

**TOWARD CLASSIFYING
UNSTABLE THEORIES
SH500**

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ABSTRACT. We prove a consistency results saying, that for a simple (first order) theory, it is easier to have a universal model in some cardinalities, than for the theory of linear order. We define additional properties of first order theories, the n -strong order property (SOP_n in short). The main result is that a first order theory with the 4-strong order property behaves like linear orders concerning existence of universal models.

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§0 INTRODUCTION

Having finished [Sh:a], an important direction seems to me to try to classify unstable theories; i.e. to find meaningful dividing lines. In [Sh 10] two such were the strict order property and the independence property, their disjunction is equivalent to instability (see [Sh:a],4.7 = [Sh:c],4.7,p.70). For theories without the independence property, we know $S(A)$ (and $S_\Delta(A)$) are relatively small (see [Sh 10], Keisler [Ke76], [Sh:a], III,§7,7.3,7.4,II,§4,4.9,4.10. Also for $\lambda > |T|$, $\{p \in S(A) : p \text{ does not split over some } B \subseteq A \text{ of cardinality } < \lambda\}$ is λ -dense (see [Sh:a],7.5 = [Sh:c],7.5,p.140).

Later this becomes interesting in the content of analyzing monadic logic (see Baldwin Shelah [BlSh 156] representation Baldwin [Bl]. By [Sh 197] if “no monadic expansion of T has the independence property” is a significant dividing line.

Lately, some model theorist have become interested in finitary versions called VC dimensions, see Laskowski [Lw92], Macintyre [Mcxx] (good bound for the case of expansion the real field).

More relevant to the present work is the tree property, which is weaker than the strict order property (in [Sh:c],III,p.171).

In [Sh:93] we try to investigate theories without the tree property, so called simple. This can be looked at as a weakening of stable, so: simple $\Leftrightarrow \kappa_{\text{cdt}}(T) < \infty \Leftrightarrow$ failure of the tree property \Leftrightarrow suitable local ranks $< \infty$ are parallel of stable. We try to do the parallel to (parts of) Ch. II,III of [Sh:a], forking being generalized in some ways. But here instead showing the number of untrafilters of the Boolean Algebras of formulas $\varphi(x, \bar{a})$ over A is small ($\leq |A|^{|T|}$) we show that it can be decomposed to few subalgebras satisfying a strong chain condition. In this context we also succeed to get averages; but the Boolean algebras we get were derived from normal ones with a little twist. We did not start with generalizing the rest of [Sh:a] like supersimple (i.e. $\kappa_{\text{cdt}}(T) = \aleph_0$, equivalently suitable rank is $< \infty$). The test problem in [Sh:93] was trying to characterize the class of pairs

$$SP(T) = \left\{ (\lambda, \kappa) : \text{every model of } T \text{ of cardinality } \lambda \text{ has a } \kappa\text{-saturated elementary extensions of cardinality } \lambda \right\}.$$

For simplicity we consider there only $\lambda = \lambda^{|T|} > 2^{|T|+\kappa}$, $\kappa > |T|$ and $(\exists \mu)(\mu = \mu^{<\kappa} \leq \lambda \leq 2^\mu)$ (if this fails, see [Sh 576]). So by [Sh:93] for non-simple T , such (λ, κ) is in $SP(T)$ iff $\lambda = \lambda^{<\kappa}$. If $\mu = \mu^{<\mu} < \lambda = \lambda^{<\lambda}$, after suitable forcing preserving $\mu = \mu^{<\mu}$, not collapsing cardinals and making $2^\mu = \lambda$, we have under suitable generalization of MA, so $\kappa < \mu < \lambda < 2^\mu \Rightarrow (\lambda, \kappa) \in SP(T)$.

It seems much better to use just the cardinal arithmetic assumptions (not the generalizations of MA). This call to investigate problem of $\mathcal{P}^-(n)$ -amalgamation (see [Sh 87b], [Sh:c],XII,§5). For the case of $n = 3$ this means that

- (*)₃ if $p_0(\bar{x}, \bar{y}), p_1(\bar{x}, \bar{z}), p_2(\bar{y}, \bar{z})$, complete types over A , each saying the two sequences of variables are “independent” in suitable ways (like nonforking) then we extend the union of the three (preserving “independence”).

Now $(*)_3$ can be proved. [Sh:93], Claim 7.8, p.201, (3.5, p.187). But the proof does not work for higher n , naturally counterexamples for the amalgamation should give counterexample to membership in SP. This was carried out by finding counterexamples in a wider framework: saturation inside P in [Sh 126]; but we could still hope that for the “true” one there is a positive one.

For long, I was occupied elsewhere and not look into it, but eventually Hrushovski becomes interested (and through him, others) and we try to explain the relevant research below. Also, it could be asked if simple unstable theories “occurs in nature”, “are important to algebraic applications”. The works cited below forms a positive answer (note that, quite natural, those examples concentrate on the lower part of hierarchy, like strongly minimal or finite Morley rank).

On the one hand, Hrushovski, continuing [Sh 126], prove that there are simple theories with bad behaviour for $\mathcal{P}(n)$ so in the result above the cardinal arithmetic are not enough.

On the other hand, by Hrushovski Pillay [HrPi] in specific cases (finite ranks) relevant cases of $(*)_n$ are proved, for $n > 3$ under very specific conditions: for $n = 3$ more general; but the relationship with [Sh:93], 7.8 of $(*)$ was not clarified (in both cases the original rank does not work; the solution in [Sh:93] is to use dnwd (= “do not weakly divide”), Hrushovski changes the rank replacing “contradictory” by having small rank; this seems a reasonable approach only for supersimple theories and was carried only for ones with finite rank, and it gives more information in other respects.

In Hrushovski [Hr1] let \mathfrak{C}_0 be the monster model for a strongly minimal theory with elimination of imaginaries, $A \subseteq \mathfrak{C}$, $A = \text{dcl } A$, such that every $p \in S^m(A)$ with multiplicity 1 is finitely satisfiable in A , now $\text{Th}(\mathfrak{C}, A)$ is simple (of rank 1) and we can understand PAC in general content. Hrushovski [Hr2] does parallel thing for finite rank.

* * *

We turn to the present work. First section deals with the existence of universal models. Note that existence of saturated models can be characterized nicely by stability (see [Sh:c]).

By Kojman Shelah [KjSh 409], the theory of linear order and more generally theories without the strict order property has universal models in “few” cardinals.

By [Sh 457] give a sufficient conditions for a consistency of “there is in μ^{++} a model of T universal for models of T of cardinality μ^+ ”, which we use below.

The main aim is to show that all simple theories behave “better” in this respect than the theory of linear order. Specifically, it is consistent that $\aleph_0 < \lambda = \lambda^{<\lambda}$, $2^\lambda > \lambda^{++}$, moreover, there is a club guessing $\langle C_\delta : \delta < \lambda^+, \text{cf}(\delta) = \lambda \rangle$, and every simple T of cardinality $< \lambda$ has a model in λ^{++} universal for λ^+ . For this we represent results of [Sh:93] and do the things needed specifically for the use of [Sh 457].

See 1.4A(2).

§1 SIMPLE THEORIES HAVE MORE UNIVERSAL MODELS

We quote [Sh 457],5.1.

1.1 Lemma. *Suppose*

- (A) *T is first order, complete for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).*
- (B) *K_{ap} is a simple λ -approximation system such that every $M \in K_{ap}$ is a model of T hence every M_Γ , where for $\Gamma \in K_{md}$ we let $M_\Gamma = \cup\{M : M \in \Gamma\}$.*
- (C) *Every model M of T of cardinality λ^+ can be embedded into M_Γ for some $\Gamma \in K_{md}$.*

Then

- (a) *in 4.9 in V^P , there is a model of T of cardinality λ^{++} universal for models of T of cardinality λ^+ .*
- (b) *So in V^P , $univ(\lambda^+, T) \leq \lambda^{++} < 2^\lambda$.*

Proof. Straightforward.

1.1 Fact. 1) Assume $M \prec N$, $\bar{a} \in {}^\omega N$, and Δ a finite set of formulas possibly with parameters from M . Then there are a formula $\psi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}, N)$ such that:

- (*) for any $\bar{a}' \in M$ realizing $\psi(\bar{x}, \bar{b})$, we can find a Δ -2-indiscernible sequence $\langle \bar{a}_i : i \leq \omega \rangle$ such that: $\bar{a}_0 = \bar{a}'$, $\bar{a}_{\omega+1} = \bar{a}$; hence we can find an indiscernible sequence $\langle \bar{a}'_i : i < \omega \rangle$ (in \mathfrak{C}) such that the Δ -type of $\bar{a}'_0 \hat{\ } \bar{a}'_1$ is the same as that of $\bar{a}' \hat{\ } \bar{a}$.

2) Assume $2^{\theta+|T|} \leq \kappa$ and $M \prec N$, moreover

- ⊗ if $A \subseteq M$, $|A| \leq \kappa$, $\bar{a} \in {}^\theta N$ then some $\bar{a}' \in M$ realizes $\text{tp}(\bar{a}, A, N)$.

Then for any $\bar{a} \in {}^\theta N$ and $B \subseteq M$, $|B| \leq \theta$, there is $A \subseteq M$, $|A| \leq \kappa$, such that for every $\bar{a}' \in {}^\theta M$ realizing $\text{tp}(\bar{a}, A, N)$ there is a sequence $\langle \bar{a}_i : i \leq \kappa \rangle$ which is 2-indiscernible over B , $\bar{a}_0 = \bar{a}'$, $\bar{a}_\kappa = \bar{a}$, hence there is an indiscernible sequence $\langle \bar{a}'_i : i < \omega \rangle$ such that $\bar{a}'_0 \hat{\ } \bar{a}'_1$ realizes the same type as $\bar{a}' \hat{\ } \bar{a}$ over B .

Proof. Obvious [notes on combination set theory].

1) Let $\langle p_i : i < k \rangle$ list the possible Δ -types of sequences of length $\ell g(\bar{a}) + \ell(\bar{a})$, so $k < \omega$. For each p_i , $\psi_i(\bar{x}, \bar{b}_i) \in \text{tp}(\bar{a}, M, N)$ such that, if possible for no $\bar{a}' \in {}^{\ell g(\bar{a})} M$ realizing $\psi_i(\bar{x}, \bar{b}_i)$ do we have: $\bar{a}' \hat{\ } \bar{a}$ realizes p_i ; (if there..)

Now $\psi(\bar{x}, \bar{b}) =: \bigwedge_{i < k} \psi_i(\bar{x}, \bar{b}_i)$ is as required.

2) Similar.

3) $\langle p_i : i < k \rangle$ list the complete $2\ell g(\bar{a})$ -types over B . Use $p_i \subseteq \text{tp}(\bar{a}, M, N)$, $|p_i| \leq \theta + |T|$ instead $\psi_i(\bar{x}, \bar{b}_i)$, $A = \bigcup_{i < k} \text{Dom } p_i$.

1.2 Theorem. *If T is a complete simple (f.o.) theory, $|T| < \lambda$ then T satisfies the assumption of 1.1 (hence its conclusions).*

1.2A Remark. 1) We can get results for a theory T of cardinality $\leq \lambda$ under stronger assumptions on T .

2) Though not always necessary, in this section we'll assume T is simple.

3) Also this section is not written in a way focused on Theorem 1.2, but leisurely relook at simple theories.

Proof. Without loss of generality T has elimination of quantifiers.

We first represent (in 1.3 - 1.10) the needed definitions and facts on simple theories from [Sh:93] (adding notation and some facts), then say a little more and prove the theorem. So for a while we work in a fixed $\bar{\kappa}$ -saturated model \mathfrak{C} of T , $\bar{\kappa}$ big enough. So M, N denotes elementary submodels of \mathfrak{C} of cardinality $< \bar{\kappa}$, A, B, C, D denote subsets of \mathfrak{C} of cardinality $< \bar{\kappa}$ and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ denote sequences of elements of \mathfrak{C} of length $< \bar{\kappa}$, usually finite. Let $\bar{a}/B = \text{tp}(\bar{a}, B) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^{\omega}B, \varphi \in L(T) \text{ is first order and } \mathfrak{C} \models \varphi[\bar{a}, \bar{b}]\}$.

1.3 Definition. 1) We say that “ $p(\bar{x})$ does not weakly divide over (r, B) ” (in short p dnwd (that is does not weakly divide) over (r, B)); we write over B when $r = p \upharpoonright B$, we write over r if $B = \text{Dom}(r)$, where $r = r(\bar{x})$ is a type over B (and \bar{x} may be infinite) when: if $\bar{b} \in B$ and $\psi = \psi(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ a formula (where $\ell g(\bar{x}^\ell) = \ell g(\bar{x}), \bar{x}^\ell$ with no repetition, $\langle \bar{x}^\ell : \ell = 1, n \rangle \hat{\ } \langle \bar{y} \rangle$ pairwise disjoint) and (see definition 1.3(2) below) $[r]^\psi$ is finitely satisfiable (in \mathfrak{C}) then so is $[r \cup p]^\psi$ (see Definition 1.3(2) below).

2) If $\psi = \psi(\bar{x}^1, \dots, \bar{x}^n)$ (possibly with parameters), $q = q(\bar{x})$ then

$$[q]^\psi = \{\psi\} \cup \bigcup_{\ell=1}^n q(\bar{x}^\ell).$$

3) $p(\bar{x})$ divides over A if for some formula $\psi(\bar{x}, \bar{a})$ we have $p \vdash q(\bar{x}, \bar{a})$ and for some indiscernible sequence $\langle \bar{a}_\ell : \ell < \omega \rangle$ over $A, \bar{a} = \bar{a}_0$, and $\{\varphi(\bar{x}, \bar{a}_i) : i < \omega\}$ is $(< \omega)$ -contradictory where a set p of formulas is n -contradictory if for any distinct $\varphi_1, \dots, \varphi_n \in p, \{\varphi_1, \dots, \varphi_n\}$ is not realized (in \mathfrak{C}), and $(< \omega)$ -contradictory means: n -contradictory for some n . We write dnd for “does not divide”.

4) The type p fork over the set A if for some $n < \omega$ and formulas $\varphi_\ell(\bar{x}, \bar{a}_\ell)$ for $\ell < n$ we have: $p \vdash \bigvee_{\ell < n} \varphi_\ell(\bar{x}, \bar{a}_\ell)$ and for each $\ell < n$ the formula $\varphi_\ell(\bar{x}, \bar{a}_\ell)$ divides over A .

We use “dnf” as shortening for “does not fork”.

5) The type p is finitely satisfiable (finitely satisfiable) in A (or in \mathbf{I}) if every finite subset p' of p is realized by some sequence from A (or a member of \mathbf{I}).

6) If D is an ultrafilter on $\text{Dom}(D) = \mathbf{I}$ (where all members of \mathbf{I} have the same length, say m) then $Av(B, D) = \{\varphi(\bar{x}, \bar{a}) : \{\bar{b} \in \text{Dom}(D) : \varphi(\bar{b}, \bar{a})\} \in D\}$.

1.4 Definition. We say “ \bar{a}/A (or $\text{tp}(\bar{a}, A)$) weakly divides over B ” if $B \subseteq A$ and $\text{tp}(\bar{a}, A)$ weakly divides over $(\text{tp}(\bar{a}, B), B)$, (similarly for does not weakly divide).

1.4A Remark. 1) An equivalent formulation is “ \bar{a}/A is an extension of \bar{a}/B with the same degree for most (Δ, \aleph_0, k) ”; see 1.5(8) below.

2) On “divides”, “fork”, “weakly divide” see [Sh:93], Def.1.1,1.2, 2.7(2) respectively. On the first two also [Sh:a], but there the focus is on stable theories. On “finitely satisfiable” see [Sh:a], Ch.VII, §4. We present here most of their properties, ignoring mainly the connections with suitable degrees and indiscernibility and the derived Boolean algebras of formulas (satisfying chain conditions). For stable T the notions of Def. 1.3 collapse becoming equivalent (finitely satisfiable - only when the set is a model, see [Sh:a], Ch.III).

Basic properties are (mostly check directly, but 0A,6,8,9 are quoted).

1.5 Claim. 0) [Implications] If p divides over A then p forks over A .

0A) If p forks over A then p weakly divides over $(p \upharpoonright A, \emptyset)$, (by [Sh:93], 2.11(1), p.184, in its proof we have rely not only of [Sh:93], 2.10(2) + 2.9(3) but also on [Sh:93], 2.4(3)).

0B) If a type p is finitely satisfiable in A then p does not fork over A .

1) [Monotonicity] If $B \subseteq A_1 \subseteq A_2 (\subseteq \mathfrak{C})$, \bar{a}/A_2 does not widely divide over B .

1A) If p does not divide over A , $A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then p_1 does not divide over A_1 .

1B) If p does not fork over A , $A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then p_1 does not fork over A_1 .

1C) If p does not widely divide over (r, A) , $A \subseteq A_1$, $r_1 \vdash r$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then p_1 does not widely divide over (r_1, A_1) .

2) [Local character] \bar{a}/A does not widely divide over B iff for every finite subsequence \bar{a}' of \bar{a} and finite subset A' of A , $\bar{a}'/(A' \cup B)$ does not widely divide over B .

2A) The type p does not weakly divide over (r, B) iff every finite $p' \subseteq p$ dnwd over (r, B) .

2B) The type p does not divide over A iff for every finite $p' \subseteq p$ does not divide over A , iff some finite conjunction φ of members of p satisfies the requirement in Definition 1.3(1).

2C) The type p does not fork over A iff every finite $p' \subseteq p$ does not fork over A .

3) [More monotonicity] Assume $\text{Rang}(\bar{a}') = \text{Rang}(\bar{a}'')$, then: \bar{a}'/A dnwd over B iff \bar{a}''/A dnwd over B .

3A) If $B \subseteq A$, $\text{Rang } \bar{a}'' \subseteq \text{acl}(B \cup \bar{a}')$ and $\bar{a}' \setminus \bar{A}$ dnwd over B then \bar{a}''/A dnwd over B .

3B) Similarly to 3), 3A) for “does not divide” and for “does not fork” and for “dnwd over (r, B) ”.

4) [Transitivity] If $A_0 \subseteq A_1 \subseteq A_2$ and $\bar{a}/A_{\ell+1}$ dnwd over A_ℓ for $\ell = 0, 1$ then \bar{a}/A_2 dnwd over A_0 .

5) [Extendability] If $B \subseteq A \subseteq A^+$, p an m -type over A and p does not fork over B then p has an extension $q \in S^m(A^+)$ which does not fork over B (clear or see [Sh:93], 2.11(3)).

5A) If p is finitely satisfiable in A and $(\text{Dom } p) \cup A \subseteq B$ then we can extend p to a complete type over B finitely satisfiable in A .

6) [Trivial nice behaviour] \bar{a}/A does not fork over A (by [Sh:93], 2.11(2)).

6A) For a set A and an m -type p we have: p does not widely divide over (p, A) (check).

6B) Every m -type over M is finitely satisfiable in M .

7) [Continuity] If p_i does not widely divide over (r_i, B_i) for $i < \delta$ and

$i < j < \delta \Rightarrow p_i \subseteq p_j \ \& \ r_i \subseteq r_j \ \& \ B_i \subseteq B_j$ then $\bigcup_{i < \delta} p_i$ does not widely divide

over $(\bigcup_{i<\delta} r_i, \bigcup_{i<\delta} B_i)$.

7A) If $\langle A_i : i < \delta \rangle$ is increasing, $\langle B_i : i < \delta \rangle$ is increasing and C/B_i is finite satisfiable in A_i for each $i < \delta$ then $C/\bigcup_{i<\delta} B_i$ is finite satisfiable in $\bigcup_{i<\delta} A_i$.

[Why? if $p \subseteq C/\bigcup_{i<\delta} B_i$ is finite then for some j it is over B_j hence $\subseteq C/B_j$ is satisfiable in A_j hence is satisfiable in $\bigcup_{i<\delta} A_i$].

8) [Degree] Let $\bar{x}_m = \langle x_\ell : \ell < m \rangle$, E_m be an ultrafilter on $\Omega_m =: \{(\Delta, k) : \Delta = \Delta(\bar{x}_m) \subseteq L(T) \text{ finite, } k < \omega\}$ such that for every $(\Delta_0, k_0) \in \Omega_m$ the following set belongs to E_m :

$$\{(\Delta, k) \in \Omega_m : \Delta \subseteq \Delta_0 \text{ and } k_0 < k\}.$$

If $p(\bar{x})$ is a type over A , $lg(\bar{x}) = m$, then for some complete type $q(\bar{x})$ over A extending p for the E_m -majority of $(\Delta(\bar{x}), k)$ we have $D(q(\bar{x}), \Delta, \aleph_0, k) = D(p(\bar{x}), \Delta, \aleph_0, k)$ (by [Sh:93], 2.2(5), p.182; of course, we can use infinite \bar{x}).

In such a case we say: $q(\bar{x})$ is an E_m -nonforking extension of $p(\bar{x})$ or $q(\bar{x})$ E_m -does not fork over $p(\bar{x})$. If $p(\bar{x}) \in S^m(A)$ (so $p = q \upharpoonright A$) we may replace “over $p(\bar{x})$ ” by “over A ”.

9) [Additivity] If for every $\alpha < \alpha^*$ the type $tp(\bar{b}^\alpha, \bar{a} \cup A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta)$ does not divide over

$A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta$ then $tp_*(\bigcup_{\beta < \alpha^*} \bar{b}^\beta, A \cup \bar{a})$ does not divide over A (by [Sh:93], 1.5, p.181).

10) [Finitely satisfiable is average] Let $lg(\bar{x}) = m$ and $p = p(\bar{x})$ a type. Then p is finitely satisfiable in \mathbf{I} iff for some ultrafilter D over \mathbf{I} we have $p \subseteq Av(D, \text{Dom } p)$.

11) If D is an ultrafilter on \mathbf{I} , then $Av(D, A)$ belongs to $S^m(A)$ and is a finitely satisfiable in \mathbf{I} .

1.6 Claim. 1) [small basis] If $p \in S^\varepsilon(A)$ and $B_0 \subseteq A$ then for some B we have:

- (α) $B_0 \subseteq B \subseteq A$
- (β) $|B| \leq |\varepsilon| + |T| + |B_0|$
- (γ) p does not wind downward over $(p \upharpoonright B, B)$.

2) If $\bar{a}/(A \cup \bar{b})$ dnwd over A and $A \subseteq A^+ \subseteq \text{acl}[A \cup \{\bar{a}' \in \mathfrak{C} : \bar{a}' \text{ realizes } \bar{a}/A\}]$ then there is \bar{b}' (of the same length as \bar{b}) such that:

- (a) if $\bar{a}' \subseteq A^+$ and $\bar{a}'/A = \bar{a}/A$ then $\bar{b}' \hat{\ } \bar{a}'/A = \bar{b}' \hat{\ } \bar{a}/A$.

3) [weak symmetry] If $\bar{a}/(A \cup \bar{b})$ dnwd over A and then $\bar{b}/(A \cup \bar{a})$ dnd over A .

4) Assume $A \subseteq B \cap C$ (all $\subseteq \mathfrak{C}$) and C/B is finitely satisfiable in A hence $A = B \cap C$. Then B/C dnwd over $(B/A, A)$.

Proof. 1) By [Sh:93], 3.3, p.186.

2) By [Sh:93], 2.13, p.185 we can get clause (a).

3) By [Sh:93], 2.14, p.185 it dnd.

4) Straightforward (e.g. use 1.5(10)).

□_{1.6}

1.7 Theorem. *If $M \prec N \prec \mathfrak{C}$, $\|M\| = \mu$, $\|N\| = \mu^+$, $|T| < \kappa$, $\mu = \mu^{<\kappa}$, then there are $M^+ \prec N^+$ such that $N \prec N^+$, $M \prec M^+$, $\|M^+\| = \mu$, $\|N^+\| = \mu^+$ and:*

- (*)₁ *if $B \subseteq A \subseteq N$, $B \subseteq M$, $|A| < \kappa$, $m < \omega$, $p \in S^m(A)$ and p dnwd over $(p \upharpoonright B, \emptyset)$ then p is realized in M^+ .*
- (*)₂ *if $B \subseteq A \subseteq N$, $B \subseteq M$, $|A| < \kappa$, $C \subseteq \mathfrak{C}$, $|C| < \kappa$ and $A/(B \cup C)$ dnwd over B then there is $C' \subseteq M^+$ realizing $C/(B \cup A)$.*

Proof. Clearly we can prove (*)₁, (*)₂ separately. Now (*)₂ is immediate from 1.6(2). As for (*)₁, this is proved in [Sh:93],§4 (read [Sh:93],4.13,4.14,4.15,p.193 there, so we use [Sh:92],Theorem 3.1 which says that a Boolean algebra of cardinality λ^+ satisfying the κ -c.c., $\lambda^{<\kappa} = \lambda$ is λ -centered, i.e. is the union of $\leq \lambda$ ultrafilters, so if $\kappa > 2^{|T|}$ we are done which is enough for our main theorem (1.2 when $\lambda > |T|$). Actually repeating the proof of [Sh:93],Theorem 3.1 in the circumstances of [Sh:93],§4 show that $\kappa > |T|$ is enough. $\square_{1.7}$

1.8 Definition. 1) K_λ^0 be

$$\left\{ \bar{M} : \bar{M} = \langle M_i : i < \lambda^+ \rangle \text{ is } \prec\text{-increasing continuous, each } M_i \text{ a model of } T \text{ of cardinality } \lambda \text{ and } |M_0| = \emptyset \text{ (we stipulate such a model } \prec M \text{ for every } M \models T) \right\}.$$

2) \leq^0 is the following partial order on K_λ^0 : $\bar{M}^1 \leq^0 \bar{M}^2$ if for $i < j < \lambda^+$ we have $M_i^1 \prec M_i^2$.

1.9 Observation. 1) If $\langle \bar{M}^\alpha : \alpha < \delta \rangle$ is an \leq^0 -increasing chain (in K_λ^0) and $\delta < \lambda^+$ then it has a lub $\bar{M} : M_i = \bigcup_{\alpha < \delta} M_i^\alpha$.

2) If M is a model of T of cardinality λ^+ , then for some $\bar{M} \in K_\lambda^0$, $M = \bigcup_{i < \lambda^+} M_i$.

3) If $\bar{M}, \bar{N} \in K_\lambda^0$ and $\bigcup_{\alpha < \lambda^+} M_\alpha \prec \bigcup_{\alpha < \lambda^+} N_\alpha$ then for some club E of λ^+ for every $\alpha \in E$: $M_\alpha \prec N_\alpha$ and $N_\alpha / \bigcup_{\beta < \lambda^+} M_\beta$ dnwd over M_β .

Proof. 1) Immediate.

2) Use 1.5(2) + 1.6(1). $\square_{1.9}$

Using 1.7 and 1.5(5A) $\lambda^+ \times \lambda$ we get

1.10 Observation. Assume $\lambda = \lambda^{<\kappa}$. For every $\bar{M} \in K_\lambda^0$ there is an \leq^0 -increasing continuous sequence $\langle \bar{N}_\zeta : \zeta \leq \lambda \rangle$, in K_λ^0 , (so $\bar{N}_\zeta = \langle N_\alpha^\zeta : \alpha < \lambda^+ \rangle$), $\bar{N}_0 = \bar{M}$

such that (letting $N_\zeta = \bigcup_{\alpha < \lambda^+} N_\alpha^\zeta$) and [fixing χ , letting $\bar{a}_{\zeta, \alpha}$ an enumeration of $|N_\alpha^\zeta|$ of length λ] we can add: every type definable in $(H(\chi), \in, <_\chi^*)$ from $\langle \bar{N}_\varepsilon : \varepsilon \leq \zeta \rangle, \langle N_\alpha^{\zeta+1} : \alpha \leq \beta \rangle, \langle \bar{a}_{\varepsilon, \alpha} : \alpha < \lambda^+ \rangle : \varepsilon \leq \zeta, \langle \bar{a}_{\zeta, \alpha} : \alpha \leq \beta \rangle$ and $\langle \alpha_{\zeta, \alpha, j} : j < i \rangle$ and finitely many ordinals $< \lambda$ is realized in $N_{\beta+1}^{\zeta+1}$, hence:

- (*)₁ if $\alpha < \lambda^+, \zeta \leq \lambda, \text{cf}(\zeta) \in \{\lambda, 1\}, \text{cf}(\alpha) \in \{\lambda, 1\}, B \subseteq A \subseteq N_{\zeta-1}, B \subseteq N_\alpha^{\zeta-1}, |A| < \kappa, p \in S^m(A)$ and p dnwd over $(p \upharpoonright B, \emptyset)$ then p is realized in N_α^ζ .
(Note: $\lambda - 1 = \lambda$).
- (*)₂ if $\alpha \leq \beta < \gamma < \lambda^+, \gamma$ is non-limit, $B \subseteq A \subseteq A^+ \subseteq M_\gamma, |A^+| < \kappa, \bar{a} \in M_\gamma, \text{tp}(\bar{a}, A \cup M_\alpha)$ dnwd over B then either for some $\bar{a}' \in N_\gamma$ we have: $\text{tp}(\bar{a}, A^+ \cup M_\alpha) = \text{tp}(\bar{a}', A \cup M_\alpha)$ and $\text{tp}(\bar{a}', A \cup N_\beta)$ dnwd over B or there is no such $\bar{a}' \in \mathfrak{C}$.
- (*)₃ for γ non-limit, $\zeta \leq \lambda, \text{cf}(\zeta) \in \{1, \lambda\}$ we have:
 N_γ^ζ is κ -saturated (so when $\kappa = \lambda$ it is saturated).

1.11 Definition. Let A, B, C be given ($\subseteq \mathfrak{C}$).

- 0) $A \leq_B^{-1} C$ means that for every $\bar{b} \subseteq B, \bar{b}/(A \cup C)$ dnwd over $(\frac{\bar{b}}{A}, A)$.
- 1) $A \leq_B^0 C$ means that for every $\bar{c} \subseteq C, \bar{c}/(A \cup B)$ dnwd over $(\frac{\bar{c}}{A}, \emptyset)$.
- 2) $A \leq_B^1 C$ means there is an increasing continuous sequence $\langle A - \alpha : \alpha \leq \beta \rangle$ such that: $A = A_0, A \cup C = A_\beta$ and

$$\alpha \text{ an even ordinal } < \beta \Rightarrow A_\alpha \leq_B^0 A_{\alpha+1}$$

$$\alpha \text{ an odd ordinal } < \beta \Rightarrow A_\alpha \leq_B^{-1} A_{\alpha+1}.$$

- 3) $A \leq_B^2 C$ means that for some $C', C \subseteq C'$ and $A \leq_B^1 C'$.
- 4) $A \leq_B^3 C$ means that for some increasing continuous sequence $\langle A_\alpha : \alpha \leq \beta \rangle$ we have $A = A_0, A \cup C = A_\beta$ and $A_\alpha \leq_B^2 A_{\alpha+1}$.

1.12 Claim. 0) $A \leq_B^e A$ (for $e = -1, 0, 1, 2, 3$).

1) $A \leq_B^e C$ iff $A \leq_{A \cup B}^e A \cup C$ (for $e = -1, 0, 1, 2, 3$).

[Why? For $e = -1$ by 1.5(3A) for $e = 0$ by the definition of \leq_B^0 ; then continued].

2) If $A \subseteq B_1 \subseteq B \cup A$ and $A \leq_B^e C$ then $A \leq_{B_1}^e C$ (for $e = -1, 0, 1, 2, 3$).

[Why? By part (1) and: for $e = -1$ trivially, for $e = 0$ by 1.5(1C) and for $e = 1, 2, 3$ use earlier cases].

3) For $e = 1, 3$ we have: \leq_B^e is a partial order.

[Why? Read their definition].

4) If $e = 1, 3$ and $\langle A_\alpha : \alpha \leq \beta \rangle$ is increasing continuous and $A_\alpha \leq_B^e A_{\alpha+1}$ for $\alpha < \beta$ then $A_0 \leq_B^e A_\beta$.

[Why? Check].

5) For $(e^1, e^2) \in \{(-1, 1), (0, 1), (1, 2), (2, 3)\}$, we have: $A \leq_B^{e^1} C$ implies $A \leq_B^{e^2} C$.

[Why? Read].

6) If for every $\bar{b} \subseteq B, \bar{b}/(A \cup C)$ is finitely satisfiable in A then $A \leq_B^0 C$.

[Why? By 1.6(4) (and Definition 1.11(1))].

7) If $A \leq_B^2 C$ and $C' \subseteq C$ then $A \leq_B^2 C'$.

[Why? Read Definition 1.11(2)].

8) $A \leq_B^0 C$ iff $A \leq_C^{-1} B$.

[Why? Read the definitions].

1.13 Claim. Let $M \prec N$ and $M \subseteq A$. Then the following are equivalent:

- (a) $M \leq_N^3 A$
- (b) there are $M_0 \prec M_1 \prec M_2$ such that:
 - (i) $M = M_0$
 - (ii) the type $tp_*(N, M_1)$ is finitely satisfiable in M_0 and the type $tp_*(M_2, M_1 \cup N)$ is finitely satisfiable in M_1
 - (iii) for some elementary map $f, f(A) \subseteq M_2$ and $f \upharpoonright N = \text{identity}$
- (c) like (b) with $\|M_2\| \leq |T| + |A|$
- (d) $M \leq_N^2 A$.

13A Remark. 1) Clause (ii) of (b) implies $M_0 \leq_n^0 M_1 \leq_N^{-1} M_2$.

2) An equivalent formulation of (b) is

(b)' for some M_0, M_1, M_2, f we have $M = M_0 \leq_{f(N)} M_1 \leq_{f(N)} M_2, f \upharpoonright M_0 = \text{id}_{M_0}, f(A) \subseteq M_2$.

3) Another formulation is

(b)'' like (b)' but $f = \text{id}_{A \cup N}$.

Proof. (c) \Rightarrow (b) Trivial.

(b) \Rightarrow (c).

By the Lowenkeim Skolem argument.

(b) \Rightarrow (d)

By 1.12(6) clearly $M_0 \leq_N^0 M_1$ and similarly $M_1 \leq_{M_2}^0 N$, hence by 1.2(8) we have $M_1 \leq_N^{-1} M_2$. Hence by 1.12(5), $M_e \leq_N^1 M_{e+1}$ (for $e = 0, 1$), so by 1.12(3) $M_0 \leq_N^1 M_2$ hence by Definition 1.11(3) (and clause (iii) of 1.13(b)), $M = M_0 \leq_N^2$ as required.

(d) \Rightarrow (a) Trivial (by 1.12(5)).

So the only (and main) part left is:

(a) \Rightarrow (b) We know $M \leq_N^3 A$, by 1.12(1) without loss of generality $M \subseteq A$, hence there is an increasing continuous sequence $\langle A_\varepsilon : \varepsilon \leq \zeta \rangle$ such that: $A_0 = M, A_\zeta = A$ and $A_\varepsilon \leq_N^2 A_{\varepsilon+1}$. By the Definition of \leq_n^2, \leq_N^1 there is an increasing continuous sequence $\langle B_{\varepsilon+i} : i \leq i_\varepsilon \rangle$ such that $B_{\varepsilon,0} = A_\varepsilon, A_{\varepsilon+1} \subseteq B_{\varepsilon, i_\varepsilon}$ and $B_{\varepsilon, i} \leq_N^{\ell(\varepsilon)} B_{\varepsilon, i+1}$ (where for $i < i_\varepsilon$ we have $\ell(\varepsilon) \in \{-1, 0\}$ and $\varepsilon = \ell(\varepsilon) \bmod 2$). Let $\theta = 2^{|T|} + |N| + \sum_{\varepsilon < \zeta} (|i_\varepsilon| + |B_{\varepsilon, i_\varepsilon}|)^+$ and choose regular $\mu = \mu^\theta$.

We choose by induction on $\alpha < \mu^+, M_\alpha, N_\alpha$ such that: medskip

- (i) $M_\alpha \prec \mathfrak{C}$ is increasing continuous
- (ii) $\|M_\alpha\| = \mu, M \subseteq M_0,$
- (iii) f_α is an elementary mapping, $\text{Dom}(f_\alpha) = N, \text{Rang}(f_\alpha) = N_\alpha$ and $f_\alpha \upharpoonright M = \text{id}_M$
- (iv) $\text{tp}_*(N_\alpha, M_\alpha)$ is finitely satisfiable in M
- (v) $N_\alpha \subseteq M_{\alpha+1}$
- (vi) $M_{\alpha+1}$ is θ^+ -saturated.

There is no problem to carry the definition. (First choose M_α : if $\alpha = 0$ to satisfy (i) + (ii), if α is a limit ordinal, as $\bigcup_{\beta < \alpha} M_\beta$, and if $\alpha = \beta + 1$ to satisfy (i) + (ii) + (v).

Second choose f_α, N_α satisfying (iii) + (iv) which exists by 1.5(10) + (11)).

By using 1.7, λ^+ times we can find $\bar{M}^+ = \langle M_\alpha^+ : \alpha < \lambda^+ \rangle$ such that:

- (A) \bar{M}^+ is an increasing continuous sequence of elementary submodels of \mathfrak{C}
 - (B) $\|M_\alpha^+\| \leq \mu, M_\alpha \prec M_\alpha^+$
 - (C)₁ if $\alpha < \beta < \mu^+, B_1 \subseteq M_\alpha$ and $\text{cf}(\alpha) = \mu, B_1 \subseteq B_2 \subseteq M_\beta, |B_2| < \theta, C \subseteq \mathfrak{C}$ and C/B_2 dnwd over $(C/B_2, \emptyset)$ (equivalently, for every finite $\bar{c} \subseteq C, \bar{c}/B_2$ dnwd over $(\bar{c}/B_1, \emptyset)$) then C/B_2 is realized in M_α
 - (C)₂ similarly, but we replace the dnwd assumption by “ $B_2/(C_1 \cup C)$ dnwd over $(B_2/B_1, B_1)$ ”.
- [Note: we use $(*)_1$ from 1.7 for (C)₁ and $(*)_2$ from 1.7 for (C)₂].

Now let $M = \bigcup_{\alpha < \mu} M_\alpha, M^+ = \bigcup_{\alpha < \mu^+} M_\alpha^+$; and let

$E = \{\delta < \mu^+ : \delta \text{ a limit ordinal and } (M_\delta^+, M_\delta) \prec (M^+, M)\}$. Clearly E is a club of μ^+ and

- (*) $\delta \in E \Rightarrow \text{tp}(M_\delta^+, M)$ is finitely satisfiable in M_δ .

Choose $\delta \in E$ of cofinality μ . Now we choose g_ε by induction on $\varepsilon \leq \zeta$ such that:

- (α) g_ε an elementary mapping
- (β) $\text{Dom}(g_\varepsilon) = N \cup A_\varepsilon$
- (γ) g_ε is increasing continuous in ε
- (δ) $g_\varepsilon \upharpoonright N = f_\delta$
- (ε) $\text{Rang}(g_\varepsilon \upharpoonright A_\varepsilon) \subseteq M_\delta^+.$

If we succeed, then we get the desired conclusion (i.e. prove clause (b)).

[Why? First note that in clause (b) we can omit $f \upharpoonright N = \text{id}$ by $f \upharpoonright M =$ the identity if in clause (ii) we use $f(N)$; we call this (b)'. Now (b)' holds with $M, M_\delta, M_\delta^+, g_\zeta$ here standing to M_0, M_1, M_2, f there). So it is enough to carry the induction on ε . For $\varepsilon = 0$ let $g_\varepsilon = f_\delta$, and for ε a limit ordinal let $g_\varepsilon = \bigcup_{\xi < \varepsilon} g_\xi$; lastly for ε a successor ordinal say $\varepsilon = \xi + 1$, we choose $g_{\varepsilon, i}$ by induction on $i \leq i_\varepsilon$ such that:

- (α)' $g_{\varepsilon, i}$ an elementary mapping
- (β)' $\text{Dom}(g_{\varepsilon, i}) = N \cup B_{\varepsilon, i}$
- (γ)' $g_{\varepsilon, i}$ is increasing continuous in i
- (δ)' $g_{\varepsilon, 0} = g_\varepsilon$
- (ε)' $\text{Rang}(g_{\varepsilon, i}) \subseteq M_\delta^+.$

If we succeed then $g_{\varepsilon, i_\varepsilon} \upharpoonright A_{\varepsilon+1}$ is as required. So it is enough to carry the induction on i . For $i = 0$ let $g_{\varepsilon, i} = g_\varepsilon$, for i limit let $g_{\varepsilon, i} = \bigcup_{j < i} g_{\varepsilon, j}$ and for i a successor ordinal say $j + 1$, use clause $(C)_1$ in the choice of M_α^+ if j even, remembering Definition 1.11(1) and use clause $(C)_2$ in the choice of M_α^+ if j is odd remembering Definition 1.11(0). $\square_{1.13}$

1.13A Claim. *If $M \prec N, M \leq_N^2 A_\ell$ for $\ell = 1, 2$ then there are M^+, f_1, f_2 such that: $M \prec M^+, M \leq_N^2 M^+$ and for $\ell = 1, 2, f_\ell$ is an elementary mapping, $\text{Dom}(f_\ell) = N \cup A_\ell, f_\ell \upharpoonright N = \text{id}_N, f_\ell(A_\ell) \subseteq M^+$.*

Proof. Same proof as 1.13 (just shorter).

1.14 Definition. 1) Let $K_0^{pr} = \{(M, N) : M \prec N < \mathfrak{C}\}$ and $(M_1, N_1) \leq^* (M_2, N_2)$ iff $((M_e, N_e) \in K_0^{pr}$ for $e = 1, 2$ and $M_1 \prec M_2, N_1 \prec N_2$ and $M_1 \leq_{N_1}^2 M_2$ (equivalently, $M_1 \leq_{N_1}^3 M_2$ (by 1.13)).
2) We define $(M_1, N_1) \leq_{fs} (M_2, N_2)$ similarly replacing “ $M_1 \leq_{N_1}^2 M_2$ ” by “ N_1/M_2 is finitely satisfiable in M_1 ”.

1.15 Claim. 1) \leq^* is a partial order on K_0^{pr} .
2) If $\langle (M_\alpha, N_\alpha) : \alpha \leq \beta \rangle$ is increasing continuous and $(M_\alpha, N_\alpha) \leq^* (M_{\alpha+1}, N_{\alpha+1})$ for $\alpha < \beta$ then $(M_0, N_0) \leq^* (M_\beta, N_\beta)$.
3) If $M \prec N$ and $M \leq_N^2 A$ then for some (M_1, N_1) we have: $A \subseteq M_2$ and $(M, N) \leq^* (M_1, N_1) \in K_0^{pr}$.

Proof. 1) If $(M_0, N_0) \leq^* (M_1, N_1) \leq^* (M_2, N_2)$ then

- (i) $M_0 \subseteq M_1 \subseteq M_2$ and $N_0 \subseteq N_1 \subseteq N_2$
- (ii) $M_0 \leq_{N_0}^2 M_1$ and
- (iii) $M_1 \leq_{N_1}^2 M_2$
- (iv) $M_\ell \subseteq N_\ell$.

By 1.12(2) + (1) and clause (iii) above

- (v) $M_1 \leq_{N_0}^2 M_2$,

by 1.12(5) we have (by (ii) and (v) respectively)

- (ii)' $M_0 \leq_{N_0}^3 M_1$
- (v)' $M_1 \leq_{N_0}^3 M_2$

hence by 1.13

- (vii) $M_0 \leq_{N_0}^2 M_2$

hence $(M_0, N_0) \leq^* (M_2, N_2)$ holds by (i), (iv) and (vii).

2) Similarly using 1.12(4) + (1.13).

3) Use 1.13 (see 1.13a(3)) so there are $M_0 \prec M_1 \prec M_2$ such that $M = M_0, N/M_1$ fs in $M_0, M_2/(M_1 \cup N)$ fs in M_1 and $A \subseteq M_2$. So by 1.x $M_0 \leq_N^{-1} M_1, M_1 \leq_N^0 M_1$ hence (see 1.12(x) $M_0 \leq_N^3 M_0 \leq_N^3 M_1$ hence (see 1.12(4)) $M_0 \leq_N^3 M_2$ hence for any $N^*, M_2 \cup N \subseteq N^* \prec \mathfrak{C}$ we have $(M, n) \leq^* (M_2, N^*) \in K_0^{pr}$. $\square_{1.15}$

1.16 Definition. $K_2^{pr} = \{(M, N) : \text{the pair } (M, N) \in K_0^{pr} \text{ and if } (M, N) \leq^* (M', N') \in K_0^{pr} \text{ then } M'/N \text{ is fs in } M\}$.

1.17 Claim. *If $(M, N) \in K_0^{pr}$ then for some (M', N') we have:*

- (a) $(M, N) \leq (M', N') \in K_0^{pr}$
- (b) $\|N'\| \leq \|N\| + |T|$
- (c) $(M', N') \in K_1^{pr}$ i.e.
if $(M', N') \leq^* (M'', N'')$ then M''/N' is fs in M' .

Proof. Let $\mu = \|N\| + |T|$, assume the conclusion fails. We now choose by induction on $\alpha < \mu^+$, (M_α, N_α) such that:

- (i) $(M_0, N_0) = (M, N)$
- (ii) $(M_\alpha, N_\alpha) \in K_0^{pr}$, $\|N_\alpha\| \leq \mu$
- (iii) $\beta < \alpha \Rightarrow (M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha)$
- (iv) for limit δ we have $(M_\delta, N_\delta) = \left(\bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right)$
- (v) $M_{\alpha+1}/N_\alpha$ is not fs in M_α .

For $\alpha = 0$ see (i) for α limit see (iv) and 1.15(2) if $\alpha = \beta + 1$ find (M_α, N_α) satisfying $(M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha) \in K_1^{pr}$ and satisfying (v). By Lowenheim Skolem argument without loss of generality $\|N_\alpha\| \leq \mu$ and by 1.15(1) also clause (iii) holds. For a club of $\delta < \mu^+$ we get contradiction to clause (v). $\square_{1.17}$

1.18 Fact. 1) If $(M, N) \in K_1^{pr}$ and $(M', N') \in K_0^{pr}$ and $(M, N) \leq^* (M', N')$ then $(M, M') \leq^* (N, N')$.
2) If $(M, N) \in K_1^{pr}$ and $M \leq_N^2 A$ then A/N is fs in M .

Proof. 1) By 1.17 we know M'/N is fs in M hence by 1.13, (b) \Rightarrow (d) we know $M \leq_{M'}^2 N$ which give the desired conclusion.

2) By 1.13A.

1.19 Claim. 1) *If $(M_\alpha, N_\alpha) \in K_1^{pr}$ for $\alpha < \delta$ and $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$ is $<^*$ -increasing then for $\alpha < \delta$*

$$(M_\alpha, N_\alpha) \leq^* \left(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \in K_1^{pr}.$$

2) *If $(M_\alpha, N_\alpha) \in K_1^{pr}$ and $\langle (M_\alpha, N_\alpha) : \alpha \leq \delta \rangle$ is $<^*$ -increasing then*

$$\left(\bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \in K_1^{pr} \text{ and } \left(\bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \leq^* (M_\delta, N_\delta).$$

Proof. We prove both together by induction on δ .

0) By the induction hypothesis without loss of generality $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$ is

increasing continuous.

1) Clearly $(M_\alpha, N_\alpha) \leq^* \left(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \in K_0^{pr}$ (see 1.17(2)). Suppose

$\left(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \leq^* (M, N)$. So by 1.17(1), for $\alpha < \delta$, $(M_\alpha, N_\alpha) \leq^* (M, N)$, but $(M_\alpha, N_\alpha) \in K_1^{pr}$ hence M/N_α is fs in M_α . But this implies $M/\bigcup_{\alpha < \delta} N_\alpha$ is fs in

$\bigcup_{\alpha < \delta} M_\alpha$ by 1.5(7A).

2) As we are proving by induction on δ ; without loss of generality $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$

is \leq^* -increasing continuous, so by part (1), $(M_\alpha, N_\alpha) \leq^* \left(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \in K_1^{pr}$

for $\alpha < \delta$. Now for $\alpha < \delta$, $(M_\alpha, N_\alpha) \leq^* (M_\delta, N_\delta)$ and $(M_\alpha, N_\alpha) \in K_1^{pr}$ clearly M_δ/N_α is fs in M_α hence by 1.5(7A), $M_\delta/\bigcup_{\alpha < \delta} N_i$ is fs in $\bigcup_{i < \delta} M_i$, hence by 1.13

$\bigcup_{i < \delta} M_i \leq^3 \bigcup_{\alpha < \delta} N_i \bigcup_{i < \delta} N_i$ hence $\left(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \leq^* (M_{\delta+1}, N_{\delta+1})$. □_{1.19}

Now we want to apply 1.1. Toward this (for λ as there) we define:

1.20 Definition. 1) $K_{ap}^0 = K_{ap}^0 = K_{ap}^0[T] = K_{ap}^0[T, \lambda]$ is the set of models M of T with universe $\subseteq \lambda^+$ and cardinality $< \lambda$ such that: $M \cap \lambda \neq \emptyset$ and $0 < \alpha < \lambda^+$ implies $M \upharpoonright (\lambda \times \alpha) \prec M$. For such M let $\text{Dom}(M) = \{\alpha < \lambda^+ : [\lambda \times \alpha, \lambda \times \alpha + \lambda) \cap M \neq \emptyset\}$. We now define $<_{K_{ap}^0}$ by: $M \leq_{K_{ap}^0} N$ if (both are in K_{ap}^0 and $M \prec N$ and): for every $\alpha \in (0, \lambda^+)$, $M \upharpoonright (\lambda \times \alpha) <_{M \upharpoonright (\lambda \times \alpha + \lambda)}^2 N \upharpoonright (\lambda \times \alpha)$.

1.21 Observation. So $M \leq_{K_{ap}^0} N$ iff both are in K_{ap}^0 , $M < N$ and for $\alpha \in (0, \lambda^+)$ we have $(M \upharpoonright (\lambda \times \alpha), M \upharpoonright (\lambda \times \alpha + \lambda)) \leq^* (N \upharpoonright (\lambda \times \alpha), N \upharpoonright (\lambda \times \alpha + \lambda))$.

1.22 Claim. 1) $\leq_{K_{ap}^0}$ is a partial order on K_{ap}^0 .

[Why? By 1.15(1)].

2) If $\langle M_i : i < \delta \rangle$ is $\leq_{K_{ap}^0}$ -increasing, $\sum_{i < \delta} \|M_i\| < \lambda$ then $M_i \leq_{K_{ap}^0} \bigcup_{j < \delta} M_j \in K_{ap}^0$.

[Why? By 1.15(2)].

1.23 Claim. Let \bar{N}^ζ, N_ζ (for $\zeta \leq \lambda$) be as in 1.10. Let $E \subseteq \lambda^+$ be a thin enough club of λ^+ , $\{\varepsilon(\alpha) : \alpha < \lambda^+\}$ enumerate $\{0\} \cup E$, H a 1-to-1 map from N_λ onto λ^+ mapping $N_{\varepsilon(\alpha)}^\lambda$ onto $\lambda \times \alpha$. Let $N_\alpha^* = H(N_{\varepsilon(\alpha)}^\lambda)$, $N^* = \bigcup_{\alpha < \lambda} N_\alpha^*$.

1) If $M \in K_{ap}^0$ then there is a lawful f (see [Sh 457], 4.1) which is an elementary embedding of M into N^* such that for $\alpha \in \text{Dom}(M)$, $f(M \upharpoonright (\lambda \times \alpha)) <_{f(M \upharpoonright (\lambda \times \alpha + \lambda))}^2 N^* \upharpoonright (\lambda \times \alpha)$.

Proof. Straightforward.

[Saharon: put old proof of 1.27 from AP here?]

But we want more, not only universality but also homogeneity.

1.24 Definition. $K_{ap}^1 = K_{ap}^1[T, \lambda]$ is the set of $M \in K_{ap}^0$ such that for every $\alpha \in (0, \lambda^+)$, if $\neg(M \subseteq \lambda \times \alpha)$ then $(M \upharpoonright (\lambda \times \alpha), M) \in K_1^{pr}$.

Let $\leq_{K_{ap}^1}$ be $\leq_{K_{ap}^0} \upharpoonright K_{ap}^1$.

1.25 Claim. 1) $\leq_{K_{ap}^1}$ is a partial order on K_{ap}^1 .

[Why? By 1.22 and Definition 1.2].

2) If $\langle M_i : i < \delta \rangle$ is $\leq_{K_{ap}^1}$ -increasing, $\sum_{1 < \delta} \|M_i\| < \lambda$ then $M_i \leq_{K_{ap}^1} \bigcup_{j < \delta} M_j \in K_{ap}^1$.

[Why? By 1.23(1) and 1.19(1)].

1.26 Claim. Let \bar{N}^ζ, N_ζ (for $\zeta \leq \lambda$) be as in 1.10. Let $E \subseteq \lambda^+$ be a thin enough club of λ^+ , $\{\varepsilon(\alpha) : \alpha < \lambda^+\}$ enumerate $\{0\} \cup E$, H a 1-to-1 map from N_λ onto λ^+ mapping $N_{\varepsilon(\alpha)}^\lambda$ onto $\lambda \times \alpha$. Let $N_\alpha^* = H(N_{\varepsilon(\alpha)}^\lambda)$, $N^* = \bigcup_{\alpha < \lambda} N_\alpha^*$.

1) If $M \in K_{ap}^1$ then there is a lawful f (see [Sh 457], 4.1) which is an elementary embedding of M into N^* such that for $\alpha \in \text{Dom}(M)$, $f(M \upharpoonright (\lambda \times \alpha)) <_{f(M \upharpoonright (\lambda \times \alpha + \lambda))}^2 N^* \upharpoonright (\lambda \times \alpha)$.

2) If $M_0 \leq_{K_{ap}^1} M_1$ and (M_0, f_0) is as in part (1) then we can find $f_1, f_0 \subseteq f_1$ such that (M_1, f_1) is as in part (1). Moreover, if $f_0 \cup (f_1 \upharpoonright (M_1 \upharpoonright (M_1 \upharpoonright (\lambda \times \alpha))))$ has been determined we can continue.

1.27 Amalgamation Claim. Assume $M_0 \leq_{K_{ap}^0} M_\ell$ for $\ell = 1, 2$ and (for simplicity) $|M_1| \cap |M_2| = |M_0|$. Then there is $M \in K_{ap}^0$ such that $M_1 \leq_{K_{ap}^0} M$ and $M_2 \leq_{K_{ap}^0} M$.

Proof. Follows from 1.26(1) + (2) (Q: domain?)

1.28 Claim. (K_{ap}, \leq^*) is a λ -system (see [Sh 457], §4).

Proof. Check.

1.29 Claim. (K_{ap}, \leq^*) is simple (see [Sh 457], §4).

Proof. Included in the proof of amalgamation (see last clause of 1.26(2)).

1.30 Claim. If M is a model of T of cardinality λ^+ then for some $\Gamma \in K_\lambda^{md}$, M can be elementarily embedded into M_Γ .

Proof. Use 1.10 with $M = \bigcup_{\alpha < \lambda} M_\alpha$ so we get $N^*, N_\alpha^*(\alpha < \lambda)$ as in 1.18. Check.

1.31 Proof of 1.2. Use the above claims.

§2 ON THE STRONG ORDER PROPERTIES AND FINITARY VERSIONS

2.0 Discussion. By [Sh 457], for some non-simple (first order complete) the answer to the following is yes:

- \bigoplus_T if $\lambda = \lambda^{<\lambda} > |T|$, $2^\lambda = \lambda^+$, is there a $(\lambda$ -complete), λ^+ -c.c. forcing notion Q , \Vdash_Q “ $\text{univ}(\lambda^+, T) \leq \lambda^{++} < 2^\lambda$ ”?
- \bigoplus'_T and by [?] if $\lambda = \lambda^{<\lambda} > |T|$, $2^\lambda = \lambda^+$ is there a λ -complete λ^+ -c.c. forcing notion Q , \Vdash_Q “ $\text{univ}(\lambda^+, T) = 1$, $\lambda^+ < 2^\lambda$ ”?

We know that for theories T with the strict order property the answer is no (by [KjSh 409], or see [Sh 457], §3). We would like to characterize the answer by a natural property of T (hence show that the answer to all reasonable variants is the same, e.g. does not depend on λ , $\bigoplus_T \equiv \bigoplus'_T$, etc.) So the results we mention above give a lower bound (simple theories $+T_{qef} + T_{trf}$) and an upper bound (failure of the strict order property) to the family of T 's with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of citeSh:a, [Sh:c] was to find natural dividing lines for the family of first order theories (so the main ones there were stable, superstable and also NTOP, deepness NOTOP). Now [Sh:93] suggests another one: simplicity. Note that the negation of simple, the tree property has been touched upon in [Sh:a] but there were conclusions only for one side. [Sh:93] establishes this dividing line by having consequences for both the property and its negation and having “semantical characterization” for T simple: when $|T| \leq \kappa < \lambda = \lambda^{<\lambda} < \mu = \mu^\kappa$ we can force by a λ^+ -c.c. λ -complete forcing notion Q that $2^\lambda > \mu$ and every model of T of cardinality μ can be extended to a κ^+ -saturated one, and the tree property implies a strong negation. Of course, both the inner theory and such “outside”, “semantical” characterization are much weaker than those for stable theories.

The strict order property has no such results only several consequences. We suggest below weaker properties (first the strong order property then the n -version of it for $n < \omega$) which has similar consequences and so may be the right dividing line (for some questions). Remember (this is in equivalent formulations).

2.1 Definition. T has the strict order property if some formula $\varphi(\bar{x}, \bar{y})$ (with $lg\bar{x} = lg\bar{y}$) define in some model M of T , a partial order with infinite chains.

2.2 Definition. 1) A first order complete T has the strong order property if some sequence $\bar{\varphi} = \langle \varphi_n(\bar{x}^n; \bar{y}^n) : n < \omega \rangle$ of formulas exemplifies it which means that for every λ :

(*) $_{\bar{\varphi}}^\lambda(a)$ $lg\bar{x}^n = lg\bar{y}^n$ are finite, \bar{x}^n an initial segment of \bar{x}^{n+1}

\bar{y}^n an initial segment of \bar{y}^{n+1}

- (b) $T, \varphi_{n+1}(\bar{x}^{n+1}, \bar{y}^{n+1}) \vdash \varphi_n(\bar{x}^n, \bar{y}^n)$
(c) for $m \leq n$, $\neg(\exists \bar{x}^{n,0} \dots \bar{x}^{n,m-1})[\bigwedge\{\varphi_n(\bar{x}^{n,k}, \bar{x}^{n,\ell}) : k = \ell + 1 \text{ mod } m\}]$ belongs to T
(d) there is a model M of T and $\bar{a}_\alpha^n \in M$ (of length \bar{y}^n , for $n < \omega$, $\alpha < \lambda$) such that $\bar{a}_\alpha^n = \bar{a}_\alpha^{n+1} \upharpoonright lg\bar{y}^n$ and $M \models \varphi_n[\bar{a}_\beta^n, \bar{a}_\alpha^n]$ for $n < \omega$ and $\alpha < \beta < \lambda$.

- 2) The finitary strong order property is defined similarly but $\bar{x}^n = \bar{x}, \bar{y}^n = \bar{y}$.
- 3) We use the shorthand SOP, FSOP and for the negation NSOP, NFSOP (similarly later for NSOPn).

2.3 Claim. 1) *The strict order property implies finitary strong order property which implies the strong order property.*

2) *There is a first order complete T , which has the strong order property (even the finitary one) but not the strict order property.*

3) *Also some first order complete T has the strong order property but not the finitary strong order property, i.e. no $\langle \varphi_n(\bar{x}, \bar{y}) : n < \omega \rangle$ exemplifies it (i.e. with $lg \bar{x}_n$ constant).*

Proof. 1) Immediate.

2) For $\ell \leq n < \omega$ let $<_{n,\ell}$ be a two-place relation. Let $<_n = <_{n,0}$. Let T_0 say:

- (a) $x <_{n,m-1} y \Rightarrow x <_{n,m} y$
- (b) $x <_{n,n} y$
- (c) $\neg(x <_{n,n-1} x)$
- (d) if $\ell + k + 1 = m \leq n$ then $x <_{n,\ell} y \ \& \ y <_{n,k} z \Rightarrow x <_{n,m} z$.

We shall now prove that T_0 has the amalgamation property; it also has the point embedding property (as the latter is easier we leave its checking to the reader).

Now suppose $M_i \models T_0, M_0 \subseteq M_i$ for $i = 0, 1, 2$ and $M_1 \cap M_2 = M_0$. We define a model M : its universe is $M_1 \cup M_2$ and

$$\begin{aligned}
 <_{n,m}^M = \left\{ (a, b) \in M \times M : \text{if } m < n \text{ then for some } i \in \{1, 2\} \text{ we have :} \right. \\
 & \quad (a, b) \in <_{n,m}^{M_i} \text{ or } a \in M_i \setminus M_0, b \in M_{3-i} \setminus M_0 \\
 & \quad \text{and for some } c \in M_0 \text{ and } \ell, k \text{ we have :} \\
 & \quad \left. m = \ell + k + 1, (a, c) \in <_{n,\ell}^{M_i}, (c, b) \in <_{n,k}^{M_{3-i}} \right\}.
 \end{aligned}$$

Now clearly M extends M_1 and M_2 : trivially $<_{n,m}^M \upharpoonright M_i = <_{n,m}^{M_i}$. Is M a model of T_0 ? Let us check.

Clause (a) holds: For $x, y \in M_i$ as $M_i \subseteq M$; for $i = 1, 2$ and $x \in M_i \setminus M_0, y \in M_{3-i} \setminus M_0$, without loss of generality $m < n$; let $c \in M$ witness $(a, b) \in <_{n,m-1}^M$ i.e. for some ℓ, k we have $\ell + k + 1 = m - 1, (a, c) \in <_{n,\ell}^{M_i}$ and $(c, b) \in <_{n,k}^{3-i}$. Now by clause (a) applied to $M_i, (a, c) \in <_{n,\ell+1}^{M_i}$ now apply the definition to get $(a, b) \in <_{n,(\ell+1)+k+1}^M = <_{n,m}^M$.

Clause (b) holds: Check as defining $<_{n,m}^M$ we say: “if $m < n$ then ...” so if $n = m$ there is no requirement.

Clause (c): As $M_i \subseteq M$ and $M_i \models T_0$.

Clause (d): Check by cases, i.e. for some $i \in \{1, 2\}$ one of the following cases hold.

- (1) $\{x, y, z\} \subseteq M_i$:
use “ M_i is a model of T_0 and M_i a submodel of M ”.
- (2) $\{x, y\} \subseteq M_i, \{y, z\} \subseteq M_{3-i}$:
use the definition of $<_{n,m}^M$.
- (3) $y \in M_i \setminus M_0, \{x, z\} \subseteq M_{3-i} \setminus M_0$.

As $x <_{n,\ell} y$ there are ℓ_1, ℓ_2 and $x_1 \in M_0$ such that $x <_{n,\ell_1} x_1, x_1 \leq_{n,\ell_2} y$ and $\ell_1 + \ell_2 + 1 = \ell$.

As $y <_{n,k} z$ there are k_1, k_2 and $z_1 \in M_0$ such that $y \leq_{n,k_1}^{M_i} z_1, z_1 \leq_{n,k_2}^{M_{3-i}} z, k_1 + k_2 + 1 = k$. In M_i we have $x_1 <_{n,\ell_2}^{M_i} y <_{n,k_1}^{M_i} z_1$ hence $x_1 \leq_{n,\ell_2+k_1+1}^{M_i} z_1$ and as $\{x_1, z_1\} \subseteq M_0 \subseteq M_i$ clearly $x_1 <_{n,\ell_2+k_1+1}^{M_0} z_1$. Now in M_{3-i} we have $x <_{n,\ell_1}^{M_{3-i}} x_1 <_{n,\ell_1}^{M_{3-i}} x_1 <_{n,\ell_2+k_1+1}^{M_{3-i}} z_1$ hence $x \leq_{n,\ell_1+\ell_2+k_1+2}^{M_{3-i}} z_1$ so $x <_{n,\ell_1+\ell_2+k_1+2}^{M_{3-i}} z_1 <_{n,k_2}^{M_{3-i}} z$ hence $x <_{n,\ell_1+\ell_2+k_1+k_2+3}^{M_{3-i}} z$ but $\ell_1 + \ell_2 + k_1 + k_2 + 3 = \ell + k + 1 = m$ so $x <_{n,m}^{M_{3-i}} z$ as required.

- (4) $y \in M_i \setminus M_0, x \in M_{3-i} \setminus M_0, z \in M_0$.
Similar to case (3) but with no x_1 .
- (5) $y \in M_i \setminus M_0, x \in M_0, z \in M_{3-i} \setminus M_i$.
Similar to case (3) but with no z_1 .

Let T be the model completion of T^0 ; easy to check that it exists and has elimination of quantifiers. Let $\varphi_n(x, y) = \bigwedge_{\ell \leq n} x <_{n,\ell} y$ (remember $x <_{n,\ell} y$ means $x <_{n,\delta} y$) now $\langle \varphi_n : n < \omega \rangle$ exemplifies that T has the (finitary) strong order property. On the other hand we shall show that for every $n(*) < \omega$ the theory $T_{n(*)} =: T \upharpoonright \{<_{n,\ell} : \ell \leq n \leq n(*)\}$ does not have the strict order property (as $T = \bigcup_{n < \omega} T_n$, this clearly implies that T does not have the strict order property).

First note that also $n(*)$ has elimination of quantifiers and then check directly.

(3) Let T^0 say:

- (a) P_n (for $n < \omega$) are pairwise disjoint (P_n unary predicates)
- (b) F_n a partial one place function from P_{n+1} into P_n
- (c) $<_{n,\ell}$ are two-place relations on P_n for $\ell \leq n < \omega$; and let $<_n = <_{n,0}$
 - (α) $x <_{n,m-1} y \rightarrow x <_{n,m} y$
 - (β) $P_n(x) \ \& \ P_n(y) \rightarrow x <_{n,n} y$
 - (γ) $\neg(x <_{n,n-1} x)$
 - (δ) if $\ell + k + 1 = m \leq n$ then: $x <_{n,\ell} y \ \& \ y <_{n,k} z \rightarrow x <_{n,m} z$
- (d) $x <_{n+1,\ell} y \rightarrow F_n(x) <_{n,\ell} F_n(y)$.

Again T will be the model completion of T^0 and it has elimination of quantifiers and we shall use $\bar{x}_n = \langle x_i : i < n \rangle, \bar{y}_n = \langle y_i : i < n \rangle$ and $\varphi_n(\bar{x}_n, \bar{y}_n) = \bigwedge_{i < n} F_i(x_{i+1}) = x_i \ \& \ \bigwedge_{i < n} F_i(y_{i+1}) = y_i \ \& \ \bigwedge_{i < n} x_i <_i y_i$. □_{2.3}

2.4 Claim. 1) *The following are equivalent (for $\lambda \geq |T|$):*

- (A) *T has the strong order property*
- (B) $_{\lambda}$ *there is a λ^+ -saturated model M of T , a L_{∞, λ^+} -formula $\varphi = \varphi(\bar{x}, \bar{y}), \varepsilon = \ell g \bar{x} = \ell g \bar{y} \leq \lambda$, possible with $\leq \lambda$ parameters, such that in M, φ defines a partial linear order with a chain of length $\geq \beth_2(\lambda)^+$.*

2) *The following are equivalent ($\lambda \geq |T|$):*

- (A)' *T has the finitary strong order property*
- (B) $'_{\lambda}$ *like (B) $_{\lambda}$ but $\varepsilon < \omega$.*

Proof. 1) (A) \Rightarrow (B) $_{\lambda}$ Straight: for a given $\bar{\varphi} = \langle \varphi_n(\bar{x}_n, \bar{y}_n) : n < \omega \rangle$, let \bar{x}, \bar{y} be the limit of \bar{x}_n, \bar{y}_n respectively and write $\psi^*(\bar{x}, \bar{y}) = \bigvee_m (\exists \bar{z}_0, \dots, \bar{z}_m)[\bar{x} = \bar{z}_0 \ \& \ \bar{y} = \bar{z}_m \ \& \ \bigwedge_{\ell < m} \varphi_{\omega}(\bar{z}_{\ell}, \bar{z}_{\ell+1})]$ where $\varphi_{\omega}(\bar{x}, \bar{y}) =: \bigwedge_n \varphi_n(\bar{x}, \bar{y})$.

(B) $_{\lambda}$ \Rightarrow (A) Let $\bar{a}_{\alpha} \in {}^{\varepsilon}M$ for $\alpha < \beth_2(\lambda)^+$ form a chain. Without loss of generality the order φ defines is strict (i.e. $\vdash \varphi(\bar{x}, \bar{x})$) and no parameters (just add them to the \bar{a}_{α} 's). By Erdos Rado theorem without loss of generality for some type $q = q(\bar{x}, \bar{y})$ for all $\alpha < \beta < \omega$ the sequence $\bar{a}_{\alpha} \hat{\ } \bar{a}_{\beta}$ realizes q .

For every $n, \bigcup \{q(\bar{x}_{\ell}, \bar{x}_k) : k = \ell + 1 \bmod n \text{ and } k, \ell < n\}$ cannot be realized in M (as if $\bar{b}_0 \hat{\ } \dots \hat{\ } \bar{b}_{n-1}$ realizes if we get a contradiction to “ $\varphi(\bar{x}, \bar{y})$ defines a strict partial order”). By saturation there is $\varphi_n^0(\bar{x}, \bar{y}) \in q(\bar{x}, \bar{y})$ such that $\{\varphi_n^0(\bar{x}_{\ell}, \bar{x}_k) : k = \ell + 1 \bmod n \text{ and } k, \ell < n\}$ is not realized in M . The rest should be clear.

2) Left to the reader. □_{2.4}

2.5 Definition. 1) T has the n -stronger order property (SOP $_n$) if there is a formula $\varphi(\bar{x}, \bar{y})$ having this property for T which means: $\ell g \bar{x} = \ell g \bar{y}$ (allowing parameters changes nothing) and there is a model M of T and $\bar{a}_k \in {}^{\ell g \bar{x}}M$ for $k < \omega$ such that:

- (a) $M \models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$
- (b) $M \models \neg \exists \bar{x}_0 \dots \bar{x}_{n-1} (\bigwedge \{\varphi(\bar{x}_{\ell}, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\})$.

2) “ $T, \varphi(\bar{x}, \bar{y})$ have the SOP $_{\leq n}$ ” is defined similarly except that in (b) we replace n by each $m \leq n$.

2.6 Claim. $SOP \Rightarrow SOP_{n+1}, SOP_{n+1} \Rightarrow SOP_n, SOP_{\leq n+1} \Rightarrow SOP_{\leq n}$ and $SOP_n \Leftrightarrow SOP_{\leq n}$ for any given T , (we did not say “for any φ ”).

Proof. The first clause is immediate. The second clause is straight too: let $\varphi(\bar{x}, \bar{y}), M, \langle \bar{a}_m : m < \omega \rangle$ exemplify SOP $_{n+1}$ and without loss of generality the sequence $\langle \bar{a}_m : m < \omega \rangle$ is an indiscernible sequence. Does $M \models (\exists \bar{x}_0, \dots, \bar{x}_{n-1})[\bar{x}_0 = \bar{a}_1 \ \& \ \bar{x}_{n-1} = \bar{a}_0 \ \& \ \bigwedge \{\varphi(\bar{x}_{\ell}, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\}]$? If the answer is yes we can replace \bar{a}_1 by \bar{a}_2 (by indiscernability), let $\bar{c}_0, \dots, \bar{c}_{n-1}$ be as required above on $\bar{x}_0, \dots, \bar{x}_{n-1}$ and $\bar{b}_0 =: \bar{a}_1, \bar{b}_1 =: \bar{a}_2 (= \bar{c}_0), \bar{b}_2 =: \bar{c}_1, \dots, \bar{b}_{n-1} =: \bar{c}_{n-2}, \bar{b}_n =: \bar{c}_{n-1} = a_0$; now they satisfy the requirement mentioned in (b) of 2.5(1) on

\bar{x}_0, \dots, x_n (for SOP_{n+1}), contradicting clause (b) of 2.5(1). So assume “no” and now $\varphi'(\bar{x}, \bar{y})$ have SOP_n for T where: $\varphi'(\bar{x}, \bar{y}) =: \varphi(\bar{x}, \bar{y}) \ \& \ \neg(\exists \bar{x}_0, \dots, x_{n-1})[\bar{x}_0 = \bar{x} \ \& \ \bar{x}_1 = \bar{y} \ \& \ \bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell \bmod n\}]$.

As for $SOP_n \Leftrightarrow SOP_{\leq n}$, the implications \rightarrow is really included in the proof above, (i.e. by it, if $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_n$ exemplifies SOP_n , for some φ_{n-1} we have $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_{n-1}$ exemplifies SOP_{n-1} (with $n, n-1$ here corresponding to $n+1, n$ there), and we can define φ_{n-2}, \dots similarly; now $\langle \bar{a}_\ell : \ell < \omega \rangle, \bigwedge_{i \leq n} \varphi_i$ exemplifies $SOP_{\leq n}$. The implication \Leftarrow is trivial. Now the third clause $SOP_{\leq n+1} \Rightarrow SOP_{\leq n}$ is trivial (read the definition). $\square_{2.6}$

2.7 Claim. *Let T be complete. If T has SOP_3 then T has the tree property (i.e. is not simple).*

Proof. Let $\kappa = \text{cf}(\kappa) > |T|$ and $\lambda > \kappa$ be a strong limit singular cardinal of cofinality κ . Let $J = {}^\kappa \lambda, I = \{\eta \in {}^\kappa \lambda : \eta(i) = 0 \text{ for every } i < \kappa \text{ large enough}\}$. Let $\varphi(\bar{x}, \bar{y})$ exemplify the SOP_3 . By the definition we can find a model M of T and $\bar{a}_\eta \in M$ (for $\eta \in J$) such that:

$$(*) \ \eta <_{\ell x} \nu \text{ in } I \Rightarrow M \models \varphi[\bar{a}_\eta, \bar{a}_\nu].$$

Without loss of generality $\|M\| \geq \lambda$, M is κ^+ -saturated. So for every $\eta \in {}^\kappa(\lambda \setminus \{0\}) \setminus I$ we can find $\bar{a}_\eta \in M$ such that it realizes $p_\eta = \{\varphi(a_{(\eta \upharpoonright i) \frown 0_{[i, \kappa]}}), \bar{x}\} \ \& \ \varphi(\bar{x}, a_{(\eta \upharpoonright i) \frown (\eta(i)+1) \frown 0_{[i, \kappa]}}) : i < \kappa\}$. But if $\eta_1 <_{\ell x} \eta_2 \in {}^\kappa(\lambda \setminus \{0\})$ then we can find $\nu, \rho \in I$ such that: $\eta_1 <_{\ell x} \nu <_{\ell x} \rho <_{\ell x} \eta_2$ and $\varphi(\bar{x}, \bar{a}_\nu) \in p_{\eta_1}, \varphi(\bar{a}_\rho, \bar{x}) \in p_{\eta_2}$ and by (*) we have $M \models \varphi[\bar{a}_\nu, \bar{a}_\rho]$, so $p_{\eta_1} \cup p_{\eta_2}$ is contradictory (by clause (b) of 2.5(1) for “ φ have the SOP_3 ”). So $\langle p_\eta : \eta \in {}^\kappa(\lambda \setminus \{0\}) \rangle$ are pairwise contradictory, $|p_\eta| = \kappa$, and $\lambda^\kappa > \lambda = \lambda^{< \kappa} > 2^{|T|}$ and $\bigcup \{\text{Dom } p_\eta : \eta \in {}^\kappa(\lambda \setminus \{0\})\}$ has cardinality $\leq \lambda$ and $\kappa > |T|$.

By [Sh:a],III,7.7 = [Sh:c],III,7.7,p.141 this implies that T has the tree property. $\square_{2.7}$

2.8 Claim. 1) *The theory $T_n =: T \upharpoonright \{\langle n, \ell : \ell \leq n \rangle$ from 2.3(2) has SOP_n but not SOP_{n+1} .*

2) *$T_{\text{trf}}^{\text{mc}}$, the model completion of the theory of triangle free graphs has SOP_3 but not SOP_4 .*

3) *For $n \geq 3$ the model completion $T_n^{\text{mc}} = T_{\text{dcf}(n)}^{\text{mc}}$ of the theory $T_n = T_{\text{dcf}(n)}$ of graphs (= directed graphs, no loops or multiple edge for simplicity) with no directed circle of length $\leq n$ has SOP_n but not SOP_{n+1} .*

4) *For odd $n \geq 3$, the model completion $T_n^{\text{mc}} = T_{\text{ocf}(n)}^{\text{mc}}$ of the theory $T_n = T_{\text{ocf}(n)}$ of graphs with no odd circle of length $\leq n$, has SOP_n but not SOP_{n+1} .*

5) *For $n \geq 3$, the model completion $T_{\text{cf}(n)}^{\text{mc}}$ of the theory $T_n = T_{\text{cf}(n)}$ of graphs with no circles of length $\leq n$, has SOP_3 but not SOP_4 .*

6) *The theory T_{qcf} (see [Sh 457] does not have SOP_3 (but is not simple).*

2.8A Remark. 1) Note that $\text{univ}(\lambda, T_{\text{cf}(n)}^{\text{mc}}) = \text{univ}(\lambda, T_{\text{cf}(n)})$.

2) For those theories, $D(T^{\text{mc}})$ is an uncountable; they have no universal model in $\lambda < 2^{\aleph_0}$.

Proof. 1) Proved really in 2.3.

2) This is included in part (5).

3), 4), 5) We discuss the existence of model completion later; note that the meaning of T_n depends on the part we are proving.

Let xRy mean (x, y) is an edge; when we say (x, y) is an edge, for graphs we mean $\{x, y\}$ is an edge. Let $\bar{y} = \langle y_\ell : \ell < n \rangle$, $\varphi(\bar{x}, \bar{y}) = \bigwedge_{\ell < n-1} x_\ell R y_{\ell+1} \ \& \ x_{n-1} R y_0$. First

we note there $T_n \vdash \neg(\exists \bar{x}_0, \dots, \bar{x}_{n-1}) \bigwedge \{\varphi(\bar{x}_i, \bar{x}_k) : \ell, k < n, k = \ell + 1 \text{ mod } n\}$, otherwise there are $M \models T_n$ and $\bar{a}_\ell = \langle a_{\ell,0}, \dots, a_{\ell,n-1} \rangle \in {}^n M$ as forbidden but then $a_{0,0}, a_{1,1}, \dots, a_{n-1,n-1}$ is a circle, so in all cases this is impossible.

For parts 3), 4) let M be the following model of T_n ; elements a_i^ℓ ($i < \omega, \ell < n$), $R = \{(a_i^\ell, a_j^{\ell+1}) : i < j < \omega, \ell < n-1\} \cup \{(a_i^{n-1}, a_j^0) : i < j < \omega\}$ (but for graphs we put all such pairs and the inverted pair as R should be symmetric and irreflexive relation). Clearly for $i < j < \omega$, $M \models \varphi[\bar{a}_i, \bar{a}_j]$ where $\bar{a}_i = \langle a_i^0, \dots, a_i^{n-1} \rangle$. Lastly $M \models T$: for part (3) as R is not symmetric the absence of any circle should be clear, $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow i(1) < i(2)$; for part (4) there are circles but even or long and $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow \ell(1) = \ell(2) + 1 \text{ mod } n$, so $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$ (and $T_{\text{ocf}(n)}^{\text{mc}}$) has even or long $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow \ell(1) = \ell(2) + 1 \text{ mod } n$. So $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$ (and $T_{\text{dcf}(n)}^{\text{mc}}$ and $T_{\text{ocf}(n)}^{\text{mc}}$) has SOF_n .

Let $n = 3$. Now $T_{\text{cf}(3)} = T_{\text{ocf}(3)}$ so we can ignore part (5). Also $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$ has the amalgamation property and joining embedding property. Thus, it is enough to show that $T_{\text{dcf}(n)}^{\text{mc}}, T_{\text{ocf}(n)}^{\text{mc}}$ fails the SOP_4 . As T^{md} has elimination of quantifiers the reader can check directly that T_n does not have SOP_4 .

Let $n > 3$. Though T_n^{mc} does not have elimination of quantifiers, every formula is equivalent to a Boolean combination of formulas of the form: $x = y, xRy$ for $m < n$, $\varphi_m(x, y) =: (\exists x_0, \dots, x_m)[x = x_0 \ \& \ y = x_m \ \& \ \bigwedge_{\ell < m} x_\ell R x_{\ell+1}]$ (i.e. the distance from x to y is $\leq m$, directed from x to y in the case of di-graphs). For part (5) of 2.8, we should add for $\ell < m < n/2, \ell > 0$ a partial function $F_{m,\ell}$ defined by: $F_{m,\ell}(x, y) = z$ iff there are t_0, \dots, t_m with no repetition such that $x = t_0, y = t_m, z = t_\ell$ and $\bigwedge_{\ell < m} t_\ell R t_{\ell+1}$ and lastly $\psi_{m,\ell}(x, y) =: (\exists z)[F_{m,\ell}(x, z) = z]$. Let T_n^2

be the set of obvious (universal) axioms for those relations. Then easily T_n^2 has amalgamation and has model completion, T_n^1 which has elimination of quantifiers (but the closure of a finite set under those functions may be infinite). Moreover, assume $M \models T_n^2$, $\langle \bar{a}_m : m < \omega \rangle$ is an indiscernible sequence in M , $\bar{a}_m = \langle a_\ell^m : \ell < k \rangle$, with $k < \omega$. Then there is $w \subseteq k$ such that $[a_\ell^m = a_\ell^{m+1} \Leftrightarrow \ell \in w]$ and without loss of generality $[\ell_1 < \ell_2 \Rightarrow a_{\ell_1}^m \neq a_{\ell_2}^m]$. Let for $u \subseteq \omega$, M_u be the submodel of μ generated by $\bigcup_{m \in u} \bar{a}_m$ for parts (3), (4), $M_u = \bigcup \{M_v : v \subseteq u \text{ and } |v| \leq 1\}$ so

things are simple. By the indiscernibility (increasing the \bar{a}_m 's e.g. taking ω blocks) without loss of generality

(*) $M_u \cap M_\nu = M_{u \cap \nu}$ and the universe of $M_{\{m\}}$ is the range of \bar{a}_m .

Let $m = n$ for parts (3), (4) of 2.8, $m = 3$ for part (5). For part (5) note: the distance between $a_{\ell_0}^0, a_{\ell_1}^1$ is $> \frac{n}{4}$.

[Why? If not there is a path $C^{i,j}$ of length $\leq \frac{n}{4}$ for $a_{\ell_0}^i$ to $a_{\ell_1}^j$, now $C^{0,3} \cup C^{1,2} \cup C^{1,4} \cup C^{0,4}$ is a circle of length $\leq n$, may cross itself but still there is a too small circle].

We can now define models $N_{\{\ell\}}$ (for $\ell < n + 1$), $N_{\{\ell, \ell+1\}}$ ($\ell < n$) and $N_{\{n, 0\}}$ and isomorphisms h_ℓ, g_ℓ ($\ell < n + 1$) such that:

- (a) for $\ell < n + 1$ h_ℓ an isomorphism from $M_{\{\ell\}}$ onto $N_{\{\ell\}}$
- (b) for $\ell < n$, g_ℓ an isomorphism $M_{\{\ell, \ell+1\}}$ onto $N_{\{\ell, \ell+1\}}$ extending $h_\ell, h_{\ell+1}$.
- (c) for $\ell = n$, g_ℓ an isomorphism from $M_{\{n, n+1\}}$ onto $N_{\{n, 0\}}$ extending h_n and $h_0 \circ f$ where f is the isomorphism from $M_{\{n+1\}}$ onto $M_{\{0\}}$ taking \bar{a}_{n+1} onto \bar{a}_0 .
- (d) $N_\emptyset =: g_\ell(M_\emptyset)$ does not depend on ℓ .
- (e) $N_u \cap N_\nu = N_{u \cap \nu}$ if u, ν are among $\emptyset, \{\ell\}, \{m, m+1\}, \{n, 0\}$ ($\ell < n+1, m < n$).

Now,

⊗ There is a model of T_n^2 extending all $N_{\{\ell, \ell+1\}}, N_{\{n, 0\}}$ ($\ell < n$).

This is enough for showing that T_n^1 lacks the SOP_{m+1} .

Lastly the reader can check that $T_{\text{cf}(n)}^{\text{mc}}$ has SOP_3 [choose $k \in (\frac{n}{3}, n)$, $\bar{a}_\ell = \langle a_{\ell, 0} \rangle, \ell_1 < \ell_2 \Rightarrow \varphi_k(a_{\ell, 0}, a_{\ell_2, 0})$]. □_{2.8}

2.9 Theorem. *Let T be first order complete, $\lambda \geq |T|$ and T has the SOP_3 . Then:*

- 1) T is maximal in the Keisler order \triangleleft_λ , i.e. for a regular filter D on λ and some (= every) model $M \models T$ we have M^λ/D is λ^+ -saturated iff D is a good ultrafilter.
- 2) Moreover, in 2.10 T is \triangleleft^ℓ -maximal, (see Definition 2.10 below).

We delay the proof.

Remark. The order \triangleleft was introduced and investigated by Keisler [Ke76]; further investigated in [Sh 42], [Sh:a], CH.VI, new version [Sh:c], CH.VI. The following is a generalization.

2.10 Definition. 1) For models M_0, M_1 we say $M_0, \triangleleft_\lambda^* M_1$ if the following holds: for some model \mathfrak{B}_0 in which M_0, M_1 are interpreted (so $M_i = M_i^{\mathfrak{B}_0}$), for every elementary extension \mathfrak{B} of \mathfrak{B}_0 , which is $(\aleph_0 + |\tau(M_0)| + |\tau(M_1)|)^+$ -saturated we have: [$M_1^{\mathfrak{B}}$ is λ^+ -saturated $\Rightarrow M_0^{\mathfrak{B}}$ is λ^+ -saturated].

2) $M_0 \triangleleft^* M_1$ if for every $\lambda \geq \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$ we have $M_0 \triangleleft_\lambda^* M_1$.

3) Using the superscript ℓ instead of $*$ means in the saturation we use only φ -types for some $\varphi = \varphi(\bar{x}, \bar{y})$ (so any φ is O.K., but for each type φ is constant) and omit the saturation demand on \mathfrak{B} .

4) For complete theories T_1, T_2 we say $T_1 \triangleleft_\lambda^* T_2$ iff for every model M_1 of T_1 for some model M_2 of T_2 , $M_1 \triangleleft_\lambda^* M_2$. Similarly for $T_1 \triangleleft_\lambda^\ell T_2, T_1 \triangleleft_{(\lambda)}^\ell T_2$.

2.11 Observation. 1) In 2.11(1) we can just use \mathfrak{B}_0 of the form

$(H(\chi), \in, <_\chi^*, M_0, M_1)$ with χ strong limit.

2) \triangleleft_λ^* is a partial order, also $\triangleleft_\ast^\ell, \triangleleft^\ell$ are partial orders; $M \triangleleft_\lambda^* M$, and if M_0 is interpretable in M_1 then $M_0 \triangleleft_\lambda^* M_1$.

2A) For models of countable vocabulary, similar statements hold for \triangleleft^* (without the countability if $|\tau(M_1)| > |\tau(M_0)| + |\tau(M_2)| + \aleph_0$, we can get a silly situation).¹

3) If $\lambda \geq \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$ then: $M_0 \triangleleft_\lambda^* M_1$ iff for every finite $\tau \subseteq \tau(M_0), M_0 \upharpoonright \tau \triangleleft_\lambda^* M_1$.

4) $M_1 \otimes_\lambda^\ell M_2 \Rightarrow M_1 \triangleleft_\lambda^* M_2$.

5) Parallel results hold for theories.

6) Any (complete first order) theory of any infinite linear order is \triangleleft^ℓ -maximal hence \triangleleft_λ^* -maximal for every $\lambda \geq |T| + \aleph_0$.

7) All countable stable theories without the f.c.p. (e.g. $T = Th(\omega_1 =)$) are \triangleleft^ℓ -equivalent.

8) All countable stable theories with the f.c.p. are equivalent

(e.g. $T_{\text{eq}} = Th(\bigcup_n (\{n\} \times n), E)$ where E is equally of first coordinates).

9) If T_1 is countable unstable, then $T = \frac{\triangleleft^\ell}{\neq} T_{\text{eq}} \frac{\triangleleft^\ell}{\neq} T$ moreover $\lambda > \aleph_0 \Rightarrow T_{\text{eq}} \triangleleft_\lambda^* T, \lambda \geq 2^{\aleph_0} \Rightarrow T \triangleleft_\lambda^* T_{\text{eq}}$.

Proof. 1) - 4) Obvious.

5) The proof of [Sh:a], VI, 2.6 = [Sh:c], VI, 2.6, p.337 gives this, too.

6), 7), 8) As there [references]. □_{2.11}

2.12 Proof of Theorem 2.9(1),(2). Without loss of generality $\tau(T)$ is finite. Remember: if T' has infinite linear orders as models then it is \triangleleft^ℓ -maximal. Let J be a dense linear order, such that:

(a) J has a closed interval which is I

(b) for any regular $\mu_1, \mu_2 \leq |J|$, J has an interval isomorphic to $[(\{1\} \times \mu_1) \cup (\{2\} \times \mu_2)]$ ordered by $i_1, \alpha_1 <_J (i_2, \alpha_2) \Leftrightarrow (i_1 = 1 \ \& \ i_2 = 2) \vee (i_1 = 1, i_2 \ \& \ \alpha_1 < \alpha_2) \vee (i_1 = 2 = i_2 \ \& \ \alpha_1 > \alpha_2)$.

Let $\varphi(x, y)$ exemplify the SOP₃. Let M be a model of T and $F : I \rightarrow {}^{\ell g \bar{x}} M$ be such that $I \models \eta < \nu \Rightarrow M \models \varphi[F(\eta), F(\nu)]$ and for every $c \in M^{\mathfrak{B}}$ we can find a finite $I' \subseteq I$ such that: if $[t_1, t_2 \in (I \setminus I')] \ \& \ \bigwedge_{s \in I'} [s <_I t \equiv s <_I t_2]$

then $M^{\mathfrak{B}} \models \varphi[F(t_1), c] \equiv \varphi[F(t_2), c]$ and $M^{\mathfrak{B}} \models \varphi[c, F(t_1)] \equiv \varphi[c, F(t_2)]$. Let $\mathfrak{B}_0 = (H(\chi), \in, <_\chi^*, J, I, F, M)$, and \mathfrak{B} be a model, \mathbf{j} an elementary embedding of \mathfrak{B}_0 into \mathfrak{B} such that $M^* = M^{\mathfrak{B}} \upharpoonright L(T)$ is locally λ^+ -saturated but $I^{\mathfrak{B}} = \mathbf{j}(I)$ is not λ^+ -saturated (for 2.9: $\mathfrak{B}^* = \mathfrak{B}^\lambda/D$).

¹so to overcome this, we may in Definition 2.10(2) replace “every $\lambda > \dots$ ” by “every large enough λ ”

As $\mathbf{j}(I)$ is not λ^+ -saturated, we can find $\lambda_0, \lambda_1 \leq \lambda$ and $\alpha_i^\ell \in \mathbf{j}(I)$ (for $i < \lambda_\ell$, $\ell < 2$) such that:

- (α) $I^\mathfrak{B} \models a_i^0 < a_j^0$ for $i < j < \lambda_0$
- (β) $I^\mathfrak{B} \models a_i^1 > a_j^1$ for $i < j < \lambda_1$
- (γ) $I^\mathfrak{B} \models a_i^0 < a_j^1$ for $i < \lambda_0, j < \lambda_1$
- (δ) $I^\mathfrak{B} \models \neg(\exists x)[\bigwedge_{\substack{i < \lambda_0 \\ j < \lambda_1}} a_i^0 < x < a_j^1]$.

Clearly $\{\varphi(\bar{a}_i^0, \bar{x}), \varphi(\bar{x}, \bar{a}_j^1) : i < \lambda_0, j < \lambda_1\}$ is finitely satisfiable in $\mathbf{j}(M)$. Now as $\mathbf{j}(M)$ is locally λ^+ -saturated there is $\bar{a} \in {}^{\ell g \bar{x}}(M^*)$ such that $M \models \text{“}\varphi(\bar{a}_i^0, \bar{a}) \ \& \ \varphi(\bar{a}, \bar{a}_j^1)\text{”}$ for $i < \lambda_0, j < \lambda_1$. In \mathfrak{B} we can define:

$$I_-^\mathfrak{B}[\bar{a}] =: \left\{ \eta \in I^\mathfrak{B} : \text{there is } \nu \in I^\mathfrak{B} \text{ such that : } I^\mathfrak{B} \models \text{“}\eta \leq \nu\text{” and } \mathbf{j}(M) \models \varphi(\bar{a}_\nu, \bar{a}) \right\}$$

$$I_+^\mathfrak{B}[\bar{a}] =: \left\{ \eta \in I^\mathfrak{B} : \text{there is } \nu \in I^\mathfrak{B} \text{ such that : } I^\mathfrak{B} \models \text{“}\nu \leq \eta\text{”} \right. \\ \left. \text{and } \mathbf{j}(M) \models \varphi[\bar{a}, \bar{a}_\nu] \right\}.$$

Clearly

- (a) $I_-^\mathfrak{B}[\bar{a}]$ is an initial segment of $\mathbf{j}(I)$ which belongs to \mathfrak{B} .
- (b) $I_+^\mathfrak{B}[\bar{a}]$ is an end segment of $\mathbf{j}(I)$ which belongs to \mathfrak{B} .
- (c) For every $i < \lambda_0$
 - (*) $_0$ $a_i^0 \in I_-^\mathfrak{B}[\bar{a}]$
- (d) for every $j < \lambda_1$
 - (*) $_1$ $a_j^1 \notin I_-^\mathfrak{B}[\bar{a}]$
- (e) By the choice of φ
 - (*) $_3$ $I_-^\mathfrak{B}[\bar{a}] \cap I_+^\mathfrak{B}[\bar{a}] = \emptyset$.

If for some $c \in \mathfrak{B}$, $\mathfrak{B} \models \text{“}c \in \mathbf{j}(J)\text{”}$ and $(\forall x \in I_-[t]) < t \leq_I c$ and $(\forall x \in I_+[t])[c \leq_I x]$ we are done. So \mathfrak{B} thinks $I_-^\mathfrak{B}[\bar{a}], I_+^\mathfrak{B}[\bar{a}]$ is a Dedekind cut, so let $\mathfrak{B} \models \text{“}cf(I_-^\mathfrak{B}[\bar{a}], <_I) = t_-, cf(I_+^\mathfrak{B}[\bar{a}], >_I) = t_+\text{”}$ and the (outside) cofinalities of t_-, t_+ are μ_1, μ_2 respectively. If μ_1, μ_2 are infinite, we use clause (b) of the choice of J (and the choice of μ). We are left with the case where $\mu_1 = 1 < \mu_2$ (the other case is the same). Use what \mathfrak{B} “thinks” is a (t_1, t_2) Dedekind cut of J to show $\mu_2 \geq \mu^+$ a contradiction. $\square_{2.9}$

2.13 Theorem. 1) *The theorems on non-existence of a universal model in λ for linear order from [KjSh 409], [Sh 457], §3 hold for any theory with SOP_4 .*
 2) *We can use embedding (not necessarily elementary) if $\varphi(\bar{x}, \bar{y})$ is quantifier free or even existential.*

Proof. We concentrate on the case λ is regular and part (1). We will concentrate on the new part relative to [KjSh 409]. Let $\varphi(\bar{x}, \bar{y})$ exemplify $SOP_{\leq 4}$ (exists by 2.6(1)) in a complete first order theory T . Without loss of generality $lg\bar{x} = lg\bar{y} = 1$ and $T \vdash \neg\varphi(\bar{x}, \bar{x})$.

Let M be a model of T with universe λ , I a linear order, $a_s \in M$, $M \models \varphi[a_s, a_t]$ for $s <_I t$ (from I).

We do not have a real Dedekind cut (as $\varphi(x, y)$ is not transitive), but we use replacements. Now for every $b \in M$, let $I^-[b] = \{t : M \models \varphi[a_t, b]\}$ and $I^+[b] = \{t : M \models \varphi[b, a_t]\}$. As φ exemplifies also $SOP_{\leq 3}$ clearly the following is satisfied:

$$(*) \quad s \in I^-[b] \ \& \ t \in I^+[t] \Rightarrow s < t$$

(if $t < s$ from a counterexample, b, t, s gives a contradiction).

Note: $I^-[a_s] = \{t : t <_I s\}$, $I^+[a_s] = \{t : s <_I t\}$.

Let $P = \{a_s : s \in I\}$, $<^* = \{(a_s, a_t) : s <_I t\}$ and

$J^-[t, a] = \{s \in I : s <_I t, a_s \in \alpha\}$, $J^+[t, \alpha] = \{s \in I : t <_I s, a_s \in \alpha\}$

(remember: $|M|$, the universe of M , is λ , $\alpha = \{\beta : \beta < \alpha\}$).

Hence $C =: \{\delta < \lambda : (M, P, <^*, <) \upharpoonright \delta \prec M^+ =: (M, P, <^*, <)\}$ is a club of λ .

Clearly:

$$(**) \quad \text{let } \delta \in C, b \in M \cap \delta;$$

- (i) if $(I^-[b], <_I)$ has cofinality $< \lambda$, then $I^-[b] \cap M_\delta$ is $<_I$ -cofinal in it
- (ii) if $(I^+[b], >_I)$ has cofinality $< \lambda$, then $I^+[b] \cap M_\delta$ is $(>_I)$ -cofinal in it
- (iii) if there is t such that $I^-[b] \leq_I t \leq_I I^+[b]$ then there is such $t \in M \cap \delta$.

Now suppose that $\delta_1 < \delta_2$ are in C , $t(*) \in I$, $a_{t(*)} \in P \setminus \delta_2$.

Case 1: For some $s(*) \in M \cap \delta_2$ we have $(\forall s \in J^-[t(*), \delta_1])(s <_I s(*) <_I t(*))$. Let $b =: a_{s(*)}$. Hence for every $c \in M \cap \delta_1$: if $\varphi(c, a_{t(*)})$ then for every t', t'' satisfying $t'' <_I t(*) <_I t'$, $t' \in I$, $a_{t'} \in \delta_1$, $t'' \in I$, $a_{t''} \in \delta_1$ we have $M^+ \models (\exists x)[x \in P \ \& \ \varphi(c, x) \ \& \ a_{t'} <^* x <^* a_{t''}]$. Clearly (or see the middle of the proof of case 2 below) necessarily for arbitrary $<_I$ -large $t \in J^-[t(*), \delta_1]$ we have $\varphi[c, a_t]$ but for any such t , $\varphi[a_t, a_{s(*)}]$ i.e. $\varphi[a_t, b]$ hence

$$(*)_1 \quad (\forall c \in M \cap \delta_1)[\varphi(c, a_{t(*)}) \rightarrow (\exists y \in M \cap \delta_1)[\varphi(c, y) \ \& \ \varphi(y, b)].$$

Of course,

$$(*)_2 \quad b \in M \cap \delta_2$$

$$(*)_3 \quad \varphi[b, a_{t(*)}].$$

Note: those three properties speak on $M, \delta_1, \delta_2, a_{t(*)}, b$ but not on $I, <^*, P, <$.

Case 2: For no $s(*) \in I, a_{s(*)} \in \delta_2$ do we have $(\forall s \in J^-[t(*), \delta_1])[s <_I s(*) <_I t(*)]$ we assume:

(A) $\{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$ is not definable in $M^+ \upharpoonright \delta_2$.

We shall now show that for no $b \in M \cap \delta_2$ do we have $(*)_1 + (*)_2 + (*)_3$, so assume b is like that and we shall get a contradiction.

Without loss of generality (A) holds. By $(*)_3$ we have $\varphi[b, a_{t(*)}]$ hence for arbitrarily $<_I$ -large $t \in J^-[t(*), \delta_2]$ we have $\varphi[b, a_t]$; choose such t_0 . [Why? Otherwise $I^+[b] \cap J^-[t(*), \delta_2]$ is bounded say by some t^* , so $\theta(x, b, a_{t^*}) =: x \in P \ \& \ (\exists y)[y \in P \ \& \ y \leq^* x \ \& \ \varphi(b, y) \ \& \ t^* <^* x]$ define in M^+ a set which is an end segment of $(P, <^*)$, include t^* (check) but so $s \in \delta_2, s <^* t^*$. So in $M^+ \upharpoonright \delta_2$ it defines the set $\{a_s : s \in J^+[t^*), \delta_2]\}$ hence $\neg\theta(x, b, a_{t^*})$ define in $M^+ \upharpoonright \delta_2$ the set $\{a_s : s \in J^-[t^*), \delta_2]\}$, hence by the assumption of the case, $M^+ \upharpoonright \delta_2$ satisfies:

$$(\forall z)[z \in P \ \& \ \neg\theta(z, b, a_{t^*}) \rightarrow (\exists y < \delta_1)(y \in P \ \& \ z \leq^* y \ \& \ \neg\theta(z, b, a_{t^*}))]$$

contradicting (A) above.

So by the assumption of the case (i.e. that $t_0 < \delta_1$ cannot serve as $s(*)$ and $t_0 <_I t(*)$) for some $t_1 \in J^-[t(*), \delta_1]$ we have $t_0 <_I t_1$ and clearly $t_1 <_I t(*)$ hence $\varphi[a_{t_1}, a_{t(*)}]$. So by $(*)_1$ applied with a_{t_1} standing for c for some $y \in M \cap \delta_1$ we have $\varphi[a_{t_1}, y] \ \& \ \varphi[y, b]$. Now b, a_{t_0}, a_{t_1}, y contradicts “ $\varphi(x, y)$ exemplifies SOP₄”.

Hence we get together

\bigoplus if $\delta_1 < \delta_2 \in C, a_{t(*)} \in P \setminus \delta_2$ (so $t(*) \in I$) then the following conditions are equivalent:

(α) for some $s(*) \in \delta_2$, (so necessarily $s(*) \neq t(*)$) we have $(\forall s \in I)[a_s \in \delta_1 \Rightarrow s <_I s(*) \equiv s <_I t(*)]$ we have:

(A) $\{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$ is definable in $M^+ \upharpoonright \delta_2$
(with parameters)

(B) $\{a_s : s \in I, a_s \in \delta_2, t(*) <_I s\}$ is definable in $M^+ \upharpoonright \delta_2$
(with parameters)

(β) for some $b \in \delta_2$ the conditions $(*)_1, (*)_2, (*)_3$ above holds for $\varphi(x, y)$ or for $\varphi^-(x, y)$ where $\varphi^-(x, y) = \varphi(y, x)$.

Proof. If clause (α) holds and $s(*) <_I t(*)$ holds use Case 1 above. If clause (α) holds and $s(*) <_I t(*)$ fails, then $t(*) <_I s(*)$ inverts the order of I , use φ^- and now apply Case 1 above. So assume $\neg(\alpha)$. We first want to apply Case 2 to prove there is no b satisfying $(*)_1, (*)_2, (*)_3$. For this we need clause (A) there. We claim it holds.

[Why? Assume $\bar{d} \in (M^2 \upharpoonright \delta_2), \psi$ a first order formula (in the vocabulary of M^+), such that for every $e \in M^+ \upharpoonright \delta_2$ we have: $M^+ \upharpoonright \delta_2 \models \psi[e, \bar{d}]$ iff $e \in \{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$. So $M^+ \models (\exists z)[P(z) \ \& \ (\forall y)(y < \delta_1 \ \& \ P(y) \Rightarrow y <^* z \equiv \psi[y, \bar{d}])]$

as $z \mapsto a_{t(*)}$ satisfies it, but $M^+ \upharpoonright \delta_2 \prec M^+$ hence there is $z^* \in \delta_2$ satisfying this. So $z^* \in P$ hence for some $s(*)$, $z^* = a_{s(*)}$, so $s(*)$ contradicts the assumptions $\neg(\alpha)$. So we have proved the failure of the first possibility from clause (β) . The second is proved similarly inverting the order of I , using φ^- (noting that this transformation preserves the statement (A) from Case 2].

2.14 Definition. Let M be a model with universe λ and $\varphi(x, y)$ a formula exemplifying SOP_4 (possibly with parameters) let $\varphi^+(x, y) =: \varphi(x, y)$, $\varphi^-(x, y) =: \varphi(y, x)$. Assume $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a club system, $S \subseteq \lambda$ stationary, guessing club² (i.e. for every club E of λ for stationarily many $\delta < \lambda$, $\delta \in S$, $C_\delta \subseteq E$)

(a) for $x \in |M|$ and $\delta \in S$ let

$$\text{inv}_\varphi(x, C_\delta, M) = \left\{ \alpha \in \text{nacc } C_\delta : \text{letting } \delta_2 = \alpha, \delta_1 = \sup(C_\delta \cap \alpha) \text{ (well defined),} \right. \\ \left. \begin{array}{l} \text{for some } b \text{ conditions } (*)_1, (*)_2, (*)_3 \text{ of Case 1} \\ \text{holds for } \varphi^+ \text{ or for } \varphi^- \end{array} \right\}$$

(b) $\text{Inv}_\varphi(C_\delta, M) = \{\text{inv}(x, C_\delta, M) : x \in M\}$
 $\text{INV}_\varphi(M, \bar{C}) = \langle \text{Inv}_\varphi(C_\delta, M) : \delta \in S \rangle$
 $\text{INV}_\varphi(M < \bar{C}) = \text{INV}_\varphi(M, \bar{C})/\text{id}^\alpha(\bar{C})$ where:

2.15 Definition.

$\text{id}^\alpha(\bar{C}) = \{S' \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ the set of } \delta \in S' \cap S \text{ for which } C_\delta \subseteq E \text{ is not stationary}\}$.

2.16 Observation. If $M' \cong M''$ are models of T and both have universe λ in M then $\text{INV}_\varphi(M', \bar{C}) = \text{INV}_\varphi(M'', \bar{C})$ so $\text{INV}_\varphi(M, \bar{C})$ can be defined for any model of cardinality λ .

Proof. Let f be from M' onto M'' , so f is a permutation of λ . So $E_0 = \{\delta < \lambda : \delta \text{ a limit ordinal, } f \text{ maps } \delta \text{ onto } \delta\}$. Assume $C_\delta \subseteq E$, then for $x \in M' \setminus \delta$, $\delta \in S$ we have $\text{inv}(x, C_\delta, M') = \text{inv}_\varphi(f(x), C_\delta, M'')$. [Why? Read $(*)_1, (*)_2, (*)_3$]. Hence $\text{Inv}_\varphi(C_\delta, M') \in \text{Inv}_\varphi(C_\delta, M'')$. By the definition of $\text{id}^\alpha(\bar{C})$ we are done. $\square_{2.16}$

2.17 Observation. If $\bar{C} = \langle C_\delta : \delta \in S \rangle$, $S \subseteq \lambda$ stationary, $C_\delta \subseteq \delta = \sup(C_\delta)$, C_δ closed, I a linear order with the set of elements being λ we let:

- (a) for $x \in \lambda$, $\delta \in S$, $\text{inv}(x, C_\delta, I) = \{\alpha \in \text{nacc}(C_\delta) : \text{there are } y, z \in \alpha \text{ such that } y <_I x <_I z \text{ such that } (\forall s)[s < \sup(C_\delta \cap \alpha) \Rightarrow s <_I y \vee z <_I s]\}$
- (b) $\text{Inv}(C_\delta, I) = \{\text{inv}(x, C_\delta, I) : x \in M\}$
- (c) $\text{INV}(I, \bar{C}) = \langle \text{Inv}(C_\delta, I) : \delta \in S \rangle$
- (d) $\text{INV}(I, \bar{C}) = \text{INV}(I, \bar{c})/\text{id}^\alpha(\bar{C})$

²otherwise dull

2.18 Observation. $\text{INV}(I, \bar{C}) = \text{INV}(I', \bar{C})$ if $I \cong I'$, so actually it is well defined for any linear order with cardinality λ .

2.19 Observation. If M is a model with universe λ and $\varphi, \langle a_s : s \in I \rangle$ as above and, $\emptyset \notin \text{id}^\alpha(\bar{C})$ then $\text{INV}(I, \bar{C}) \leq \text{INV}_\varphi(M, \bar{C})$ i.e. for some club E of $\lambda, \delta \in S$ & $C_\delta \subseteq E \Rightarrow \text{Inv}(C_\delta, I) \subseteq \text{Inv}_\varphi(C_\delta, M)$.

Proof. By \oplus above.

Conclusion of the proof of 2.13. As in [KjSh 409]. □_{2.13}

2.20 Claim. For a complete T , the following are equivalent:

- (a) T does not have SOP_3
- (b) if in $\mathfrak{C}, \langle \bar{a}_i : i < \alpha \rangle$ is an indiscernible sequence, α infinite and $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ contradictory and for each j for some \bar{b}_j we have $i \leq j \Rightarrow \models \varphi[\bar{a}_i, \bar{b}_j]$ and $i > j \Rightarrow \models \psi[\bar{b}_j, \bar{a}_i]$ then for $i < j$ we have $(\exists \bar{x})(\varphi(\bar{x}, \bar{a}_i) \ \& \ \psi(\bar{x}, \bar{a}_j))$
- (c) in clause (b) we replace the conclusion: for every finite disjoint $u, v \subseteq \omega$

we have $(\exists \bar{x}) \left(\bigwedge_{i \in u} \varphi(\bar{x}, \bar{a}_i) \ \& \ \bigwedge_{j \in v} \psi(\bar{x}, \bar{a}_j) \right)$.

Proof. (c) \Rightarrow (b): Trivial.

$\neg(c) \Rightarrow \neg(b)$: Choose counterexample with $|u \cup v|$ minimal, assume $\alpha > \omega + |u \cup v|$.

$\neg(a) \Rightarrow \neg(b)$: Straight by the Definition of SOP_3 , etc.

$\neg(b) \Rightarrow \neg(a)$: Without loss of generality $\langle \bar{a}_i \hat{\ } \bar{b}_i : i < \alpha \rangle$ is an indiscernible sequence. Now we cannot find $\bar{c}_0, \bar{c}_1, \bar{c}_2$ such that $\bar{c}_0 \hat{\ } \bar{c}_1, \bar{c}_1 \hat{\ } \bar{c}_2, \bar{c}_2 \hat{\ } \bar{c}_0$ realizes the same type as $(\bar{a}_0 \hat{\ } \bar{b}_0) \hat{\ } (\bar{a}_1 \hat{\ } \bar{b}_1)$, so SOP_3 is exemplified. □_{2.20}

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