

The Strength Of The Isomorphism Property¹

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Abstract

In §1 of this paper, we characterize the isomorphism property of nonstandard universes in terms of the realization of some second-order types in model theory. In §2, several applications are given. One of the applications answers a question of D. Ross in [R] about infinite Loeb measure spaces.

0. INTRODUCTION

We always use *V for a nonstandard universe. We refer to [CK] or [SB] for the definition of nonstandard universes.

In the book [SB], there is an interesting example (see [SB, Theorem 1.2.12.(e)]) for illustrating the unusual behavior of infinite Loeb measure spaces. The example of [SB] says that in a nonstandard universe called a polyenlargement, the statement (\dagger) is true, where the statement (\dagger) is the following:

Every infinite Loeb measure space has a subset S such that S has infinite Loeb outer measure, but the intersection of S with any finite Loeb measure set has Loeb measure zero.

Under certain definition, the set S is called measurable but has infinite outer measure and zero inner measure (see [SB]). The diagonal argument for constructing S in [SB] depends on the construction of polyenlargements, say, an iterated ultrapower (or ultralimit) construction. During the preparation of the book [SB], K. D. Stroyan asked (see [R]) whether or not (\dagger) can be proved by some nice general properties of nonstandard universes without mentioning any particular construction. The first natural candidate would be C. W. Henson's *isomorphism property* [H1].

Let \mathcal{L} be a first-order language. An \mathcal{L} -model \mathfrak{A} is called internally presented in *V if the base set A and every interpretation under \mathfrak{A} of a symbol in \mathcal{L} are internal in *V . (For any \mathcal{L} -model \mathfrak{A} and symbol P in \mathcal{L} we write $P^{\mathfrak{A}}$ for the interpretation of P

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under \mathfrak{A} . We sometime use $\mathcal{L}_{\mathfrak{A}}$ for the language of \mathfrak{A} .) Let κ be an infinite cardinal. A nonstandard universe *V is said to satisfy the κ -isomorphism property (IP_{κ} for short) if

$$\begin{aligned} &\text{for any two internally presented } \mathcal{L}\text{-models } \mathfrak{A} \text{ and } \mathfrak{B} \text{ with } |\mathcal{L}| < \kappa, \\ &\mathfrak{A} \equiv \mathfrak{B} \text{ implies } \mathfrak{A} \cong \mathfrak{B}, \end{aligned}$$

where “ \equiv ” means *to be elementarily equivalent to* and “ \cong ” means *to be isomorphic to*. It is easy to see that IP_{κ} implies $IP_{\kappa'}$ when $\kappa' \leq \kappa$.

Instead of using the isomorphism property, D. Ross in [R] proved that a property called *the κ -special model axiom* for any infinite cardinal κ , which is stronger than IP_{κ} , implies (\dagger) . In [R], Ross showed also that κ -special model axiom has many new consequences, which hadn't been proved by IP_{κ} then. In his paper, Ross asked which of those results can or cannot be proved by IP_{κ} . The most important question among them is that if we can or cannot prove (\dagger) by IP_{κ} for some infinite cardinal κ . Basically, it was not known back then whether or not the κ -special model axiom is strictly stronger than IP_{κ} (see [R]).

The first author then answered the most of Ross's questions in [J]. In that paper, Jin showed that IP_{κ} for arbitrary large κ does not imply some consequences of the \aleph_0 -special model axiom. As a corollary IP_{κ} is strictly weaker than the κ -special model axiom. He also showed that many of the consequences of the κ -special model axiom in [R] are also the consequences of IP_{κ} . Unfortunately, [J] didn't answer Ross's question about (\dagger) .

In the another direction, the authors of [JK] proved that (\dagger) is true in some ultrapowers of the standard universe. Since we need iterated ultrapower construction to build the nonstandard universes of the κ -special model axiom while we need only one-step ultrapower construction to build the nonstandard universes of IP_{κ} (see [H2]), the result of [JK] seems to suggest that IP_{κ} have the right strength to prove (\dagger) .

The main purpose of this paper is to solve Ross's question about (\dagger) . In §1, we characterize IP_{κ} in terms of the realization of some second-order types. By applying Theorem 1 of §1, we show in §2 that Ross's question about (\dagger) has a positive answer, *i.e.* (\dagger) can be proved by IP_{\aleph_0} . In §2, we reprove also three known results in [J] by

using the same method in a uniform way. The new method simplifies significantly the original proofs in [J].

Notation for model theory in this paper will be consistent with [CK].

1. CHARACTERIZATION OF THE ISOMORPHISM PROPERTY

We use always \mathcal{L} for a first-order language. Let X be an n -ary predicate symbol which is not in \mathcal{L} . We call $\Gamma(X)$ an n - $\Delta_0^1(\mathcal{L})$ type iff $\Gamma(X)$ is a consistent set of $\mathcal{L} \cup \{X\}$ -sentences. Let \mathfrak{A} be an \mathcal{L} -model with base set A and let $\Gamma(X)$ be an n - $\Delta_0^1(\mathcal{L})$ type. We say that $\Gamma(X)$ is consistent with \mathfrak{A} iff $\Gamma(X) \cup Th(\mathfrak{A})$ is consistent, where $Th(\mathfrak{A})$ is the set of all \mathcal{L} -sentences which are true in \mathfrak{A} . We say that \mathfrak{A} realizes $\Gamma(X)$ iff there exists an $S \subseteq A^n$ such that the $\mathcal{L} \cup \{X\}$ -model $\mathfrak{A}_S = (\mathfrak{A}, S)$, where S is the interpretation of X under \mathfrak{A}_S , is a model of $\Gamma(X)$.

Let *V be a nonstandard universe. Let \mathfrak{A} be an \mathcal{L} -model with base set A in the standard universe. We write ${}^*\mathfrak{A}$ for an internally presented \mathcal{L} -model in *V with base set *A and the interpretation $P^{*\mathfrak{A}} = {}^*(P^{\mathfrak{A}})$ for every symbol $P \in \mathcal{L}$. It is not hard to see that $\mathfrak{A} \equiv {}^*\mathfrak{A}$. In fact, \mathfrak{A} can be considered as an elementary submodel of ${}^*\mathfrak{A}$.

Main Theorem *Let $\kappa < \beth_\omega$ be a regular cardinal. Then the following are equivalent:*

- (1) IP_κ ,
- (2) *For any first-order language \mathcal{L} with fewer than κ many symbols, for any n - $\Delta_0^1(\mathcal{L})$ type $\Gamma(X)$ and for any internally presented \mathcal{L} -model \mathfrak{A} in *V , if $\Gamma(X)$ is consistent with \mathfrak{A} , then \mathfrak{A} realizes $\Gamma(X)$.*

We will break the main theorem into following two theorems.

Theorem 1. *Assume $\kappa < \beth_\omega$ is a regular cardinal. Let *V be a nonstandard universe which satisfies IP_κ . For any first-order language \mathcal{L} with fewer than κ many symbols, for any n - $\Delta_0^1(\mathcal{L})$ type $\Gamma(X)$ and for any internally presented \mathcal{L} -model \mathfrak{A} in *V , if $\Gamma(X)$ is consistent with \mathfrak{A} , then \mathfrak{A} realizes $\Gamma(X)$.*

Proof: Let *V , \mathcal{L} , $\Gamma(X)$ and \mathfrak{A} are as described in the theorem. We want to show that \mathfrak{A} realizes $\Gamma(X)$.

Since $\Gamma(X)$ is consistent with \mathfrak{A} , there exists an \mathcal{L} -model \mathfrak{B} with base set B and an $S' \subseteq B^n$ such that the $\mathcal{L} \cup \{X\}$ -model $\mathfrak{B}_{S'} = (\mathfrak{B}, S')$ is a model of $Th(\mathfrak{A}) \cup \Gamma(X)$. We can assume $|B| \leq \kappa$ by the Downward Löwenheim–Skolem–Tarski theorem. Furthermore we can assume that \mathfrak{B} is in the standard universe because $\kappa < \beth_\omega$. Let ${}^*\mathfrak{B}_{S'} = ({}^*\mathfrak{B}, {}^*S')$ be the internally presented $\mathcal{L} \cup \{X\}$ -model in *V defined above. It is easy to see now that $\mathfrak{A} \equiv {}^*\mathfrak{B}$. By IP_κ , there is an isomorphism i from ${}^*\mathfrak{B}$ to \mathfrak{A} . Let

$$S = \{(i(b_1), i(b_2), \dots, i(b_n)) : (b_1, b_2, \dots, b_n) \in {}^*S'\}.$$

Then i is an isomorphism from $({}^*\mathfrak{B}, {}^*S')$ to (\mathfrak{A}, S) in $\mathcal{L} \cup \{X\}$. Since $\mathfrak{B}_{S'} \models \Gamma(X)$, then ${}^*\mathfrak{B}_{S'} \models \Gamma(X)$. Since ${}^*\mathfrak{B}_{S'} \cong \mathfrak{A}_S$, we conclude that \mathfrak{A} realizes $\Gamma(X)$. \square

Remark: If we replace the predicate symbol X in the definition of $n\text{-}\Delta_0^1(\mathcal{L})$ types by a new constant symbol c , the proof of Theorem 1 can still go through. So, as a corollary of Theorem 1, IP_κ implies κ -saturation.

Next we are going to prove the converse of Theorem 1. Before going further, we need to introduce more notation. The first is about model pairs. Let \mathcal{L} be a language. We call a language \mathcal{L}' *the language for \mathcal{L} -model pairs* if

- (1) \mathcal{L} and \mathcal{L}' have same function symbols,
- (2) every relation symbol in \mathcal{L} is in \mathcal{L}' and \mathcal{L}' contains two additional unary relation symbols P and Q ,
- (3) for every constant symbol c in \mathcal{L} , \mathcal{L}' contains exactly two copies of c , say, c_0 and c_1 .

Let \mathfrak{A} and \mathfrak{B} be two \mathcal{L} -models. A model pair $\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}$ is an \mathcal{L}' -model with base set $A \cup B$ (we assume $A \cap B = \emptyset$) such that

- (1) for every function symbol or relation symbol R in \mathcal{L} , $R^{\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$,
- (2) $P^{\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}} = A$ and $Q^{\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}} = B$,
- (3) $c_0^{\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}} = c^{\mathfrak{A}}$ and $c_1^{\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}} = c^{\mathfrak{B}}$.

Let \mathcal{L} be a language and let R be an unary predicate symbol. For any \mathcal{L} -formula ϕ , we write ϕ^R , the relativization of ϕ under R , for the formula defined inductively by

- (1) $\phi^R = \phi$ if ϕ is an atomic formula,
- (2) if $\phi = \neg\psi$, then $\phi^R = \neg\psi^R$,
- (3) if $\phi = \psi \wedge \chi$, then $\phi^R = \psi^R \wedge \chi^R$,
- (4) if $\phi = \exists x\psi$, then $\phi^R = \exists x(R(x) \wedge \psi^R)$.

Theorem 2. *Let $*V$ be a nonstandard universe. Let κ be a regular cardinal. If for any language \mathcal{L} with fewer than κ -many symbols, for any internally presented \mathcal{L} -model \mathfrak{A} in $*V$, and for any $2-\Delta_0^1(\mathcal{L})$ type $\Gamma(X)$ which is consistent with \mathfrak{A} , the model \mathfrak{A} realizes $\Gamma(X)$, then $*V$ satisfies IP_κ .*

Proof: Let \mathcal{L} be a language with fewer than κ -many symbols. Let \mathfrak{A} and \mathfrak{B} be two internally presented \mathcal{L} -models in $*V$ such that $\mathfrak{A} \equiv \mathfrak{B}$. We want to show that $\mathfrak{A} \cong \mathfrak{B}$.

Let \mathcal{L}' be the language for \mathcal{L} -model pairs and let $\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}$ be the model pair of \mathfrak{A} and \mathfrak{B} . We want now to define a $2-\Delta_0^1(\mathcal{L}')$ type $\Gamma(X)$ which will be used to force an isomorphism between \mathfrak{A} and \mathfrak{B} . Let

$$\Gamma(X) = \{\phi_n(X) : n = 0, 1, 2, 3, 4\} \cup \{\psi_\varphi(X) : \varphi \text{ is an } \mathcal{L}\text{-formula.}\},$$

where

$$\phi_0(X) = \forall x \forall y (X(x, y) \rightarrow P(x) \wedge Q(y))$$

$$\phi_1(X) = \forall x (P(x) \rightarrow \exists y X(x, y))$$

$$\phi_2(X) = \forall y (Q(y) \rightarrow \exists x X(x, y))$$

$$\phi_3(X) = \forall x \forall y \forall z (X(x, z) \wedge X(y, z) \rightarrow x = y)$$

$$\phi_4(X) = \forall x \forall y \forall z (X(z, x) \wedge X(z, y) \rightarrow x = y)$$

$$\psi_\varphi(X) = \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n \left(\bigwedge_{k=1}^n X(x_k, y_k) \rightarrow (\varphi^P(x_1, \dots, x_n) \leftrightarrow \varphi^Q(y_1, \dots, y_n)) \right).$$

We can see that the sentences $\{\phi_n(X) : n = 0, 1, 2, 3, 4\}$ say that X is a one to one correspondence between P and Q . Hence $\Gamma(X)$ says that the one to one correspondence X is actually an isomorphism between \mathfrak{A} and \mathfrak{B} . It is easy to check that for any two \mathcal{L} -models \mathfrak{A}' and \mathfrak{B}' , the model pair $\mathfrak{C}_{\mathfrak{A}', \mathfrak{B}'}$ realizes $\Gamma(X)$ if and only if $\mathfrak{A}' \cong \mathfrak{B}'$. We need now only to show that $\Gamma(X)$ is consistent with $\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}$. Since \mathfrak{A} and \mathfrak{B} are elementarily equivalent, there exists an ultrafilter \mathcal{F} on some cardinal λ such that the

ultrapower of \mathfrak{A} and the ultrapower of \mathfrak{B} modulo \mathcal{F} are isomorphic (see [S]). Hence the ultrapower of $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ modulo \mathcal{F} , which is the model pair of the ultrapower of \mathfrak{A} and the ultrapower of \mathfrak{B} modulo \mathcal{F} , realizes $\Gamma(X)$. On the other hand, the ultrapower of $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ is elementarily equivalent to $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$. So $\Gamma(X)$ is consistent with $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$. \square

Remarks: (1) As a corollary we have that in a nonstandard universe, the realizability for all $2-\Delta_0^1(\mathcal{L})$ types is equivalent to the realizability of all $n-\Delta_0^1(\mathcal{L})$ types for every n . (2) We didn't required that $\kappa < \beth_\omega$ in Theorem 2.

2. THE APPLICATIONS

The first application will give an answer to Ross's question about (\dagger) . In order to avoid dealing with the lengthy definition of Loeb measure we are going to express (\dagger) in an internal version as Ross did (see [R]).

We use the words *finite* or *infinite* for externally finite or externally infinite, respectively. We use **finite* or **infinite* for internally finite or internally infinite, respectively. For example, if $n \in {}^*\mathbb{N} \setminus \mathbb{N}$, where \mathbb{N} is the set of all standard natural numbers, then the set $\{0, 1, \dots, n\}$ is both **finite* and infinite. We use \mathbb{R} for the set of all standard reals.

Let *V be a nonstandard universe. Let $r \in {}^*\mathbb{R}$. We say that r is finite if there is a standard $n \in \mathbb{N}$ such that $|r| < n$. Otherwise we call r infinite. We say that r is an infinitesimal if $|r| < \frac{1}{n}$ for every standard $n \in \mathbb{N}$.

Application 1. (IP_{\aleph_0}) Suppose Ω is an infinite internal set and \mathcal{B} is an internal subalgebra of ${}^*\mathcal{P}(\Omega)$ which contains all singletons. Let $\mu : \mathcal{B} \rightarrow {}^*[0, \infty)$ be an internal, finitely additive measure with $\mu(\Omega)$ infinite and $\mu(\{x\})$ infinitesimal for every $x \in \Omega$. Then there exists a subset $S \subseteq \Omega$ such that

- (1) for any $D \in \mathcal{B}$ with $\mu(D)$ finite, for any $n \in \mathbb{N}$, there exists an $E \in \mathcal{B}$ such that $D \cap S \subseteq E$ and $\mu(E) < \frac{1}{n}$,
- (2) for any $D \in \mathcal{B}$, if $S \subseteq D$, then $\mu(D)$ is infinite.

Proof: Let Ω , \mathcal{B} and μ be as described in the Application 1. Let ${}^*\mathbb{R}$ be the set of hyperreal numbers. Assume that Ω , \mathcal{B} and ${}^*\mathbb{R}$ are all disjoint.

We form first an internally presented $\mathcal{L}_{\mathfrak{A}}$ -model \mathfrak{A} with base set $A = \Omega \cup \mathcal{B} \cup {}^*\mathbb{R}$ such that

$$\mathfrak{A} = (A ; \Omega, \mathcal{B}, {}^*\mathbb{R}, \in, \mu, \cap, \setminus, +, *, \leq, 0, 1),$$

where Ω , \mathcal{B} and ${}^*\mathbb{R}$ are three unary relations on A , $\in \subseteq \Omega \times \mathcal{B}$ is the membership relation, $\mu : \mathcal{B} \mapsto {}^*\mathbb{R}$ is the finite additive measure, \cap is the set intersection and \setminus is the set subtraction on \mathcal{B} , and $({}^*\mathbb{R} ; +, *, \leq, 0, 1)$ is the usual hyperreal ordered field. For simplicity, we do not distinguish a symbol in $\mathcal{L}_{\mathfrak{A}}$ from its interpretation under \mathfrak{A} .

We form next a $1-\Delta_0^1(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi_n(X), \chi_n(X) : n = 1, 2, \dots\},$$

where

$$\phi(X) = \forall x(X(x) \rightarrow \Omega(x))$$

$$\psi_n(X) = \forall U(\mathcal{B}(U) \wedge \mu(U) < n \rightarrow \exists V(\mathcal{B}(V) \wedge \forall x(X(x) \wedge x \in U \rightarrow x \in V) \wedge \mu(V) < \frac{1}{n}))$$

$$\chi_n(X) = \forall U(\mathcal{B}(U) \wedge \forall x(X(x) \rightarrow x \in U) \rightarrow \mu(U) > n).$$

Notice that in \mathfrak{A} , the element 1 is definable, so do n and $\frac{1}{n}$ for every $n \in \mathbb{N}$.

The sentence $\phi(X)$ says that X is a subset of Ω . The sentence $\psi_n(X)$ says that the intersection of X with any U in \mathcal{B} with measure less than n has outer measure less than $\frac{1}{n}$. The sentence $\chi_n(X)$ says that X has outer measure greater than n . So the application 1 is true if and only if \mathfrak{A} realizes $\Gamma(X)$. Hence, by the Theorem 1, it suffices to show that \mathfrak{A} is consistent with $\Gamma(X)$.

Let $T = Th(\mathfrak{A})$.

Claim: $T \cup \Gamma(X)$ is consistent.

Proof of Claim: By Downward Löwenheim-Skolem Theorem we can find a countable model $\mathfrak{A}_0 \preceq \mathfrak{A}$ with base set $A_0 = \Omega_0 \cup \mathcal{B}_0 \cup \mathbb{R}_0$. Since

$$\exists U(\mathcal{B}(U) \wedge \forall x(\Omega(x) \rightarrow x \in U))$$

is true in \mathfrak{A} , it is true in \mathfrak{A}_0 . Hence $\Omega_0 \in \mathcal{B}_0$. Since $\mu(\Omega) > n$ for all $n \in \mathbb{N}$ are true in \mathfrak{A} , they are also true in \mathfrak{A}_0 . Hence $\mu(\Omega_0)$ is infinite in \mathfrak{A}_0 . Since

$$\forall U \forall x \forall y (\mathcal{B}(U) \wedge \Omega(x) \wedge \Omega(y) \wedge (x \in U \wedge y \in U \rightarrow x = y) \rightarrow \mu(U) < \frac{1}{n})$$

for all $n \in \mathbb{N}$ are true in \mathfrak{A} , they are also true in \mathfrak{A}_0 . Hence the measure of every singleton is infinitesimal in \mathfrak{A}_0 . Let

$$\{B \in \mathcal{B}_0 : \mu(B) \text{ is finite}\} = \{B_n : n \in \mathbb{N}\}.$$

It is now easy to pick

$$x_n \in \Omega_0 \setminus \left(\bigcup_{k=0}^{n-1} B_k \cup \{x_k : k < n\} \right)$$

because Ω_0 has infinite measure and the measure of $\bigcup_{k=0}^{n-1} B_k \cup \{x_k : k < n\}$ is finite. Also notice that the measure of a finite set $\{x_k : k < n\}$ for $n \in \mathbb{N}$ is infinitesimal because the sum of finitely many infinitesimals is an infinitesimal and \mathcal{B}_0 is closed under finite union.

Let $S_0 = \{x_n : n \in \mathbb{N}\}$. It is obvious that (\mathfrak{A}_0, S_0) is a model of $T \cup \Gamma(X)$. \square

Next three applications are also the questions of [R] and were proved in [J]. The purpose of including them here with simplified proofs is to illustrate that IP_κ is an “easy to use” tool in nonstandard analysis.

Application 2. (IP_{\aleph_0}) *Suppose that $(P, <_P)$ and $(Q, <_Q)$ are two internal linear orders without endpoints. There is an order-preserving map $f : P \mapsto Q$ such that $f[P]$ is cofinal in Q .*

Proof: Without loss of generality, we can assume that $P \cap Q = \emptyset$. Let \mathfrak{A} be an internally presented $\mathcal{L}_{\mathfrak{A}}$ -model with base set $A = P \cup Q$ such that

$$\mathfrak{A} = (A ; P, Q, \leq_P, \leq_Q),$$

where P and Q are two binary relations on A , and $<_P$ and $<_Q$ are the correspondent orders on P and Q . We define a $2\text{-}\Delta_0^1(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi(X), \chi(X), \pi(X)\}$$

where

$$\phi(X) = \forall x \forall y (X(x, y) \rightarrow P(x) \wedge Q(y))$$

$$\psi(X) = \forall x \exists! y (P(x) \rightarrow X(x, y))$$

$$\chi(X) = \forall x_0 \forall x_1 \forall y_0 \forall y_1 (x_0 <_P x_1 \wedge X(x_0, y_0) \wedge X(x_1, y_1) \rightarrow y_0 <_Q y_1)$$

$$\pi(X) = \forall y_0 \exists x \exists y_1 (Q(y_0) \rightarrow P(x) \wedge X(x, y_1) \wedge y_0 <_Q y_1).$$

In $\Gamma(X)$ the sentence $\phi(X)$ says that X is a relation between P and Q , the sentence $\psi(X)$ says that X is the graph of a function from P to Q , the sentence $\chi(X)$ says that the function is order-preserving, and $\pi(X)$ says that the function is a cofinal embedding. If there exists an $S \subseteq P \times Q$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then it is easy to see that the map f defined by its graph S is the order-preserving map we are looking for. By Theorem 1, we need only to show that $T \cup \Gamma(X)$ is consistent, where $T = Th(\mathfrak{A})$.

Claim: $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, Q_0, \leq_{P_0}, \leq_{Q_0})$ be a countable elementary submodel of \mathfrak{A} . By the Compactness Theorem and Löwenheim–Skolem Theorem \mathfrak{A}_0 can be elementarily extended to a countable model $\mathfrak{A}_1 = (A_1, P_1, Q_1, \leq_{P_1}, \leq_{Q_1})$ such that the set of all rational numbers in $[0, 1)$, together with the usual order, can be order-isomorphically embedded into the set $\{q \in Q_1 : \forall x \in Q_0 (x \leq_{Q_1} q)\}$. Let i_0 be that embedding. By the same argument we can find an elementary chain of length ω of countable models $\mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \dots$ and a sequence of maps $\{i_n : n \in \omega\}$ such that i_n is an order-preserving map from the set of all rational numbers in $[n, n+1)$ with the usual order to $\{q \in Q_{n+1} : \forall x \in Q_n (x \leq_{Q_{n+1}} q)\}$. Let $\mathfrak{A}_\omega = \bigcup_{n \in \omega} \mathfrak{A}_n$ and let $i_\omega = \bigcup_{n \in \omega} i_n$. Since \mathfrak{A}_0 is elementarily equivalent to both \mathfrak{A} and \mathfrak{A}_ω , then \mathfrak{A}_ω is a model of T . It is easy to see that i_ω is an order-preserving map from the set of all positive rational numbers cofinally into Q_ω . Since every countable order without a right endpoint can be cofinally embedded into the set of all positive rational numbers, then P_ω can be cofinally embedded into Q_ω . Hence \mathfrak{A}_ω realizes $\Gamma(X)$. This proves the consistency of $T \cup \Gamma(X)$. \square

Application 3. (IP_{\aleph_0}) Let $(P, <_P)$ be an internal partial order with no right endpoints. There is an external subset $S \subseteq P$ such that $\{s \in S : s <_P p\}$ is internal for every $p \in P$.

Proof: Let $\mathcal{Q} = {}^*\mathcal{P}(P)$. Let \mathfrak{A} be an internally presented $\mathcal{L}_{\mathfrak{A}}$ -model with base set $A = P \cup \mathcal{Q}$ such that

$$\mathfrak{A} = (A ; P, \mathcal{Q}, \in, <_P, \cap, \setminus),$$

where P and \mathcal{Q} are two unary relations, $\in \subseteq P \times \mathcal{Q}$ is the membership relation, $<_P$ is the order on P , \cap is the set intersection on \mathcal{Q} and \setminus is the set subtraction on \mathcal{Q} . We define a $1-\Delta_0^1(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi(X), \chi(X)\}$$

where

$$\phi(X) = \forall x(X(x) \rightarrow P(x))$$

$$\psi(X) = \forall x \exists U(P(x) \rightarrow \mathcal{Q}(U) \wedge \forall y(y <_P x \wedge X(y) \leftrightarrow y \in U))$$

$$\chi(X) = \forall U \exists x(\mathcal{Q}(U) \rightarrow (x \in U \wedge \neg X(x)) \vee (X(x) \wedge \neg x \in U)).$$

The sentence $\phi(X)$ says that X is a subset of P , the sentence $\psi(X)$ says that for every x in P there exists a U in ${}^*\mathcal{P}(P)$ such that $U = \{y \in X : y <_P x\}$, and the sentence $\chi(X)$ says that for all U in ${}^*\mathcal{P}(P)$, the set X is different from U . It is easy to see that if there is an $S \subseteq P$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then S the set we are looking for. By Theorem 1, it suffices to show that $T \cup \Gamma(X)$ is consistent, where $T = Th(\mathfrak{A})$.

Claim: $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, \mathcal{Q}_0, \dots)$ be a countable elementary submodel of \mathfrak{A} . It suffices to construct a set $S = \{s_n : n \in \omega\} \subseteq P_0$ such that $(\mathfrak{A}_0, S) \models \Gamma(X)$. Let $P_0 = \{p_n : n \in \omega\}$ and let $\mathcal{Q}_0 = \{Q_n : n \in \omega\}$. Since P has no right endpoints, then P_0 has no right endpoints. Now we can pick the elements s_k and t_k from P_0 for every $k \in \omega$ such that

$$(1) s_0 < t_0 < s_1 < t_1 < \dots,$$

and for every $k \in \omega$

$$(2) \text{ we have } s_k \not\prec p_k \text{ and}$$

$$(3) \text{ either both } s_k \text{ and } t_k \text{ are in } Q_k \text{ or both } s_k \text{ and } t_k \text{ are not in } Q_k.$$

Let $S = \{s_n : n \in \omega\}$. Then S differs from every element in \mathcal{Q}_0 . For every $p \in P_0$ the set $\{s \in S : s \leq p\}$ is finite, and hence is in \mathcal{Q}_0 because for every finite set

$\{a_1, \dots, a_n\} \subseteq P_0$ the sentence

$$\exists U(\mathcal{Q}(U) \wedge \forall x(x \in U \leftrightarrow \bigvee_{i=1}^n x = a_i))$$

is true in \mathfrak{A} and therefore, it is true in \mathfrak{A}_0 .

The arguments above showed that $(\mathfrak{A}_0, S) \models \Gamma(X)$. \square

Application 4. (IP_{\aleph_0}) *Let P and Q be two *infinite internal sets. There is a bijection $f : P \mapsto Q$ such that for every *finite $b \subseteq P$ and for every *finite $c \subseteq Q$, the restriction of f to b and the restriction of f^{-1} to c are internal.*

Proof: Without loss of generality, we can assume that $P \cap Q = \emptyset$. Let \mathfrak{A} be an internally presented $\mathcal{L}_{\mathfrak{A}}$ -model with base set $A = P \cup Q \cup F$, where $F = \{f : f \text{ is an internal bijection from some *finite subset of } P \text{ to } Q\}$, such that

$$\mathfrak{A} = (A ; P, Q, F, R)$$

where P, Q and F are three unary relations on A and $R \subseteq P \times Q \times F$ is defined by

$$(a, b, f) \in R \text{ iff } (a, b) \in f.$$

We now define a $2-\Delta_0^1(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi_n(X) : n = 0, 1, 2, 3, 4, 5\},$$

where

$$\phi_0(X) = \forall x \forall y (X(x, y) \rightarrow P(x) \wedge Q(y))$$

$$\phi_1(X) = \forall x \forall y \forall z (X(x, z) \wedge X(y, z) \rightarrow x = y)$$

$$\phi_2(X) = \forall x \forall y \forall z (X(z, x) \wedge X(z, y) \rightarrow x = y)$$

$$\phi_3(X) = \forall x \exists y ((P(x) \rightarrow X(x, y)) \wedge (Q(x) \rightarrow X(y, x)))$$

$$\phi_4(X) =$$

$$\forall g \exists f (F(g) \rightarrow F(f) \wedge \forall x (\exists y R(x, y, g) \leftrightarrow \exists y R(x, y, f)) \wedge \forall x \forall y (R(x, y, f) \rightarrow X(x, y)))$$

$$\phi_5(X) =$$

$$\forall g \exists f (F(g) \rightarrow F(f) \wedge \forall y (\exists x R(x, y, g) \leftrightarrow \exists x R(x, y, f)) \wedge \forall x \forall y (R(x, y, f) \rightarrow X(x, y))).$$

The sentences $\phi_0(X), \phi_1(X), \phi_2(X), \phi_3(X)$ say that X is a one to one onto correspondence between P and Q . The sentence $\phi_4(X)$ says that the restriction of X on any

*finite set of P (as the domain of an element g in F) coincides with an element f in F . The sentence $\phi_5(X)$ says that the restriction of X on any *finite subset of Q (as the range of an element g in F) coincides also with an element f in F . It is easy to see that if there exists an $S \subseteq P \times Q$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then the bijection induced by S is the map we are looking for. Let $T = Th(\mathfrak{A})$. By Theorem 1, we need only to show that

Claim: $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, Q_0, F_0, R_0)$ be a countable elementary submodel of \mathfrak{A} . It suffices to find a relation $S \subseteq P_0 \times Q_0$ such that S is the graph of a bijection i from P_0 to Q_0 and for every C which is the domain of a function in F_0 and for every D which is the range of a function in F_0 , both $i \upharpoonright C$ and $(i^{-1} \upharpoonright D)^{-1}$ are functions in F_0 .

Let $F_0 = \{f_n : n \in \omega\}$. We want to construct a sequence $\{i_m : m \in \omega\} \subseteq F_0$ such that

$$(1) i_0 \subseteq i_1 \subseteq i_2 \subseteq \dots,$$

(2) for every $f \in F_0$ there is an $m \in \omega$ such that $dom(f) \subseteq dom(i_m)$ and $range(f) \subseteq range(i_m)$.

The claim follows from the construction because we can let $i = \bigcup_{m \in \omega} i_m$. It is easy to check that

(a) i is one to one function,

(b) $dom(i) = P_0$ and $range(i) = Q_0$,

(c) for every $f \in F_0$ there exists an $m \in \omega$ such that $i \upharpoonright dom(f) = i_m \upharpoonright dom(f) \in F_0$ and $(i^{-1} \upharpoonright range(f))^{-1} = (i_m^{-1} \upharpoonright range(f))^{-1} \in F_0$.

(c) is true because both sentences

$$\forall g \in F \forall f \in F (dom(g) \subseteq dom(f) \rightarrow \exists h \in F (h = f \upharpoonright dom(g)))$$

and

$$\forall g \in F \forall f \in F (range(g) \subseteq range(f) \rightarrow \exists h \in F (h = (f^{-1} \upharpoonright range(g))^{-1}))$$

are true in \mathfrak{A} . Then let S be the graph of i and we have now $(\mathfrak{A}_0, S) \models \Gamma(X)$.

Before constructing $\{i_m : m \in \omega\}$ let's observe two facts.

Fact 1. Suppose $f, g \in F_0$. If $\text{dom}(f) \cap \text{dom}(g) = \emptyset$ and $\text{range}(f) \cap \text{range}(g) = \emptyset$, then $f \cup g \in F_0$.

Fact 2. For any $f, g \in F_0$ there are $h, j \in F_0$ such that $\text{dom}(f) = \text{dom}(h)$, $\text{range}(g) \cap \text{range}(h) = \emptyset$ and $\text{range}(f) = \text{range}(j)$, $\text{dom}(g) \cap \text{dom}(j) = \emptyset$.

Fact 1 and *Fact 2* above are true in \mathfrak{A}_0 because they are true in \mathfrak{A} .

Let $i_0 = f_0$. Assume that we have constructed $\{i_m : m < k\} \subseteq F_0$ such that

$$i_0 \subseteq i_1 \subseteq \cdots \subseteq i_{k-1},$$

$$\text{dom}(f_m) \subseteq \text{dom}(i_{2m}) \text{ when } 2m < k$$

and

$$\text{range}(f_m) \subseteq \text{range}(i_{2m+1}) \text{ when } 2m + 1 < k.$$

Case 1: $k = 2n$.

Let $C = \text{dom}(f_n) \setminus \text{dom}(i_{k-1})$. If $C = \emptyset$, then let $i_k = i_{k-1}$. Otherwise let $h \in F_0$ such that $\text{dom}(h) = C$ and $\text{range}(h) \cap \text{range}(i_{k-1}) = \emptyset$. The function h exists by *Fact 2*. Let $i_k = i_{k-1} \cup h$. The function $i_k \in F_0$ by *Fact 1*.

Case 2: $k = 2n + 1$.

Let $D = \text{range}(f_n) \setminus \text{range}(i_{k-1})$. If $D = \emptyset$, then let $i_k = i_{k-1}$. Otherwise let $h \in F_0$ such that $\text{range}(h) = D$ and $\text{dom}(h) \cap \text{dom}(i_{k-1}) = \emptyset$. Again the function h exists by *Fact 2* and $i_k = i_{k-1} \cup h \in F_0$ by *Fact 1*. \square

Remark: The claims in above four applications could also be shown by quoting simply four results from [R] or other papers. We present our own proofs here because these proofs use only countable models so that the reader can read the paper without knowing special models.

REFERENCES

- [CK] Chang C. C. and Keisler H. J., **Model Theory**, North-Holland, 1973. (Third edition, 1990.)
- [H1] Henson C. W., “The isomorphism property in nonstandard analysis and its use in the theory of Banach space”, *The Journal of Symbolic Logic*, Vol. 39, No. 4 (1974), 717–731.
- [H2] ———, “When do two Banach spaces have isometrically isomorphic nonstandard hulls”, *Israel Journal of Mathematics*, Vol. 22 (1975), pp. 57–67.
- [H3] ———, “Unbounded Loeb measures”, *Proceedings of the American Mathematical Society*, Vol. 74, (1979), pp. 143–150.
- [J1] Jin R., “The isomorphism property versus the special model axiom”, *The Journal of Symbolic Logic*, Vol. 57, No. 3 (1992), pp. 975–987.

- [J2] _____, “A theorem on the isomorphism property”, *The Journal of Symbolic Logic*, Vol. 57, No. 3 (1992), pp. 1011—1017.
- [JK] Jin R. & Keisler H. J., “Game sentences and ultrapowers”, *Annals of Pure and Applied Logic*, Vol. 60 (1993), pp. 261—274.
- [L] Loeb P., “Conversion from nonstandard to standard measure space and applications in probability theory”, *Transactions of the American Mathematical Society*, 211 (1975), 113—122.
- [R] Ross D., “The special model axiom in nonstandard analysis”, *The Journal of Symbolic Logic*, Vol. 55, No. 3 (1990), 1233—1242.
- [S] S. Shelah, “Every two elementarily equivalent models have isomorphic ultrapowers”, *Israel Journal of Mathematics*, Vol. 10 (1971), pp. 224—233.
- [SB] Stroyan K. D. and Bayod J. M., **Foundation of Infinitesimal Stochastic Analysis**, North-Holland, 1986.

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