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ON TREE IDEALS

Abstract. Let ℓ^0 and m^0 be the ideals associated with Laver and Miller forcing, respectively. We show that $\mathbf{add}(\ell^0) < \mathbf{cov}(\ell^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ are consistent. We also show that both Laver and Miller forcing collapse the continuum to a cardinal $\leq \mathfrak{h}$.

Introduction and Notation In this paper we investigate the ideals connected with the classical tree forcings introduced by Laver [La] and Miller [Mi]. Laver forcing \mathbb{L} is the set of all trees p on ${}^{<\omega}\omega$ such that p has a stem and whenever $s \in p$ extends $\mathit{stem}(p)$ then $\mathit{Succ}_p(s) := \{n : s \hat{\ } n \in p\}$ is infinite. Miller forcing \mathbb{M} is the set of all trees p on ${}^{<\omega}\omega$ such that p has a stem and for every $s \in p$ there is $t \in p$ extending s such that $\mathit{Succ}_p(t)$ is infinite. The set of all these splitting nodes in p we denote by $\mathit{Split}(p)$. For any $t \in \mathit{Split}(p)$, $\mathit{Split}_p(t)$ is the set of all minimal (with respect to extension) members of $\mathit{Split}(p)$ which properly extend t . For both \mathbb{L} and \mathbb{M} the order is inclusion.

The Laver ideal ℓ^0 is the set of all $X \subseteq {}^\omega\omega$ with the property that for every $p \in \mathbb{L}$ there is $q \in \mathbb{L}$ extending p such that $X \cap [q] = \emptyset$. Here $[q]$ denotes the set of all branches of q . The Miller ideal m^0 is defined analogously, using conditions in \mathbb{M} instead of \mathbb{L} . By a fusion argument one easily shows that ℓ^0 and m^0 are σ -ideals.

The additivity (**add**) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (**cov**) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined – ${}^\omega\omega$ in our case. Clearly $\omega_1 \leq \mathbf{add}(\ell^0) \leq \mathbf{cov}(\ell^0) \leq \mathfrak{c}$ and $\omega_1 \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{c}$ hold.

The main result in this paper says that there is a model of ZFC where $\mathbf{add}(\ell^0) < \mathbf{cov}(\ell^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ hold. The motivation was that by a result of Plewik [Pl] it was known that the additivity and the covering number of the ideal connected with Mathias forcing are the same and they are equal to the cardinal invariant \mathfrak{h} – the least cardinality of a family of maximal antichains of $\mathcal{P}(\omega)/\mathit{fin}$ without a common refinement. On the other hand, in [JuMiSh] it was shown that $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ is consistent, where

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s^0 is Marczewski's ideal – the ideal connected with Sacks forcing \mathbb{S} . Intuitively, \mathbb{L} and \mathbb{M} sit somewhere between Mathias forcing and \mathbb{S} . In [GoJoSp] it was shown that under Martin's axiom $\mathbf{add}(\ell^0) = \mathbf{add}(m^0) = \mathfrak{c}$, whereas this is false for s^0 (see [JuMiSh]).

The method of proof for $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ in [JuMiSh] is the following: For a forcing P denote by $\kappa(P)$ the least cardinal to which forcing with P collapses the continuum. In [JuMiSh] it is shown that $\mathbf{add}(s^0) \leq \kappa(\mathbb{S})$. In [BaLa] it was shown that in $V^{\mathbb{S}_{\omega_2}}$ $\kappa(\mathbb{S}) = \omega_1$ holds – where \mathbb{S}_{ω_2} is the countable support iteration of length ω_2 of \mathbb{S} . Hence $V^{\mathbb{S}_{\omega_2}} \models \mathbf{add}(s^0) = \omega_1$. On the other hand, a Löwenheim-Skolem argument shows that $V^{\mathbb{S}_{\omega_2}} \models \mathbf{cov}(s^0) = \omega_2$.

Our method of proof is similar. Denoting by P_{ω_2} a countable support iteration of length ω_2 of \mathbb{L} and \mathbb{M} (each occuring on a stationary set), in §2 we prove the following:

Theorem

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(\ell^0) = \mathbf{add}(m^0) < \mathbf{cov}(\ell^0) = \mathbf{cov}(m^0) = \omega_2$$

The crucial steps in the proof are to show that $\kappa(\mathbb{L}), \kappa(\mathbb{M})$ equal ω_1 and $\mathbf{add}(\ell^0) \leq \kappa(\mathbb{L}), \mathbf{add}(m^0) \leq \kappa(\mathbb{M})$ holds.

We will use the standard terminology for set theory and forcing. By \mathfrak{b} we denote the least cardinality of a family of functions in ${}^\omega\omega$ which is unbounded with respect to eventual dominance and \mathfrak{d} will be the least cardinality of a dominating family in ${}^\omega\omega$. Moreover \mathfrak{p} is the least cardinality of a filter base on $([\omega]^\omega, \subseteq^*)$ without any lower bound, and \mathfrak{t} is the least cardinality of a decreasing chain in $([\omega]^\omega, \subseteq^*)$ without any lower bound. It is easy to see that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

1. Upper and lower bounds

- 1.1 Theorem** (1) $\mathfrak{t} \leq \mathbf{add}(\ell^0) \leq \mathbf{cov}(\ell^0) \leq \mathfrak{b}$
 (2) $\mathfrak{p} \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{d}$

Proof of (1): We have to prove the first and the third inequality. For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ be an unbounded family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\exists^\infty k) f(k) < f_\alpha(k)\}$$

Clearly $\bigcup\{X_\alpha : \alpha < \mathfrak{b}\} = {}^\omega\omega$. We claim $X_\alpha \in \ell^0$. Let $p \in \mathbb{L}$. We define $q \in \mathbb{L}$ as follows: $stem(q) := stem(p)$, and for any s extending $stem(q)$ we have $s \in q$ if and only if $s \in p$ and $(\forall k)$ if $|stem(q)| \leq k < |s|$ then $s(k) \geq f_\alpha(k)$. Then clearly $q \in \mathbb{L}$, q extends p and $[q] \cap X_\alpha = \emptyset$.

In order to prove the first inequality we use the following notation from [JuMiSh]: Let $Q := \{\bar{A} = \langle A_s : s \in {}^{<\omega}\omega \rangle : (\forall s) A_s \in [\omega]^\omega\}$. For $\bar{A} \in Q$ we define a sequence of Laver trees $\langle p_s(\bar{A}) : s \in {}^{<\omega}\omega \rangle$ as follows: $p_s(\bar{A})$ is the unique Laver tree such that $\text{stem}(p_s(\bar{A})) = s$ and if $t \in p_s(\bar{A})$ extends s then $\text{Succ}_{p_s(\bar{A})}(t) = A_t$.

For $\bar{A}, \bar{B} \in Q$ we define:

$$\bar{A} \subseteq \bar{B} \Leftrightarrow (\forall s) A_s \subseteq B_s$$

$$\bar{A} \subseteq^* \bar{B} \Leftrightarrow (\forall s) A_s \subseteq^* B_s$$

$$\bar{A} \leq^* \bar{B} \Leftrightarrow (\forall s) A_s \subseteq^* B_s \wedge (\forall^\infty s) A_s \subseteq B_s$$

Here \leq^* is a slight but important modification of \subseteq^* from [JuMiSh].

Fact 1.2 (Q, \leq^*) is \mathfrak{t} -closed.

Proof of 1.2 Suppose $\langle \bar{A}_\alpha : \alpha < \gamma \rangle$ where $\gamma < \mathfrak{t}$ is a decreasing sequence in (Q, \leq^*) . Let $\bar{A}_\alpha := \langle A_s^\alpha : s \in {}^{<\omega}\omega \rangle$. Since $\gamma < \mathfrak{t}$ there is $\bar{B}' = \langle B'_s : s \in {}^{<\omega}\omega \rangle \in Q$ such that $(\forall \alpha < \gamma) \bar{B}' \subseteq^* \bar{A}_\alpha$. Define $f_\alpha : {}^{<\omega}\omega \rightarrow \omega$ such that $(\forall s) B'_s \setminus f_s(\alpha) \subseteq A_s^\alpha$. Since $\mathfrak{t} \leq \mathfrak{b}$ there exists $f : {}^{<\omega}\omega \rightarrow \omega$ such that $(\forall \alpha)(\forall^\infty s) f_\alpha(s) \leq f(s)$. Now let $B_s := B'_s \setminus f(s)$ and $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega \rangle$. It is easy to check that $(\forall \alpha < \gamma) \bar{B} \leq^* \bar{A}_\alpha$.

Fact 1.3 Suppose $X \in \ell^0$ and $\bar{A} \in Q$. There exists $\bar{B} \in Q$ such that $\bar{B} \subseteq \bar{A}$ and $(\forall s \in {}^{<\omega}\omega) [p_s(\bar{B})] \cap X = \emptyset$.

Proof of 1.3: First note that if $D := \{p \in \mathbb{L} : [p] \cap X = \emptyset\}$ then D is open dense and even 0-dense, i.e. for every $p \in \mathbb{L}$ there exists $q \in D$ extending p such that $\text{stem}(q) = \text{stem}(p)$. The proof of this is similar to Laver's proof in [La] that the set of Laver trees deciding a sentence in the language of forcing with \mathbb{L} is 0-dense: Suppose $p \in \mathbb{L}$ has no 0-extension whose branches are not in X . Then inductively we can construct $q \in \mathbb{L}$ extending p such that every extension of q has a branch in X , contradicting $X \in \ell^0$.

Using this it is straightforward to construct \bar{B} as desired.

Fact 1.4: Suppose $X \subseteq {}^\omega\omega$, $\bar{A}, \bar{B} \in Q$, $\bar{B} \leq^* \bar{A}$ and $(\forall s) [p_s(\bar{A})] \cap X = \emptyset$. Then $(\forall s) [p_s(\bar{B})] \cap X = \emptyset$.

Proof of 1.4: Clearly, if $F \subseteq p_s(\bar{B})$ is finite, then

$$[p_s(\bar{B})] = \bigcup \{[p_t(\bar{B})] : t \in p_s(\bar{B}) \setminus F\}$$

But for almost all $t \in p_s(\bar{B})$, $p_t(\bar{B})$ extends $p_t(\bar{A})$. So clearly $[p_s(\bar{B})] \subseteq [p_s(\bar{A})]$ and hence $[p_s(\bar{B})] \cap X = \emptyset$.

End of the proof of 1.1(1). Suppose we are given $\langle X_\alpha : \alpha < \gamma \rangle$ and $q \in \mathbb{L}$, where $\gamma < \mathfrak{t}$ and $(\forall \alpha) X_\alpha \in \ell^0$. Choose $\bar{A} \in Q$ such that $p_{stem(q)}(\bar{A}) = q$ and let \bar{B}_0 be the \bar{B} given by 1.3 for \bar{A} and X_0 . If $\langle \bar{B}_\alpha : \alpha < \beta \rangle$ has been constructed for $\beta \leq \gamma$ and β is a successor, then choose \bar{B}_β as given by 1.3 for $\bar{A} = \bar{B}_{\beta-1}$ and $X = X_\beta$. If β is a limit, then by 1.2 choose first \bar{A} such that $(\forall \alpha < \beta) \bar{A} \leq^* \bar{B}_\alpha$ and then find $\bar{B}_\beta \subseteq \bar{A}$ as given by 1.3 for \bar{A} and $X = X_\beta$. Finally, if we have constructed $\bar{B}_\gamma = \langle B_s^\gamma : s \in {}^{<\omega}\omega \rangle$ define $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega \rangle$ by $B_s := B_s^\gamma \cap Succ_q(s)$ if $s \in q$ extends $stem(q)$ and $B_s := B_s^\gamma$ otherwise. It is easy to check that $\bar{B} \in Q$, $p_{stem(q)}(\bar{B})$ extends q and $(\forall \alpha < \gamma) [p_{stem(q)}(\bar{B})] \cap X_\alpha = \emptyset$.

Proof of 1.1(2) The proof is similar to (1). For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$ be a dominating family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\forall^\infty k) f(k) < f_\alpha(k)\}$$

Then $\bigcup \{X_\alpha : \alpha < \mathfrak{d}\} = {}^\omega\omega$ and in an analogous way as in (1) it can be seen that $X_\alpha \in m^0$.

In order to prove the first inequality we need the following concept from [GoJoSp]. Let R be the set of all $\bar{P} = \langle P_s : s \in {}^{<\omega}\omega \rangle$ where each $P_s \subseteq {}^{<\omega}\omega$ is infinite, $t \in P_s$ implies $s \subset t$ and if $t, t' \in P_s$ are distinct then $t(|s|) \neq t'(|s|)$. Given $\bar{P} \in R$ we can define $\langle p_s(\bar{P}) : s \in {}^{<\omega}\omega \rangle$ as follows: $p_s(\bar{P})$ is the unique Miller tree with stem s such that if $t \in Split(p_s(\bar{P}))$ then $Split_{p_s(\bar{P})}(t) = P_t$.

Define the following relations on R :

$$\begin{aligned} \bar{P} \leq \bar{Q} &\Leftrightarrow (\forall s) p_s(\bar{P}) \leq p_s(\bar{Q}) \\ \bar{P} \approx \bar{Q} &\Leftrightarrow (\forall s) P_s =^* Q_s \wedge (\forall^\infty s) P_s = Q_s \\ \bar{P} \leq^* \bar{Q} &\Leftrightarrow (\exists \bar{P}') \bar{P} \approx \bar{P}' \wedge \bar{P}' \leq \bar{Q} \end{aligned}$$

Fact 1.5 [GoJoSp, 4.14] *Assume $MA_\kappa(\sigma\text{-centered})$. If $\langle \bar{P}_\alpha : \alpha < \kappa \rangle$ is a \leq^* -decreasing sequence in R , then there exists $\bar{Q} \in R$ such that $(\forall \alpha < \kappa) \bar{Q} \leq^* \bar{P}_\alpha$.*

The following two facts have similar proofs as 1.3 and 1.4.

Fact 1.6 *Suppose $X \in m^0$ and $\bar{P} \in R$. There exists $\bar{Q} \leq \bar{P}$ such that $(\forall s) [p_s(\bar{Q})] \cap X = \emptyset$.*

Fact 1.7 *Suppose $X \in m^0$, $\bar{P}, \bar{Q} \in R$, $\bar{P} \leq^* \bar{Q}$ and $(\forall s) [p_s(\bar{Q})] \cap X = \emptyset$. Then $(\forall s) [p_s(\bar{P})] \cap X = \emptyset$.*

Now using 1.5, 1.6, 1.7 and the well-known result that for all $\kappa < \mathfrak{p}$ $MA_\kappa(\sigma\text{-centered})$ holds, a similar construction as in 1.1(1) shows that $\mathfrak{p} \leq \mathbf{add}(m^0)$.

2 add and cov are distinct

Definition 2.1 A set $A \subseteq {}^\omega\omega$ is called *strongly dominating* if and only if

$$(\forall f \in {}^\omega\omega)(\exists \eta \in A)(\forall^\infty k) f(\eta(k-1)) < \eta(k)$$

Definition 2.2 For any set $A \subseteq {}^\omega\omega$, we define the domination game $D(A)$ as follows: There are two players, GOOD and BAD. GOOD plays first. The game lasts ω moves.

GOOD	BAD
s	n_0
m_0	n_1
m_1	\vdots
\vdots	\vdots

The rules are: s is a sequence in ${}^{<\omega}\omega$, and the n_i and m_i are natural numbers. (Whoever breaks these rules first, loses immediately).

The GOOD player wins if and only if

- (a) For all i , $m_i > n_i$.
- (b) The sequence $s \frown m_0 \frown m_1 \frown \dots$ is in A .

Lemma 2.3 Let $A \subseteq {}^\omega\omega$ be a Borel set. Then the following are equivalent:

- (1) There exists a Laver tree p such that $[p] \subseteq A$.
- (2) A is strongly dominating.
- (3) GOOD has a winning strategy in the game $D(A)$.

Remark: Strongly dominating is not the same as dominating. For example, the closed set

$$A := \{\eta \in {}^\omega\omega : (\forall k) \eta(2k) = \eta(2k+1)\}$$

is dominating but is not strongly dominating.

Proof of 2.3 We consider the following condition:

$$(4) \text{ (For all } F : {}^{<\omega}\omega \times \omega \rightarrow \omega)(\exists \eta \in A)(\forall^\infty k)(\forall i \leq k) \eta(k) > F(\eta \upharpoonright k, i).$$

We will show (1) \rightarrow (2) \rightarrow (4) \rightarrow (3) \rightarrow (1).

(1) \rightarrow (2) is clear.

(2) \rightarrow (4): Given F , define f by

$$f(m) := \max\{F(s, i) : i \leq m, s \in m^{\leq m+1}\} + m$$

f is increasing, $f(m) \geq m$ for all m .

Find η such that $\forall^\infty k \eta(k) > f(\eta(k-1))$. Then η is increasing. For almost all k we have, letting $m := \eta(k-1)$:

$m \geq k-1$, so $\eta \upharpoonright k \in m+1^{m+1}$, so by the definition of f we get $f(m) \geq F(\eta \upharpoonright k, i)$ for any $i \leq k$. So $\eta(k) > f(\eta(k-1)) \geq F(\eta \upharpoonright k, i)$.

(4) \rightarrow (3): Assume that GOOD has no winning strategy. Then BAD has a winning strategy σ (since the game $D(A)$ is Borel, hence determined.)

We can find a function $F : {}^{<\omega}\omega \times \omega \rightarrow \omega$ such that for all s, m_0, \dots, m_k we have

$$\sigma(s, m_0, \dots, m_k) = F(s \frown m_0 \frown \dots \frown m_k, |s|)$$

Find $\eta \in A$ as in (4). So there is k_0 such that $\forall k \geq k_0 \eta(k) \geq F(\eta \upharpoonright k, k_0)$. So in the play

GOOD	BAD
$s := \eta \upharpoonright k_0$	$n_0 := \sigma(s) = F(\eta \upharpoonright k_0, k_0)$
$m_0 := \eta(k_0 + 1)$	$n_1 := \sigma(s, m_0) = F(\eta \upharpoonright (k_0 + 1), k_0)$
$m_1 := \eta(k_0 + 2)$	\vdots
\vdots	\vdots

player BAD followed the strategy σ , but player GOOD won, a contradiction.

(3) \rightarrow (1): Let B be the set of all sequences $s \frown m_0 \frown m_1 \frown \dots$ that can be played when GOOD follows a specific winning strategy. Clearly $B \subseteq A$, and for some Laver tree p , $B = [p]$.

Lemma 2.4 [Ke] *Let $A \subseteq {}^\omega\omega$ be an analytic set. Then the following are equivalent:*

- (1) *There exists a Miller tree p such that $[p] \subseteq A$.*
- (2) *A is unbounded in $({}^\omega\omega, \leq^*)$.*

Lemma 2.5 (1) *Suppose $\mathfrak{b} = \mathfrak{c}$. For every dense open $D \subseteq \mathbb{L}$ there exists a maximal antichain $A \subseteq D$ such that*

$$\forall q \in \mathbb{L} ([q] \subseteq \bigcup \{[p] : p \in A\}) \Rightarrow \exists A' \in [A]^{<\mathfrak{c}} \forall p \in A \setminus A' p \perp q \quad (*)$$

(2) *The same is true for \mathbb{M} .*

Proof: Let $\mathbb{L} = \{q_\alpha : \alpha < \mathfrak{c}\}$. Inductively we will define a set $S \subseteq \mathfrak{c}$ and sequences $\langle x_\gamma : \gamma < \mathfrak{c} \rangle$ and $\langle p_\gamma : \gamma \in S \rangle$. Finally we will let $A = \{p_\gamma : \gamma \in S\}$.

Let $0 \in S$ and choose $x_0 \in [q_0]$ arbitrarily.

It can be easily seen that every Laver tree contains \mathfrak{c} extensions such that every two of them do not contain a common branch. So clearly we may find $p_0 \in D$ such that $x_0 \notin [p_0]$.

Now suppose that $\langle x_\gamma : \gamma < \alpha \rangle$ and $\langle p_\gamma : \gamma \in S \cap \alpha \rangle$ have been constructed for $\alpha < \mathfrak{c}$.

First choose $x_\alpha \in [q_\alpha]$ arbitrarily, but such that, if $[q_\alpha] \not\subseteq \bigcup \{[p_\gamma] : \gamma < \alpha\}$ then $x_\alpha \notin \bigcup \{[p_\gamma] : \gamma < \alpha\}$.

In order to decide whether $\alpha \in S$ or not we distinguish the following two cases:

Case 1: q_α is compatible with some p_γ , $\gamma < \alpha$. In this case $\alpha \notin S$.

Case 2: q_α is incompatible with all p_γ , $\gamma < \alpha$. Now we let $\alpha \in S$, and we define p_α as follows:

By Lemma 2.3 for each $\gamma \in \alpha$ we may find $f_\gamma : \omega \rightarrow \omega$ such that

$$(\forall \eta \in [p_\gamma] \cap [q_\alpha])(\exists^\infty k) \eta(k) \leq f_\gamma(\eta(k-1)) \quad (**)$$

By our assumption on \mathfrak{b} there exists a strictly increasing f which dominates all the f_γ 's. Now define $p'_\alpha \in \mathbb{L}$ as follows: $stem(p'_\alpha) = stem(q_\alpha)$, and for $t \in p'_\alpha$, if $t \supseteq stem(p'_\alpha)$ and $|t| =: n$ we require

$$Succ_{p'_\alpha}(t) = Succ_{q_\alpha}(t) \cap [f(t(n-1)), \infty)$$

Clearly $p'_\alpha \in \mathbb{L}$, $p'_\alpha \subseteq q_\alpha$, and by $(**)$ and our assumption on f we conclude $[p_\gamma] \cap [p'_\alpha] = \emptyset$ for every $\gamma < \alpha$.

By the remark above that every Laver tree contains \mathfrak{c} extensions such that every two of them do not contain a common branch, we may find $p_\alpha \in D$ such that p_α extends p'_α and $[p_\alpha]$ and $\{x_\gamma : \gamma \leq \alpha\}$ are disjoint.

This finishes the construction. Now let $A := \{p_\gamma : \gamma \in S\}$.

Since every q_α is either compatible with some p_γ , $\gamma < \alpha$ (in case 1) or contains the condition p_α (in case 2) and for $\alpha \neq \gamma$ with $\alpha, \gamma \in S$ we have $[p_\alpha] \cap [p_\gamma] = \emptyset$ we conclude that A is a maximal antichain.

A also satisfies condition $(*)$: Let $q = q_\alpha$. By construction, if $[q_\alpha] \not\subseteq \bigcup \{[p_\gamma] : \gamma \in S \cap \alpha\}$ then $[q_\alpha] \not\subseteq \bigcup \{[p_\gamma] : \gamma \in S\}$.

The proof of (2) is analogous, but instead of Lemma 2.3 we use 2.4.

Lemma 2.6 *Suppose $\mathfrak{b} = \mathfrak{c}$. Then $\mathbf{add}(\ell^0) \leq \kappa(\mathbb{L})$ and $\mathbf{add}(m^0) \leq \kappa(\mathbb{M})$.*

Proof: We may assume $\kappa(\mathbb{L}) < \mathfrak{c}$. Let \dot{f} be a \mathbb{L} -name such that $\Vdash_{\mathbb{L}} \text{“}\dot{f} : \kappa(\mathbb{L}) \rightarrow \mathfrak{c} \text{ is onto”}$. For $\alpha < \kappa(\mathbb{L})$ let

$$D_\alpha := \{p \in \mathbb{L} : (\exists \beta) p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta\}$$

For $p \in D_\alpha$ will write $\beta_p = \beta_p(\alpha)$ for the unique β satisfying $p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta$.

Clearly D_α is dense and open. So we may choose a maximal antichain $A_\alpha \subseteq D_\alpha$ as in Lemma 2.5. Let

$$X_\alpha := {}^\omega\omega \setminus \bigcup\{[p] : p \in A_\alpha\}$$

Then $X_\alpha \in \ell^0$. We claim that $X = \bigcup_{\alpha < \kappa(\mathbb{L})} X_\alpha \notin \ell^0$. Suppose on the contrary $X \in \ell^0$. So we may find $q \in \mathbb{L}$ such that $[q] \cap X = \emptyset$ and hence $[q] \subseteq \bigcup\{[p] : p \in A_\alpha\}$ for each α . By the choice of A_α each of the sets

$$B_\alpha := \{\beta_p(\alpha) : p \in A_\alpha, p \text{ compatible with } q\}$$

is bounded in \mathfrak{c} . Since \mathfrak{c} is regular by our assumption $\mathfrak{b} = \mathfrak{c}$ we can find $\nu < \mathfrak{c}$ such that for all $\alpha < \kappa(\mathbb{L})$, $B_\alpha \subseteq \nu$. So easily conclude that

$$q \Vdash_{\mathbb{L}} \text{“range}(\dot{f}) \subseteq \nu < \mathfrak{c}\text{”}$$

This is a contradiction.

The proof for \mathbb{M} is similar.

Theorem 2.7 $\kappa(\mathbb{L}) \leq \mathfrak{h}$ and $\kappa(\mathbb{M}) \leq \mathfrak{h}$.

Proof: We prove it only for \mathbb{L} . The proof for \mathbb{M} is very similar. We work in V . Let $\langle \mathcal{A}_\alpha : \alpha < \mathfrak{h} \rangle$ be a family of maximal almost disjoint families such that,

- (1) if $\alpha < \beta < \mathfrak{c}$ then \mathcal{A}_β refines \mathcal{A}_α
- (2) there exists no maximal almost disjoint family refining all the \mathcal{A}_α
- (3) $\bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ is dense in $([\omega]^\omega, \subseteq^*)$

That such a sequence exists was shown in [BaPeSi].

Since \mathfrak{h} is regular, for every $p \in \mathbb{L}$ there exists $\alpha < \mathfrak{h}$ such that for each $s \in \text{Split}(p)$ there is $A \in \mathcal{A}_\alpha$ with $A \subseteq^* \text{Succ}_p(s)$. Hence, writing \mathbb{L}_α for the set of those $p \in \mathbb{L}$ for which α has the property just stated, we conclude $\mathbb{L} = \bigcup\{\mathbb{L}_\alpha : \alpha < \mathfrak{h}\}$.

For each $A \in \mathcal{A}_\alpha$ choose $\mathcal{B}_A = \{B^A(p) : p \in \mathbb{L}\}$, an almost disjoint family on A .

Now we will define $\mathbb{L}'_\alpha := \{q^\alpha(p) : p \in \mathbb{L}_\alpha\}$ such that $q^\alpha(p)$ extends p for every $p \in \mathbb{L}_\alpha$ and $p_1 \neq p_2$ implies $q^\alpha(p_1) \perp q^\alpha(p_2)$. For $p \in \mathbb{L}_\alpha$, $q^\alpha(p)$ will be defined as follows:

For each $s \in \text{Split}(p)$ let $C_s^\alpha(p) := \text{Succ}_p(s) \cap B^A(p)$ where $A \in \mathcal{A}_\alpha$ is such that $A \subseteq^* \text{Succ}_p(s)$. So clearly $C_s^\alpha(p)$ is infinite. Now $q^\alpha(p)$ is the unique Laver tree $\subseteq p$ satisfying $\text{stem}(q^\alpha(p)) = \text{stem}(p)$ and for each $s \in \text{Split}(q^\alpha(p))$ we have $\text{Succ}_{q^\alpha(p)}(s) = C_s^\alpha(p)$.

It is not difficult to see that \mathbb{L}'_α has the stated properties.

Now we are ready to define a \mathbb{L} -name \dot{f} such that $\Vdash_{\mathbb{L}} \text{“}\dot{f} : \mathfrak{h}^V \rightarrow \mathfrak{c}^V \text{ is onto”}$: For each $p \in \mathbb{L}_\alpha$, let $\{r_\xi^\alpha(p) : \xi < \mathfrak{c}\} \subseteq \mathbb{L}$ be a maximal antichain below $q^\alpha(p)$, and define \dot{f}

in such a way that $r_{\xi}^{\alpha}(p) \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \xi$. As $\bigcup\{\mathbb{L}'_{\alpha} : \alpha < \mathfrak{h}\}$ is dense in \mathbb{L} , it is easy to check that \dot{f} is as desired.

Theorem 2.8 *Let $\omega_2 = S_{\mathbb{M}} \dot{\cup} S_{\mathbb{L}}$, where the sets $S_{\mathbb{M}}$ and $S_{\mathbb{L}}$ are disjoint and stationary. Let $(P_{\alpha}, Q_{\alpha} : \alpha < \omega_2)$ be a countable support iteration of length ω_2 such that for all α we have $\Vdash_{P_{\alpha}} Q_{\alpha} = \mathbb{M}$ whenever $\alpha \in S_{\mathbb{M}}$, and $\Vdash_{P_{\alpha}} Q_{\alpha} = \mathbb{L}$ otherwise. Also suppose that V satisfies CH. Then in V^P , $\mathfrak{h} = \omega_1$ holds.*

Proof: Both \mathbb{M} and \mathbb{L} have the property $(*)_1$ of [JuSh]. (For \mathbb{L} , this was proved in [JuSh] and for \mathbb{M} this was proved in [BaJuSh].) [JuSh] also showed that this property is preserved under countable support iterations, so also P_{ω_2} has this property. Hence the reals of V do not have measure zero in V^P , so from $\mathfrak{h} \leq \mathfrak{s} \leq \mathbf{unif}(\mathcal{L})$ (where \mathfrak{s} is the splitting number and $\mathbf{unif}(\mathcal{L})$ is the smallest cardinality of a set of reals which is not null) we get the desired conclusion.

Theorem 2.9 *Let P_{ω_2} be as in 2.8. Then*

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(\ell^0) = \mathbf{add}(m^0) < \mathbf{cov}(\ell^0) = \mathbf{cov}(m^0) = \omega_2$$

Proof: Since \mathbb{L} adds a dominating real, we have $V^{P_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$, so by 2.6, 2.7 and 2.8, it suffices to prove that the covering coefficients are ω_2 in the respective models. The proof of this is similar to the proof of [JuMiSh, Thm1.2] that \mathbf{cov} of the Marczewski ideal is ω_2 in the iterated Sacks' forcing model.

We give the proof only for ℓ^0 . Suppose $\langle X_{\alpha} : \alpha < \omega_1 \rangle \in V^{P_{\omega_2}}$ is a sequence of ℓ^0 -sets. In $V^{P_{\omega_2}}$ let $f_{\alpha} : \mathbb{L} \rightarrow \mathbb{L}$ be such that for every $p \in \mathbb{L}$, $f_{\alpha}(p)$ extends p and $[f_{\alpha}(p)] \cap X_{\alpha} = \emptyset$. Since P_{ω_2} has the ω_2 -chain condition, by a Löwenheim-Skolem argument it is possible to find $\gamma < \omega_2$ such that

$$\langle f_{\alpha} \upharpoonright \mathbb{L}^{V_{\gamma}} : \alpha < \omega_1 \rangle \in V^{P_{\gamma}}$$

where $V_{\gamma} := V^{P_{\gamma}}$. Moreover, it is possible to find such a γ in $S_{\mathbb{L}}$. We claim that the Laver real x_{γ} (which is added by $Q_{\gamma} = \mathbb{L}^{V_{\gamma}}$) is not in $\bigcup_{\alpha < \omega_1} X_{\alpha}$, which will finish the proof. Otherwise, for some $p \in \mathbb{L}_{\gamma\omega_2}$ where $\mathbb{L}_{\gamma\omega_2} := \mathbb{L}_{\omega_2}/G_{\gamma}$ and some $\alpha < \omega_1$ we would have: $p \Vdash x_{\gamma} \in X_{\alpha}$. But letting $q := p(\gamma) \in \mathbb{L}$ and letting $r(\gamma) := f_{\alpha}(q)$ and $r(\beta) := p(\beta)$ for $\beta > \gamma$ we see that $r \Vdash x_{\gamma} \notin X_{\alpha}$, a contradiction.

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