

# Subalgebras of Cohen algebras need not be Cohen

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## 0 Introduction

Let us denote by  $\mathbb{C}_\kappa$  the standard Cohen algebra of  $\pi$ -weight  $\kappa$ , i.e. the complete Boolean algebra adjoining  $\kappa$  Cohen reals, where  $\kappa$  is an infinite cardinal or 0. More generally, we call a Boolean algebra  $A$  a Cohen algebra if (for technical convenience in Theorems 0.3 and 0.4 below) it satisfies the countable chain condition and forcing with  $A$  (more precisely with the partial ordering  $A \setminus \{0\}$ ) is equivalent to Cohen forcing, i.e. if every generic extension of the universe of set theory arising from forcing with  $A$  arises from forcing with some standard Cohen algebra. Since forcing with an arbitrary Boolean algebra is equivalent to forcing with its completion and forcing with a product of algebras is equivalent to forcing with one of the factors, an algebra is Cohen iff its completion is isomorphic to a product of at most countably many standard Cohen algebras; we will use this description as the definition of a Cohen algebra in the rest of the paper.

Cohen algebras are among the most important objects to be studied in the realm of Boolean algebras or forcing. There is a general feeling that more or less “every” algebraic property of the standard Cohen algebras is well-known; similarly, the effect of adding Cohen reals to a given model of set theory is, generally, quite well understood. It is therefore quite surprising that the answer to an apparently innocent question was open, up to now; cf. Problem 5.2 in [Koppelberg, 1993].

**Definition 0.1 (Problem).** If  $B$  is a regular subalgebra of some Cohen algebra  $A$ , does it follow that  $B$  is Cohen?

The first reference to this problem we are aware of is in Kamburelis’s paper [Kamburelis, 1989]. By Example 5.5 in [Koppelberg, 1993], the assumption that  $B$  be regular in  $A$  cannot be disposed with.

We make some simple observations which somewhat restrict the problem (cf. [Koppelberg, 1993]). First, both  $A$  and  $B$  may be assumed to be complete; moreover  $A$  may be assumed to be a standard Cohen algebra

$\mathbb{C}_\kappa$ . Finally,  $\kappa$  may be assumed to be at least  $\omega_2$ , by Proposition 5.4 in [Koppelberg, 1993]. Thus the following theorem, the unique result of the paper, is the strongest result one can hope for.

**Theorem 0.2** *For every  $\kappa \geq \omega_2$ ,  $\mathbb{C}_\kappa$  has a complete regular subalgebra of  $\pi$ -weight  $\kappa$  which is not Cohen.*

Let us mention that several beautiful internal descriptions of Cohen algebras were proved in recent years, based on Shapiro's theorem (cf. [Shapiro, 1986], [Shapiro, 1987]) that every subalgebra of a free Boolean algebra is Cohen; see Section 1 for all unexplained notions. Two of these are given below; they will, however, not be applied in the present paper.

**Theorem 0.3** ([Koppelberg, 1993], 0.3) *A Boolean algebra is Cohen iff it is the union of a continuous chain  $(A_\alpha)_{\alpha < \rho}$  where  $\rho$  is any ordinal,  $\pi(A_0) \leq \omega$ ,  $A_\alpha$  is a regular subalgebra of  $A_{\alpha+1}$  and  $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$ .*

The following is a reformulation of a result due to Bandlow ([Bandlow, 1994]).

**Theorem 0.4** *A Boolean algebra  $A$  is Cohen iff there is a club subset  $\mathbb{S}$  of  $[A]^\omega$  such that the elements of  $\mathbb{S}$  are subalgebras of  $A$  and, for every subset  $T$  of  $\mathbb{S}$ , the subalgebra of  $A$  generated by  $\bigcup T$  is regular in  $A$ .*

We now give a survey of the proof of our theorem and explain the organization of the paper. In fact, what we show is a result on forcing: we find a Boolean extension  $V^{Q^0}$  of the universe  $V$  of set theory which is not Cohen, but some Boolean extension  $V^{Q^0 * Q^1}$  of  $V^{Q^0}$  is. — After reviewing some material on Boolean algebras and forcing in Section 1, we will define forcings  $Q^0$ ,  $Q^1$ ,  $P^0$ , and  $P^1$ , most of which depend on the cardinal  $\kappa$  given in the Main Theorem as a parameter. More precisely, we define  $Q^0$  in Section 2 and list some of its basic properties. In Section 3, we prove that for  $\kappa \geq \omega_2$ ,  $Q^0$  respectively its associated complete Boolean algebra  $B(Q^0)$  is not Cohen. We define  $Q^1$  in Section 4,  $P^0$  and  $P^1$  in Section 5; moreover, we find dense subsets  $D_Q$  of the iteration  $Q^0 * Q^1$  of  $Q^0$  and  $Q^1$ , respectively  $D_P$  of  $P^0 * P^1$ , and prove that  $P^0 * P^1$  is Cohen. Finally in Section 6, we prove that  $D_Q$  and  $D_P$  are isomorphic.

This proves the Theorem, because of the following well-known facts on the connection between partial orderings  $P$  and their associated Boolean algebras  $B(P)$ . For  $D$  a dense subset of  $P$ ,  $B(D)$  is isomorphic to  $B(P)$ ; thus  $B(Q^0 * Q^1)$  is isomorphic to  $B(P^0 * P^1)$  and  $B(Q^0 * Q^1)$  is Cohen.  $Q^0$  is completely contained in the iteration  $Q^0 * Q^1$  and thus  $B(Q^0)$  is completely embeddable into the Cohen algebra  $B(Q^0 * Q^1)$ , but  $B(Q^0)$  was not Cohen.

Both iterations  $Q^0 * Q^1$  and  $P^0 * P^1$  will adjoin the same generic objects ( $\underline{f}$  and, for each  $\alpha \in \kappa$ , functions  $\underline{t}_\alpha : \omega \rightarrow \omega$  and  $\underline{x}_\alpha : \omega \rightarrow 2$ ), but in different order; this is why  $B(P^0 * P^1)$  is isomorphic to  $B(Q^0 * Q^1)$ . The functions  $\underline{t}_\alpha$ ,  $\alpha \in \kappa$ , will be almost disjoint in the sense that for  $\alpha \neq \beta$ ,  $\underline{t}_\alpha(i) \neq \underline{t}_\beta(i)$  will hold for almost all  $i$ . And the generic objects will be

connected as follows. Denote by  $B$  the binary tree of height  $\omega$  and by  $\text{lev}_i B$  its  $i$ 'th level. For  $i \in \omega$ , let  $a_i$  be a subset of  $\omega$  of size  $|\text{lev}_i B| = 2^i$  (for ease of notation,  $a_i$  will later be the set  $\{0, \dots, 2^i - 1\}$ , but this is neither important nor necessary for the proof).  $f$  will be a function from  $B$  into  $\omega$  mapping  $\text{lev}_i B$  onto  $a_i$ ; for every  $\alpha \in \kappa$ ,  $\underline{t}_\alpha$  and  $\underline{x}_\alpha$  will be connected by  $f$  in such a way that for almost all  $i$ ,  $\underline{t}_\alpha(i) = f(\underline{x}_\alpha \upharpoonright i)$ . Now the branches  $\{\underline{x}_\alpha \upharpoonright i : i \in \omega\}$  through  $B$  induced by the  $\underline{x}_\alpha$  are almost disjoint, and this causes the  $\underline{t}_\alpha$  to be almost disjoint. In fact, this line of argument reflects the standard construction of a family of  $2^\omega$  almost disjoint subsets of  $\omega$  out of the branches of a binary tree: if we choose the sets  $a_i$  to be pairwise disjoint, then even the sets  $\text{ran } \underline{t}_\alpha$ ,  $\alpha \in \kappa$ , will be almost disjoint.

The fact that  $Q^0$  is not Cohen can be partially explained by a combinatorial principle forced by  $Q^0$ , as observed by Soukup ([Juhász *et al.*, 1996]). Call an almost disjoint family  $A$  of subsets of  $\omega$  a  $\kappa$ -Luzin gap if  $|A| = \kappa$  and there is no  $X \subseteq \omega$  such that both  $\{a \in A : a \text{ is almost contained in } X\}$  and  $\{a \in A : a \text{ is almost contained in } \omega \setminus X\}$  have size  $\kappa$ . Now  $Q^0$  forces that  $\{\text{ran } \underline{t}_\alpha : \alpha \in \kappa\}$  is a  $\kappa$ -Luzin gap. On the other hand, if  $\kappa = \omega_2$  and CH holds in the ground model, then Cohen's partial order  $Fn(\kappa, 2)$  forces that there is no  $\kappa$ -Luzin gap.

The paper uses basic notions and results on forcing respectively two-step iterated forcing in the proof of Proposition 3.2 and in Sections 4 and 5. It is however possible to give an elementary proof which deals only with partial orderings and their associated Boolean algebras: Proposition 3.2 is provable without forcing, as explained in Section 3. In Section 4, one can define the partial ordering  $D_Q$  by Proposition 5.5 (plus Proposition 5.4) and then check that the map  $e : Q^0 \rightarrow B(D_Q)$  given by  $e(p) = \sum^{B(D_Q)} \{(p', q') \in D_Q : p' \text{ extends } p, \text{ in } Q^0\}$  is a complete embedding. A similar argument gives a complete embedding from  $P^0$  into  $B(D_P)$ , and it can be checked in an elementary way that  $B(D_P)$  is Cohen.

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## 1 Preliminaries

For unknown results or unexplained notions, cf. [Jech, 1978] and [Jech, 1989] in set theory, [Koppelberg, 1989] in Boolean algebras.

**Definition 1.1 (Boolean algebras).** The finitary Boolean operations are denoted by  $+$ ,  $\cdot$ , and  $-$ , the infinitary ones by  $\sum$  and  $\prod$ . 0 and 1 are the distinguished elements.

For a Boolean algebra  $D$ ,  $D^+$  is the set  $D \setminus \{0\}$  of non-zero elements of

$D$ . For  $X \subseteq D$ ,  $\langle X \rangle$  (respectively  $\langle X \rangle^{cm}$ ) is the subalgebra of  $D$  generated by  $X$  (respectively completely generated by  $X$ , if  $D$  is complete).

$B \leq D$  denotes that  $B$  is a subalgebra of  $D$ .  $B$  is a regular subalgebra of  $D$  if all infinite sums and products of subsets of  $B$  that happen to exist in  $B$  are preserved in  $D$ .

**Definition 1.2 (Dense subsets of Boolean algebras).** A subset  $P$  of  $D$  is dense in  $D$  if for every  $d \in D^+$  there is  $p \in P$  such that  $0 < p \leq d$ , i.e. every element of  $D$  is the least upper bound of some subset of  $P$ .  $\pi(D)$ , the  $\pi$ -weight of  $D$ , is the minimal size of a dense subset of  $D$ . More generally for  $B \leq D$ , the relative  $\pi$ -weight of  $D$  over  $B$ ,  $\pi(D/B)$ , is defined as  $\min \{|A| : A \subseteq D \text{ and } A \cup B \text{ generates a dense subalgebra of } D\}$ ; replacing  $A$  by a subalgebra of  $D$  including  $A$ , we can assume that  $A$  is a subalgebra of  $D$ . In this case  $\{a \cdot b : a \in A, b \in B\} \setminus \{0\}$  is dense in  $\langle A \cup B \rangle$ , so  $\langle A \cup B \rangle$  is dense in  $D$  iff, for every  $d \in D^+$ , there are  $a \in A$  and  $b \in B$  such that  $0 < a \cdot b \leq d$ . Moreover, if  $A$  and  $B$  are regular subalgebras of  $D$  and  $P_A \subseteq A$ ,  $P_B \subseteq B$ ,  $P_D \subseteq D$  are dense and  $P_A \cup P_B \subseteq P_D$ , then  $\langle A \cup B \rangle$  is dense in  $D$  iff for every  $q \in P_D$  there are elements  $p \in P_A$  and  $p' \in P_B$  such that  $p \cdot p' > 0$  and for every  $r \in P_D$ , if  $r \leq p$  and  $r \leq p'$ , then  $r \leq q$ ; this is because every element of  $A$  respectively  $B$  is the sum of a subset of  $P_A$  respectively  $P_B$ .

If  $A$  and  $B$  are complete and regular subalgebras of  $D$  and  $A \leq B \leq D$ , then  $\pi(B/A) \leq \pi(D/A)$ . This is proved as follows. Choose a subalgebra  $E$  of  $D$  such that  $|E| = \pi(D/A)$  and  $\langle A \cup E \rangle$  is dense in  $D$ . Then consider the set  $F = \{h(e) : e \in E\}$  where  $h : D \rightarrow B$  denotes the projection map given by  $h(d) = \min \{b \in B : b \geq d\}$  from  $D$  to  $B$ . Then  $|F| \leq \pi(D/A)$  and it is easily checked that  $\langle A \cup F \rangle$  is dense in  $B$ .

For a partially ordered set  $(P, \leq_P)$ , we write  $B(P)$  for its associated Boolean algebra or completion, i.e.  $B(P)$  is the unique complete Boolean algebra  $B$  such that there is an embedding  $i : P \rightarrow B$  with  $i[P]$  dense in  $B$ ; cf. [Kunen, 1980] II.3.3. Caused by the notation on forcing used in Definition 1.3, we assume that  $i$  is order-reversing. Moreover,  $i$  is one-one and satisfies  $p \leq_P q$  iff  $i(q) \leq_B i(p)$ , for all  $p, q \in P$ , iff  $P$  is separative, i.e. for  $p$  and  $q$  satisfying  $p \not\leq_P q$ , there is  $r \in P$  such that  $q \leq_P r$  and  $r$  is incompatible with  $p$ . In this case, we will think about  $P$  as being a dense subset of  $B(P)$ .

**Definition 1.3 (Forcing).** When dealing with notions of forcing  $(P, \leq)$ ,  $p \leq q$  means that the condition  $q$  is stronger than  $p$ . For an arbitrary cardinal  $\kappa$ ,  $F_n(\kappa, 2)$  is the forcing which adjoins  $\kappa$  Cohen reals, i.e. a condition in  $F_n(\kappa, 2)$  is a function  $p$  from some finite subset of  $\kappa$  into  $2$ , and  $p \leq q$  holds iff  $p \subseteq q$ . We call its completion  $\mathbb{C}_\kappa = B(F_n(\kappa, 2))$  the standard Cohen algebra of  $\pi$ -weight  $\kappa$ , since  $\pi(\mathbb{C}_\kappa) = \kappa$ .  $\mathbb{C}_\kappa$  is also the completion of the free Boolean algebra over  $\kappa$  generators; this is the definition used in

[Koppelberg, 1993]. As usual in the literature, we will also call a forcing  $P$  Cohen if  $B(P)$  is isomorphic to some standard Cohen algebra.

**Definition 1.4 (Two-step iterated forcing).** Let us recall the general version of two-step iterated forcing. If  $(P, \leq_P)$  is a partial ordering in the ground model  $V$  and  $(Q, \leq_Q) \in V^P$  is a  $P$ -name for a partial ordering (i.e. if  $P \Vdash \text{“}(Q, \leq_Q) \text{ is a partial ordering”}$ ), then the iteration  $P * Q$  of  $P$  and  $Q$  is the partial ordering (in  $V$ ) defined as follows. The elements of  $P * Q$  are certain pairs  $(p, q)$  where  $p \in P$ ,  $q \in \text{dom } Q$  (thus  $q \in V^P$ ) and  $p \Vdash q \in Q$ . And  $(p, q) \leq (p', q')$  holds in  $P * Q$  if  $p \leq_P p'$  and  $p' \Vdash q \leq_Q q'$ .

When applying this in Sections 4 and 5, we will deal with a simpler situation: we will have a set  $N$  in  $V$  such that  $P \Vdash Q \subseteq \check{N}$ . If  $(p, q) \in P * Q$ , let us say that  $p$  decides  $q$  if, for some  $n \in N$ ,  $p \Vdash q = \check{n}$ . Here  $\check{n}$  is the canonical name for  $n \in V$  in  $V^P$ ; we will usually write  $n$  for  $\check{n}$ . We call the subset

$$\text{stp}_N(P * Q) = \{(p, q) : p \in P, q \in N, p \Vdash q \in Q\}$$

of  $P * Q$  the standard part of  $P * Q$  relative to  $N$ . This is a dense subset of  $P * Q$ , hence their associated Boolean algebras are isomorphic. We will omit the subscript  $N$  and, if convenient, tacitly pass to a dense subset of  $\text{stp}(P * Q)$  and still call it the standard part of  $P * Q$ .

Note that if  $p$  decides  $q (= \check{n})$  and if, e.g.,  $n$  is, in  $V$ , a function, then  $p$  also decides the domain, the range, and the values of  $q$  since the statements “ $u = \text{dom } n$ ”, “ $v = \text{ran } n$ ”, “ $i \in u = \text{dom } n$  and  $j = n(i)$ ” are  $\Delta_0$ , hence absolute for  $V$  and  $V^P$ .

**Definition 1.5 (Some definitions for Sections 2 to 6).** We fix some notation which will be used throughout the paper.

For  $n \in \omega$ , let  $a_n = 2^n = \{0, \dots, 2^n - 1\} \subseteq \omega$ .

$T$  is the tree of height  $\omega$  with  $n$ 'th level  $\text{lev}_n T$  the set of those functions  $t$  from  $n$  to  $\omega$  such that, for  $i < n$ ,  $t(i) \in a_i$ , i.e.  $\text{lev}_n T = a_0 \times \dots \times a_{n-1}$ .  $B$  is the binary tree of height  $\omega$  with  $\text{lev}_n B$  the set of all functions from  $n$  to 2. In both cases, the tree ordering is set-theoretic inclusion.

For  $M$  a set of sequences with common domain some  $n \leq \omega$  and  $k \leq n$ , we say that the elements of  $M$  are disjoint above  $k$  if for every  $i \in [k, n)$ , the values  $m(i)$ ,  $m \in M$ , are pairwise distinct.

## 2 Simple properties of $Q_X^0$

Our investigation of the forcing  $Q^0$  will use, more generally, the forcings  $Q_X^0$  where  $X$  is an arbitrary set (or, in section 3, a subset of some cardinal  $\kappa \geq \omega_2$ ).  $Q^0$  will simply be the special case  $Q_X^0$  where  $X = \kappa$ . In this section, we collect some basic properties of the  $Q_X^0$ .

**Definition 2.1** For any set  $X$ , the forcing  $Q_X^0$  is defined as follows. An element of  $Q_X^0$  is a function  $p$  with  $\text{dom } p$  a finite subset of  $X$  such that, for some  $n \in \omega$  (the *height* of  $p$ ,  $\text{ht } p$ ):

- (a)  $|\text{dom } p| \leq a_n$
- (b) writing  $t_\alpha^p$  for  $p(\alpha)$ : each  $t_\alpha^p$ ,  $\alpha \in \text{dom } p$ , is an element of  $\text{lev}_n T$ , i.e.  $t_\alpha^p(i) \in a_i$  for  $i < n$
- (c)  $\text{dom } p = \emptyset$  implies  $\text{ht } p = 0$  and  $\text{dom } p \neq \emptyset$  implies  $\text{ht } p \geq 2$  (this is only for technical convenience).

For  $p$  and  $q$  in  $Q_X^0$ ,  $p \leq q$  iff

- (d)  $\text{dom } p \subseteq \text{dom } q$  and  $\text{ht } p \leq \text{ht } q$
- (e) for  $\alpha \in \text{dom } p$ ,  $t_\alpha^p \subseteq t_\alpha^q$
- (f) the  $t_\alpha^q$ ,  $\alpha \in \text{dom } p$ , are disjoint above  $\text{ht } p$ .

**Lemma 2.2 (and Definition).** *For every  $k \in \omega$ , the set  $\{q : \text{ht } q \geq k\}$  is dense in  $Q_X^0$ . Also for every  $\alpha \in X$ ,  $\{q : \alpha \in \text{dom } q\}$  is dense in  $Q_X^0$ . It follows that, if  $G \subseteq Q_X^0$  is  $Q_X^0$ -generic over the ground model  $V$  and we put  $\underline{t}_{\alpha G} = \bigcup \{t_\alpha^p : p \in G \text{ and } \alpha \in \text{dom } p\}$  for  $\alpha \in X$ , each  $\underline{t}_{\alpha G}$  is an element of the cartesian product  $\prod_{i \in \omega} a_i$ .*

PROOF. Obvious. For the first claim, note that this is where (a) of Definition 2.1 is used; for the second one, enlarge first the height of a given condition in  $Q_X^0$ , if necessary, and then the domain. ■

The subsequent propositions will use the following criterion for compatibility in  $Q_X^0$ .

**Proposition 2.3** *Assume  $p$  and  $q$  are in  $Q_X^0$  and  $\text{ht } p \leq \text{ht } q$ . Then  $p$  and  $q$  are compatible in  $Q_X^0$  iff*

- (a) for  $\alpha \in \text{dom } p \cap \text{dom } q$ ,  $t_\alpha^p \subseteq t_\alpha^q$
- (b) the  $t_\alpha^q$ ,  $\alpha \in \text{dom } p \cap \text{dom } q$ , are disjoint above  $\text{ht } p$ .

PROOF. Obvious. ■

**Corollary 2.4**  $Q_X^0$  *satisfies the countable chain condition.*

PROOF. By the usual  $\Delta$ -system lemma argument. ■

The next proposition says that we can think of  $Q_X^0$  as being a dense subset of its associated Boolean algebra  $B_X = B(Q_X^0)$ .

**Proposition 2.5**  $Q_X^0$  *is separative.*

PROOF. Assume  $p$  and  $q$  are in  $Q_X^0$  and  $p \not\leq q$ . We will find  $r \in Q_X^0$  such that  $q \leq r$  and  $r$  is incompatible with  $p$ . Without loss of generality, we may assume that  $p$  and  $q$  are compatible.

*Case 1.*  $\text{dom } p \not\subseteq \text{dom } q$ . Fix  $\alpha \in \text{dom } p \setminus \text{dom } q$ . We will choose  $r \geq q$  such that  $\text{dom } r = \text{dom } q \cup \{\alpha\}$ ,  $\text{ht } r > \text{ht } q$ , and  $|\text{dom } r| \leq a_{\text{ht } r}$ . More

precisely, take  $t_\alpha^r \in a_0 \times \dots \times a_{\text{ht } r-1}$  such that  $t_\alpha^r(1) \neq t_\alpha^p(1)$  (recall Definition 2.1(c) and  $a_1 = 2$ ). For  $\beta \in \text{dom } q$ , let  $t_\beta^r \in \text{lev}_{\text{ht } r}$  extend  $t_\beta^q$  in such a way that all  $t_\beta^r(i)$ ,  $\text{ht } \leq i < \text{ht } r$ , are distinct.

*Case 2.*  $\text{dom } p \subseteq \text{dom } q$ , but  $\text{ht } q < \text{ht } p$ .  $\text{ht } p > 0$ , so by Definition 2.1(c), also  $\text{dom } p \neq \emptyset$ ; fix  $\alpha \in \text{dom } p$ . Now take  $r \geq q$  such that  $\text{dom } r = \text{dom } q$ ,  $\text{ht } r = \text{ht } p$ , and  $t_\alpha^r(\text{ht } q) \neq t_\alpha^p(\text{ht } q)$ ; thus  $r$  and  $p$  are incompatible. This is possible since  $\text{dom } q \neq \emptyset$  (so  $\text{ht } q \geq 2$ ) and  $a_{\text{ht } q} \geq 2$ .

*Case 3.*  $\text{dom } p \subseteq \text{dom } q$  and  $\text{ht } p \leq \text{ht } q$ .  $p$  and  $q$  are compatible, so by Proposition 2.3, we have that for every  $\alpha \in \text{dom } p$ ,  $t_\alpha^p \subseteq t_\alpha^q$  and the  $t_\alpha^q$ ,  $\alpha \in \text{dom } p$ , are disjoint above  $\text{ht } p$ . But then  $p \leq q$ , a contradiction. ■

In the two subsequent propositions, we use the following construction. Every permutation  $h$  of  $X$  induces an automorphism  $\bar{h}$  of the partial ordering  $Q_X^0$  by letting, for  $p \in Q_X^0$ ,  $\text{dom } \bar{h}p = h[\text{dom } p]$ ,  $\text{ht } \bar{h}p = \text{ht } p$ , and, for  $\alpha \in \text{dom } p$ ,  $t_{h(\alpha)}^{\bar{h}p} = t_\alpha^p$ . Moreover, we call a forcing  $Q$  *weakly homogeneous* if for arbitrary  $p$  and  $q$  in  $Q$ , there are  $p'$  and  $q'$  in  $Q$  such that  $p \leq p'$ ,  $q \leq q'$ , and  $Q \upharpoonright p'$  is isomorphic to  $Q \upharpoonright q'$ , where  $Q \upharpoonright r = \{x \in Q : x \geq r\}$ , for  $r \in Q$ .

**Proposition 2.6**  $Q_X^0$  is weakly homogeneous, for infinite  $X$ .

PROOF. Let  $p$  and  $q$  in  $Q_X^0$  be given. Fix a permutation  $h$  of  $X$  such that  $h[\text{dom } q]$  is disjoint from  $\text{dom } p$ . By Proposition 2.3, there is a common extension  $r$  of  $p$  and  $\bar{h}(q)$ . Now  $r \geq \bar{h}(q)$ ,  $\bar{h}^{-1}(r) \geq q$ , and  $Q_X^0 \upharpoonright r$  is isomorphic to  $Q_X^0 \upharpoonright \bar{h}^{-1}(r)$ . ■

**Proposition 2.7** Assume  $X$  is infinite and  $X \subseteq Y$ . Then  $Q_X^0$  is completely contained in  $Q_Y^0$  (cf. [Kunen, 1980] VII.7.1 for this notion).

PROOF. Clearly,  $Q_X^0$  is a subordering of  $Q_Y^0$ , and, by Proposition 2.3, two elements of  $Q_X^0$  are compatible in  $Q_X^0$  iff they are in  $Q_Y^0$ . Thus assume  $p' \in Q_Y^0$  with the aim of finding  $p \in Q_X^0$  such that every extension of  $p$  in  $Q_X^0$  is compatible with  $p'$ .

Write  $\text{dom } p' = r \cup s'$  where  $r \subseteq X$  and  $s' \subseteq Y \setminus X$ . Then choose a subset  $s$  of  $X$  disjoint from  $r$  such that  $|s| = |s'|$  and a permutation  $h$  of  $Y$  satisfying  $h \upharpoonright r = \text{id}$  and  $h \upharpoonright s' = s$ . We will show that  $p = \bar{h}(p')$  works for our claim; note that  $\text{dom } p = r \cup s$ .

In fact, assume that  $q \in Q_X^0$  extends  $p$ ; say  $\text{dom } q = r \cup s \cup u$  where  $u$  is disjoint from  $r \cup s$ . Choose another permutation  $k$  of  $Y$  such that  $k$  and  $h^{-1}$  coincide on  $\text{dom } p$  and  $k$  maps  $u$  onto a subset  $u'$  of  $Y$  disjoint from  $\text{dom } q$ . Clearly  $q' = \bar{k}(q)$  extends  $p'$ ; thus it suffices to prove that  $q$  and  $q'$  are compatible. But  $\text{ht } q = \text{ht } q'$ ,  $\text{dom } q \cap \text{dom } q' = r$ , and for every  $\alpha \in r$ , we have  $k(\alpha) = \alpha$  and thus  $t_\alpha^q = t_\alpha^{q'}$ . ■

### 3 $Q^0$ is not Cohen

We prove in this section that the forcing  $Q^0 = Q_\kappa^0$  is not Cohen, for  $\kappa \geq \omega_2$ , i.e. its associated Boolean algebra  $B(Q^0)$  is not Cohen. The ideas lying behind the proof are from [Koppelberg, 1993] (cf. Theorem 0.1 in the introduction), i.e. essentially from Shapiro's proof that subalgebras of free algebras are Cohen, but we give a completely self-contained presentation here. The main argument in the proof is the following lemma. We have not tried to minimize its assumptions since they are so naturally satisfied in the intended application.

**Lemma 3.1** *Assume that  $\kappa \geq \omega_2$  is a cardinal and that, for every subset  $X$  of  $\kappa$ , we are given two separative partial orderings  $P_X$  and  $Q_X$  with the following properties. We write  $A_X = B(P_X)$ ,  $B_X = B(Q_X)$ , and assume that  $P_X$  is a dense subset of  $A_X$ ; similarly for  $Q_X$  and  $B_X$ .*

- (a)  $P_X$  and  $Q_X$  satisfy the countable chain condition
  - (b)  $X \subseteq Y \subseteq \kappa$  implies that  $P_X \subseteq P_Y$  and  $P_X$  is completely contained in  $P_Y$  (so without loss of generality,  $A_X$  is a regular subalgebra of  $A_Y$ ); similarly for  $Q_X \subseteq Q_Y$  and  $B_X \subseteq B_Y$
  - (c)  $P_X = \bigcup \{P_e : e \subseteq X \text{ finite}\}$ ; similarly for  $Q_X$
  - (d)  $|P_X| \leq |X|$  for infinite  $X$ ; similarly for  $Q_X$
  - (e) for  $X, Y \subseteq \kappa$ ,  $A_{X \cup Y}$  is completely generated by  $A_X \cup A_Y$ ; similarly for  $B_{X \cup Y}$
  - (f) if  $Y$  is countable, then  $\pi(A_{X \cup Y}/A_X) \leq \omega$ .
- Assume  $B_\kappa$  is isomorphic to  $A_\kappa$ . Then there is a club subset  $\mathbb{C}$  of  $[\kappa]^{\omega_1}$  such that*
- (g) for  $X \in \mathbb{C}$  and  $Y \subseteq \kappa$  countable,  $\pi(B_{X \cup Y}/B_X) \leq \omega$ .

We will apply Lemma 3.1 to the situation where  $P_X = Fn(X, 2)$  is standard Cohen forcing and  $Q_X = Q_X^0$  as defined in Definition 2.1. (a) through (f) of Lemma 3.1 are clearly satisfied for the forcings  $P_X$ , and (a) through (d) hold for  $Q_X$ , by the results of Section 2. We prove in the subsequent lemmas that the  $Q_X$  satisfy (e), but not (g) — hence  $B(Q_\kappa^0)$  is not isomorphic to  $B(Fn(\kappa, 2))$ .

**PROOF** of Lemma 3.1. For convenience of notation, we assume that  $A_\kappa$  and  $B_\kappa$  are the same Boolean algebra  $D$ ; so for  $X \subseteq \kappa$ ,  $A_X$  and  $B_X$  are regular subalgebras of  $D$ .

Call a subset  $M$  of  $D$  nice if there is a regular complete subalgebra  $C$  of  $D$  such that  $M$  is a dense subset of  $C$ . (E. g.,  $P_X$  and  $Q_X$  are nice.) In this case,  $C$  is uniquely determined by  $M$  since  $C = \{\sum^D M_0 : M_0 \subseteq M\}$  and we write  $C = C(M)$ .



For  $M$  and  $N$  subsets of  $D$ , we say that  $M$  is dense for  $N$  if for every  $n \in N$ , there is  $M_0 \subseteq M$  such that  $n = \sum^D M_0$ . Thus if  $M$  and  $N$  are nice,  $C(M) = C(N)$  iff  $M$  is dense for  $N$  and  $N$  is dense for  $M$ .

Define

$$\mathbb{C} = \{X \subseteq \kappa : |X| = \omega_1, P_X \text{ is dense for } Q_X \text{ and } Q_X \text{ is dense for } P_X\}.$$

It follows from the assumptions (a), (c), and (d) that  $\mathbb{C}$  is club in  $[\kappa]^{\omega_1}$ . And for  $X \in \mathbb{C}$ , we have  $A_X = B_X$ .

To check (g), assume  $X$  is an element of  $\mathbb{C}$  and  $Y \subseteq \kappa$  is countable. By (a), (c), and (d) again, we find a countable  $Z \subseteq \kappa$  such that  $P_Z$  is dense for  $Q_Y$ ; so  $B_Y \leq A_Z$ . Now by (e),

$$A_X = B_X \leq B_{X \cup Y} = \langle B_X \cup B_Y \rangle^{\text{cm}} \leq \langle A_X \cup A_Z \rangle^{\text{cm}} = A_{X \cup Z}$$

and  $\pi(A_{X \cup Z}/A_X) \leq \omega$  holds by (f). It follows from 1.2 that

$$\pi(B_{X \cup Y}/B_X) \leq \omega.$$

■

For the rest of this section, fix a cardinal  $\kappa \geq \omega_2$  and write, for  $X \subseteq \kappa$ :

$$B_X = B(Q_X^0),$$

a regular subalgebra of  $B_\kappa$ . We proceed to show that assumption (e) of Lemma 3.1 holds for the  $B_X$ , but (g) fails. In  $B_\kappa$ , note that  $B_X$  has  $Q_X^0$  as set of complete generators, since  $Q_X^0$  is a dense subset of  $B_X$  and thus every element of  $B_X$  is (in  $B_\kappa$ ) the join of some subset of  $Q_X^0$ .

The details of the following proof may be unnecessary for a reader experienced with forcing. On the other hand, the use of forcing can be avoided, at the price of a little more computation and less insight: simply define  $b_{\alpha ij}$  to be  $\sum N_{\alpha ij}$  (where  $N_{\alpha ij}$  is as in the proof of Claim 1 below); this makes Claim 1 trivial. For Claim 2, just prove that, for  $p \in Q_X^0$  with domain  $u$  and height  $n$ :

$$p = \prod \{b_{\alpha ij} : \alpha \in u, i < n, t_\alpha^p(i) = j\} \cdot \prod \{-(b_{\beta ij} \cdot b_{\gamma ij}) : \beta \neq \gamma \text{ in } u, i \geq n, j \in a_i\}.$$

**Proposition 3.2** For  $X, Y \subseteq \kappa$ ,  $B_{X \cup Y}$  is completely generated by  $B_X \cup B_Y$ .

PROOF. For  $\alpha \in \kappa$ ,  $i \in \omega$ , and  $j \in a_i$ , let  $\sigma_{\alpha ij}$  be the sentence “ $\underline{t}_\alpha(i) = j$ ” of the forcing language over  $Q_\kappa^0$  and let  $b_{\alpha ij}$  be its Boolean truth value  $\|\sigma_{\alpha ij}\|$ , computed in  $B_\kappa$ . Here  $\underline{t}_\alpha$  is (the canonical name for) the generic object introduced in Lemma 2.2; we do not distinguish notationally between

$\alpha, i, j$  in the ground model and their canonical names  $\check{\alpha}, \dots$  in the forcing language.

For  $X \subseteq \kappa$ , consider the set  $M_X = \{b_{\alpha ij} : \alpha \in X, i \in \omega, j \in a_i\}$ , and the subsequent claims.

*Claim 1.*  $M_X \subseteq B_X$ .

*Claim 2.*  $M_X$  completely generates  $B_X$ .

Now clearly  $M_{X \cup Y} = M_X \cup M_Y$ , and thus the proposition follows from the claims.

To prove the claims, recall the basic fact on the connection between forcing and Boolean-valued models (cf. [Jech, 1978] p. 166): for  $p \in Q_\kappa^0$  and  $\sigma$  a sentence of the forcing language over  $Q_\kappa^0$ ,

$$p \Vdash \sigma \text{ iff } p \leq_B \|\sigma\|.$$

Here we write  $\leq_B$  for the Boolean partial ordering. All joins and meets below are computed in  $B_\kappa$ .

*Proof of Claim 1.* We prove that for  $\alpha \in X$ ,  $i \in \omega$ ,  $j \in a_i$ , we have  $b_{\alpha ij} = \sum N_{\alpha ij}$  where  $N_{\alpha ij} = \{p \in Q_{\{\alpha\}}^0 : \text{dom } p = \{\alpha\}, \text{ht } p > i, t_\alpha^p(i) = j\}$ . Here,  $\geq$  is obvious since every  $p \in N_{\alpha ij}$  forces  $\sigma_{\alpha ij}$ . Assume for contradiction that  $b_{\alpha ij}$  is (in  $B_\kappa$ ) strictly greater than  $\sum N_{\alpha ij}$ . Then there is some  $q \in Q_\kappa^0$  forcing  $\sigma_{\alpha ij}$  but incompatible with all  $p \in N_{\alpha ij}$ . By extending  $q$ , we may assume that  $\alpha \in \text{dom } q$  and  $i < \text{ht } q$ . Consider  $k = t_\alpha^q(i)$ . Now  $k \neq j$ : otherwise, let  $p$  be the restriction of  $q$  with domain  $\{\alpha\}$  and height  $i+1$ . Then  $p \in N_{\alpha ij}$  and  $q$  extends  $p$  in  $Q_\kappa^0$ ; a contradiction since  $q$  was incompatible with all  $p \in N_{\alpha ij}$ . - It follows that  $q \Vdash \underline{t}_\alpha(i) = k \neq j$  and thus  $q \Vdash \neg \sigma_{\alpha ij}$ , a contradiction.

*Proof of Claim 2.* It suffices to show that  $Q_X^0 \subseteq B_X$  is completely generated by  $M_X$ . So let  $p \in Q_X^0$ , say with domain  $u$  and height  $n$ , and consider the set of sentences of the forcing language

$$\Sigma_p = \{\sigma_{\alpha ij} : \alpha \in u, i < n, t_\alpha^p(i) = j\} \cup \{\neg(\sigma_{\beta ij} \wedge \sigma_{\gamma ij}) : \beta \neq \gamma \text{ in } u, i \geq n, j \in a_i\}.$$

The Boolean value of each  $\sigma \in \Sigma_p$  is clearly generated by  $M_u \subseteq M_X$ , thus it suffices to prove that  $p = \prod \{\|\sigma\| : \sigma \in \Sigma_p\}$ , i. e. that for each  $q \in Q_\kappa^0$ ,  $q$  extends  $p$  iff  $q \Vdash \Sigma_p$  (where  $q \Vdash \Sigma_p$  means that  $q \Vdash \sigma$ , for every  $\sigma \in \Sigma_p$ ). Here,  $\Rightarrow$  is clear since  $p \Vdash \Sigma_p$ , by (f) of Definition 2.1. Conversely, assume for contradiction that  $q \Vdash \Sigma_p$  but  $q$  does not extend  $p$ . By applying Proposition 2.5 and extending  $q$ , we may assume that  $q$  is incompatible with  $p$ ,  $u \subseteq \text{dom } q$  and  $n \leq \text{ht } q$ . By Proposition 2.3, we have to distinguish two cases. Either there are  $\alpha \in u$  and  $i < n$  such that  $t_\alpha^p(i) \neq t_\alpha^q(i)$ ; then  $q \Vdash \neg \sigma_{\alpha ij}$  where  $j = t_\alpha^p(i)$ , contradiction. Or there are  $i \in [n, \text{ht } q]$  and  $\beta \neq \gamma$  in  $u$  such that  $t_\beta^q(i) = t_\gamma^q(i)$ ; then  $q \Vdash \sigma_{\beta ij} \wedge \sigma_{\gamma ij}$  where  $j = t_\beta^q(i)$ , a contradiction again. ■

The example given in Proposition 3.4 below is the crucial fact responsible for the failure of (g) in Lemma 3.1 for the algebra  $B_\kappa$ . We need another easy lemma on the forcings  $Q_X^0$  for this.

**Lemma 3.3** *Let  $X$  and  $Y$  be arbitrary sets and assume that  $p \in Q_X^0$  and  $p' \in Q_Y^0$  are compatible in  $Q_{X \cup Y}^0$ ,  $k \in \omega$ ,  $\alpha \in X$ , and  $\beta \in Y$ . Then there are compatible  $q \in Q_X^0$  and  $q' \in Q_Y^0$  such that:  $p \leq q$ ,  $p' \leq q'$ ,  $\alpha \in \text{dom } q$ ,  $\beta \in \text{dom } q'$ , and  $\text{ht } q = \text{ht } q' \geq k$ .*

PROOF. In  $Q_{X \cup Y}^0$ , take a common extension  $r$  of  $p$  and  $p'$ . By extending  $r$ , we may assume that  $\text{ht } r \geq k$  and  $\alpha, \beta \in \text{dom } r$ . Then let  $q$  respectively  $q'$  be the restrictions of  $r$  to  $\text{dom } p \cup \{\alpha\}$  respectively  $\text{dom } p' \cup \{\beta\}$ . ■

**Proposition 3.4** *Let  $T$  be a proper subset of  $X$  and let  $Y$  be a non-empty set disjoint from  $X$ . Then, for some  $q \in Q_{X \cup Y}^0$ , there are no compatible*

$p \in Q_X^0$  and  $p' \in Q_{T \cup Y}^0$  such that every common extension of  $p$  and  $p'$  extends  $q$ .

PROOF. Fix elements  $\alpha \in X \setminus T$ ,  $\beta \in Y$ ; thus  $\alpha \in X$ ,  $\beta \in Y$ , and  $\alpha \neq \beta$ . Let  $q$  be an arbitrary element of  $Q_{X \cup Y}^0$  satisfying  $\text{dom } q = \{\alpha, \beta\}$ .

Assume for contradiction that we have compatible  $p \in Q_X^0$  and  $p' \in Q_{T \cup Y}^0$  such that every common extension of  $p$  and  $p'$  extends  $q$ . Applying Lemma 3.3 and extending  $p$  and  $p'$  if necessary, we may assume that  $\alpha \in \text{dom } p$ ,  $\beta \in \text{dom } p'$ ,  $\text{ht } p = \text{ht } p' = m \geq \text{ht } q$ , and  $w = \text{dom } p \cup \text{dom } p'$  has size at most  $a_m$ .

We choose a common extension  $r \in Q_{X \cup Y}^0$  of  $p$  and  $p'$  as follows: put  $\text{dom } r = w$  and  $\text{ht } r = m + 1$ . We are left with defining  $t_\gamma^r(m)$ , for all  $\gamma \in w$ . Simply define these values such that: all  $t_\gamma^r(m)$ ,  $\gamma \in \text{dom } p$ , are distinct; all  $t_\gamma^r(m)$ ,  $\gamma \in \text{dom } p'$ , are distinct; but  $t_\alpha^r(m) = t_\beta^r(m)$ . This is possible since  $\alpha \in \text{dom } p \setminus \text{dom } p'$  and  $\beta \in \text{dom } p' \setminus \text{dom } p$ .

By our assumption above,  $r$  must extend  $q$ . But this is not the case, since  $\alpha, \beta$  are distinct elements of  $\text{dom } q$ ,  $m \in [\text{ht } q, \text{ht } r)$ , and  $t_\alpha^r(m) = t_\beta^r(m)$ . ■

**Corollary 3.5** *Assume  $X \subseteq \kappa$  is uncountable and  $Y \subseteq \kappa$  is nonempty and disjoint from  $X$ . Then  $\pi(B_{X \cup Y}/B_X)$  is uncountable.*

PROOF. If not, we can find a countable subset  $C$  of  $B_{X \cup Y}$  such that  $B_X \cup C$  generates a dense subalgebra of  $B_{X \cup Y}$ . Choose a countable subset  $T$  of  $X$  such that  $C \subseteq B_{T \cup Y}$ ; the subalgebra generated by  $B_X \cup B_{T \cup Y}$  is still dense in  $B_{X \cup Y}$ . A remark in 1.2, applied to  $B_X, B_{T \cup Y} \leq B_{X \cup Y}$ , gives that for every  $q \in Q_{X \cup Y}^0$ , there are compatible  $p \in Q_X^0$  and  $p' \in Q_{T \cup Y}^0$  such that every common extension of  $p$  and  $p'$  (in  $Q_{X \cup Y}^0$ ) extends  $q$ . But this contradicts Proposition 3.4. ■

**Theorem 3.6**  $B_\kappa (= B(Q_\kappa^0))$  is not Cohen, for  $\kappa \geq \omega_2$ .

PROOF. Assume it is. It is easily checked that  $\pi(B_\kappa) = \kappa$ . Also  $B_\kappa$  is weakly homogeneous, by Proposition 2.6; thus  $B_\kappa$  must be isomorphic to the standard Cohen algebra  $\mathbb{C}_\kappa$  of  $\pi$ -weight  $\kappa$ .

By Lemma 3.1, there is some  $X \in [\kappa]^{\omega_1}$  such that for every countable  $Y \subseteq \kappa$ ,  $\pi(B_{X \cup Y}/B_X) \leq \omega$ , contradicting Corollary 3.5. ■

## 4 $Q^1$ and a dense subset of $Q^0 * Q^1$

For the remaining sections, let  $\kappa$  be an arbitrary cardinal and put, as before,  $Q^0 = Q_\kappa^0$ . Following the plan in the introduction, we will define a forcing

$Q^1$  in  $V^{Q^0}$ , i.e. a  $Q^0$ -name  $Q^1$  such that  $Q^0 \Vdash "Q^1$  is a partial ordering", describe the standard part of  $Q^0 * Q^1$  and find a dense subset of the standard part. The definition of  $Q^1$  will use the generic functions  $\underline{t}_{\alpha G} : \omega \rightarrow \omega$  defined in Lemma 2.2 for  $\alpha \in \kappa$  respectively their canonical  $Q^0$ -names  $\underline{t}_\alpha$ . Recall from Definition 1.5 the definitions concerning the trees  $B$  and  $T$  and the numbers  $a_n$ .

**Definition 4.1** (In  $V^{Q^0}$ ) An element of  $Q^1$  is a pair  $q = (f^q, (x_\alpha^q)_{\alpha \in u})$  such that for some finite  $u \subseteq \kappa$  (the domain of  $q$ ,  $\text{dom } q$ ) and some  $m \in \omega$  (the height of  $q$ ,  $\text{ht } q$ ):

(a)  $f^q$  maps  $\bigcup_{i < m} \text{lev}_i B$  into  $\omega$ ; for  $i < m$ ,  $f^q \upharpoonright \text{lev}_i B$  is a bijection from  $\text{lev}_i B$  onto  $a_i$  (note  $a_i = 2^i = |\text{lev}_i B|$ )

(b) the  $x_\alpha^q$ ,  $\alpha \in u = \text{dom } q$ , are pairwise distinct elements of  $\text{lev}_m B$  (and thus  $|\text{dom } q| \leq 2^m = a_m$ )

(c) the  $\underline{t}_\alpha$ ,  $\alpha \in u = \text{dom } q$ , are disjoint above  $m = \text{ht } q$ .

For  $q$  and  $q'$  in  $Q^1$ ,  $q \leq q'$  iff

(d)  $\text{dom } q \subseteq \text{dom } q'$  and  $\text{ht } q \leq \text{ht } q'$

(e)  $f^q \subseteq f^{q'}$  and, for  $\alpha \in \text{dom } q$ ,  $x_\alpha^q \subseteq x_\alpha^{q'}$

(f) for  $\alpha \in \text{dom } q$  and  $i \in [\text{ht } q, \text{ht } q')$ ,  $\underline{t}_\alpha(i) = f^{q'}(x_\alpha^{q'} \upharpoonright i)$ .

For  $H$   $Q^1$ -generic over  $V^{Q^0}$ , we obtain the generic objects  $\underline{f}_H = \bigcup \{f^q : q \in H\}$ , a map from the binary tree  $B$  into  $\omega$  which maps the  $i$ 'th level of  $B$  in a one-one manner onto  $a_i$ , and, for  $\alpha \in \kappa$ ,  $\underline{x}_{\alpha H} = \bigcup \{x_\alpha^q : q \in H \text{ and } \alpha \in \text{dom } q\} : \omega \rightarrow 2$ . They are related to the generic objects  $\underline{t}_{\alpha G}$  adjoined by  $Q^0$  by the fact that, for almost all  $i \in \omega$  (we omit the subscripts  $G$  and  $H$ ),  $\underline{t}_\alpha(i) = \underline{f}(\underline{x}_\alpha \upharpoonright i)$ .

Let us describe the standard part  $\text{stp}(Q^0 * Q^1)$  of  $Q^0 * Q^1$ . All conditions defining the elements of respectively the partial order on  $Q^1$  in Definition 4.1 deal with objects in  $V$  or are absolute, except (c). But for  $p \in Q^0$ ,  $u \subseteq \text{dom } p$  finite, and  $m \leq \text{ht } p$ ,  $p$  forces the  $\underline{t}_\alpha$ ,  $\alpha \in u$ , to be disjoint above  $m$  iff the  $t_\alpha^p$ ,  $\alpha \in u$ , are disjoint above  $m$  (i.e. disjoint on the interval  $[m, \text{ht } p)$ ). This is because  $p$  forces that the  $\underline{t}_\alpha$ ,  $\alpha \in u$ , are disjoint above  $\text{ht } p$ .

**Proposition 4.2** *The elements of  $\text{stp}(Q^0 * Q^1)$  are those pairs  $(p, q)$  such that  $p \in Q^0$  and  $q$  is a pair  $(f^q, (x_\alpha^q)_{\alpha \in u})$  where  $u \subseteq \text{dom } p$ ,  $m = \text{ht } q \leq \text{ht } p$  satisfying (in  $V$ ) (a) and (b) of Definition 4.1, plus*

(c) the  $t_\alpha^p$ ,  $\alpha \in u$ , are disjoint above  $m$ .

For  $(p, q)$  and  $(p', q')$  in  $\text{stp}(Q^0 * Q^1)$ ,  $(p, q) \leq (p', q')$  iff (d) and (e) of Definition 4.1 hold, plus

(f) for  $\alpha \in \text{dom } q$  and  $i \in [\text{ht } q, \text{ht } q')$ ,  $\underline{t}_\alpha^{p'}(i) = f^{q'}(x_\alpha^{q'} \upharpoonright i)$ .

**Proposition 4.3** *The following subset of  $\text{stp}(Q^0 * Q^1)$  is dense in  $\text{stp}(Q^0 * Q^1)$ , hence in  $Q^0 * Q^1$ .*

$$D_Q = \{(p, q) \in \text{stp}(Q^0 * Q^1) : \text{dom } p = \text{dom } q \text{ and } \text{ht } p = \text{ht } q\}.$$

PROOF. Let  $(p, q) \in \text{stp}(Q^0 * Q^1)$  be given; we find  $q'$  such that  $(p, q')$  is in  $D_Q$  and extends  $(p, q)$ . Write  $v = \text{dom } p \supseteq u = \text{dom } q$ ,  $n = \text{ht } p \geq m = \text{ht } q$ .

First pick, for  $\alpha \in v$ , an element  $x_\alpha^{q'}$  of  $\text{lev}_n B$  such that:

- (a) the  $x_\alpha^{q'}$ ,  $\alpha \in v$ , are pairwise distinct
- (b) for  $\alpha \in u$ ,  $x_\alpha^q \subseteq x_\alpha^{q'}$ .

This is possible since the  $x_\alpha^q$ ,  $\alpha \in u$ , are distinct in  $\text{lev}_m B$  and  $|\text{lev}_n B| = 2^n = a_n \geq |v|$  (cf. Definition 2.1(a)). Then define, for  $m \leq i < n$ , the bijection  $f^{q'} \upharpoonright \text{lev}_i B \rightarrow a_i$  such that, if  $x \in \text{lev}_i B$  happens to be  $x_\alpha^{q'} \upharpoonright i$  for some  $\alpha \in u$ ,  $f^{q'}(x) = t_\alpha^p(i)$ . This works since the  $x_\alpha^{q'} \upharpoonright i$ ,  $\alpha \in u$ , are distinct (recall  $m \leq i$  and the  $x_\alpha^q$ ,  $\alpha \in u$  are distinct), and the  $t_\alpha^p(i)$ ,  $\alpha \in u$ , are distinct by Proposition 4.2.(c).  $\blacksquare$

## 5 $P^0$ , $P^1$ , and a dense subset of $P^0 * P^1$

We define here the forcings  $P^0$  (in  $V$ ) and  $P^1$  (in  $V^{P^0}$ ), describe the standard part of  $P^0 * P^1$  and find a dense subset of the standard part. The central property of the construction is that, on one hand,  $P^0 * P^1$  is easily seen to be Cohen and, on the other hand,  $P^0$  and  $P^1$  adjoin the same generic objects (a function  $\underline{f}$  and, for each  $\alpha \in \kappa$ , functions  $\underline{t}_\alpha : \omega \rightarrow \omega$  and  $\underline{x}_\alpha : \omega \rightarrow 2$ ) as  $Q^0$  and  $Q^1$ ; this is why  $B(P^0 * P^1)$  is isomorphic to  $B(Q^0 * Q^1)$ .

**Definition 5.1** An element of  $P^0$  is a function  $f$  such that for some  $n \in \omega$  (the height of  $f$ ,  $\text{ht } f$ ):

- (a)  $f$  maps  $\bigcup_{i < n} \text{lev}_i B$  into  $\omega$  and for  $i < n$ ,  $f \upharpoonright \text{lev}_i B$  is a bijection from  $\text{lev}_i B$  onto  $a_i$ .

For  $f$  and  $f'$  in  $P^0$ :

- (b)  $f \leq f'$  iff  $f \subseteq f'$  (and hence  $\text{ht } f \leq \text{ht } f'$ ).

Thus for  $K$   $P^0$ -generic over  $V$ ,  $\underline{f}_K = \bigcup K$  is a map from the tree  $B$  into  $\omega$ , mapping the  $i$ 'th level of  $B$  in a one-one manner onto  $a_i$ . Using the canonical name  $\underline{f}$  for the generic function  $\underline{f}_K$ , we can define the forcing  $P^1$  in  $V^{P^0}$ .

**Definition 5.2** (In  $V^{P^0}$ )  $P^1$  is the finite-support product of the forcings  $P_\alpha^1$ ,  $\alpha \in \kappa$ , defined as follows. An element of  $P_\alpha^1$  is a pair  $q_\alpha = (x_\alpha^{q_\alpha}, t_\alpha^{q_\alpha})$  such that for some  $m_\alpha \in \omega$  (the height of  $q_\alpha$ ,  $\text{ht } q_\alpha$ ):

- (a)  $x_\alpha^{q_\alpha} \in \text{lev}_{m_\alpha} B$  and  $t_\alpha^{q_\alpha} \in \text{lev}_{m_\alpha} T$ .

For  $q_\alpha$  and  $q'_\alpha$  in  $P_\alpha^1$ ,  $q_\alpha \leq q'_\alpha$  iff

- (b)  $\text{ht } q_\alpha \leq \text{ht } q'_\alpha$ ,  $x_\alpha^{q_\alpha} \subseteq x_\alpha^{q'_\alpha}$  and  $t_\alpha^{q_\alpha} \subseteq t_\alpha^{q'_\alpha}$
- (c) for  $i \in [\text{ht } q_\alpha, \text{ht } q'_\alpha)$ ,  $t_\alpha^{q'_\alpha}(i) = \underline{f}(x_\alpha^{q'_\alpha} \upharpoonright i)$ .

**Remark 5.3** The forcing  $P^0$  is Cohen, since it is countable and every element has two incompatible extensions. For the same reason, each  $P_\alpha^1$  is Cohen (in  $V^{P^0}$ ), hence  $P^1$  is Cohen in  $V^{P^0}$ . It follows that  $P^0 * P^1$  is Cohen.

**Proposition 5.4** *The elements of  $\text{stp}(P^0 * P^1)$  are those pairs  $(f, q)$  such that  $f \in P^0$ ,  $q = (x_\alpha^q, t_\alpha^q)_{\alpha \in w}$  where  $w (= \text{dom } q)$  is a finite subset of  $\kappa$ , and there are natural numbers  $n_\alpha$ ,  $\alpha \in w$ , such that, for  $\alpha \in w$ :*

(a)  *$\text{ht } f \geq n_\alpha$ ,  $x_\alpha^q \in \text{lev}_{n_\alpha} B$ , and  $t_\alpha^q \in \text{lev}_{n_\alpha} T$  (we write  $n_\alpha = \text{ht } x_\alpha^q = \text{ht } t_\alpha^q$ ).*

*For  $(f, q)$  and  $(f', q')$  in  $\text{stp}(P^0 * P^1)$ ,  $(f, q) \leq (f', q')$  iff*

(b)  *$\text{ht } f \leq \text{ht } f'$ ,  $\text{dom } q \subseteq \text{dom } q'$ , and, for  $\alpha \in \text{dom } q$ ,  $\text{ht } x_\alpha^q \leq \text{ht } x_\alpha^{q'}$*

(c)  *$f \subseteq f'$*

(d) *for  $\alpha \in \text{dom } q$ ,  $x_\alpha^q \subseteq x_\alpha^{q'}$  and  $t_\alpha^q \subseteq t_\alpha^{q'}$*

(e) *for  $\alpha \in \text{dom } q$  and  $i \in [\text{ht } x_\alpha^q, \text{ht } x_\alpha^{q'})$ ,  $t_\alpha^{q'}(i) = f'(x_\alpha^{q'} \upharpoonright i)$ .*

**Proposition 5.5** *The following subset of  $\text{stp}(P^0 * P^1)$  is dense in  $\text{stp}(P^0 * P^1)$ , hence in  $P^0 * P^1$ .*

$$D_P = \{(f, q) \in \text{stp}(P^0 * P^1) : \text{for all } \alpha \in \text{dom } q, \text{ht } x_\alpha^q (= \text{ht } t_\alpha^q) = \text{ht } f, \\ \text{and the } x_\alpha^q, \alpha \in \text{dom } q, \text{ are pairwise distinct}\}.$$

PROOF. Let  $(f, q) \in \text{stp}(P^0 * P^1)$  be given; say with  $\text{ht } f = n$ ,  $\text{dom } q = w$ , and  $\text{ht } x_\alpha^q = \text{ht } t_\alpha^q = n_\alpha$ , for  $\alpha \in w$ . We will find  $(f', q') \in D_P$  extending  $(f, q)$  such that  $\text{dom } q' = w$  and  $\text{ht } f' = \text{ht } x_\alpha^{q'} = \text{ht } t_\alpha^{q'} = N$ , where  $N$  is sufficiently large.

To this end, put  $m = \max \{n_\alpha : \alpha \in w\}$  and take  $N$  so large that  $m \leq N$  and  $|w| \leq 2^{N-m}$ . Thus we can choose, for  $\alpha \in w$ ,  $x_\alpha^{q'} \in \text{lev}_N B$  such that  $x_\alpha^q \subseteq x_\alpha^{q'}$  and the  $x_\alpha^{q'}$ ,  $\alpha \in w$ , are pairwise distinct. Fix an arbitrary extension  $f'$  of  $f$  in  $P^0$  of height  $N$ . Finally define  $t_\alpha^{q'} \supseteq t_\alpha^q$  for  $\alpha \in w$  by  $t_\alpha^{q'}(i) = f'(x_\alpha^{q'} \upharpoonright i)$ , for  $i \in [n_\alpha, N)$ . ■

## 6 Conclusion

According to the sketch of proof given in the introduction, we are left with showing that the dense subsets  $D_P$  of  $P^0 * P^1$  and  $D_Q$  of  $Q^0 * Q^1$  given in Sections 4 and 5 are isomorphic. This is straightforward, since  $D_P$  is the following partial order (cf. Propositions 5.4, 5.5). An element  $\rho$  of  $D_P$  is, for some finite  $u \subseteq \kappa$  and some  $m \in \omega$ , a sequence  $\rho = (f, (x_\alpha)_{\alpha \in u}, (t_\alpha)_{\alpha \in u})$  where

1.  $f$  maps  $\bigcup_{i < m} \text{lev}_i B$  into  $\omega$ ; for  $i < m$ ,  $f \upharpoonright \text{lev}_i B$  is a bijection from  $\text{lev}_i B$  onto  $a_i$
2. the  $x_\alpha$ ,  $\alpha \in u$ , are pairwise distinct elements of  $\text{lev}_m B$

3. the  $t_\alpha$ ,  $\alpha \in u$ , are elements of  $\text{lev}_m T$ .

And for  $\rho = (f, (x_\alpha)_{\alpha \in u}, (t_\alpha)_{\alpha \in u})$  and  $\sigma = (g, (y_\alpha)_{\alpha \in v}, (s_\alpha)_{\alpha \in v})$  (with domain  $v$  and height  $n$ ) in  $D_P$ ,  $\rho \leq \sigma$  iff the following hold.

4.  $u \subseteq v$  and  $m \leq n$

5.  $f \subseteq g$

6. for  $\alpha \in u$ ,  $x_\alpha \subseteq y_\alpha$  and  $t_\alpha \subseteq s_\alpha$

7. for  $\alpha \in u$  and  $i \in [m, n)$ ,  $s_\alpha(i) = g(y_\alpha \upharpoonright i)$ .

Note that 2. implies that  $|u| \leq a_m$ ; similarly, 2., 1., and 7. imply that the  $s_\alpha$ ,  $\alpha \in u$ , are disjoint above  $m$ . Thus, up to permutation of coordinates,  $D_Q$  is the same partial order (cf. Definition 4.1 and Propositions 4.2, 4.3).

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