } 2^\mu\text{-c.c.})$?
 2) Can we get e.g. $CON(\text{if } P \text{ is } 2^{\aleph_0}\text{-c.c., } Q \text{ is } \aleph_2\text{-c.c. then } P \times Q \text{ is } 2^{\aleph_0}\text{-c.c.})$
 3) Can we get e.g. $CON(2^{\aleph_0} > \lambda > \aleph_0, \text{ and if } P \text{ is } \lambda\text{-c.c., } Q \text{ is } \aleph_2\text{-c.c. then } P \times Q \text{ is } \lambda\text{-c.c.})$

On A1 see [Sh 546].

Discussion Maybe the solution to (A1) is by using squared demand and if in $\delta < \lambda, cf(\delta) > \mu$ we guess $\langle N_s : s \in [B]^{\leq 2} \rangle, \mathcal{C}$, try to by Q_δ to add a large subset of B on which only two colours appear; but we want to do it also when $otp(B) > \mu^+$. Naturally we assume that if $\delta' < \delta, cf(\delta') > \mu^+, \delta' = \text{Sup}(B \cap \delta')$ this was done $\langle N_s : s \in [B \cap \delta']^{\leq 2} \rangle$, but we need more: including dividing B to μ set on each only two colours (by Q_δ). To do this and have λ, k_3^2 -Mahlo (rather than measurable or $(\lambda \rightarrow^+ (\omega_1)_2^{\leq \omega})$) we have to use a very strong diamond.

For problem (A2) the natural thing is to use systems $\bar{N} = \langle N_s : s \in [B]^{\leq 2} \rangle$ which are not end extension systems. Then it is natural to use the forcing on this

stronger; “specializing” not only the colouring but all $P_{N_s \cap \lambda}$ -names of ordinals (as defined in §8). This required a suitable squared diamond; this has not yet been clarified (actually we need somewhat less than \bar{N}, S).

But for problem (A3) a weaker version of this suggest itself. As in the solution of 1, λ is k_2^2 -Mahlo $\langle \langle N_s^\delta : s \in [B^\delta]^{\leq 2} \rangle : \delta \in S \rangle$ is such that $N_s^\delta \subseteq H(\delta^+)$, (we think of N_s^δ as guessing the isomorphism type over $H(\delta)$). Now we have to define a preliminary forcing R , λ -complete or at least strategically λ -complete, satisfying the λ^+ -c.c. So we have “copies” of $\langle N_s^\delta : s \in [B^\delta]^{\aleph_2} \rangle$ which behave like Δ -systems. [Saharon].

But if want to get tree like systems (ease requirement on forcing) we need more (enough dependency). For simplicity $\lambda = cf(\chi) = \chi^\lambda$ and use the following instead forcing and do it with.

We can have in V (or force),

$\bar{C} = \langle C_\delta : \delta \in S \rangle$ a square, $S \subseteq \chi, \bar{S} =: \{\delta \in S : cf(\delta) = \lambda\} \subseteq \chi$ stationary, $[\delta \in S \Rightarrow otp(C_\delta) \leq \lambda]$, we have squared diamond $\bar{C} = \langle C_\delta : \delta \in S \rangle$, and we choose for $\delta \in S, B_\delta \prec \mathfrak{A}_\delta, \|B_\delta\| \leq |\omega + otp(C_\delta)|, \delta_1 \in C_{\delta_2} \Rightarrow B_{\delta_1} \prec B_{\delta_2}$ & $\langle B_\delta : \delta \in C_\delta \cup \{\delta_1\} \rangle \in B_{\delta_1}, \delta_2 = \sup C_{\delta_2} = B_{\delta_1} = \bigcup_{\delta \in C_{\delta_1}} B_\delta^*$.

Now we can copy the squared diamond $\langle \langle N_{\delta,s} : s \in [B_\delta]^{\leq 2} \rangle : \delta < \lambda \rangle$

getting $\langle \langle N_{\delta,s}^* : s \in [B_\delta^*]^{\leq 2} \rangle : \delta \in S \rangle$. We then define $\langle P_i, Q_i, A_i : i < \chi \rangle, |a_i| \leq \chi_2$ (or $|a_i| < \kappa$).

Concerning (B1) the expected theorem holds. For 2-place function, note that the Sierpinski colouring can be viewed as a function from ${}^\mu 2$ to $\{0_\mu, 1_\mu\} \subseteq {}^\mu 2$. So the best we can hope for is

(B2)' can we get the consistency of $(*)_\mu$ for any 2-place function f from ${}^\mu 2$ to ${}^\mu 2$ there are (everywhere) non-meagre $A \subseteq {}^\mu 2$ and continuous functions f_0, f_2 from A to ${}^\mu 2$ such that $(\forall \eta, \nu \in A)[f(\eta, \nu) \in \{f_0(\eta, \nu), f_1(\eta, \nu)\}]$.

So we have to put together the proofs of [Sh 473] (continuity on non-meagre), [Sh 288] ($2^{\aleph_0} \rightarrow [\aleph_1]_3^2$), using the k_2^2 - Mahlo only and replace \aleph_0 by μ .

For problem (B3), $\mu = \aleph_0$ we have to generalize [Sh 288],§3. But also for $\mu > \aleph_0$, we have to consider what can be said on the partition of trees (see [Sh 288],§4 for a positive answer for large cardinal (indestructible measurable $n^* < \omega$)).

Concerning (C2), the problem with the approach to (A1) is “why should Q_δ from [Sh 481],1.7 satisfies Q_δ^2 is c.c.c.”

Similarly (C3)(A3). A natural approach is to consider $\langle N_s : s \in [B]^{< \aleph_0} \rangle$ and use a subset $X \in [B]^{otp B}$ such that for different uses we use almost disjoint X 's. This was not completed but we restrict ourselves to “not only P satisfies the 2^{\aleph_0} - c.c. but even P^n (for each $n < \omega$).

Concerning (C1) we cannot replace ${}^\mu n$ elements of B by 1, but we can use a directed system, so “ P satisfies the 2^μ -c.c.”, is replaced by “for $\sigma < \mu, P^\sigma$ satisfies the 2^μ -c.c.” (or slightly less).

Another question is Velickovic’s question answered for Borel c.c.c. forcing in [Sh 480]; i.e. (C4).

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Preliminaries

0.A. Let $<^*_\chi$ be a well ordering of

$H(\chi) = \{x : \text{the transitive closure of } x \text{ has cardinality } < \chi\}$ agreeing with the usual well ordering of the ordinals,

P (and Q, R) will denote forcing notions, i.e. partial order with a minimal element $\emptyset = \emptyset_P$.

A forcing notion P is λ -closed if every increasing sequence of members of P , of length less than λ , has an upper bound.

0.B. For sets of ordinals, A and B , define $H_{B,A}^{OP}$ as the maximal order preserving bijection between initial segments of A and B , i.e., it is the function with domain $\{\alpha \in A : otp(\alpha \cap A) < otp(B)\}$ and $H_{A,B}^{OP}(\alpha) = \beta$ if and only if $\alpha \in A$, $\beta \in B$ and $otp(\alpha \cap A) = otp(\beta \cap B)$.

Definition 0.1. $\lambda \rightarrow^+ (\alpha)_\mu^{<\omega}$ holds provided that: if whenever F is a function from $[\lambda]^{<\omega}$ to λ , $F(w) < \min(w)$, $C \subseteq \lambda$ is a club then there is $A \subseteq C$ of order type α such that $\left[w_1, w_2 \in [A]^{<\omega}, |w_1| = |w_2| \Rightarrow F(w_1) = F(w_2) \right]$.

(See [Sh:f],XVII,4.x).

0.1A Remark. 1) If λ is Ramsey cardinal then $\lambda \rightarrow^+ (\lambda)_\mu^{<\omega}$.

2) If $\lambda = \text{Min}\{\lambda : \lambda \rightarrow (\alpha)_\mu^{<\omega}\}$ then λ is regular and $\lambda \rightarrow^+ (\alpha)_\mu^{<\omega}$.

Definition 0.2. $\lambda \rightarrow [\alpha]_{\kappa,\theta}^n$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{F(w) : w \in [A]^n\}$ has power $\leq \theta$.

Definition 0.3. A forcing notion P satisfies the Knaster condition (has property K) if for any $\{p_i : i < \omega_1\} \subseteq P$ there is an uncountable $A \subseteq \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

§1 CONSISTENCY OF “C.C.C. $\times 2^{\aleph_0}$ -C.C. = 2^{\aleph_0} - C.C.”

The a_i 's are not really necessary but (hopefully) clarify.

1.1 Definition. 1) $\mathcal{K}_{\mu,\kappa}$ is the family of $\bar{Q} = \langle P_\gamma, \underline{Q}_\beta, a_\beta : \gamma \leq \alpha, \beta < \alpha \rangle$, where:

- (a) $\langle P_\gamma, \underline{Q}_\beta : \gamma \leq \alpha, \beta < \alpha \rangle$ is a finite support iteration
- (b) every $P_\gamma, \underline{Q}_\gamma$ satisfies the c.c.c.
- (c) \underline{Q}_β is a P_β -name which depends just on $G_{P_\beta} \cap P_{a_\beta}^*$ (see below; hence it is in $V[G_{P_\beta}^*]$), and $|\underline{Q}_\beta| \leq \kappa$ and its set of members $\subseteq V$ (for simplicity)
- (d) $a_\beta \subseteq \beta, |a_\beta| \leq \mu$ and $\gamma \in a_\beta \Rightarrow a_\gamma \subseteq a_\beta$.

2) For such \bar{Q} we call $a \subseteq \ell g(\bar{Q}), \bar{Q}$ -closed if $[\beta \in a \Rightarrow a_\beta \subseteq a]$ and let

$$P_a^* = P_a^{\bar{Q}} =: \left\{ p \in P_\alpha : \text{Dom}(p) \subseteq a \text{ and for all } \beta \in \text{Dom}(p) : p(\beta) \in V \right. \\ \left. \text{(not a name) and } p \upharpoonright a_\beta \Vdash “p(\beta) \in Q_\beta” \right\}$$

(so we are defining P_a^* by induction on $\text{sup}(a)$) ordered by the order of $P_{\text{sup}(a)}$.

3) $\mathcal{K}_{\mu,\kappa}^k$ is the class of $\bar{Q} \in \mathcal{K}_{\mu,\kappa}$ such that if $\beta < \gamma \leq \ell g(\bar{Q}), \text{cf}(\beta) \neq \aleph_1$ then P_γ/P_β satisfies the Knaster condition (actually we can use somewhat less). Let $\mathcal{K}_{\mu,\kappa}^n = \mathcal{K}_{\mu,\kappa}$.

4) If defining \bar{Q} we omit P_α we mean $\bigcup_{\beta < \alpha} P_\beta$ if α is limit, $P_\beta * \underline{Q}_\beta$ if $\alpha = \beta + 1$.

5) We do not lose, if we assume $\underline{Q}_\beta \subseteq [\kappa]^{<\aleph_0}$ and the order \subseteq ; (then 1.2(1)(g) becomes trivial as for closed $p, q \in P_j^*, p \upharpoonright a \leq q \upharpoonright a$).

1.2 Claim. 1) Assume $x \in \{n, k\}$ and $\bar{Q} = \langle P_\gamma, \underline{Q}_\beta, a_\beta : \beta < \alpha, \gamma \leq \alpha \rangle \in \mathcal{K}_{\mu,\kappa}^x$.

Then

- (a) for $\alpha^* < \alpha, \bar{Q} \upharpoonright \alpha^* =: \langle P_\gamma, \underline{Q}_\beta, a_\beta : \beta < \alpha^*, \gamma \leq \alpha^* \rangle$ belongs to $\mathcal{K}_{\mu,\kappa}^x$
- (b) P_α^* is a dense subset of P_α
- (c) for any \bar{Q} -closed $a \subseteq \alpha, P_a^* \leq P_\alpha$ (in particular P_α^* is a dense subset of P_α); moreover, if $p \in P_\alpha^*$ then $p \upharpoonright a \in P_a^*$ and $[p \upharpoonright a \leq q \in P_a^* \Rightarrow r =: q \cup p \upharpoonright (\alpha \setminus a) \in P_\alpha \ \& \ p \leq r \ \& \ q \leq r]$
- (d) for a \bar{Q} -closed $a \subseteq \alpha, \langle P_{a \cap \gamma}^*, \underline{Q}_\beta, a_\beta : \beta \in a, \gamma \in a \rangle$ belongs to $\mathcal{K}_{\mu,\lambda}^x$
(except renaming; not used)
- (e) if \underline{Q}_α is a P_a^* -name of a c.c.c. forcing notion of cardinality $\leq \kappa$, each member of \underline{Q}_α is from $V, a \subseteq \alpha$ is \bar{Q} -closed, $|a| \leq \mu$ and $P_{\alpha+1} = P_\alpha * \underline{Q}_\alpha$ and \underline{Q}_α satisfies the Knaster condition or at least $\beta < \alpha \Rightarrow P_\alpha * \underline{Q}_\alpha / P_{\beta+1}$ satisfies the Knaster condition then
 $\langle P_\gamma, \underline{Q}_\beta, a_\beta : \beta < \alpha + 1, \gamma \leq \alpha + 1 \rangle \in \mathcal{K}_{\mu,\lambda}^x$

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- (f) if $n < \omega$, $p_1, \dots, p_n \in P_{\alpha^*}$ and
- (*) for every $\beta \in \bigcup_{\ell=1}^n \text{Dom}(p_\ell)$ for some $m = m_{\beta,\ell} \in \{1, \dots, n\}$ we have
- $$p_m \upharpoonright \beta \Vdash \text{“} p_\ell(\beta) \leq_{Q_\beta} p_m(\beta) \text{ for } \ell = \{1, \dots, n\}\text{”}$$
- then p_1, \dots, p_n has a least common upper bound p which is defined by:
- $$\text{Dom}(p) = \bigcup_{\ell=1}^n \text{Dom}(p_\ell), p_\ell(\beta) = p_{m_{\beta,\ell}}(\beta), \text{ so in particular } p \in P_{\alpha^*} \text{ and}$$
- $$\bigwedge_{\ell=1}^n p_\ell \in P_{\alpha^*} \Rightarrow p \in P_{\alpha^*}$$
- (g) if $p_\ell \leq p$ and $p_\ell \in P_\gamma^*$ for $\ell < n$, and a_k is \bar{Q} -closed for $k < m$ then there is $p' \in P_\gamma^*$, such that $p \leq p'$ and $P_{a_k}^* \models p_\ell \upharpoonright a_k \leq p' \upharpoonright a_k$ for $\ell < n, k < m$.

2) If $x \in \{n, k\}$ and $\delta < \lambda$ is a limit ordinal, for $\alpha < \delta$ we have

$$\langle P_\gamma, Q_\beta, a_\beta : \beta < \alpha, \gamma \leq \alpha \rangle \in \mathcal{K}_{\mu,\lambda}^x \text{ and } P_\delta = \bigcup_{\gamma < \delta} P_\gamma \text{ then } \langle P_\gamma, Q_\beta, a_\beta : \beta < \delta, \gamma \leq \delta \rangle$$

belongs to $\mathcal{K}_{\mu,\lambda}^x$.

Proof. Straightforward.

Essentially by [Sh 289],2.4(2),p.176 (which is slightly weaker and its proof left to the reader, so we give details here).

1.3 Claim. Assume $\lambda \rightarrow^+ (\omega\alpha^*)_\mu^{<\omega}$ (e.g. λ a Ramsey cardinal, $\alpha^* = \lambda$) $\chi > \lambda$, $x \in H(\chi)$.

- 1) There is an end extension strong $(\chi, \lambda, \alpha^*, \mu, \aleph_0, \omega)$ -system for x (see Definition 1.3A).
- 2) There is an end extension $(\chi, \lambda, \alpha, \mu, \aleph_0, \omega)$ -system for x if x is Ramsey or $\lambda = \text{Min}\{\lambda : \lambda \rightarrow (\omega\alpha^*)_\mu^{<\omega}\}$ (also then the condition holds for every $\mu' < \mu$).

1.3A Definition. 1) We say $\bar{N} = \langle N_s : s \in [B]^{<1+n} \rangle$ is a $(\chi, \lambda, \alpha, \theta, \sigma, n)$ -system if:

- (a) $N_s \prec (H(\chi), \in)$ (or of some expansion)
 $\theta + 1 \subseteq N_s, \|N_s\| = \theta, \sigma \succ (N_s) \subseteq (N_s)$
- (b) $B \subseteq \lambda, \text{otp}(B) = \alpha$
- (c) $n \leq \omega$ (equally is allowed but $1 + \omega = \omega$ so s is always finite)
- (d) $N_s \cap N_t \subseteq N_{s \cap t}$
- (e) $N_s \cap B = s$
- (f) if $|s| = |t|$ then $N_s \cong N_t$ say $H_{s,t}$ is an isomorphism from N_s onto N_t (necessarily $H_{s,t}$ is unique)
- (g) if $s' \subseteq s, t' = \{\alpha \in t : (\exists \beta \in s') [|\beta \cap s| = |\alpha \cap t|]\}$ then $H_{s',t'}, H_{s,t}$ are compatible functions; $H_{s,s} = \text{id}$,
 $H_{s,t} \supseteq H_{s,t}^{OP}, H_{s_0,s_1} \circ H_{s_1,s_2} = H_{s_0,s_2}, H_{t,s} = (H_{s,t})^{-1}$
- (h) $\text{sup}(N_s \cap \lambda) < \text{Min}\{\alpha \in B : \bigwedge_{\gamma \in s} \gamma < \alpha\}$.

2) We add the adjective “strong” if in strengthen clause (d) by

$$(d)^+ N_s \cap N_t = N_{s \cap t} \text{ (so in clause (g), } H_{s',t'} \subseteq H_{s,t} \text{).}$$

3) We add the adjective “end extension” if

$$(i) s \triangleleft t \Rightarrow N_s \cap \lambda \triangleleft N_t \cap \lambda \text{ (where } A \triangleleft B \text{) means } A = B \cap \min(B \setminus A)$$

4) We add “for x ” if $x \in N_s$ for every $s \in [B]^{<1+n}$, and $H_{s,t}(x) = x$.

1.3B Remark. If λ is a Ramsey cardinal (or much less see [Sh:f],XVII,4.x,[Sh 289],§4) then we have if $\gamma \in s \cap t$, $s \cap \gamma = t \cap \gamma$ and $y \in N_s$ then in $(H(\chi), \in, <^*_\chi)$ the elements y and $H_{t,s}(y)$ realizes the same type over $\{i : i < \gamma\}$. [prove?]

Proof. 1) Let $C = \{\delta < \lambda : \text{for every } \alpha < \delta \text{ there is } N \prec (H(\chi), \in, <^*_\chi) \text{ such that } \mu + 1 + \alpha \subseteq N \text{ and } \sup(N \cap \lambda) < \delta\}$. Clearly C is a club of λ .

Let $B_0 = \{\alpha_i : i < \omega\alpha^*\} \subseteq C$, $(\alpha_i \text{ strictly increasing})$ be indiscernible in $(H(\chi), \in, <^*_\chi, x)$ (see Definition 0.1). Let $B = \{\alpha_i : i < \omega\alpha^* \text{ limit}\}$. For $s \in [B_0]^{<\aleph_0}$ let N_s^0 = the Skolem Hull of $s \cup \{i : i \leq \mu\} \cup \{x, \lambda\}$ under the definable functions of $(H(\chi), \in, <^*_\chi)$ and

$$N_s = \cup \left\{ N_{t_1}^0 \cap N_{t_2}^0 : t_1, t_2 \in [B_0]^{<\aleph_0} \text{ and } s = t_1 \cap t_2 \right\}.$$

Clearly

$$(*) \quad \|N_s\| \leq \mu \text{ and } \{x, \lambda\} \subseteq N_s.$$

Now we shall show

$$(*)_1 \text{ if } s \in [B]^{<\aleph_0}, y \in N_s \text{ then for every finite } t \subseteq B_0 \text{ there is } u \in [B_0]^{<\aleph_0} \text{ such that } s \subseteq u, u^* \cap t \subseteq s \text{ and } y \in N_u^0.$$

As $y \in N_s$ there are $s_1, s_2 \in [B_0]^{<\aleph_0}$ such that $y \in N_{s_1}^0 \cap N_{s_2}^0$ and $s = s_1 \cap s_2$. Let $s_1 \cup s_2 = \{\alpha_{i_0}, \dots, \alpha_{i_{m-1}}\}$ (increasing), and let $n^* = \sup\{n : \text{for some } \beta, \beta + n \in t\} + 1$, and define for $\ell \leq m$ a function f_ℓ with domain $s_1 \cup s_2$, such that

$$f_\ell(\alpha_{i_k}) = \begin{cases} \alpha_{i_{k+1}} & \text{if } k \geq m - \ell \text{ and } i_k \notin s \\ \alpha_{i_k} & \text{otherwise} \end{cases}$$

Note that

- ⊗₁ for $\ell < m$, $f_\ell \upharpoonright s_1 = f_{\ell+1} \upharpoonright s_1$ or $f_\ell \upharpoonright s_2 = f_{\ell+1} \upharpoonright s_2$ (or both) [why? as $i_\ell \in s_2 \setminus s_1 \setminus s$ or $i_\ell \in s_2 \setminus s_1 \setminus s$ or $i_\ell \in s = s_1 \cap s_2$].
- ⊗₂ f_ℓ is order preserving with domain $s_0 \cup s_1$, $f_\ell \upharpoonright s = \text{the identity}$.

As $y \in N_{s_1}^0 \cap N_{s_2}^0$ there are terms τ_1, τ_2 such that

$$y = \tau_1(\dots, \alpha_{i_k}, \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, \alpha_{i_k}, \dots)_{\alpha_{i_k} \in s_2}.$$

Using the indiscernibility of B_0 we can prove by induction on $\ell \leq m$ that

$$\otimes_{3,\ell} y = \tau_1(\dots, f_\ell(i_{\alpha_{i_k}}, \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, f_\ell(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_2}.$$

[Why? For $\ell = 0$ this is given by the choice of τ_1, τ_2 . For $\ell + 1$ note that by \otimes_2 , $f_{\ell+1} \circ f_\ell^{-1}$ is an order preserving function from $\text{Rang}(f_\ell)$ onto $\text{Rang}(f_{\ell+1})$.

By $\otimes_{3,\ell}$ and “ B_0 is indiscernible” we know

$\tau_1(\dots, f_\ell(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, f_\ell(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_2}$. By the last two sentences and the indiscernibility of B_0

$$\tau_1(\dots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \dots)_{\alpha_{i_k} \in s_2}.$$

But $(f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})) = f_{\ell+1}(\alpha_{i_k})$ so

$$\tau_1(\dots, f_{\ell+1}(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, f_{\ell+1}(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_2}.$$

But by \otimes_1 for some $e \in \{1, 2\}$ we have $f_\ell \upharpoonright s_e = f_{\ell+1} \upharpoonright s_e$, so

$\tau_e(\dots, f_{\ell+1}(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_e} = \tau_e(\dots, f_\ell(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_e}$ but the latter is equal to y (by the induction hypothesis), hence the former so by the last sentence

$$y = \tau_1(\dots, f_{\ell+1}(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, f_{\ell+1}(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_2}.$$

So we have carried the induction on $\ell \leq m$, and for $\ell = m$ we get $y \in N_{f_m(s_1)}^0$, but by the choice of n^* and f_m clearly $f_m(s_1) \cap t \subseteq s$, and we have proved $(*)_1$.

Now we can note

$(*)_2$ if $s \in [B]^{<\aleph_0}$ and $y_1, \dots, y_n \in N_s$ then for some $s_1, s_2 \in [B_0]^{<\aleph_0}$ we have:
 $s = s_1 \cap s_2$ and $y_1, \dots, y_n \in N_{s_1}^0 \cap N_{s_2}^0$.

[Why? We can find $u_1, \dots, u_n \in [B_0]^{<\aleph_0}$ such that $s \subseteq u_\ell, y_\ell \in N_{u_\ell}^0$ (as $y_\ell \in N_s$). Now by $(*)_1$ for each $\ell = 1, 2, \dots, n$ we can find $v_\ell \in [B_0]^{<\aleph_0}$ such that $s \subseteq v_\ell, s = v_\ell \cap (\bigcup_{m=1}^n u_m)$ and $y_\ell \in N_{v_\ell}^0$. Let $u = \bigcup_{i=1}^n u_i, v = \bigcup_{\ell=1}^n v_\ell$, clearly $y_1, \dots, y_n \in N_u^0 \cap N_v^0$ and $u \cap v = s$, as required].

Now as we have Skolem functions $(*)_2$ implies

$(*)_3$ $N_s \prec (H(\chi), \in, <^*_\chi)$

Also trivially

$(*)_4$ $N_s^0 \prec N_s$ hence $\mu + 1 \subseteq N_s$

$(*)_5$ $s \subseteq t \Rightarrow N_s \prec N_t$.

Also

$(*)_6$ $N_{s_1} \cap N_{s_2} = N_{s_1 \cap s_2}$ for $s_1, s_2 \in [B]^{<\aleph_0}$.

[Why? The inclusions $N_{s_1 \cap s_2} \subseteq N_{s_1} \cap N_{s_2}$ follows from $(*)_5$; for the other direction let $y \in N_{s_1} \cap N_{s_2}$. By $(*)_1$ as $y \in N_{s_1}$ there is t_1 such that $s_1 \subseteq t_1 \in [B_0]^{<\aleph_0}, t_1 \cap (s_1 \cup s_2) = s_2$ and $y \in N_{t_1}^0$. By $(*)_1$, as $y \in N_{s_2}$ there is t_2 such that $s_2 \subseteq t_2 \in [B_0]^{<\aleph_0}, t_2 \cap (s_1 \cup s_2 \cup t_1) = s_1$ and $y \in N_{t_2}^0$. So $y \in N_{t_1}^0 \cap N_{t_2}^0$, but easily $t_1 \cap t_2 = s_1 \cap s_2$].

$(*)_7$ $\sup(N_s \cap \lambda) < \text{Min}\{\alpha \in B : \bigwedge_{\gamma \in s} \gamma < \alpha\}$.

[why? as $B_0 \subseteq C$ and see the Definition of C].

Now check that (a)-(h) of Definition 1.3A holds.

Now $\langle N_s : s \in [B]^{<\aleph_0} \rangle$ is as required.

2) If λ is Ramsey, without loss of generality $\text{otp}(B_0) = \lambda$ and it is easy to check 1.3A(i). The other case is like [Sh 289],§4. $\square_{1.3}$

1.4 Theorem. *Assume $\aleph_0 < \mu \leq \kappa < \lambda = \text{cf}(\lambda), \lambda$ strongly inaccessible, λ a Ramsey cardinal, and $\diamond_{\{\delta < \lambda : \text{cf}(\delta) = \aleph_1\}}$ (can be added by a preliminary forcing). Then we have P such that:*

- (a) P is a c.c.c. forcing of cardinality λ adding λ reals (so the cardinals and cardinal arithmetic in V^P should be clear), in particular in V^P we have $2^{\aleph_0} = \lambda$
- (b) \Vdash_P “MA holds for c.c.c. forcing notions of cardinality $\leq \mu$ and $< \lambda$ dense sets (and even for c.c.c. forcing notions of cardinality $\leq \kappa$ which are from $V[A]$ for some $A \subseteq \mu$)”
- (c) \Vdash_P “if B is a λ -c.c. Boolean algebra, $x_i \in B \setminus \{0\}$ for $i < \lambda$ then for some $Z \subseteq \lambda$, $|Z| = \aleph_1$ and $\{x_i : i \in Z\}$ generates a proper filter of B (i.e. no finite intersection is 0_B)”
- (d) \Vdash_P “if B_1 is a c.c.c. Boolean algebra, B_2 is a λ -c.c. Boolean algebra then $B_1 \times B_2$ is a λ -c.c. Boolean algebra.”

Proof. Let $\langle A_\delta : \delta < \lambda, \text{cf}(\delta) = \aleph_1 \rangle$ exemplifies the diamond. We choose by induction on $\alpha < \lambda$, $\bar{Q}^\alpha = \langle P_\gamma, \bar{Q}_\beta, a_\beta : \gamma \leq \alpha, \beta < \alpha \rangle \in \mathcal{K}_{\mu, \kappa}^n$ such that $\alpha^1 < \alpha \Rightarrow \bar{Q}^{\alpha^1} = \bar{Q}^\alpha \upharpoonright \alpha^1$. In limits α use 1.2(2), for $\alpha = \beta + 1$, $\text{cf}(\beta) \neq \aleph_1$ take care of (b) by suitable bookkeeping using 1.2(1)(e). If $\alpha = \beta + 1$, $\text{cf}(\beta) = \aleph_1$ and A_β codes $p \in P_\beta$ and P_β -names of a Boolean algebra \bar{B}_β and sequence $\langle x_i^\beta : i < \beta \rangle$ of non-zero members of \bar{B}_β , and p forces (\Vdash_{P_β}) that there is in $V[G_{P_\beta}]$ some c.c.c. forcing notion Q of cardinality $\leq \mu$ adding some $Z \subseteq \beta$, $|Z| = \aleph_1$ with $\{x_i^\beta : i \in Z\}$ generating a proper filter of \bar{B}_β then we choose \bar{Q}_β , if $p \in G_{P_\beta}$, as such Q . If $p \notin G_{P_\beta}$ or there is no such Q in $V[G_{P_\beta}]$, then \bar{Q}_β is e.g. Cohen forcing.

So every \bar{Q}^α is defined, let $P = \bigcup_{\gamma < \lambda} P_\gamma$. Clearly (a) + (b) holds and (d) follows

by (c). So the rest of the proof is dedicated to proving (c).

So let $p \in P$, $p \Vdash$ “ \bar{B} a λ -c.c. Boolean algebra, $x_i \in \bar{B} \setminus \{0_B\}$ for $i < \lambda$ ” without loss of generality the set of members of \bar{B} is λ .

Let $x = \langle P, p, \bar{B}, \langle x_i : i < \lambda \rangle \rangle$, $\chi = \lambda^+$, by Claim 1.3 there are $A \in [\lambda]^\lambda$ and $\langle N_s : s \in [A]^{<\aleph_0} \rangle$ as there (for $\kappa = \mu + \kappa$ here standing for μ there). Let

$$C = \left\{ \delta < \lambda : \delta \text{ a strong limit cardinal } > \kappa + \mu, [\alpha < \delta \Rightarrow \bar{Q} \upharpoonright \alpha \in H(\delta)], \right. \\ \delta = \sup(A \cap \delta), s \in [A \cap \delta]^{<\aleph_0} \Rightarrow \sup(\lambda \cap N_s) < \delta, \\ \left. \bar{B} \upharpoonright \delta \text{ a } P_\delta\text{-name, and for } i < \delta \text{ we have } x_i \text{ a } P_\delta\text{-name} \right\}.$$

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For some accumulation point δ of C , $cf(\delta) = \aleph_1$ and A_δ codes $\langle p, \underline{B} \upharpoonright \delta, \langle x_i : i < \delta \rangle \rangle$. We shall show that for some $q, p \leq q \in P_\delta$ and $q \Vdash_{P_\delta}$ “there is Q as required above”. By the inductive choice of Q_δ this suffices.

Let $A^* \subseteq A \cap \delta$, $\text{otp}(A^*) = \omega_1$, $\delta = \sup(A^*)$ and $\langle \delta_i : i < \omega_1 \rangle$ increasing continuous, $\delta = \bigcup_{i < \omega_1} \delta_i$, $\delta_i \in C$, $A^* \cap \delta_0 = \emptyset$, $|A^* \cap [\delta_i, \delta_{i+1})| = 1$.

In V^{P_δ} we define:

$$Q = \left\{ u : u \in [A^*]^{<\aleph_0}, \text{ and } \underline{B} \models \text{“} \bigcap_{i \in u} x_i \neq 0_B \text{”} \right\}$$

ordered by inclusion. It suffices to prove that some $q, p \leq q \in P_\delta$, q forces that: Q is c.c.c. with $\bigcup G_Q$ an uncountable set; now clearly q forces that $\{x_i : i \in \bigcup G_Q\}$ generates a proper filter of \underline{B} .

If not, we can find q_i, u_i such that:

$$p \leq q_i \in P_\delta^* \text{ and } q_i \Vdash_{P_\delta} \text{“} u_i \in Q \text{” (where } u_i \in [A^*]^{<\aleph_0}\text{)}$$

and $\langle (q_i, u_i) : i < \omega_1 \rangle$ are pairwise incompatible in $P_\delta * Q$.

Let v_i be a finite subset of A^* such that: $u_i \subseteq v_i$, and

$$(*) [v \subseteq A^* \ \& \ v \text{ finite} \ \& \ \gamma \in (\text{Dom } q_i) \cap N_v \Rightarrow \gamma \in (\text{Dom } q_i) \cap N_{v \cap v_i}].$$

By Fodor’s Lemma for some stationary, $S \subseteq \omega_1, u^*, v^*, n^*$ and $i(*)$ we have: for $i < j$ in S ,

$$v_i \cap \delta_i = v^* \subseteq \delta_{i(*)}, v_i \subseteq \delta_j, |v_i| = n^*, u_i \cap \delta_i = u^*$$

$$i(*) = \text{Min}(S)$$

$\{|\gamma \cap v_i| : \gamma \in u_i\}$ does not depend on i

$$q_i \upharpoonright \delta_i \in P_{\delta_{i(*)}}^*$$

$$q_i \in P_{\delta_j}^*.$$

Let $b_i =: N_{v_i} \cap \lambda$, so b_i is necessarily \bar{Q}^δ -closed and $|b_i| = \kappa$. Let $q_i^1 = q_i \upharpoonright b_i$, so necessarily $q_i^1 \in P_{b_i}^*$ (see 2.2(1)(c)). Easily $P_{b_i}^* \subseteq N_{v_i}$ (though do not belong to it) so $q_i^1 \in N_{v_i}$.

Let $q_i^2 =: H_{v_{i(*)}, v_i}(q_i^1)$, so $q_i^2 \in P_{b_{i(*)}}^*$; let $q_i^3 =: (q_i \upharpoonright \delta_{i(*)}) \cup \left[q_i^2 \upharpoonright (N_{v_{i(*)}} \cap \lambda \setminus \delta_{i(*)}) \right]$ by 1.2(1)(c) we know $q_i^3 \in P_{\sup(b_{i(*)})+1}^*$ and $q_i^2 \leq q_i^3$, even without loss of generality $q_i^2 \leq q_i^3 \upharpoonright b_{i(*)}$. As $P_{\sup(b_{i(*)})+1}^* \triangleleft P_\delta$ and P_δ satisfies the c.c.c. clearly for

some $i < j$ from S , q_i^3, q_j^3 , are compatible in $P_{\sup(b_{i(*)})+1}^*$, so let $r \in P_{\sup(b_{i(*)})+1}^*$ be a common upper bound. So $q_i^3 \upharpoonright (\delta_{i(*)} \cap b_{i(*)}) \leq r \upharpoonright (\delta_{i(*)} \cap b_{i(*)})$ and $q_j^3 \upharpoonright (\delta_{i(*)} \cap b_{i(*)}) \leq r \upharpoonright (\delta_{i(*)} \cap b_{i(*)})$ and $q_i^3 \upharpoonright b_{i(*)} \leq r \upharpoonright b_{i(*)}, q_j^3 \upharpoonright b_{i(*)}$.

Without loss of generality $\text{Dom}(r) \subseteq b_{i(*)} \cup \delta_{i(*)}$ (allowed as $b_{i(*)}$ and $\delta_{i(*)}$ are closed, see 1.2(1)(c)); let $r_i = H_{v_i, v_{i(*)}}(r \upharpoonright b_{i(*)})$ and similarly $r_j = H_{v_j, v_{i(*)}}(r \upharpoonright b_{i(*)})$.

Note that $r_i \in P_{\delta_j}^*, r_j \in P_{\delta_j}^*, r_j \upharpoonright \delta_j = r_i \upharpoonright \delta_i = r \upharpoonright \delta_{i(*)}$. Hence $r_i \cup r_j \in P_{\delta_j}^*$.

Case 1. $r_i \cup r_j$ do not force (i.e. $\Vdash_{P_{\delta_j}}^*$) that

$$\bar{B} \Vdash \text{“} \bigcap_{\alpha \in u_i \cup u_j} \bar{x}_\alpha = 0_{\bar{B}} \text{”}.$$

Then there is $r' \in P_{\delta_j}, r_i \leq r', r_j \leq r'$ forcing the negation. So without loss of generality $r' \in P_{\delta_j}^*$, and (as all parameters appearing in the requirements on r' are in $N_{v_i \cup v_j}$ also) $r' \in P_{\lambda \cap (N_{v_i \cup v_j})}^*$. Now

r', r, q_i, q_j has an upper bound $r'' \in P_{\delta_j}$.

[Why? By 1.2(1)(f), we have to check the condition (*) there, so let

$\beta \in \text{Dom}(r') \cup \text{Dom}(r) \cup \text{Dom}(q_i) \cup \text{Dom}(q_j)$].

Subcase a. $\beta \in \delta_{i(*)} \setminus N_{v_i \cup v_j}$. Note that $N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$ (see choice of the N_u 's and definition of the b_ε 's) but $\text{Dom}(r') \subseteq N_{v_i \cup v_j} \cap \lambda$, so $\beta \notin \text{Dom}(r')$. Now

$$q_i \upharpoonright \delta_i = q_i \upharpoonright \delta_{i(*)} = q_i^3 \upharpoonright \delta_{i(*)} \leq r$$

$$q_j \upharpoonright \delta_j = q_j \upharpoonright \delta_{i(*)} = q_j^3 \upharpoonright \delta_{i(*)} \leq r.$$

So $r \upharpoonright \beta \Vdash_{P_\beta} \text{“} q_i(\beta) \leq r(\beta), q_j(\beta) \leq r(\beta) \text{”}$ and $\beta \notin \text{Dom}(r')$. So we have confirmed (*) from 1.2(1)(f) for this subcase.

Subcase b. $\beta \in \delta_{i(*)} \cap N_{v_i \cup v_j}$.

Exactly as above:

$N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$, so $\beta \in N_{v^*}, \beta \in \delta_{i(*)} \cap b_{i(*)}$. Also

$q_i \upharpoonright b_{i(*)} = q_i^1 \upharpoonright \delta_{i(*)} = q_i^2 \upharpoonright \delta_{i(*)} = q_i^3 \upharpoonright (\delta_{i(*)} \cap b_{i(*)}) \leq r \upharpoonright (\delta_{i(*)} \cap b_{i(*)})$ and

$q_j \upharpoonright b_{i(*)} = q_j^1 \upharpoonright \delta_{i(*)} = q_j^2 \upharpoonright \delta_{i(*)} = q_j^3 \upharpoonright (\delta_{i(*)} \cap b_{i(*)}) \leq r \upharpoonright (\delta_{i(*)} \cap b_{i(*)})$ and

$r \upharpoonright (\delta_{i(*)} \cap b_{i(*)}) \leq r'$ (as $H_{v_i, v_{i(*)}}$ is the identity on $\delta_{i(*)} \cap b_{i(*)}$).

The last three inequalities confirm the requirement in 1.2(1)(f) (as $\beta \in \delta_{i(*)} \cap b_{i(*)}$, see above).

Subcase c. $\beta \in (\delta \setminus \delta_{i(*)}) \setminus N_{v_i \cup v_j}$.

In this case $\beta \notin \text{Dom}(r')$ (as $r' \in N_{v_i \cup v_j}$). Also $\delta_{i(*)} < \delta_i < \delta_j < \delta$ and:

$$\text{Dom}(r) \setminus \delta_{i(*)} \subseteq (b_{i(*)} \cup \delta_{i(*)}) \setminus \delta_{i(*)} \subseteq [\delta_{i(*)}, \delta_i)$$

$$\text{Dom}(q_i) \setminus \delta_{i(*)} \subseteq [\delta_i, \delta_j)$$

$$\text{Dom}(q_j) \setminus \delta_{i(*)} \subseteq [\delta_j, \delta).$$

So β belongs to at most one of $\text{Dom}(r')$, $\text{Dom}(r)$, $\text{Dom}(q_i)$, $\text{Dom}(q_j)$ so the requirement $(*)$ from 1.2(1)(f) holds trivially.

Subcase d. $\beta \in (\delta \setminus \delta_{i(*)}) \cap N_{v_i \cup v_j}$.

Clearly $\beta \notin \text{Dom}(r)$.

We know $q_i \upharpoonright b_i = q_i^1$, $r_i \leq r'$, $H_{v_i(*), v_i}(q_i^1) = q_i^2 \leq q_i^3 \upharpoonright b_{i(*)} \leq r \upharpoonright b_{i(*)}$ hence $q_i^1 \leq H_{v_i(*), v_i}^{-1}(r \upharpoonright b_{i(*)}) = H_{v_i, v_i(*)}(r \upharpoonright b_{i(*)}) = r_i$ but $r_i \leq r'$, so together $q_i^1 \leq r'$, and similarly $q_j^1 \leq r'$. As we have noted $\beta \notin \text{Dom}(r)$ we have finished confirming condition $(*)$ from 1.2(1)(f).

So really r, r', q_i, q_j has a least common upper bound, hence $(r'', u_i \cup u_j) \in P_\delta * Q$ exemplified $(q_i, u_i), (q_j, u_j)$ are compatible, as required.

Case 2. Not 1.

Let $\langle s_\beta : \beta < \lambda \rangle$ be such that:

$$s_\beta \in [A]^{<\aleph_0}, v^* \subseteq s_\beta, |s_\beta \setminus v^*| = |v_i \setminus v^*|, \sup(v^*) < \delta_{i(*)} < \min(s_\beta \setminus v^*)$$

$$\delta < \min(s_\beta \setminus v^*) \text{ (for simplicity)}$$

$$\beta < \gamma \Rightarrow \max(s_\beta) < \min(s_\gamma \setminus v^*).$$

As the truth value of $\bigcap_{\alpha \in u_i} x_\alpha$ is a P_a^* -name for some closed $a \in N_{v_i}$ of cardinality $\leq \mu$, and $q_i \Vdash [B \Vdash \text{“} \bigcap_{\alpha \in u_i} x_\alpha \neq 0_B \text{”}]$ clearly $q_i^1 \Vdash [B \Vdash \text{“} \bigcap_{\alpha \in u_i} x_\alpha \neq 0_B \text{”}]$.

For $\beta < \lambda$ let $\gamma^\beta = H_{s_\beta, v_i(*)}(r \upharpoonright b_{i(*)})$, and $u'_\beta = H_{s_\beta, v_i(*)}(u_0)$. Let

$$Y = \{\beta < \lambda : r^\beta \in \underline{G}_P\}.$$

Clearly:

$$r^\beta \Vdash_{P_\lambda} [B \Vdash \text{“} \bigcap_{i \in u'_\beta} x_i \neq 0_B \text{”}].$$

Clearly $p \leq r^\beta$ and for some β we have $r^\beta \Vdash \text{“} Y \in [\lambda]^\lambda \text{ (and } p \in \underline{G}_P \text{)”}$ and by the assumption of the case:

$$p \Vdash \text{“} \left\{ \bigcap_{i \in u'_\beta} x_i : \beta \in Y \right\} \text{ is a set of non-zero members of } B$$

any two having zero intersection in B ”.

This contradicts an assumption on B . □_{1.4}

* * *

We can phrase the consistency result as one on colouring.

1.5 Lemma. 1) In 1.4 we can add:

(e) if c is a symmetric function from $\left[2^{\aleph_0}\right]^{<\omega}$ to $\{0, 1\}$ then at least one of the following holds:

(α) we can find pairwise disjoint $w_i \subseteq 2^{\aleph_0}$ for $i < 2^{\aleph_0}$ such that:
 $c \upharpoonright [w_i] < \aleph_0$ is constantly zero but

$$\bigwedge_{i < j} (\exists u \subseteq w_i, \exists v \subseteq w_j) \left[c[u \cup v] = 1 \right]$$

(β) we can find an unbounded $B \subseteq 2^{\aleph_0}$ such that $c \upharpoonright [B]^{<\omega}$ is constantly 0.

It is natural to ask:

1.6 Question. Can we replace 2^{\aleph_0} by $\lambda < 2^{\aleph_0}$? \aleph_1 by $\mu < \lambda$? What is the consistency strength of the statements we prove consistent? (see later). Does λ strongly inaccessible k_2^2 -Mahlo (see [Sh289]) suffice?

1.7 Discussion. Of course, 1.5(e) \Rightarrow 1.4(c) \Rightarrow 1.4(d). Starting with λ weakly compact we can get a c.c.c. forcing notion P of cardinality λ , such that in V^P , $2^{\aleph_0} = \lambda$ and (e) of 1.5 holds for $c : \left[2^{\aleph_0}\right]^2 \rightarrow \{0, 1\}$ (so $c(u) = 0$ if $|u| \neq 2$) and this suffices for the result. Also we can generalize to higher cardinals. We shall deal with this elsewhere.

1.8 Theorem. Concerning the consistency strength, in 1.4 it suffices to assume

(*) λ is strongly inaccessible and for every $F : [\lambda]^{<\aleph_0} \rightarrow \mu$ and club C we can find $B \subseteq C$, (or just $B \subseteq \lambda$) otp(B) = ω_1 such that

- (a) B is F -indiscernible i.e. if $n < \omega$, $u, v \in [B]^n$ then $F(u) = F(v)$
- (b) for every $n < \omega$ there is $B' \in [C]^\lambda$ such that:

$$\text{if } u \in [B']^n \text{ and } v \in [B]^n \text{ then } F(u) = F(v)$$

Proof. Let $R = \{\bar{Q} : \bar{Q} \in H(\lambda), \bar{Q} \in \mathcal{K}_{\mu, \kappa}^n\}$ ordered by $\bar{Q}^1 < \bar{Q}^2$ if $\bar{Q}^1 = \bar{Q}^2 \upharpoonright \ell g(\bar{Q}^1)$. Clause (b) takes care also of “the end extension” clause and for 1.3(A)(4), Clause (b) the proof is the same.

A somewhat less natural property though suffices.

(Note: Clause (b) also helps to get rid of the club C).

1.9 Claim. *In 1.4 it suffices to assume*

- (*)' *if $F : [\lambda]^{<\aleph_0} \rightarrow \mu$ then there is $B \subseteq \lambda$, $otp(B) = \omega$ such that*
- (a) *$F \upharpoonright [B]^n$ is constant for $n < \omega$*
 - (b) *if $u \triangleleft v^\ell \in [B]^{<\aleph_0}$ for $\ell = 1, 2$ then we can find $v_i \in [\lambda]^n$ for $i < \lambda$, $u \subseteq v_i$, $\min(v_i \setminus u) \geq i$, and $i < j \Rightarrow F(v^1 \cup v^2) = F(v_i \cup v_j)$.*

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