

# Universal graphs without large cliques

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*Dedicated to the memory of Alan Mekler*

## 0. Introduction

The theory of universal graphs originated from the observation of R. Rado [4,5] that a universal countable graph  $X$  exists, i.e.,  $X$  is countable and isomorphically embeds every countable graph. He also showed that under GCH, there is a universal graph in every infinite cardinal. Since then, several results have been proved about the existence of universal elements in different classes of graphs. For example, a construction similar to Rado's shows, that for every natural number  $n \geq 3$ , there is a universal  $K(n)$ -free countable graph, or, if GCH is assumed, there is one in every infinite cardinal (here  $K(n)$  denotes the complete graph on  $n$  vertices). This result also follows from the existence theorem of universal and special models.

The following folklore observation shows that this cannot be extended to  $K(\omega)$ . Assume that  $X = (V, E)$  is a  $K(\omega)$ -free graph of cardinal  $\lambda$  that embeds every  $K(\omega)$ -free graph of cardinal  $\lambda$ . Let  $a \notin V$ , and define the graph  $X'$  on  $V' = V \cup \{a\}$  as follows.  $X'$  on  $V$  is identical with  $X$ ,  $a$  is joined to every vertex of  $V$ . Clearly,  $X'$  is  $K(\omega)$ -free. So, by assumption, there is an embedding  $g: V' \rightarrow V$  of  $X'$  into  $X$ . Put  $a_0 = a$ , and, by induction,  $a_{n+1} = g(a_n)$ . As  $g$  is edge preserving, we get, by induction on  $n$ , that  $a_n$  is joined to every  $a_t$  with  $t > n$ , so they are distinct, and form a  $K(\omega)$  in  $X'$ , a contradiction.

In Section 1 we give some existence/nonexistence statements on universal graphs, which under GCH give a necessary and sufficient condition for the existence of a universal graph of size  $\lambda$  with no  $K(\kappa)$ , namely, if either  $\kappa$  is finite or  $\text{cf}(\kappa) > \text{cf}(\lambda)$ . The special case when  $\lambda^{<\kappa} = \lambda$  was first proved by F. Galvin.

In Section 2 we investigate the question that if there is no universal  $K(\kappa)$ -free graph of size  $\lambda$  then how many of these graphs embed all the other. It was proved in [1], that if  $\lambda^{<\lambda} = \lambda$  (e.g., if  $\lambda$  is regular and the GCH holds below  $\lambda$ ), and  $\kappa = \omega$ , then this number is  $\lambda^+$ . We show that this holds for every  $\kappa \leq \lambda$  of countable cofinality. On the other hand, even for  $\kappa = \omega_1$ , and any regular  $\lambda \geq \omega_1$  it is consistent that the GCH holds below  $\lambda$ ,  $2^\lambda$  is as large as we wish, and the above number is either  $\lambda^+$  or  $2^\lambda$ , so both extremes can actually occur. Similar results when the excluded graphs are disconnected, were proved in [2] and [3].

**Notation.** We use the standard axiomatic set theory notation. If  $X$  is a set,  $\kappa$  a cardinal,  $[X]^\kappa = \{Y \subseteq X: |Y| = \kappa\}$ ,  $[X]^{<\kappa} = \{Y \subseteq X: |Y| < \kappa\}$ . A *graph* is a pair  $X = (V, E)$  where  $V$  is some set, and  $E \subseteq [V]^2$ , i.e., we exclude loops and parallel edges. If  $|V| = \lambda$ , we call  $X$  a  $\lambda$ -*graph*, and whenever possible, we outright assume that  $V = \lambda$ . A graph  $X = (V, E)$  is  $K(\kappa)$ -*free*, if there is no clique of cardinal  $\kappa$ , i.e.,  $[T]^2 \not\subseteq E$  holds for every  $T \in [V]^\kappa$ . A  $(\lambda, \kappa)$ -*graph* is a  $K(\kappa)$ -free  $\lambda$ -graph. If  $X_i = (V_i, E_i)$  ( $i < 2$ ) are graphs, the one-to-one function  $f: V_0 \rightarrow V_1$  is a *weak (strong) embedding* if  $\{x, y\} \in E_0$  implies  $\{f(x), f(y)\} \in E_1$  (if  $\{x, y\} \in E_0$  iff  $\{f(x), f(y)\} \in E_1$ ). A *weakly (strongly)  $(\lambda, \kappa)$ -universal graph* is a  $(\lambda, \kappa)$ -graph  $X$  that weakly (strongly) embeds every  $(\lambda, \kappa)$ -graph.

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## 1. When GCH holds

**Lemma 1.** *If  $\lambda$  is strong limit,  $\lambda > \kappa \geq \omega$ ,  $\text{cf}(\kappa) > \text{cf}(\lambda)$  then there exists a strongly  $(\lambda, \kappa)$ -universal graph.*

**Proof.** Let  $\lambda = \sup\{\lambda_\alpha: \alpha < \text{cf}(\lambda)\}$ , where the sequence is continuous, and  $2^{\lambda_\alpha} \leq \lambda_{\alpha+1}$ ,  $\lambda_0 = 0$ . Let  $T$  be a tree of height  $\text{cf}(\lambda)$  in which every  $\alpha$ -branch has  $\lambda_{\alpha+2}$  extensions on the  $\alpha$ -th level. Clearly,  $|T| = \lambda^{<\text{cf}(\lambda)} = \lambda$ . The vertex set of the universal graph  $X$  will be the disjoint union of some sets  $\{A(t): t \in T\}$  with  $|A(t)| = \lambda_{\alpha+1}$ . No edge of  $X$  will go between  $A(t)$  and  $A(t')$  when  $t, t'$  are incomparable in  $T$ . By induction on  $\alpha < \text{cf}(\lambda)$ , we determine for each  $t \in T$  of height  $\alpha$  how to build  $X$  on  $A(t)$ , and how to join the vertices of  $A(t)$  into  $\bigcup\{A(t'): t' < t\}$ . This latter set is of cardinal  $\lambda_\alpha$ , with a graph on it, and we make sure that it will be extended to a set of cardinal  $\lambda_{\alpha+1}$ , i.e., to some  $A(t)$ , in all possible ways, such that the graph on  $A(t)$  is  $K(\kappa)$ -free. This is possible, as for every branch we have enough extensions reserved. It is immediately seen that every  $(\lambda, \kappa)$ -graph embeds into  $X$ , one only has to select the right branch.

The vertex set is of cardinal  $\leq |T|\lambda = \lambda$ . Finally, a  $K(\kappa)$  could only be produced along a branch  $\{A(t): t \in b\}$ , but as  $|b| \leq \text{cf}(\lambda) < \text{cf}(\kappa)$ , some  $A(t)$  must contain a  $K(\kappa)$ , a contradiction, i.e.,  $X$  is a  $(\lambda, \kappa)$ -graph.

**Lemma 2.** *(F. Galvin) If  $\lambda^{<\kappa} = \lambda$ , then there is no weakly  $(\lambda, \kappa)$ -universal graph.*

**Proof.** Assume that  $X = (\lambda, E)$  is  $(\lambda, \kappa)$ -universal. Let  $Y = (V, G)$  be the following graph. The elements of  $V$  are those functions  $f$  with  $\text{Dom}(f) < \kappa$  such that  $\text{Ran}(f)$  is a clique in  $E$ .  $\{f, g\} \in G$  iff  $f \subset g$ . Clearly,  $|V| = \lambda^{<\kappa} = \lambda$ . If  $\{f_\alpha: \alpha < \kappa\}$  form a  $K(\kappa)$ , then they are compatible functions, and their union  $f = \bigcup\{f_\alpha: \alpha < \kappa\}$  injects  $\kappa$  into a clique of  $X$ , a contradiction, as  $X$  is  $K(\kappa)$ -free.

Assume that  $g: V \rightarrow \lambda$  is a weak embedding of  $Y$  into  $X$ . By induction on  $\alpha < \kappa$  we define  $x_\alpha < \lambda$ ,  $f_\alpha \in V$  such that for  $\beta < \alpha$   $\{x_\beta, x_\alpha\} \in E$ ,  $f_\beta \subset f_\alpha$  (so  $\{f_\beta, f_\alpha\} \in G$ ) should hold. If we succeed, we are done, as  $\{x_\alpha: \alpha < \kappa\}$  is a clique again. If  $\{x_\beta, f_\beta: \beta < \alpha\}$  are defined, let  $f_\alpha$  be the following function:  $\text{Dom}(f_\alpha) = \alpha$ ,  $f_\alpha(\beta) = x_\beta$  ( $\beta < \alpha$ ).  $f_\alpha \in V$ , as its range,  $\{x_\beta: \beta < \alpha\}$  is a clique. Put  $x_\alpha = g(f_\alpha)$ . As by the way  $f_\alpha$  is constructed,  $f_\beta \subset f_\alpha$  ( $\beta < \alpha$ ), and  $g$  is a weak embedding,  $x_\alpha$  will indeed, be joined into  $x_\beta$  for  $\beta < \alpha$ , and so the inductive step is successfully completed.

**Lemma 3.** *If  $\lambda$  is strong limit,  $\kappa \leq \lambda$ ,  $\text{cf}(\kappa) \leq \text{cf}(\lambda)$ , then there is no weakly  $(\lambda, \kappa)$ -universal graph.*

**Proof.** We can assume that  $\kappa > \text{cf}(\lambda)$ , as otherwise Lemma 2 gives the result. Assume that  $X = (\lambda, E)$  is  $(\lambda, \kappa)$ -universal. Let  $\{\kappa_\alpha: \alpha < \text{cf}(\kappa)\}$  be an increasing sequence of regular cardinals, cofinal in  $\kappa$ , with  $\kappa_0 > \text{cf}(\lambda)$ . Let  $F$  be the set of those  $f$  functions which satisfy the following requirements.  $\text{Dom}(f) < \text{cf}(\kappa)$ , for  $\alpha \in \text{Dom}(f)$ ,  $f(\alpha)$  is a bounded subset of  $\lambda$  with  $|f(\alpha)| = \kappa_\alpha$ , and  $\bigcup\{f(\alpha): \alpha < \text{Dom}(f)\}$  is a clique in  $X$ . Let  $V$ , the vertex set of the graph  $Y = (V, G)$  be the disjoint union of the sets  $\{A(f): f \in F\}$  where  $|A(f)| = \kappa_\alpha$  if  $\text{Dom}(f) = \alpha$ . Two distinct vertices are joined iff one of them is in  $A(f)$  the other in  $A(f')$  for some  $f \subseteq f'$ .

Clearly,  $|V| \leq \kappa|F| = \lambda$ . Assume that  $T$  spans a clique in  $Y$  and  $|T| = \kappa$ . Then  $T \subseteq \bigcup\{A(f_\gamma): \gamma \in \Gamma\}$  for a collection of pairwise compatible  $f_\gamma$ 's.  $\sup(\text{Dom}(f_\gamma)) = \text{cf}(\kappa)$  as otherwise  $|T| < \kappa$ , but then  $\bigcup\{\text{Ran}(f_\gamma): \gamma \in \Gamma\}$  is a  $K(\kappa)$  in  $X$ , a contradiction. We therefore established that  $Y$  is a  $(\lambda, \kappa)$ -graph.

Assume that  $g: V \rightarrow \lambda$  is a weak embedding of  $Y$  into  $X$ . By induction on  $\alpha < \text{cf}(\kappa)$  we are going to define  $f_\alpha \in F$  such that  $\text{Dom}(f_\alpha) = \alpha$ ,  $f_{\alpha+1}(\alpha) \subseteq g''A(f_\alpha)$ , and  $f_\beta \subset f_\alpha$  whenever  $\beta < \alpha$ . If this can be carried out, we reached a contradiction as then  $\bigcup\{\text{Ran}(f_\alpha): \alpha < \text{cf}(\kappa)\}$  is a  $K(\kappa)$  in  $X$ . There is no problem with the definition of  $f_\alpha$  if  $\alpha = 0$  or limit. Assume that  $f_\alpha$  is given.  $g''A(f_\alpha)$  is a clique in  $X$  of size  $\kappa_\alpha = \text{cf}(\kappa_\alpha) > \text{cf}(\lambda)$ , so, there is a bounded (in  $\lambda$ ) subset of it of cardinal  $\kappa_\alpha$ , say,  $S$ . We can now define  $f_{\alpha+1}(\alpha) = S$ ,  $f_{\alpha+1}(\beta) = f_\alpha(\beta)$  ( $\beta < \alpha$ ), the vertices in  $f_\alpha(\beta)$  will be joined to  $S$ , as by condition,  $f_\alpha(\beta) = f_{\beta+1}(\beta) \subseteq g''A(f_\beta)$ ,  $A(f_\beta)$  is joined to  $A(f_\alpha)$  by the condition  $f_\beta \subset f_\alpha$ , and  $g$  is a weak embedding.

From the known results and Lemmas 1–3 we can deduce the following.

**Theorem 1.** (GCH) Given  $\lambda \geq \kappa$ ,  $\lambda \geq \omega$ , there is a weakly/strongly  $(\lambda, \kappa)$ -universal graph iff  $\kappa < \omega$  or  $\text{cf}(\kappa) > \text{cf}(\lambda)$ .

## 2. The structure of the class of $(\lambda, \kappa)$ -graphs

In this Section we investigate the complexity of the class of  $(\lambda, \kappa)$ -graphs when there is no universal element in it.

**Definition.** For  $\lambda \geq \kappa$ ,  $\text{CF}(\lambda, \kappa)$  is the minimal cardinal  $\mu$  such that there is a family  $\{X_\alpha: \alpha < \mu\}$  of  $(\lambda, \kappa)$ -graphs, with the property that every  $(\lambda, \kappa)$ -graph is weakly embedded into some  $X_\alpha$ .  $\text{CF}^+(\lambda, \kappa)$  is the same with strong embeddings.

Clearly,  $\text{CF}(\lambda, \kappa) \leq \text{CF}^+(\lambda, \kappa) \leq 2^\lambda$ . Also,  $\text{CF}(\lambda, \kappa) \leq \lambda$  iff  $\text{CF}(\lambda, \kappa) = 1$  iff there is a weakly  $(\lambda, \kappa)$ -universal graph, and likewise for  $\text{CF}^+(\lambda, \kappa)$ .

It was observed in [1] that  $\text{CF}^+(\omega, \omega) = \omega_1$ . We slightly extend that result.

**Theorem 2.** If  $\lambda \geq \kappa$ ,  $\lambda$  is either strong limit or of the form  $\lambda = \mu^+ = 2^\mu$ ,  $\text{cf}(\kappa) = \omega$ , then  $\text{CF}^+(\lambda, \kappa) = \lambda^+$ .

**Proof.** From Lemmas 2–3,  $\text{CF}(\lambda, \kappa) \geq \lambda^+$ . Fix an increasing sequence  $\kappa_n \rightarrow \kappa$ ,  $\kappa_0 = 0$ . Call a structure  $(A, <, X, R)$  a *ranked graph* if  $(A, <)$  is a well-ordered set,  $X$  is a graph on  $A$ , and  $R$  is a function mapping those bounded cliques of  $X$  with order-type some  $\kappa_n$  into the ordinals, with the property that if clique  $C'$  end-extends clique  $C$ , then  $R(C') < R(C)$ . Obviously, then  $X$  will be  $K(\kappa)$ -free. On the other hand, if a  $K(\kappa)$ -free graph  $X$  is given on a well-ordered set  $(A, <)$ , then the tree

$$T(X) = \{C \subseteq A: \text{type}(C) = \kappa_n \text{ (some } n), C \text{ clique} \}$$

endowed with end-extension, as the partial order, will be  $\omega$ -branchless, so an ordinal valued function  $R$  as above exists. If  $|A| = \lambda$ , then  $|T| = \lambda$ , so only  $\lambda$  ordinals are used, therefore  $R(0) < \lambda^+$  holds. We call the minimal possible  $R(0)$  the rank of  $X$ .

Assume first that  $\lambda$  is strong limit. Fix a continuous, cofinal sequence  $\{\lambda_\alpha: \alpha < \text{cf}(\lambda)\}$  of cardinals with  $\lambda_0 = 0$  and  $2^{\lambda_\alpha} \leq \lambda_{\alpha+1}$ .

For every  $\xi < \lambda^+$  we are going to construct a graph that embeds all graphs with rank  $\xi$ .

Let  $T$  be a tree with height  $\text{cf}(\lambda)$ , with one root, such that whenever  $\alpha < \text{cf}(\lambda)$ , then every  $\alpha$ -branch has  $\lambda_{\alpha+2}$  extensions to the  $\alpha$ -th level. For  $t \in T$  on the  $\alpha$ -th level, let  $A(t)$  be an ordered set of order-type  $\lambda_{\alpha+1}$ , such that the sets  $\{A(t): t \in T\}$  are pairwise disjoint. The vertex set  $V$  of our graph will be the union  $V$  of these sets. We partially order  $V$  by assuming  $A(t) < A(t')$  for  $t < t'$ , i.e., all elements of  $A(t)$  precede all elements of  $A(t')$ .

For every  $t \in T$ , put  $B(t) = \bigcup\{A(t'): t' < t\}$ . By induction on the height of  $t$  we define  $S(t)$ , a ranked graph with ranks  $\leq \xi$  on  $B(t) \cup A(t)$  such that if  $b$  is an  $\alpha$ -branch, then all possible end-extensions (if there are any) of the already defined structure on  $\bigcup\{A(t): t \in b\}$  actually occur. This is possible, as there are enough extensions of  $b$  to the  $\alpha$ -th level.

It is now obvious that all  $(\lambda, \kappa)$ -graphs of rank  $\leq \xi$  embed into our tree. One only has to select the appropriate branch through  $T$ . Also,  $|V| = |T|\lambda = \lambda^{<\text{cf}(\lambda)} = \lambda$ . We need to show that there is no  $K(\kappa)$  in the resulting graph. Assume that  $U$  is a clique,  $|U| = \kappa$ . As we joined vertices only in comparable  $A(t)$ 's,  $U \subseteq \bigcup\{A(t): t \in b\}$  for some branch  $b$ . For some  $t_n \in b$  ( $n = 0, 1, \dots$ ), it is true that the first  $\kappa_n$  elements of  $U$  are bounded in  $S(t_n)$ , so they get a decreasing sequence of ordinals as ranks, a contradiction.

The case  $\lambda = \mu^+ = 2^\mu$  is actually simpler, we need one-element  $A(t)$ 's, and having  $\mu^+$  extensions of every branch of length  $< \mu^+$ .

Finally we show that under  $\kappa^{<\kappa} = \kappa$ ,  $\text{CF}(\kappa, \omega_1)$  can be as small as  $\kappa^+$ , and as large as  $2^\kappa$ , and this latter value as large as we wish.

**Theorem 3.** *Assume that in  $V$ , a model of GCH,  $\mu, \kappa > \omega$  are cardinals,  $\text{cf}(\mu) > \kappa = \text{cf}(\kappa)$ , then in a cardinal and cofinality preserving forcing extension  $V^P$ , the GCH holds below  $\kappa$  and  $\text{CF}(\kappa, \omega_1) = 2^\kappa = \mu$ .*

**Proof.** If  $\kappa = \lambda^+$ , with  $\text{cf}(\lambda) = \omega$ , then we first add a  $\square_\lambda$ -sequence, i.e., a sequence  $\{C_\alpha: \alpha < \kappa, \text{ limit}\}$  with the following properties:

- (1)  $C_\alpha \subseteq \alpha$  is closed, unbounded ;
- (2) if  $\gamma$  is a limit point of  $C_\alpha$ , then  $C_\gamma = \gamma \cap C_\alpha$  ;
- (3)  $|C_\alpha| < \kappa$ .

It is well known that such a sequence can be added by a cardinal and cofinality preserving forcing of size  $\kappa$ , so we may assume that it exists in  $V$ . Fix such a sequence, and a sequence of cardinals  $\lambda_n \rightarrow \lambda$ , and a one-to-one mapping  $\phi_{\alpha,\beta}: [\alpha, \beta) \rightarrow \lambda$  for each  $\alpha < \beta < \kappa$ .

We call a countable set  $A \subseteq \kappa$  *low*, if  $\text{tp}(A)$  is limit, and, if we put  $\delta = \sup(A)$ ,  $C_\delta = \{c_\xi: \xi < \text{tp}(C_\delta)\}$  the increasing enumeration of  $C_\delta$ , then for some  $n < \omega$ ,  $\phi_{c_\xi, c_{\xi+1}}(a) < \lambda_n$  holds for  $a \in A$ ,  $c_\xi \leq a < c_{\xi+1}$ .

If  $\kappa > \omega_1$  is not of the form  $\kappa = \lambda^+$ , with  $\text{cf}(\lambda) = \omega$ , then we call every countable subset of limit type *low*.

**Claim 1.** *The number of low subsets of some  $\alpha < \kappa$  is  $< \kappa$ .*

**Proof.** If  $\kappa$  is not of the form  $\lambda^+$  with  $\text{cf}(\lambda) = \omega$ , then  $|\alpha|^\omega < \kappa$ . In the other case the statement follows from property (3).

**Claim 2.** If  $B \subseteq \kappa$  is of order-type  $\omega_1$ , then for some cofinal subset  $B' \subseteq B$  it is true that if  $\gamma < \sup(B')$  is a limit point of  $B'$ , then  $B' \cap \gamma$  is low.

**Proof.** Put  $\delta = \sup(B)$ . Shrink  $B$  to a cofinal  $B' \subseteq B$ , such that the elements of  $B'$  are separated by  $C_\delta$ , and there is an  $n < \omega$ , such that if  $c_\xi \leq b < c_{\xi+1}$  for some  $\xi$ , then  $\phi_{c_\xi, c_{\xi+1}}(b) < \lambda_n$  ( $b \in B'$ ). Then the Claim follows from property (2) of the  $\square$ -sequence.

If  $\kappa$  is not of the form  $\kappa = \lambda^+$  with  $\text{cf}(\lambda) = \omega$  the choice  $B' = B$  works.

The poset  $(P, \leq)$  of the proof of the Theorem will be the  $< \kappa$  support product of  $\mu$  copies of some poset  $(Q, \leq)$  to be described below.

$q \in Q$  if  $q = (\delta, X, \mathcal{A})$  where  $\delta < \kappa$ ,  $X \subseteq [\delta]^2$ ,  $X$  is  $K(\omega_1)$ -free, if  $\kappa > \omega_1$  the  $\mathcal{A}$  is a family of low subsets of  $\delta$ , if  $\kappa = \omega_1$ , then  $\mathcal{A}$  is a countable family of countable subsets of  $\delta$  of limit type. Moreover, we require that if  $A \in \mathcal{A}$ ,  $\sup(A) \leq x < \delta$ , then  $A \times \{x\} \not\subseteq X$ .

$q' = (\delta', X', \mathcal{A}') \leq q = (\delta, X, \mathcal{A})$  iff  $\delta' \geq \delta$ ,  $X = X' \cap [\delta]^2$ ,  $\mathcal{A} = \mathcal{A}' \cap [\delta]^{\aleph_0}$ .

**Claim 3.**  $|Q| = \kappa$ .

**Proof.** For every  $\delta < \kappa$  there are at most  $\kappa$  many possibilities of selecting  $X, \mathcal{A}$  such that  $(\delta, X, \mathcal{A}) \in Q$ .

**Claim 4.** Forcing with  $(Q, \leq)$  does not introduce new sequences of ordinals of length  $< \kappa$ .

**Proof.** If  $\kappa = \omega_1$ , then  $(Q, \leq)$  is  $< \omega_1$ -closed.

If  $\kappa > \omega_1$ , assume that  $q \Vdash f: \tau \rightarrow \text{OR}$ ,  $\tau < \kappa$ . We construct the decreasing sequence of conditions  $\{q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha): \alpha \leq \tau\}$  such that  $q_0 = q$ ,  $q_{\alpha+1} \Vdash f(\alpha) = g(\alpha)$ , and if  $\alpha$  is limit, then  $\delta_\alpha = \sup\{\delta_\beta: \beta < \alpha\}$ ,  $X_\alpha = \bigcup\{X_\beta: \beta < \alpha\}$ . If  $\text{cf}(\alpha) \neq \omega$  then  $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta: \beta < \alpha\}$ , otherwise we add all the low subsets that are cofinal in  $\delta_\alpha$ , to  $\mathcal{A}_\alpha$ , as well. If we can carry out the construction, we are done,  $q_\tau$  determines all values of  $f$ . The only problem is if some of the  $X_\alpha$ 's is not  $K(\omega_1)$ -free. Let  $\alpha \leq \tau$  be minimal such that there exists an uncountable clique  $T \subseteq \delta_\alpha$ . Clearly,  $\text{cf}(\alpha) = \omega_1$ . For some cofinal  $T' \subseteq T$ , if  $\gamma < \delta_\alpha$  is a limit point of  $T'$ , then  $T' \cap \gamma$  is low. There is a limit  $\beta < \alpha$  such that  $\delta_\beta$  is a limit point of  $T'$ , so by our construction  $T' \cap \delta_\beta \in \mathcal{A}_\beta$ , so  $T' \cap \delta_\beta$  may not have been later extended to an  $\omega_1$ -clique.

**Claim 5.** Forcing with  $(P, \leq)$  does not introduce new sequences of ordinals of length  $< \kappa$ .

**Proof.** Similar to the previous proof.

**Claim 6.**  $(P, \leq)$  is  $\kappa^+$ -c.c.

**Proof.** By Claim 3 and  $\Delta$ -system arguments.

If, in  $V^P$ ,  $\text{CF}(\kappa, \omega_1) < \mu$ , then a family of graphs witnessing this is in a  $< \mu$  sized subproduct of  $P$ . By the product lemma we only need to show that forcing with  $(Q, \leq)$  introduces a  $(\kappa, \omega_1)$ -graph that cannot be embedded into any ground model  $(\kappa, \omega_1)$ -graph. If  $G \subseteq Q$  is generic, put  $Y = \bigcup\{X: (\delta, X, \mathcal{A}) \in G\}$ .

**Claim 7.**  $Y$  is  $K(\omega_1)$ -free.

**Proof.** If  $\kappa = \omega_1$ ,  $q \Vdash T$  is an  $\omega_1$ -clique, select a decreasing sequence  $q = q_0 \geq q_1 \geq \dots$  such that  $q_{n+1} = (\delta_{n+1}, X_{n+1}, \mathcal{A}_{n+1}) \Vdash t_n \in T$ ,  $\delta_n < t_n < \delta_{n+1}$ , and then put  $q' =$

$(\delta, X, \mathcal{A})$  where  $\delta = \lim \delta_n$ ,  $X = \bigcup\{X_n : n < \omega\}$ , and  $\mathcal{A} = \bigcup\{\mathcal{A}_n : n < \omega\} \cup \{\{t_n : n < \omega\}\}$ . Then  $q' \Vdash T \subseteq \delta$ , a contradiction.

If  $\kappa > \omega_1$ , then by Claim 4 some  $q = (\delta, X, \mathcal{A})$  determines all elements of  $T$ , the alleged  $\omega_1$ -clique. We can assume that  $T \subseteq \delta$ , but then  $X$  is not  $K(\omega_1)$ -free, a contradiction.

**Claim 8.**  $Y$  does not embed into any ground model  $(\kappa, \omega_1)$ -graph.

**Proof.** Assume that  $q \Vdash f : \kappa \rightarrow \kappa$  is an embedding of  $Y$  into some ground model  $(\kappa, \omega_1)$ -graph,  $Z$ . By induction on  $\alpha < \omega_1$  construct the decreasing sequence  $q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha)$  such that  $q_0 = q$ ,  $q_{\alpha+1} \Vdash f(\delta_\alpha) = g(\alpha)$ , for  $\alpha$  limit  $\delta_\alpha = \lim\{\delta_\beta : \beta < \alpha\}$ ,  $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ ,  $\{\delta_\beta, \delta_\alpha\} \in X_{\alpha+1}$  for  $\beta < \alpha$ , and  $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$ . The only problem with the definition would be that  $A \subseteq \{\delta_\beta : \beta < \alpha\}$  for some  $A \in \mathcal{A}_\alpha$ . But then,  $\sup(A)$  is of the form  $\delta_\gamma$  for some limit  $\gamma \leq \alpha$ , and no set of that form was added to  $\mathcal{A}_\gamma$ .

We can therefore define the sequence, but then the range of  $g$  will be a  $K(\omega_1)$  in  $Z$ , a contradiction.

**Theorem 4.** If, in a model of GCH,  $\mu, \kappa > \omega$  are cardinals, with  $\text{cf}(\mu) > \kappa = \text{cf}(\kappa)$ , then, in some cardinal and cofinality preserving extension the GCH holds below  $\kappa$ ,  $2^\kappa = \mu$ , and  $\text{CF}^+(\kappa, \omega_1) = \kappa^+$ .

**Proof.** Again, as in the proof of Theorem 3, we can assume, that if  $\kappa = \lambda^+$ , with  $\lambda > \text{cf}(\lambda) = \omega$ , then  $\square_\lambda$  holds in the ground model. We also assume that the GCH holds below  $\kappa$  and  $2^\kappa = \mu$ .

In a  $< \kappa$ -support iteration of length  $\kappa^+$ , we add a family witnessing  $\text{CF}^+(\kappa, \omega_1) = \kappa^+$ . Factor  $Q_\alpha$  will add a  $(\kappa, \omega_1)$ -graph that strongly embeds every  $(\kappa, \omega_1)$ -graph of  $V^{P_\alpha}$ . Notice, that if the forcing does not collapse cardinals, then  $\square_\lambda$  will still hold at every stage.

We first define and investigate one step of the iteration.

Let  $(Q, \leq)$  be the following poset.  $q = (\delta, X, \mathcal{A}, \mathcal{Z}, F) \in Q$ , if  $\delta < \kappa$ ,  $X \subseteq [\delta]^2$  is a  $K(\omega_1)$ -free graph,  $\mathcal{A} \subseteq [\delta]^{\aleph_0}$  is a family of low sets ( $\kappa > \omega_1$ ), is a countable family of limit type subsets of  $\delta$  ( $\kappa = \omega_1$ ).  $\mathcal{Z}$  is a family of  $< \kappa$  many  $(\kappa, \omega_1)$ -graphs,  $F : \mathcal{Z} \times \delta \rightarrow \delta$  is a function such that if  $Z \in \mathcal{Z}$  then the mapping  $x \mapsto F(Z, x)$  is a strong embedding of  $Z \upharpoonright \delta$  into  $X$ , and the following two more conditions hold.

- (1) If  $A \in \mathcal{A}$ ,  $\sup(A) \leq x < \delta$ , then  $A \times \{x\} \not\subseteq X$ ;
- (2) if  $A \in \mathcal{A}$ ,  $Z \in \mathcal{Z}$ , then  $A \not\subseteq F''(\{Z\} \times \delta)$ .

$q' = (\delta', X', \mathcal{A}', \mathcal{Z}', F') \leq q = (\delta, X, \mathcal{A}, \mathcal{Z}, F)$  if  $\delta' \geq \delta$ ,  $X = X' \cap [\delta]^2$ ,  $\mathcal{Z}' \supseteq \mathcal{Z}$ ,  $\mathcal{A} = \mathcal{A}' \cap [\delta]^{\aleph_0}$  and, moreover,

- (3) if  $Z_0 \neq Z_1 \in \mathcal{Z}$ ,  $\delta \leq x, y < \delta'$ , then  $F'(Z_0, x) \neq F'(Z_1, y)$ .

**Claim 1.**  $(Q, \leq)$  is transitive.

**Proof.** Assume that  $q_0 \geq q_1 \geq q_2$ ,  $q_i = (\delta_i, X_i, \mathcal{A}_i, \mathcal{Z}_i, F_i)$  ( $i < 3$ ). In establishing  $q_0 \geq q_2$  only condition (3) could cause problems, but it will not: if  $Z_0 \neq Z_1 \in \mathcal{Z}_0$ ,  $\delta_0 \leq x < \delta_1 \leq y < \delta_2$ , then  $F_2(Z_0, x) \neq F_2(Z_1, y)$  as the first element is in  $[\delta_0, \delta_1)$ , the second is in  $[\delta_1, \delta_2)$ .

**Claim 2.** If  $\varepsilon < \kappa$ ,  $D = \{(\delta, X, \mathcal{A}, \mathcal{Z}, F) : \delta \geq \varepsilon\}$  is dense.

**Proof.** We can extend a given  $(\delta, X, \mathcal{A}, \mathcal{Z}, F)$  to a large enough  $\delta'$  by mapping  $Z \upharpoonright [\delta, \delta')$  ( $Z \in \mathcal{Z}$ ) onto disjoint sets, not extending  $\mathcal{A}$ ,  $\mathcal{Z}$ , and adjusting  $X$ . Condition (1) won't cause problem, as by (2) no  $A \in \mathcal{A}$  will be forced to be joined to a vertex.

**Claim 3.** If  $Z$  is a  $(\kappa, \omega_1)$ -graph, then  $D = \{(\delta, X, \mathcal{A}, \mathcal{Z}, F) : Z \in \mathcal{Z}\}$  is dense.

**Proof.** A similar argument works.

**Claim 4.** Forcing with  $(Q, \leq)$  doesn't introduce sequences of ordinals of length  $< \kappa$ .

**Proof.**  $(Q, \leq)$  is  $< \omega_1$ -closed, and this is enough if  $\kappa = \omega_1$ .

Assume that  $\kappa > \omega_1$ . Let  $q \Vdash f : \tau \rightarrow \text{OR}$ ,  $\tau < \kappa$ . By induction on  $\alpha \leq \tau$  we define the decreasing sequence  $\{q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha, \mathcal{Z}_\alpha, F_\alpha) : \alpha \leq \tau\}$  such that  $q_{\alpha+1} \Vdash f(\alpha) = g(\alpha)$ , and for limit  $\alpha$ ,  $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$ ,  $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ ,  $\mathcal{Z}_\alpha = \bigcup\{\mathcal{Z}_\beta : \beta < \alpha\}$ ,  $F_\alpha = \bigcup\{F_\beta : \beta < \alpha\}$ . If  $\text{cf}(\alpha) > \omega$ , we take  $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$ , otherwise we add all cofinal in  $\delta_\alpha$  low subsets  $A$ , for which there is no  $Z \in \mathcal{Z}_\alpha$  with  $A \subseteq F''_\alpha(\{Z\} \times \delta_\alpha)$ . The only thing we have to show is that no  $K(\omega_1)$  will be created. We may assume, that  $\alpha \leq \tau$  is limit,  $T \subseteq \delta_\alpha$  is cofinal, and  $T$  is an uncountable clique in  $X_\alpha$ . We can assume that segments of  $T$  of limit type are low sets. As  $T$  could grow, for a club subset  $C \subseteq \alpha$ , of order type  $\omega_1$ , it is true that if  $\beta \in C$ , then  $T \cap \delta_\beta \subseteq F''_\beta(\{Z\} \times \delta_\beta)$  for some  $Z \in \mathcal{Z}_\beta$ . By condition (3), there can be only one such  $Z$ . If, moreover  $\beta$  is a limit point of limit points of  $C$ , then there is a  $h(\beta) < \beta$ , such that for  $h(\beta) < \gamma \leq \beta$  this  $Z$  for  $\gamma$  is the same. By the pressing down lemma,  $h$  is bounded on an unbounded subset, so  $T \cap \delta_\beta \subseteq F''_\alpha(\{Z\} \times \delta_\beta)$  for uncountably many  $\beta < \alpha$ , but then the inverse image of  $T$  will be a  $K(\omega_1)$  in  $Z$ , a contradiction.

Let  $Y$  be the graph added by  $Q$ , i.e., if  $G \subseteq Q$  is generic, then  $Y = \bigcup\{X : (\delta, X, \mathcal{A}, \mathcal{Z}, F) \in G\}$ .

**Claim 5.**  $Y$  is  $K(\omega_1)$ -free.

**Proof.** If  $\kappa = \omega_1$ ,  $q \Vdash T$  is an  $\omega_1$ -clique in  $Y$ , then an argument as above shows that there is a decreasing sequence  $\{q_\alpha : \alpha < \omega_1\}$  determining more and more elements of  $T$ , and we can freeze  $T$  unless it is covered by  $\bigcup\{F''_\alpha(\{Z\} \times \delta_\alpha) : \alpha < \omega_1\}$  for some  $Z$ , which again gives a  $K(\omega_1)$  in  $Z$ .

If  $\kappa > \omega_1$ , by the above Claim, the supposed clique  $T$  is in the ground model, some  $q \in G$  contains in its  $X$ -part, a contradiction.

The iteration  $(P_\alpha, Q_\alpha : \alpha \leq \kappa^+)$  is defined as a  $< \kappa$ -support iteration, with  $Q_\alpha$  as the above  $Q$ , defined in  $V^{P_\alpha}$ .

In  $Q_\alpha$ , let  $D_\alpha$  be the set of those conditions of the form  $q = (\delta, X, \mathcal{A}, \mathcal{Z}, F)$  for which it is true that  $Z_0 \neq Z_1 \in \mathcal{Z}$  implies that  $Z_0|_\delta \neq Z_1|_\delta$ .

**Claim 6.**  $D_\alpha$  is dense in  $Q_\alpha$ .

**Proof.** Using Claim 1, with  $\varepsilon$  large enough.

If  $q = (\delta, X, \mathcal{A}, \mathcal{Z}, F) \in Q_\alpha$  we put  $\ell(q) = (\delta, X, \mathcal{A}, \mathcal{Z}|_\delta, F)$ . Let  $E_\alpha$  be the following subset of  $P_\alpha$ .  $p \in P_\alpha$  if for all  $\beta < \alpha$ ,  $p|_\beta$  determines  $\ell(p|_\beta)$  and forces that  $p|_\beta \in D_\beta$ .

**Claim 7.** For every  $\alpha \leq \kappa^+$

- (a)  $E_\alpha$  is dense in  $P_\alpha$  ;
- (b) forcing with  $P_\alpha$  does not add sequences of ordinals of length  $< \kappa$ .

**Proof.** Assume first that  $\kappa > \omega_1$ . The proof is by induction on  $\alpha \leq \kappa^+$ . If (b) holds for  $\alpha$ , then it holds for  $\alpha + 1$ , by Claim 4. Assume that (a) and (b) hold for  $\alpha$ , and  $p \in P_{\alpha+1}$ . We may assume that  $p|_{\alpha} \Vdash p(\alpha) \in D_\alpha$ . As (b) holds for  $\alpha$ , there is a  $q \leq p|_{\alpha}$  which determines  $p(\alpha)$ . Extend  $q$  to an  $r \in E_\alpha$ , then take  $r \cup p(\alpha) \in E_{\alpha+1}$ .

Assume that  $\alpha$  is limit,  $p \in P_\alpha$ . In order to prove (a) for  $\alpha$ , we may assume that  $\text{supp}(p)$  is cofinal in  $\alpha$ , let  $\{\alpha_\xi : \xi < \tau\}$  converge to  $\alpha$ . We define  $\{p_\xi : \xi < \tau\}$ , a decreasing sequence of conditions.  $p_0 = p$ .  $p_\xi|_{\alpha_\xi} \in E_{\alpha_\xi}$ , and  $p_\xi \leq p_\zeta$ ,  $p_\xi|_{[\alpha_\xi, \alpha)} = p_\zeta|_{[\alpha_\zeta, \alpha)}$  hold for  $\zeta < \xi$ . If  $\xi$  is limit,  $\beta \geq \alpha_\xi$ , the names  $p_\zeta(\beta)$  are identical, so we can take it as  $p_\xi(\beta)$ . If  $\beta < \alpha_\xi$ , we take  $p_\xi(\beta)$  as  $\bigcup\{p_\zeta(\beta)\}$  by adding all low subsets which can be added, as in Claim 4. We show that  $p_\xi$  is a condition. To this end, we show by induction on  $\beta < \alpha$  that  $p_\xi|_{\beta}$  is a condition. The limit case is trivial. The problem with  $p_\xi(\beta)$  can only be that its  $X$  part contains a  $K(\omega_1)$ , but then, as in the proof of Claim 4, we get that  $p_\xi|_{\beta} \Vdash Z$  is not  $K(\omega_1)$ -free for some  $Z \in \mathcal{Z}$ .

If  $\alpha$  is limit and we are to show (b) for  $\alpha$ , and  $p \Vdash f : \tau \rightarrow \text{OR}$  for some  $\tau < \kappa$ , we can define a decreasing, continuous sequence  $\{p_\xi : \xi \leq \tau\}$  with  $p_\xi \Vdash f(\xi) = g(\xi)$ ,  $p_\xi \in E_\alpha$ . This can be carried out, as above, and then  $p_\tau$  decides  $f$ .

For  $\kappa = \omega_1$ , (b) follows from the fact that we iterate a countably closed poset with countable supports, and for (a) an easy inductive proof can be given, as for the other case above.

**Claim 8.**  $P_{\kappa^+}$  is  $\kappa^+$ -c.c.

**Proof.** Given  $\kappa^+$  conditions, we can assume that they are from  $E_{\kappa^+}$ . By the usual  $\Delta$ -system arguments we can find two of them  $p$  and  $p'$  such that  $\ell(p(\alpha)) = \ell(p'(\alpha))$  holds for every  $\alpha \in \text{supp}(p) \cap \text{supp}(p')$ . We show that  $p \cup p'$  is a condition (though not necessarily in  $E_{\kappa^+}$ ).

To this end, we show that  $(p \cup p')|_{\alpha} \in P_\alpha$  by induction on  $\alpha$ . All cases are trivial, except when  $\alpha = \beta + 1$ ,  $\beta \in \text{supp}(p) \cap \text{supp}(p')$ . What we have to show is that the  $F$  part of  $(p \cup p')(\beta)$  is well-defined, i.e., if  $Z = Z'$  are from the  $\mathcal{Z}$  part, then  $F(Z, x) = F(Z', x)$  ( $x < \delta$ ). But this will hold (or, more precisely, will be forced to hold by  $(p \cup p')|_{\beta}$ ) as  $F(Z, x)$  is determined by  $Z|\delta$  and by  $x$ , and it is determined the same way in  $p$  and  $p'$ .

From the last Claim, every  $(\kappa, \omega_1)$ -graph appears in some intermediate extension, and so it is embedded into the next graph,  $Y_\alpha$ , by  $Q_\alpha$ . We still have to show that  $Y_\alpha$  remains  $K(\omega_1)$ -free under the further extensions. This follows from Claim 7(b) if  $\kappa > \omega_1$ , and from the following statement which is a special case of a well-known lemma about forcing.

**Claim 9.** *If, in  $V$ ,  $Y$  is a  $K(\omega_1)$ -free graph,  $P$  is an  $< \omega_1$ -closed forcing, then, in  $V^P$ ,  $Y$  is still  $K(\omega_1)$ -free.*

**Proof.** If  $p \Vdash T$  is an uncountable clique, select  $\{p_\alpha : \alpha < \omega_1\}$  fixing more and more elements of  $T$ ,  $p_0 = p$ .

**Remark.** With the technique of Theorem 4 it is possible to show that if  $\mu \geq \nu > \kappa$ ,  $\text{cf}(\mu) > \kappa$ , and  $\nu, \kappa$  are regular, then it is consistent that  $2^\kappa = \mu$ ,  $\text{CF}(\kappa, \omega_1) = \nu$ , and GCH holds below  $\kappa$ . Add a sequence  $\{Y_\alpha : \alpha < \nu\}$ , rather than of length  $\kappa^+$ , as in Theorem 4. One only has to observe that  $Y_\alpha$  does not embed into any  $K(\omega_1)$ -free graph in  $V^{P_\alpha}$ , this can be proved similarly to Claim 8 in Theorem 3.



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