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On uniformly antisymmetric functions

0. Introduction

Recently there has been considerable research on symmetric properties of functions, i.e., when e.g. continuity is replaced by the limit properties of $f(x+h)-f(x-h)$ $(h \to 0)$. The excellent monograph [6] surveys most of the recent developments.

The following definiton was considered by Evans and Larson (in Santa Barbara, 1984) and by Kostyrko (in Smolenice, 1991).

Definition. A *uniformly antisymmetric function* is an $f:\mathbf{R}\to\mathbf{R}$ such that for every $x\in\mathbf{R}$ there is a $d(x)>0$ so that $0<h<d(x)$ implies $|f(x+h)-f(x-h)|\geq d(x)$.

They posed the question if there exists a uniformly antisymmetric function. Kostyrko showed that no such function with a two element range exists, that is, there is no uniformly antisymmetric *set* (see [5]). This was extended to functions with 3-element ranges by Ciesielski in [1]. In [2] a uniformly antisymmetric function $f:\mathbf{R}\to\omega$ was constructed. It had the stronger property that for every $x\in\mathbf{R}$ the set $S_x=\{h>0: f(x-h)=f(x+h)\}$ is finite. [2] contains several other relevant results and questions. Kostyrko's result is extended to functions defined on any uncountable subfield of the reals. The authors of [2] ask if this can be extended to countable subfields, as well. As for functions defined on $\mathbf{R}$ they ask if there is an $f:\mathbf{R}\to\omega$ such that $|S_x|\leq 1$ for $x\in\mathbf{R}$, or if there is an f with finite range that S_x is always finite.

In this paper we solve some of those problems. We show that there is always a uniformly antisymmetric $f: A\to\{0,1\}$ if $A\subset\mathbf{R}$ is countable. We prove that the continuum hypothesis is equivalent to the statement that there is an $f:\mathbf{R}\to\omega$ with $|S_x|\leq 1$ for every $x\in\mathbf{R}$. If the continuum is at least $\aleph_n$ then there exists a point x such that S_x has at least 2^n-1 elements. We also show that there is a function $f:\mathbf{Q}\to\{0,1,2,3\}$ such that S_x is always finite, but no such function with finite range on $\mathbf{R}$ exists.

Notation. We use the standard set theory notation. Notably, ω is the set of natural numbers, ordinals are identified with the sets of smaller ordinals. $\mathbf{R}$ is the set of reals, $\mathbf{Q}$ is the set of rationals. $|A|$ denotes the cardinality of A. If A is a set, κ is a cardinal, then $[A]^\kappa=\{X\subseteq A: |X|=\kappa\}$, $[A]^{<\kappa}=\{X\subseteq A: |X|<\kappa\}$. CH denotes the continuum hypothesis, i.e., that $|\mathbf{R}|=\aleph_1$.

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1. Uniformly antisymmetric functions on countable sets

Theorem 1. *If $A \subseteq \mathbf{R}$ is countable, then there is a uniformly antisymmetric function $f : A \to \{0,1\}$.*

Proof. Enumerate A as $A = \{a_1, a_2, \ldots\}$. By induction on $n < \omega$ we define a finite set $\mathcal{I}_n = \{I_\gamma : \gamma \in \Gamma_n\}$ of open intervals such that $\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$, so $\emptyset = \mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \ldots$, each I_γ is of the form $I_\gamma = (b_\gamma - h_\gamma, b_\gamma + h_\gamma)$ with the following properties. Put $B_n = \{b_\gamma : \gamma \in \Gamma_n\}$.
(1) If $\gamma \neq \gamma'$ then either $I_\gamma \cap I_{\gamma'} = \emptyset$, or one of them contains the other;
(2) if $I_{\gamma'} \subseteq I_\gamma$ then either $I_{\gamma'} \subseteq (b_\gamma - h_\gamma, b_\gamma)$ or $I_{\gamma'} \subseteq (b_\gamma, b_\gamma + h_\gamma)$;
(3) $\{a_1, \ldots, a_n\} \subseteq B_n$;
(4) $b_\gamma \pm h_\gamma \notin A$ $(\gamma \in \Gamma_n)$;
(5) if we put $\varphi_\gamma(x) = 2b_\gamma - x$ $(x \in I_\gamma,\ x \neq b_\gamma)$, then for $I_{\gamma'} \subseteq I_\gamma$, $\varphi_\gamma(I_{\gamma'}) \in \mathcal{I}_n$ holds.

To start, we put $\Gamma_0 = \emptyset$.

If Γ_{n-1} is already given, and $a_n \in B_{n-1}$, put $\Gamma_n = \Gamma_{n-1}$. Otherwise, let I_γ be the unique shortest interval in $\mathcal{I}_{n-1}$ containing a_n if there exists one. Select $I = (a_n - h, a_n + h)$ in such a way that it is either in $(b_\gamma - h_\gamma, b_\gamma)$ or in $(b_\gamma, b_\gamma + h_\gamma)$ and $\varphi_{\gamma_1} \cdots \varphi_{\gamma_r}(a_n \pm h) \notin A$ for any (applicable) product $(\gamma_i \in \Gamma_{n-1})$. Notice that the number of those products is 2^t where t is the number of intervals in $\mathcal{I}_{n-1}$ containing a_n. Now add all $\varphi_{\gamma_1} \cdots \varphi_{\gamma_r}(I)$ to $\mathcal{I}_{n-1}$ and get $\mathcal{I}_n$. If no interval of $\mathcal{I}_{n-1}$ contains a_n then let $I = (a_n - h, a_n + h)$, $a_n \pm h \notin A$ be an arbitrary interval disjoint from those in $\mathcal{I}_{n-1}$ and add it to get $\mathcal{I}_n$.

To conclude the proof of the Theorem we are going to show that there exists a function $f : \mathbf{R} \to \{0,1\}$ such that $f(\varphi_\gamma(x)) = 1 - f(x)$ $(\gamma \in \bigcup \Gamma_n)$. As φ_γ^2 is always a partial identity it suffices to show that no $x \in \mathbf{R}$ is a fixed point of the product of odd many φ_γ.

Assume that $x = \varphi_{\gamma_1}\varphi_{\gamma_2} \cdots \varphi_{\gamma_t}(x)$, t odd. Among the intervals $I_{\gamma_1}, \ldots, I_{\gamma_t}$ there is a longest one, say I_γ and that must contain all the others. At every appearence of φ_γ in the product $\varphi_{\gamma_1}\varphi_{\gamma_2} \cdots \varphi_{\gamma_t}$ the image of x moves from one side of b_γ to the other. φ_γ therefore appears even times. In the product the interval $\varphi_\gamma \varphi_{\gamma_i} \cdots \varphi_{\gamma_j} \varphi_\gamma$ can be replaced by $\varphi_{\gamma_i'} \cdots \varphi_{\gamma_j'}$ where $I_{\gamma_r'} = \varphi_\gamma(I_{\gamma_r})$ $(i \le r \le j)$, so eventually we succeed in eliminating an even number of φ's. We got a shorter formula $x = \varphi_{\gamma_1'} \cdots \varphi_{\gamma_{t'}'}(x)$, but t' is still odd. Finally we get that $x = \varphi_\gamma^t(x)$ for some odd t which is impossible. □

2. When S_x is finite

Definition. If $f : \mathbf{R} \to \omega$ is a function, then for $x \in \mathbf{R}$, set $S_x = \{h > 0 : f(x-h) = f(x+h)\}$.

Theorem 2. *There is a function $F\colon [\omega_1]^{<\omega} \to \omega$ such that*
(a) if $F(A) = F(B)$ then $|A| = |B|$;
(b) if $F(A) = F(B)$ then $A \cap B$ is an initial segment in A, B; and
(c) there do not exist A_0, B_0, A_1, $B_1 \in [\omega_1]^{<\omega}$ such that $A_0 \cup B_0 = A_1 \cup B_1$, $F(A_0) = F(B_0)$, $F(A_1) = F(B_1)$, $A_0 \neq B_0$, $A_1 \neq B_1$, and $\{A_0, B_0\} \neq \{A_1, B_1\}$.

Proof. Let the diadic intervals of $\mathbf{R}$ be $I_0, I_1, \ldots$. For $\alpha < \omega_1$ enumerate α as $\alpha = \{\gamma(\alpha, i) \colon i < \omega\}$. (Recall that by our axiomatic set theory assumptions α is identified with the set of smaller ordinals.) Select different irrational numbers r_α for $\alpha < \omega_1$. We define a function $c\colon [\omega_1]^2 \to \omega$ as follows. We construct $c(\beta, \alpha)$ by induction on β, in the order of the enumeration of α. For $\beta < \alpha$, if $\beta = \gamma(\alpha, i)$, let $c(\beta, \alpha)$ be some $j < \omega$ such that

(1) $j > c(\gamma(\alpha, 0), \alpha), \dots, c(\gamma(\alpha, i-1), \alpha)$;

(2) $r_\beta \in I_j$;

(3) $r_\alpha \notin I_j$;

(4) $r_\xi \notin I_j$ for $\xi = \gamma(\alpha, 0), \dots, \gamma(\alpha, i-1)$.

Clearly, such a $j < \omega$ can be found. Let, for $A \in [\omega_1]^{<\omega}$, $F(A)$ be the isomorphism type of the structure $(A; <, c)$, i.e., $F(A) = F(B)$ iff $|A| = |B|$ and $c(a_i, a_j) = c(b_i, b_j)$ whenever $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$ are the monotonic enumerations of A, B, respectively.

Claim 1. *If $F(A) = F(B)$, then $A \cap B$ is an initial segment in both sets.*

Proof. Again, let $A = a_1, \dots, a_n$, $B = b_1, \dots, b_n$ be the increasing enumerations. Assume that $a_i = b_j$ is a common element. If $i \neq j$, say $i < j$, then $k = c(a_i, a_j) = c(b_i, b_j)$ has $r_{a_i} \in I_k$ (by (2)), and $r_{b_j} \notin I_k$ (by (3)), a contradiction. So we have that $i = j$. If $t < i$, then, as $c(a_t, a_i) = c(b_t, b_i) = c(b_t, a_i)$, $a_t = b_t$ by property (1). ☐

Claim 2. *There do not exist $\beta, \beta', \alpha, \alpha' < \omega_1$ such that $\max(\beta, \beta') < \min(\alpha, \alpha')$, $c(\beta, \alpha) = c(\beta', \alpha')$, and $c(\beta', \alpha) = c(\beta, \alpha')$.*

Proof. Set $i = c(\beta, \alpha)$, $j = c(\beta', \alpha)$. As $\beta, \beta' < \alpha$, $i \neq j$, say, $i < j$. Then, considering $c(\beta', \alpha)$ we get (by (4)) $r_\beta \notin I_j$ while considering $c(\beta, \alpha')$ we get that $r_\beta \in I_j$, a contradiction. If $i > j$ we argue similarly. ☐

Assume now that $F(A) = F(B)$ and we know $A \cup B$. We try to reconstruct A, B. Put $X = A \cap B$, $Y = A - X$, $Z = B - X$. We can assume that $m' = \max(Y) < \max(Z) = m$. In general, to every $x \in Z$ let x' be the element in Y corresponding to x under the (unique) order-preserving bijection between Z and Y.

For $a < b$ in A, $c(a, b)$ codes a diadic interval including r_a but excluding r_b. The structure $(A; <, c)$ gives a diadic interval for every element in A separating it from the rest of A. As $F(A) = F(B)$ this interval is the same for x and x'. We get therefore, that there is a diadic interval containing r_x, $r_{x'}$ but nothing else from $A \cup B$. This makes possible to find x' if x is given, or to find x if x' is given. Anyway, we can find m'.

Claim 3. $X = \{x \in A \cup B\colon x < m'$ *and* $c(x, m') = c(x, m)\}$.

Proof. $\subseteq$ is obvious. If, say $x \in Z$ and $c(x, m') = c(x, m)$ then $c(x, m') = c(x, m) = c(x', m')$ a contradiction to (1), as $x \neq x'$. ☐

As now X is known, we can decompose $Y \cup Z$ into the pairs $\{x, x'\}$ by the argument before Claim 3. Given such a pair $\{u, v\}$ we have to find if $u \in Z$, $v \in Y$ or vice versa. We know that $c(x', m') = c(x, m)$, so, knowing m, m' we can identify x, x' if we can show that $c(x, m') \neq c(x', m)$. But this is done in Claim 2. ☐

Theorem 3. *If CH holds, then there is a function $f\colon \mathbf{R} \to \omega$ such that for every $x \in \mathbf{R}$ S_x has at most one element.*

Proof. Let $\{b_\alpha\colon \alpha < \omega_1\}$ be a Hamel basis, $F : [\omega]^{<\omega} \to \omega$ as in Theorem 1. To

$$x = \sum_{i=1}^{n} \lambda_i b_{\alpha_i}$$

($\lambda_i \neq 0$, $\lambda_i \in \mathbf{Q}$), $\alpha_1 < \cdots < \alpha_n$ we associate some $f(x)$ that codes the ordered string $\langle \lambda_1, \ldots, \lambda_n \rangle$ as well as $F(\{\alpha_1, \ldots, \alpha_n\})$. This is possible as there are countably many possibilities for both.

Assume that $x \neq y$, $f(x) = f(y)$. We try to recover the pair $\{x, y\}$ from $x + y$. By our coding of the string of the coefficients in the Hamel basis and the properties of the function F described in the previous Theorem, x, y can be written as

$$x = \sum_{i=1}^{n} \lambda_i b_{\alpha_i}, \quad y = \sum_{i=1}^{n} \lambda_i b_{\beta_i}$$

such that $\alpha_i = \beta_i$ for $1 \leq i \leq m$ (some $m < n$), and $\{\alpha_{m+1}, \ldots, \alpha_n\} \cap \{\beta_{m+1}, \ldots, \beta_n\} = \emptyset$. $x + y$ can be written in the above basis as

$$x + y = \sum_{i=1}^{m} (2\lambda_i) b_{\alpha_i} + \sum_{i=m+1}^{n} \lambda_i b_{\alpha_i} + \sum_{i=m+1}^{n} \lambda_i b_{\beta_i}.$$

The support of $x + y$, i.e., the set of those basis vectors in which it has nonzero coefficients is

$$\{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n, \beta_{m+1}, \ldots, \beta_n\}$$

from which, by the previous Theorem $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ can be recovered. Then we can find $\lambda_1, \ldots, \lambda_n$, i.e., x and y can be reconstructed. $\square$

Before proving that if a vector space V with $|V| \geq \omega_n$ is ω-colored then $|S_x| \geq 2^n - 1$ holds for some $x \in V$ we give a proof of the combinatorial part of the theorem. We then show how to modify the proof to get the stated result.

Theorem 4. *If $2 \leq n < \omega$ and $f : [\omega_n]^{<\omega} \to \omega$ then there exists a set $s \in [\omega_n]^{<\omega}$ which can be written in $2^n - 1$ ways as the union of two different sets $s = x \cup y$ such that $f(x) = f(y)$.*

Proof. Assume that $f : [\omega_n]^{<\omega} \to \omega$. Select $\omega_{n-1} < y_n^0 < \omega_n$ such that it is not in any of the sets

$$\{x : \omega_{n-1} < x < \omega_n, f(s_1 \cup \{x\}) = j_1, \ldots, f(s_t \cup \{x\}) = j_t\}$$

(for some $s_1, \ldots, s_t \in [\omega_{n-1}]^{<\omega}$, $j_1, \ldots, j_t < \omega$) which happen to have one element. This is possible, as the number of those sets is $\aleph_{n-1}$, and they are all small enough.

Assume now that $y_{i+1}^0, \ldots, y_n^0$ are already defined. Let $\omega_{i-1} < y_i^0 < \omega_i$ be such that it is not in any of the sets of the form

$$\{x : f(s_1 \cup \{x, y_{i+1}^0, \ldots, y_n^0\}) = j_1, \ldots, f(s_t \cup \{x, y_{i+1}^0, \ldots, y_n^0\}) = j_t, \omega_{i-1} < x < \omega_i\}$$

for some $s_1, \ldots, s_t \in [\omega_{i-1}]^{<\omega}$, $j_1, \ldots, j_t < \omega$, which are singletons. Again, this choice is possible.

If $y_1^0, \ldots, y_n^0$ are given, we define y_i^1 $(1 \leq i \leq n)$ in increasing order. Select $y_1^1 \neq y_1^0$ such that $\omega < y_1^1 < \omega_1$ and $f(\{y_1^1, y_2^0, \ldots, y_n^0\}) = f(\{y_1^0, \ldots, y_n^0\})$. This is possible, as otherwise y_1^0 would be the only element in $\{x : \omega < x < \omega_1, f(\{x, y_2^0, \ldots, y_n^0\}) = j\}$ where $j = f(\{y_1^0, y_2^0, \ldots, y_n^0\})$, a contradiction to the choice of y_1^0.

If $y_1^1, \ldots, y_{i-1}^1$ have already been selected, let $y_i^1 \neq y_i^0$ be such that $\omega_{i-1} < y_i^1 < \omega_i$ and

$$f(s \cup \{y_i^1, y_{i+1}^0, \ldots, y_n^0\}) = f(s \cup \{y_i^0, y_{i+1}^0, \ldots, y_n^0\})$$

for every $s \subseteq \{y_1^0, y_1^1, \ldots, y_{i-1}^0, y_{i-1}^1\}$. This is possible by the choice of y_i^0.

For $1 \leq k \leq m \leq n$, $g : \{k, \ldots, m\} \to \{0, 1\}$ put $A = \{y_1^0, y_1^1, \ldots, y_{k-1}^0, y_{k-1}^1\}$, $B = \{y_k^0, \ldots, y_m^0\}$, $B^g = \{y_k^{g(k)}, \ldots, y_m^{g(m)}\}$, $C = \{y_{m+1}^0, \ldots, y_n^0\}$.

Claim. *$f(A \cup B \cup C) = f(A \cup B^g \cup C)$.*

Proof. By induction on m. The inductive step trivially follows from the choice of y_m^1. □

To conclude the proof of the Theorem, assume that $1 \leq k \leq n$, $g : \{k, \ldots, n\} \to \{0, 1\}$. Put $A = \{y_1^0, y_1^1, \ldots, y_{k-1}^0, y_{k-1}^1\}$, $B^g = \{y_k^{g(k)}, \ldots, y_n^{g(n)}\}$ and let $1-g$ be the function with $(1-g)(i) = 1-g(i)$ for $k \leq i \leq n$. Using the Claim we get that $f(A \cup B^g) = f(A \cup B^{1-g})$ and clearly $(A \cup B^g) \cup (A \cup B^{1-g}) = \{y_1^0, y_1^1, \ldots, y_n^0, y_n^1\}$. The number of those decompositions, i.e., that of the pairs $\{g, 1-g\}$ is 2^{n-k}, summing we get $2^{n-1} + \cdots + 1 = 2^n - 1$. □

Theorem 5. *Let V be a vector space, $|V| \geq \aleph_n$ $(2 \leq n < \omega)$ and $f : V \to \omega$ be given. Then $|S_x| \geq 2^n - 1$ for some $x \in V$.*

Proof. Assume that $\{b(\alpha) : \alpha < \omega_n\}$ is a linearly independent set. Select $\omega_{n-1} < y_n^0 < \omega_n$ outside any of the one-element sets of the form

$$\Big\{\omega_{n-1} < x < \omega_n : f\Big(\sum_{z \in s_1} b(z) + \frac{1}{2} \sum_{z \in s_1'} b(z) + b(x)\Big) = j_1, \ldots, \qquad f\Big(\sum_{z \in s_t} b(z) + \frac{1}{2} \sum_{z \in s_t'} b(z) + b(x)\Big) = j_t\Big\}$$

where $s_1, s_1', \ldots, s_t, s_t' \in [\omega_{n-1}]^{<\omega}$, $j_1 \ldots, j_t < \omega$. Given $y_{i+1}^0, \ldots, y_n^0$, let $\omega_{i-1} < y_i^0 < \omega_i$ be not in any of the one-element sets

$$\Big\{\omega_{i-1} < x < \omega_i : f\Big(\sum_{z \in s_1} b(z) + \frac{1}{2} \sum_{z \in s_1'} b(z) + b(x) + b(y_{i+1}^0) + \cdots + b(y_n^0)\Big) = j_1, \ldots, \qquad f\Big(\sum_{z \in s_t} b(z) + \frac{1}{2} \sum_{z \in s_t'} b(z) + b(x) + b(y_{i+1}^0) + \cdots + b(y_n^0)\Big) = j_t\Big\}$$

where $s_1, s_1', \ldots, s_t, s_t' \in [\omega_{i-1}]^{<\omega}$, $j_1, \ldots, j_t < \omega$. If $y_1^0, \ldots, y_n^0$ are already constructed, let $y_1^1 \neq y_1^0$ be such that $\omega < y_1^1 < \omega_1$ and $f\big(b(y_1^1) + b(y_2^0) + \cdots + b(y_n^0)\big) = f\big(b(y_1^0) + b(y_2^0) + \cdots + b(y_n^0)\big)$. With $y_1^1, \ldots, y_{i-1}^1$ defined, let $\omega_{i-1} < y_i^1 < \omega_i$, $y_i^1 \neq y_i^0$ be such that for every $s \cup s' \subseteq \{y_1^0, y_1^1, \ldots, y_{i-1}^0, y_{i-1}^1\}$, if $s \cap s' = \emptyset$, then

$$f\Big(\sum_{z \in s} b(z) + \frac{1}{2} \sum_{z \in s'} b(z) + b(y_i^1) + b(y_{i+1}^0) + \cdots + b(y_n^0)\Big) = f\Big(\sum_{z \in s} b(z) + \frac{1}{2} \sum_{z \in s'} b(z) + b(y_i^0) + b(y_{i+1}^0) + \cdots + b(y_n^0)\Big)$$

holds. This is possible by the choice of y_i^0.

For $1 \le k \le m \le n$, $g : \{k, \dots, m\} \to \{0,1\}$ we define

$$A_k = \frac{1}{2}\big(b(y_1^0) + b(y_1^1) + \cdots + b(y_{k-1}^0) + b(y_{k-1}^1)\big),$$

$$B = b(y_k^0) + \cdots + b(y_m^0), \quad B^g = b(y_k^{g(k)}) + \cdots + b(y_m^{g(m)}),$$

$$C = b(y_{m+1}^0) + \cdots + b(y_n^0).$$

Claim. $f(A_k + B + C) = f(A_k + B^g + C)$.

Proof. As in Theorem 4. □

To conclude the proof one can argue as in Theorem 4, and decompose $b(y_1^0) + b(y_1^1) + \cdots + b(y_n^0) + b(y_n^1)$ in $2^n - 1$ ways into the sum of two vectors with the same f value as $(A_k + B^g) + (A_k + B^{1-g})$ where $g : \{k, \dots, n\} \to \{0,1\}$. □

3. Finite range

Theorem 6. *There is a function $f : \mathbf{Q} \to \{0,1,2,3\}$ such that for every $x \in \mathbf{Q}$, S_x is finite.*

Proof. It suffices to find such a function assuming two values on the set $\mathbf{Q}^+ = \{x \in \mathbf{Q} : x > 0\}$. We decompose $\mathbf{Q}^+$ into the increasing union of finite sets $A_1 \subseteq A_2 \subseteq \cdots$. We also define an auxiliary graph G on $\mathbf{Q}^+$. Two points x and y will be joined in G if $(x+y)/2 \in A_n$ for some n but $x, y \notin A_{n+1}$. If, with an appropriate choice of the sets we can guarantee that the graph G is bipartite, then the bipartition of G will give a good function on $\mathbf{Q}^+$. Indeed, if $x \in A_n$ and $f(x-h) = f(x+h)$ then one of $x-h$, $x+h$ is in A_{n+1} so there are only finitely many such h's.

Let a positive rational number be in A_n if it is of the form $x = p/n!$ and $x < 2^n$. Clearly these sets are finite, constitute an increasing sequence, and their union is $\mathbf{Q}^+$.

We first show that if x, y are joined in G, then they first appear in the same A_n. Assume that $x \in A_{n+1} - A_n$, $y \in A_{m+1} - A_m$, $m \ge n$, and $z = (x+y)/2 \in A_{n-1}$. Then, the denominator of $y = 2z - x$ is (a divisor of) $(n+1)!$. Also, $y < 2z < 2^{n+1}$, so $y \in A_{n+1}$, i.e., $m = n$.

Finally, we show that G on $A_{n+1} - A_n$ does not contain odd circuits. Assume that $a_1, \dots, a_{2u+1}$ is one, i.e., $a_i + a_{i+1} = 2b_i$ for some $b_i \in A_{n-1}$ $(1 \le i \le 2u+1)$. Here, we use cyclical indexing, i.e., $a_{2u+2} = a_1$. Then again, $a_1 < 2b_1 < 2^n$, and as $a_1 = b_1 - b_2 + b_3 - \cdots + b_{2u+1}$, a_1 has denominator $(n-1)!$, so it is in A_n, a contradiction. □

Theorem 7. *If $f : \mathbf{R} \to \{1, \dots, n\}$ is a function, then S_x is infinite for some $x \in \mathbf{R}$.*

Proof. Actually the result is true for any uncountable vector space V over $\mathbf{Q}$. Assume that $f : V \to \{1, \dots, n\}$. Let $\{b(\alpha) : \alpha < \omega_1\}$ be linearly independent. For $\beta < \alpha < \omega_1$, the formula $F(\beta, \alpha) = f\big(b(\alpha) - b(\beta)\big)$ defines an n-coloring of $[\omega_1]^2$. By an old Erdős-Rado theorem (see Cor. 1, p. 459 in [3]), there are a color $1 \le k \le n$ and ordinals $\alpha(0) < \cdots < \alpha(\omega)$, such that $F\big(\alpha(i), \alpha(j)\big) = k$ for $i < j \le \omega$. But then,

$$f\big(b(\alpha(i)) - b(\alpha(0))\big) = f\big(b(\alpha(\omega)) - b\big(\alpha(i)\big)\big) = k,$$

i.e., the vector $b\left(\alpha(\omega)\right) - b\left(\alpha(0)\right)$ can be written infinitely many ways as the sum of two monocolored vectors. □

References

[1] K. Ciesielski: Notes on problem 1 from “Uniformly antisymmetric functions”, to appear.

[2] K. Ciesielski, L. Larson: Uniformly antisymmetric functions, to appear.

[3] P. Erdős, R. Rado: A partition calculus in set theory, *Bull. of the Amer. Math. Soc.* **62** (1956), 427–489.

[4] P. Komjáth: Vector sets with no repeated differences, *Coll. Math.* **64**(1993), 129–134.

[5] P. Kostyrko: There is no strongly locally antisymmetric set, *Real Analysis Exchange,* **17** (1991/92), 423–425.

[6] B. S. Thomson: *Symmetric properties of real functions*, to appear.