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A Combinatorial Principle]A Combinatorial Principle Equivalent to the Existence of Non-free Whitehead Groups

Ams classification: Primary 20K20, 20K35; Secondary 03E05

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Abstract

As a consequence of identifying the principle described in the title, we prove that for any uncountable cardinal λ, if there is a λ-free Whitehead group of cardinality λ which is not free, then there are many "nice" Whitehead groups of cardinality λ which are not free.

1 Introduction

Throughout, "group" will mean abelian group; in particular, "free group" will mean free abelian group.

Two problems which have been shown to be undecidable in ZFC (ordinary set theory) for some uncountable λ are the following:

- Is there a group of cardinality λ which is λ-free (that is, every subgroup of cardinality $< \lambda$ is free), but is not free?
- Is there a Whitehead group G (that is, $\mathrm{Ext}(G, \mathbb{Z}) = 0$) of cardinality λ which is not free?

(See [6] for the first, [7] for the second; also [2] is a general reference for unexplained terminology and further information.)

The second author has shown that the first problem is equivalent to a problem in pure combinatorial set theory (involving the important notion of a λ-system; see Theorem 3.) This not only makes it easier to prove independence results (as in [6]), but also allows one to prove (in ZFC!) group-theoretic results such as:

> ($\bigtriangledown$) if there is a λ-free group of cardinality λ which is not free, then there is a strongly λ-free group of cardinality λ which is not free.

*Thanks to Rutgers University for funding the authors' visits to Rutgers.

†The second author was partially supported by the *Basic Research Foundation* administered by The Israel Academy of Sciences and Humanities. Pub. No. 505. Final version.

(See [10] or [2, Chap. VII].) A group G is said to be *strongly λ-free* if every subset of G of cardinality $< \lambda$ is contained in a free subgroup H of cardinality $< \lambda$ such that G/H is λ-free. One reason for interest in this class of groups is that they are precisely the groups which are equivalent to a free group with respect to the infinitary language $L_{\infty\lambda}$ (see [1]).

There is no known way to prove ($\bigtriangledown$) except to go through the combinatorial equivalent.

As for the second problem, the second author has shown that for $\lambda = \aleph_1$, there is a combinatorial characterization of the problem:

> there is a non-free Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of $\aleph_1$ which satisfies 2-uniformization.

(See the Appendix to this paper; a knowledge of the undefined terminology in this characterization is not needed for the body of this paper.) Again, there are group-theoretic consequences which are provable in ZFC:

> ($\bigtriangledown\bigtriangledown$) if there is a non-free Whitehead group of cardinality $\aleph_1$, then there is a strongly $\aleph_1$-free, non-free Whitehead group of cardinality $\aleph_1$.

(See [2, §XII.3]. It is consistent with ZFC that there are Whitehead groups of cardinality $\aleph_1$ which are not strongly $\aleph_1$-free.)

Our aim in this paper is to generalize ($\bigtriangledown\bigtriangledown$) to cardinals $\lambda > \aleph_1$ by combining the two methods used to prove ($\bigtriangledown$) and ($\bigtriangledown\bigtriangledown$). Since the existence of a non-free Whitehead group G of cardinality $\aleph_1$ implies that for every uncountable cardinal λ there exist non-free Whitehead groups of cardinality λ (e.g. the direct sum of λ copies of G — which is not λ-free), the appropriate hypothesis to consider is:

- there is a λ-free Whitehead group of cardinality λ which is not free.

By the Singular Compactness Theorem (see [8]), λ must be regular. It can be proved consistent that there are uncountable $\lambda > \aleph_1$ such that the hypothesis holds. (See [4] or [11].) In particular, for many λ (for example, $\lambda = \aleph_{n+1}$, $n \in \omega$) it can be proved consistent with ZFC that λ is the smallest cardinality of a non-free Whitehead group; hence there is a λ-free Whitehead group of cardinality λ which is not free. (See [4].)

Our main theorem is then:

Theorem 1 *If there is a λ-free Whitehead group of cardinality λ which is not free, then there are 2^λ different strongly λ-free Whitehead groups of cardinality λ.*

Our proof will proceed in three steps. First, assuming the hypothesis — call it (A) — of the Theorem, we will prove

(B) there is a combinatorial object, consisting of a λ-system with a type of uniformization property.

Second, we will show that the combinatorial property (B) can be improved to a stronger combinatorial property (B+), which includes the "reshuffling property". Finally, we prove that (B+) implies

(A+): the existence of many strongly λ-free Whitehead groups.

Note that we have found, in (B) or (B+), a combinatorial property which is equivalent to the existence of a non-free λ-free Whitehead group of cardinality λ, answering an open problem in [2, p. 453]. Certainly this combinatorial characterization is more complicated than the one for groups of cardinality $\aleph_1$ cited above. This is not unexpected; indeed the criterion for the existence of λ-free groups in Theorem 3 implies that the solution to the open problem is inevitably going to involve the notion of a λ-system. A good reason for asserting the interest of the solution is that it makes possible the proof of Theorem 1.

In an Appendix we provide a simpler proof than the previously published one of the fact that the existence of a non-free Whitehead group of cardinality $\aleph_1$ implies the existence of a ladder system on a stationary subset of $\aleph_1$ which satisfies 2-uniformization.

We thank Michael O'Leary for his careful reading of and comments on this paper.

2 Preliminaries

The following notion, of a λ-set, may be regarded as a generalization of the notion of a stationary set.

Definition 2 *(1) The set of all functions*

$$\eta: n = \{0, \ldots, n-1\} \to \lambda$$

($n \in \omega$) is denoted ${}^{<\omega}\lambda$; the domain of η is denoted $\ell(\eta)$ and called the length *of η; we identify η with the sequence*

$$\langle \eta(0), \eta(1), \ldots, \eta(n-1) \rangle.$$

Define a partial ordering on ${}^{<\omega}\lambda$ by: $\eta_1 \leq \eta_2$ if and only if η_1 is a restriction of η_2. This makes ${}^{<\omega}\lambda$ into a tree. *For any $\eta = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in {}^{<\omega}\lambda$, $\eta \frown \langle \beta \rangle$ denotes the sequence $\langle \alpha_0, \ldots, \alpha_{n-1}, \beta \rangle$. If S is a subtree of ${}^{<\omega}\lambda$, an element η of S is called a* final node *of S if no $\eta \frown \langle \beta \rangle$ belongs to S. Denote the set of final nodes of S by S_f.*

*(2) A λ-*set *is a subtree S of ${}^{<\omega}\lambda$ together with a cardinal λ_η for every $\eta \in S$ such that $\lambda_\emptyset = \lambda$, and:*

(a) for all $\eta \in S$, η is a final node of S if and only if $\lambda_\eta = \aleph_0$;

(b) if $\eta \in S \setminus S_f$, then $\eta \frown \langle\beta\rangle \in S$ implies $\beta \in \lambda_\eta$, $\lambda_{\eta\frown\langle\beta\rangle} < \lambda_\eta$ and $E_\eta = \{\beta < \lambda_\eta : \eta \frown \langle\beta\rangle \in S\}$ is stationary in λ_η.

(3) A λ-system *is a λ-set together with a set B_η for each $\eta \in S$ such that $B_\emptyset = \emptyset$, and for all $\eta \in S \setminus S_f$:*

(a) for all $\beta \in E_\eta$, $\lambda_{\eta\frown\langle\beta\rangle} \leq |B_{\eta\frown\langle\beta\rangle}| < \lambda_\eta$;

(b) $\{B_{\eta\frown\langle\beta\rangle} : \beta \in E_\eta\}$ is a continuous chain of sets, i.e. if $\beta \leq \beta'$ are in E_η, then $B_{\eta\frown\langle\beta\rangle} \subseteq B_{\eta\frown\langle\beta'\rangle}$, and if σ is a limit point of E_η, then $B_{\eta\frown\langle\sigma\rangle} = \cup\{B_{\eta\frown\langle\beta\rangle} : \beta < \sigma, \beta \in E_\eta\}$;

(4) For any λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$, and any $\eta \in S$, let $\bar{B}_\eta = \cup\{B_{\eta\restriction m} : m \leq \ell(\eta)\}$. Say that a family $\mathcal{S} = \{s_\zeta : \zeta \in S_f\}$ of countable sets is based on Λ *if $\mathcal{S}$ is indexed by S_f and for every $\zeta \in S_f$, $s_\zeta \subseteq \bar{B}_\zeta$.*

(5) A family $\mathcal{S} = \{s_i : i \in I\}$ is said to be free *if it has a transversal, that is, a one-one function $T : I \to \cup\mathcal{S}$ such that for all $i \in I$, $T(i) \in s_i$. We say $\mathcal{S}$ is* λ-free *if every subset of $\mathcal{S}$ of cardinality $< \lambda$ has a transversal.*

It can be proved that a family $\mathcal{S}$ which is based on a λ-system is not free. (See [2, Lemma VII.3.6].)

The following theorem now gives combinatorial equivalents to the existence of a λ-free group of cardinality λ which is not free. (See [10] or [2, §VII.3].)

Theorem 3 *For any uncountable cardinal λ, the following are equivalent:*

1. *there is a λ-free group of cardinality λ which is not free;*
2. *there is a family $\mathcal{S}$ of countable sets such that $\mathcal{S}$ has cardinality λ and is λ-free but not free;*
3. *there is a family $\mathcal{S}$ of countable sets such that $\mathcal{S}$ has cardinality λ, is λ-free, and is based on a λ-system.*

Definition 4 *A subtree S of ${}^{<\omega}\lambda$ is said to have* height n *if all the final nodes of S have length n. A λ-set or λ-system is said to have height n if its associated subtree S has height n.*

A λ-set of height 1 is essentially just a stationary subset of λ. Not every λ-set has a height, but the following lemma implies that every λ-set contains one which has a height. It is a generalization, and a consequence, of the fact that if a stationary set E is the union of subsets E_n ($n \in \omega$), then for some n, E_n is stationary (cf. [2, Exer. 2, p. 238]).

Lemma 5 *If $(S, \lambda_\eta : \eta \in S)$ is a λ-set, and $S_f = \bigcup_{n\in\omega} S^n_f$, let $S^n = \{\eta \in S : \eta \leq \tau$ for some $\tau \in S^n_f\}$. Then for some n, $(S^n, \lambda_\eta : \eta \in S^n)$ contains a λ-set.*

If $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$ is a λ-system of height n, and $\mathcal{S} = \{s_\zeta : \zeta \in S_f\}$ is a family of countable sets based on Λ, let $s^k_\zeta = s_\zeta \cap B_{\zeta \restriction k}$ for $1 \leq k \leq n$.

The following is useful in carrying out an induction on λ-systems.

Definition 6 *Given a λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$ and a node η of S, let $S^\eta = \{\nu \in S : \eta \leq \nu\}$. We will denote by Λ^η the λ_η-system which is naturally isomorphic to $(S^\eta, \lambda_\nu, B'_\nu : \nu \in S^\eta)$ where $B'_\eta = \emptyset$ and $B'_\nu = B_\nu$ if $\nu \neq \eta$.*

(That is, we replace the initial node, η, of S^η by $\emptyset$, and translate the other nodes accordingly.)

If $\mathcal{S} = \{s_\zeta : \zeta \in S_f\}$ is a family of countable sets based on Λ *and $\zeta \in S^\eta_f$, let $s^\eta_\zeta = \cup\{s^k_\zeta : k > \ell(\eta)\}$. Let $\mathcal{S}^\eta = \{s^\eta_\zeta : \zeta \in S^\eta_f\}$; it is a family of countable sets based on Λ^η.*

In order to construct a (strongly) λ-free group from a family of countable sets based on a λ-system, we need that the family have an additional property:

Definition 7 *A family $\mathcal{S}$ of countable sets based on a λ-system Λ is said to have the* reshuffling property *if for every $\alpha < \lambda$ and every subset I of S_f such that $|I| < \lambda$, there is a well-ordering $<_I$ of I such that for every τ, $\zeta \in I$, $s_\zeta \setminus \bigcup_{\nu <_I \zeta} s_\nu$ is infinite, and $\tau(0) \leq \alpha < \zeta(0)$ implies that $\tau <_I \zeta$.*

It can be shown (in fact it is part of the proof of the theorem) that the three equivalent conditions in Theorem 3 are equivalent to:

> (iv) *there is a family $\mathcal{S}$ of countable sets such that $\mathcal{S}$ has cardinality λ, is λ-free, is based on a λ-system, and has the reshuffling property.*

Finally, for future reference, we observe the following simple fact:

Lemma 8 *Suppose that for some integers $r \geq 0$, and d^ℓ_m, and some primes q_m ($\ell < r$, $m \in \omega$), H is the abelian group on the generators $\{z_j : j \in \omega\}$ modulo the relations*

$$q_m z_{m+r+1} = z_{m+r} + \sum_{\ell < r} d^\ell_m z_\ell$$

for all $m \in \omega$. Then H is not free.

Conversely, if C is a torsion-free abelian group of rank $r+1$ which is not free but is such that every subgroup of rank $\leq r$ is free, then C contains a subgroup H which is given by generators and relations as above.

PROOF. Let H be as described in the first part. If H is free, then H is finitely generated, since it clearly has rank $\leq r+1$. Let L be the subgroup of H generated by (the images of) $z_0, ..., z_{r-1}$. By comparing coefficients of linear combinations in the free group on $\{z_j : j \in \omega\}$, one can easily verify that L

is a pure subgroup of H, and that H/L is a rank one group which is not free (because $z_r + L$ is non-zero and divisible by $q_0 q_1 \cdot \cdots \cdot q_m$ for all $m \in \omega$) and hence not finitely-generated. But this is impossible if H is free.

Conversely, let C be as stated, and let L be a pure subgroup of rank r. Then L is free (say with basis $z_0, ..., z_{r-1}$) and C/L is a non-free torsion-free group of rank 1. Thus C/L contains a subgroup with a non-zero element $z_r + L$ such that either: $z_r + L$ is divisible by all powers of p for some prime p (in which case we let $q_m = p$ for all m); or $z_r + L$ is divisible by infinitely many primes (in which case we let $\{q_m : m \in \omega\}$ be an infinite set of primes dividing z_r). It is then easy to see that H exists as desired. □

3 (A) implies (B)

Theorem 9 *For any regular uncountable cardinal λ, if*

(A) there is a Whitehead group of cardinality λ which is λ-free but not free, then

(B) there exist integers $n > 0$ and $r \geq 0$, and:

1. *a λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$ of height n;*
2. *one-one functions φ^k_ζ $(\zeta \in S_f,\ 1 \leq k \leq n)$ with $\mathrm{dom}(\varphi^k_\zeta) = \omega$;*
3. *primes $q_{\zeta,m}$ $(\zeta \in S_f,\ m \in \omega)$; and*
4. *integers $d^\ell_{\zeta,m}$ $(\zeta \in S_f,\ m \in \omega,\ \ell < r)$*

such that

(a) if we define $s_\zeta = \bigcup_{k=1}^n \mathrm{rge}(\varphi^k_\zeta)$, then $\mathcal{S} = \{s_\zeta : \zeta \in S_f\}$ is a λ-free family of countable sets based on Λ; in particular, $\mathrm{rge}(\varphi^k_\zeta) \subseteq B_{\zeta \restriction k}$;

and

(b) for any functions $c_\zeta : \omega \to \mathbb{Z}$ $(\zeta \in S_f)$, there is a function $f : \bigcup \mathcal{S} \to \mathbb{Z}$ such that for all $\zeta \in S_f$ there are integers $a_{\zeta,j}$ $(j \in \omega)$ such that for all $m \in \omega$,

$$c_\zeta(m) = q_{\zeta,m} a_{\zeta,m+r+1} - a_{\zeta,m+r} - \sum_{\ell<r} d^\ell_{\zeta,m} a_{\zeta,\ell} - \sum_{k=1}^n f(\varphi^k_\zeta(m)).$$

PROOF. We shall refer to the data in (B), which satisfies (a) and (b), as a *λ-system with data for the Whitehead problem* or more briefly a *Whitehead λ-system.* Given a Whitehead group G of cardinality λ which is λ-free but not free, we begin by defining a λ-system and a family of countable sets based on the

λ-system following the procedure given in [2, VII.3.4]; we review that procedure here.

Choose a λ-*filtration* of G, that is, write G as the union of a continuous chain

$$G = \bigcup_{\alpha<\lambda} B_\alpha$$

of pure subgroups of cardinality $< \lambda$ such that if G/B_α is not λ-free, then $B_{\alpha+1}/B_\alpha$ is not free. Since G is not free,

$$E_\emptyset = \{\alpha < \lambda\colon B_\alpha \text{ is not } \lambda\text{-pure in } G\}$$

is stationary in λ. For each $\alpha \in E_\emptyset$, let λ_α $(< \lambda)$ be minimal such that $B_{\alpha+1}/B_\alpha$ has a subgroup of cardinality λ_α which is not free; λ_α is regular by the Singular Compactness Theorem (see [8] or [2, IV.3.5]). If λ_α is countable, then let $\langle\alpha\rangle$ be a final node of the tree; otherwise choose $G_\alpha \subseteq B_{\alpha+1}$ of cardinality λ_α such that

$$H_\alpha = (G_\alpha + B_\alpha)/B_\alpha$$

is not free. Then H_α is λ_α-free, and we can choose a λ_α-filtration of G_α,

$$G_\alpha = \bigcup_{\beta<\lambda_\alpha} B_{\alpha,\beta}$$

such that for all β, $(B_{\alpha,\beta} + B_\alpha)/B_\alpha$ is pure in H_α and if $(B_{\alpha,\beta} + B_\alpha)/B_\alpha$ is not λ_α-pure in H_α, then

$$(B_{\alpha,\beta+1} + B_\alpha/B_\alpha)/(B_{\alpha,\beta} + B_\alpha/B_\alpha) \cong (B_{\alpha,\beta+1} + B_\alpha)/(B_{\alpha,\beta} + B_\alpha)$$

is not free. Since H_α is not free,

$$E_\alpha = \{\beta < \lambda_\alpha\colon B_{\alpha,\beta} + B_\alpha/B_\alpha \text{ is not } \lambda_\alpha\text{-pure in } H_\alpha\}$$

is stationary in λ_α. For each $\beta \in E_\alpha$, choose $\lambda_{\alpha,\beta}$ $(< \lambda_\alpha)$ minimal such that there is a subgroup $G_{\alpha,\beta}$ of $B_{\alpha,\beta+1}$ of cardinality $\lambda_{\alpha,\beta}$ so that

$$H_{\alpha,\beta} = (G_{\alpha,\beta} + B_{\alpha,\beta} + B_\alpha)/(B_{\alpha,\beta} + B_\alpha)$$

is not free. If $\lambda_{\alpha,\beta}$ is countable, let $\langle\alpha,\beta\rangle$ be a final node; otherwise choose a $\lambda_{\alpha,\beta}$-filtration of $G_{\alpha,\beta}$. Continue in this way along each branch until a final node is reached.

As we have just done, we will use, when convenient, the notation $G_{\alpha,\beta}$ instead of $G_{\langle\alpha,\beta\rangle}$, etc.; thus for example we will write $G_{\eta,\delta}$ instead of $G_{\eta\frown\langle\delta\rangle}$.

In this way we obtain a λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$ where for each $\zeta\in S_f$, there is a countable subgroup G_ζ of G such that

$$G_\zeta + \langle\bar{B}_\zeta\rangle/\langle\bar{B}_\zeta\rangle$$

is not free. We can assume that for each $\eta \in S \setminus S_f$ and each $\delta \in E_\eta$,

$$B_{\eta,\delta+1} + \langle \bar{B}_\eta \rangle = G_{\eta,\delta} + B_{\eta,\delta} + \langle \bar{B}_\eta \rangle.$$

We can also assume that for all $\zeta \in S_f$, G_ζ has been chosen so that $G_\zeta + \langle \bar{B}_\zeta \rangle / \langle \bar{B}_\zeta \rangle$ has finite rank $r_\zeta + 1$ for some r_ζ such that every subgroup of rank $\leq r_\zeta$ is free. By restricting to a sub-λ-set, we can assume that there is an r such that $r_\zeta + 1 = r + 1$ for all $\zeta \in S_f$ and that there is an n such that Λ has height n (cf. Lemma 5). Moreover, we can assume (easing the purity condition, if necessary) that $G_\zeta + \langle \bar{B}_\zeta \rangle / \langle \bar{B}_\zeta \rangle$ is as described in Lemma 8, that is, it is generated modulo $\langle \bar{B}_\zeta \rangle$ by the cosets of elements $z_{\zeta,j}$ which satisfy precisely the relations which are consequences of relations

$$q_{\zeta,m} z_{\zeta,m+r+1} = z_{\zeta,m+r} + \sum_{\ell<r} d^\ell_{\zeta,m} z_{\zeta,\ell}$$

(modulo $\langle \bar{B}_\zeta \rangle$) for some primes $q_{\zeta,m}$ and integers $d^\ell_{\zeta,m}$. Fix $g_{\zeta,m} \in \langle \bar{B}_\zeta \rangle$ such that in G

$$q_{\zeta,m} z_{\zeta,m+r+1} = z_{\zeta,m+r} + \sum_{\ell<r} d^\ell_{\zeta,m} z_{\zeta,\ell} + g_{\zeta,m}.$$

There is a countable subset t_ζ of $\bar{B}_\zeta$ such that $G_\zeta \cap \langle \bar{B}_\zeta \rangle$ is contained in the subgroup generated by t_ζ. Let $s_\zeta = t_\zeta \times \omega$. Then it is proved in [2, VII.3.7] that $\{s_\zeta : \zeta \in S_f\}$ is λ-free and based on the λ-system $(S, \lambda_\eta, B'_\eta : \eta \in S)$ where $B'_\eta = B_\eta \times \omega$.

Let $s^k_\zeta = s_\zeta \cap B'_{\zeta\restriction k}$ and let $\nu^k_\zeta : \omega \to s^k_\zeta$ enumerate s^k_ζ without repetition. We can write each $g_{\zeta,m}$ as a sum $\sum_{k=1}^n g^k_{\zeta,m}$ where $g^k_{\zeta,m} \in B_{\zeta\restriction k}$. Now define

$$\varphi^k_\zeta(m) = \langle \nu^k_\zeta(m), g^k_{\zeta,m} \rangle \in B'_{\zeta\restriction k} \times B_{\zeta\restriction k}.$$

Then $s''_\zeta = \bigcup_{k=1}^n \mathrm{rge}(\varphi^k_\zeta)$ is based on the λ-system $(S, \lambda_\eta, B''_\eta : \eta \in S)$ where $B''_\eta = B'_\eta \times B_\eta$. Moreover $\{s''_\zeta : \zeta \in S_f\}$ is a λ-free family because of the choice of the first coordinate of $\varphi^k_\zeta(m)$. Thus we have defined the data in (B) such that (a) holds. It remains to verify (b). So let $c_\zeta : \omega \to \mathbb{Z}$ $(\zeta \in S_f)$ be given. We are going to define a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow M \xrightarrow{\pi} G \longrightarrow 0$$

and then use a splitting of π to define the function $f : \bigcup \mathcal{S} \to \mathbb{Z}$.

We will use the lexicographical ordering, $<_\ell$, on S defined as follows: $\eta_1 <_\ell \eta_2$ if and only if either η_1 is a restriction of η_2 or $\eta_1(i) < \eta_2(i)$ for the least i such that $\eta_1(i) \neq \eta_2(i)$. Note that if $\eta_1 <_\ell \eta_2$, then $\langle \bar{B}_{\eta_1} \rangle \subseteq \langle \bar{B}_{\eta_2} \rangle$. The lexicographical ordering is a well-ordering of S, so there is an order-preserving bijection $\theta : \tau \to \langle S, <_\ell \rangle$ for some ordinal τ. If for each $\sigma < \tau$ we let $A_\sigma = \langle \bar{B}_{\theta(\sigma)} \rangle$, then $G = \cup_{\sigma<\tau} A_\sigma$ represents G as the union of a chain of subgroups. However, we must exercise caution since, as we will see, this chain is not necessarily continuous.

The kernel of π will be generated by an element $e \in M$. We will define π to be the union of a chain of homomorphisms $\pi_\sigma : M_\sigma \to A_\sigma \to 0$ with kernel $\mathbb{Z}e$. The π_σ will be defined by induction on σ. At the same time, we will also define, as we go along, a chain of set functions $\psi_\sigma : A_\sigma \to M_\sigma$ such that $\pi_\sigma \circ \psi_\sigma = \mathrm{id}_{A_\sigma}$. Let π_0 be the zero homomorphism : $\mathbb{Z}e \to A_0 = \{0\}$.

Suppose that π_ρ and ψ_ρ have been defined for all $\rho < \sigma$ for some $\sigma < \tau$; say $\theta(\sigma) = \eta$ where η=$\langle \nu, \delta \rangle$ for some $\nu \in S$, $\delta \in E_\nu$. Suppose first that σ is a limit ordinal. Let $\pi'_\sigma : M'_\sigma \to \cup_{\rho<\sigma} A_\rho$ be the direct limit of the π_ρ $(\rho < \sigma)$ and let ψ'_σ be the direct limit of the ψ_ρ. In particular, $M'_\sigma = \varinjlim\{M_\rho : \rho < \sigma\}$. If $\cup_{\rho<\sigma} A_\rho = A_\sigma$, then we can let $\pi_\sigma = \pi'_\sigma$ and $\psi_\sigma = \psi'_\sigma$; this will happen, for example, if δ is a limit point of E_ν.

But it may be that δ has an immediate predecessor $\delta_1 \in E_\nu$. (Since σ is a limit ordinal, it follows that η is not a final node of S.) Then

$$\cup_{\rho<\sigma} A_\rho = \cup_{\gamma<\lambda_{\nu,\delta_1}} B_{\nu,\delta_1,\gamma} + B_{\nu,\delta_1} + \langle \bar{B}_\nu \rangle = B_{\nu,\delta_1+1} + \langle \bar{B}_\nu \rangle.$$

Notice that $\cup_{\rho<\sigma} A_\rho$ will be a proper subgroup of A_σ if $\delta_1 + 1 < \delta$ (i.e. if $\delta_1 + 1 \notin E_\nu$). We can extend π'_σ to $\pi_\sigma : M_\sigma \to A_\sigma$ because the inclusion of $\cup_{\rho<\sigma} A_\rho$ into A_σ induces a surjection of $\mathrm{Ext}(A_\sigma, \mathbb{Z})$ onto $\mathrm{Ext}(\cup_{\rho<\sigma} A_\rho, \mathbb{Z})$. Finally, extend ψ'_σ to ψ_σ in any way such that $\pi_\sigma \circ \psi_\sigma$ is the identity on A_σ.

Now let us consider the case when $\sigma = \rho + 1$ is a successor ordinal. Recall that $\theta(\sigma) = \langle \nu, \delta \rangle$. There are two subcases. In the first, δ is the least element of E_ν, so $\theta(\rho) = \nu$ and $A_\rho = \langle \bar{B}_\nu \rangle$, $A_\sigma = B_{\nu,\delta} + \langle \bar{B}_\nu \rangle$. In this subcase, we extend π_ρ to π_σ using the surjectivity of $\mathrm{Ext}(A_\rho, \mathbb{Z}) \to \mathrm{Ext}(A_\sigma, \mathbb{Z})$.

In the second and last subcase, δ has an immediate predecessor δ_1 in E_ν; then $\theta(\rho)$ =$\langle v, \delta_1 \rangle$, a final node of S. Let ζ denote $\langle v, \delta_1 \rangle$; then $B_{\nu,\delta_1+1} + \langle \bar{B}_\zeta \rangle / \langle \bar{B}_\zeta \rangle$ is as described in Lemma 8, that is, it is generated modulo $\langle \bar{B}_\zeta \rangle$ by the cosets of elements $z_{\zeta,j}$ which satisfy the relations

$$q_{\zeta,m} z_{\zeta,m+r+1} = z_{\zeta,m+r} + \sum_{\ell<r} d^\ell_{\zeta,m} z_{\zeta,\ell} + \sum_{k=1}^{n} g^k_{\zeta,m}$$

in G for some primes $q_{\zeta,m}$, integers $d^\ell_{\zeta,m}$ and elements $g^k_{\zeta,m} \in B_{\zeta\restriction k}$. It is at this point that we use the function c_ζ. Define M'_σ to be generated over M_ρ by elements $z'_{\zeta,j}$ modulo the relations

$$q_{\zeta,m} z'_{\zeta,m+r+1} = z'_{\zeta,m+r} + \sum_{\ell<r} d^\ell_{\zeta,m} z'_{\zeta,\ell} + \sum_{k=1}^{n} \psi_\rho(g^k_{\zeta,m}) + c_\zeta(m)e$$

and define

$$\pi'_\sigma : M'_\sigma \to B_{\nu,\delta_1+1} + \langle \bar{B}_\zeta \rangle$$

to be the homomorphism extending π_ρ which takes $z'_{\zeta,j}$ to $z_{\zeta,j}$. One can verify that π'_σ is well-defined and has kernel $\mathbb{Z}e$. Extend ψ_ρ to ψ'_σ in any way such that $\pi'_\sigma \circ \psi'_\sigma$ is the identity. We extend π'_σ to $\pi_\sigma : M_\sigma \to A_\sigma = \langle \bar{B}_{\langle \nu,\delta \rangle} \rangle$ by using the surjectivity of $\mathrm{Ext}(A_\sigma, \mathbb{Z}) \to \mathrm{Ext}(B_{\nu,\delta_1+1} + \langle \bar{B}_\zeta \rangle, \mathbb{Z})$; finally we extend ψ'_σ.

This completes the definition of $\pi : M \to G$ and of the set map $\psi : G \to M$ (= the direct limit of the ψ_σ). Since G is a Whitehead group, there is a homomorphism ρ: $G \to M$ such that $\pi \circ \rho$ is the identity on G. In order to define f, consider an element x of $\cup \mathcal{S}$; x is an ordered pair equal to $\varphi^k_\zeta(m)$ (possibly for many different (ζ, k, m)). If g is the second coordinate of x, let $f(x)$ be the unique integer such that

$$\psi(g) - \rho(g) = f(x)e.$$

Also for any $\zeta \in S_f$ and $j \in \omega$ define $a_{\zeta,j}$ such that

$$z'_{\zeta,j} - \rho(z_{\zeta,j}) = a_{\zeta,j}e.$$

Then applying ρ to the equation (3) and subtracting the result from equation (3), we obtain

$$q_{\zeta,m}(z'_{\zeta,m+r+1} - \rho(z_{\zeta,m+r+1})) = (z'_{\zeta,m+r} - \rho(z_{\zeta,m+r})) + \\ \sum_{\ell<r} d^\ell_{\zeta,m}(z'_{\zeta,\ell} - \rho(z_{\zeta,\ell})) + \sum_{k=1}^{n}(\psi(g^k_{\zeta,m}) - \rho(g^k_{\zeta,m})) + c_\zeta(m)e$$

from which, comparing coefficients in $\mathbb{Z}e$, we get

$$q_{\zeta,m}a_{\zeta,m+r+1} = a_{\zeta,m+r} + \sum_{\ell<r} d^\ell_{\zeta,m} a_{\zeta,\ell} + \sum_{k=1}^{n} f(\varphi^k_\zeta(m)) + c_\zeta(m).$$

□

4 (B) implies (B+)

Now we are going to move from one combinatorial property, (B), to a stronger one, (B+), which will allow us to construct Whitehead groups that are strongly λ-free. Recall that in section 2 we defined the reshuffling property (Definition 7).

Theorem 10 *Suppose that for some regular uncountable cardinal λ, there is a Whitehead λ-system. Then the following also holds:*
(B+) there exist integers $n > 0$ and $r \geq 0$, and:

1. *a λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$ of height n ;*
2. *one-one functions φ^k_ζ $(\zeta \in S_f,\ 1 \leq k \leq n)$ with $\mathrm{dom}(\varphi^k_\zeta) = \omega$;*
3. *primes $q_{\zeta,m}$ $(\zeta \in S_f,\ m \in \omega)$;*
4. *integers $d^\ell_{\zeta,m}$ $(\zeta \in S_f,\ m \in \omega,\ \ell < r)$*

satisfying (a) and (b) as in (B), with the additional properties:

- $\mathcal{S} = \{s_\eta : \eta \in S_f\}$ *has the reshuffling property; and*
- *for all* $\zeta \in S_f$ *and* $k, i \in \omega$, $\mathrm{rge}(\varphi^i_\zeta) \cap \mathrm{rge}(\varphi^k_\zeta) = \emptyset$ *if* $i \neq k$.

PROOF. We shall refer to the data in (B+), with the given properties, as a *strong Whitehead λ-system.*

Suppose that $\Lambda' = (S', \lambda'_\eta, B'_\eta : \eta \in S')$, $\varphi'^k_\zeta, q'_{\zeta,m}$, and $d'^\ell_{\zeta,m}$ is a Whitehead λ-system (as in (B)); in particular, $\mathcal{S}' = \{s'_\zeta : \zeta \in S'_f\}$ is a family of countable sets based on Λ' , where $s'_\zeta = \bigcup_{k=1}^n \mathrm{rge}(\varphi'^k_\zeta)$. In [2, §VII.3A] is contained a proof that if there exists a family $\mathcal{S}'$ of countable sets based on a λ-system Λ' which is λ-free, then there is a family $\mathcal{S}$ of countable sets based on a λ-system Λ which has the reshuffling property. Our task is to examine the proof and show how the transformations carried out in the proof can be done in such a way that the additional data and properties given in (B) — namely the existence of the functions φ^k_ζ, primes $q_{\zeta,m}$, and integers $d^\ell_{\zeta,m}$ satisfying (b) — continue to hold. The transformations in question change the given $\mathcal{S}'$ and Λ' into $\mathcal{S}$ and Λ which are *beautiful,* that is, they satisfy the following six properties:

1. for η, $\nu \in S$, if $B_\eta \cap B_\nu \neq \emptyset$, then there are $\tau \in S$ and α, β so that $\eta = \tau \frown \langle\alpha\rangle$ and $\nu = \tau \frown \langle\beta\rangle$;
2. for ζ, $\nu \in S_f$ and k, $i \in \omega$, if $s^k_\zeta \cap s^i_\nu \neq \emptyset$ then $k = i$, $\ell(\zeta) = n = \ell(\nu)$ for some n and for all $j \neq k - 1$,[1] $\zeta(j) = \nu(j)$;
3. for each k and ζ, s^k_ζ is infinite and has a tree structure; that is, for each ζ there is an enumeration $t^{k\zeta}_0, t^{k\zeta}_1, \ldots$ of s^k_ζ so that for all ν, $\zeta \in S_f$ and $n \in \omega$, if $t^{k\zeta}_{n+1} \in s^k_\nu$, then $t^{k\zeta}_n \in s^k_\nu$;
4. $\mathcal{S}$ is λ-free;
5. for all $\alpha \in E_\emptyset$, $\Lambda^{\langle\alpha\rangle}$ and $\mathcal{S}^{\langle\alpha\rangle}$ are beautiful; and
6. one of the following three possibilities holds:

   (a) every $\gamma \in E_\emptyset$ has cofinality ω and there is an increasing sequence of ordinals $\{\gamma_n : n \in \omega\}$ approaching γ such that for all $\zeta \in S_f$ if $\zeta(0) = \gamma$ then $s^1_\zeta = \{\langle\gamma_n, t_n\rangle : n \in \omega\}$ for some t_n's; moreover, these enumerations of the s^1_ζ satisfy the tree property of (iii);

   (b) there is an uncountable cardinal κ and an integer $m > 0$ so that for all $\gamma \in E_\emptyset$ the cofinality of γ is κ and for all $\zeta \in S_f$, $\lambda_{\zeta\restriction m} = \kappa$; moreover, for each $\gamma \in E_\emptyset$ there is a strictly increasing continuous sequence $\{\gamma_\rho : \rho < \kappa\}$

[1]Note that this corrects an error in [2]. A list of errata for [2] is available from the first author.

cofinal in γ such that for all $\zeta \in S_f$ if $\zeta(0) = \gamma$ then $s^1_\zeta = \{\gamma_{\zeta(m)}\} \times X_\zeta$ for some X_ζ;

(c) each $\gamma \in E_\emptyset$ is a regular cardinal and $\lambda_{\langle\gamma\rangle} = \gamma$; moreover, for every $\zeta \in S_f$, $s^1_\zeta = \{\zeta(1)\} \times X_\zeta$ for some X_ζ.

By [2, Thm. VII.3A.6], if $\mathcal{S}$ and Λ are beautiful, then $\mathcal{S}$ has the reshuffling property. Thus it is enough to show that we can transform $\mathcal{S}'$ and Λ' into a beautiful $\mathcal{S}$ and Λ and at the same time preserve the additional structure of (B).

Let us begin with property (i). We do not change S' (the tree), but for every $\tau \in S' \setminus S_f$ and $\alpha \in E'_\tau$, we replace $B'_{\tau,\alpha}$ with $B'_{\tau,\alpha} \times \{\tau\}$. Define

$$\varphi^k_\zeta(m) = \langle \varphi'^k_\zeta(m), \zeta \restriction k-1 \rangle \in B'_{\zeta\restriction k} \times \{\zeta \restriction k-1\}.$$

The definitions of the rest of the data are unchanged. Then (B)(b) continues to hold since given the c_ζ, define f by $f(\varphi^k_\zeta(m)) = f'(\varphi'^k_\zeta(m))$, where f' is the function associated with the original data (and the same c_ζ). The function f is well-defined because if $\varphi^k_{\zeta_1}(m_1) = \varphi^k_{\zeta_2}(m_2)$, then $\varphi'^k_{\zeta_1}(m_1) = \varphi'^k_{\zeta_2}(m_2)$. Note that property (i) implies that $\mathrm{rge}(\varphi^i_\zeta) \cap \mathrm{rge}(\varphi^k_\zeta) = \emptyset$ if $i \neq k$.

Property (ii) of the definition of beautiful is handled similarly.

To obtain property (iii), we do not change S', but for all $\tau \in S'$ we replace B'_τ with ${}^{<\omega}B'_\tau$, the set of all finite sequences of elements of B'_τ. Enumerate s'^k_ζ as $\{x^k_{\zeta,j} : j \in \omega\}$. If $\varphi'^k_\zeta(m) = x^k_{\zeta,j_m}$, define

$$\varphi^k_\zeta(m) = \langle x^k_{\zeta,i} : i \leq j_m \rangle \in {}^{<\omega}B'_{\zeta\restriction k}.$$

Given c_ζ ($\zeta \in S_f$), define $f(\varphi^k_\zeta(m)) = f'(\varphi'^k_\zeta(m))$, where f' is the function associated with the original data (and the same c_ζ). Again, f is well-defined.

So we can suppose that $\Lambda' = (S', \lambda'_\eta, B'_\eta : \eta \in S')$ and $\mathcal{S}' = \{s'_\zeta : \zeta \in S'_f\}$ satisfy also properties (i), (ii) and (iii). The proof of [2, Thm. VII.3A.5] shows that one can define Λ and $\mathcal{S}$ which are beautiful and such that

> there is a one-one order-preserving map ψ of S into S' such that for all $\eta \in S$, $\lambda_\eta = \lambda'_{\psi(\eta)}$; and for each $\zeta \in S_f$, there is a level-preserving bijection $\theta_\zeta : s_\zeta \to s'_{\psi(\zeta)}$ such that for all $\zeta, \nu \in S_f$, if $x \in s'_{\psi(\zeta)}$, $y \in s'_{\psi(\nu)}$ and $x \neq y$, then $\theta^{-1}_\zeta(x) \neq \theta^{-1}_\nu(y)$.[2]

Observe that $\eta \in S_f$ if and only if $\psi(\eta) \in S'_f$ since $\lambda_\eta = \lambda'_{\psi(\eta)}$. We use the functions ψ and θ_ζ to define the additional data in (B): let $q_{\zeta,m} = q'_{\psi(\zeta),m}$ and $d^\ell_{\zeta,m} = d'^\ell_{\psi(\zeta),m}$; moreover, define $\varphi^k_\zeta(m) = \theta^{-1}_\zeta(\varphi'^k_{\psi(\zeta)}(m))$. Given c_ζ for $\zeta \in S_f$, define $c'_{\psi(\zeta)} = c_\zeta$ and let c_ν be arbitrary for $\nu \in S'_f \setminus \psi[S]$. Then since

[2]Note that this is a clarification and correction of the first paragraph of the proof of [2, Thm. VII.3A.5, p. 213]. Also, in the third paragraph of that proof, ψ should be ψ^{-1}.

the original data satisfy (B), $f' : \cup\mathcal{S}' \to \mathbb{Z}$ and $a'_{\nu,j}$ ($\nu \in S'_f$, $j \in \omega$) exist. Let $f(\varphi^k_\zeta(m)) = f'(\varphi'^k_{\psi(\zeta)}(m))$; the (contrapositive of the) final hypothesis on θ_ζ implies that f is well-defined. Let $a_{\zeta,m} = a'_{\psi(\zeta),m}$. Then for each $\zeta \in S_f$, the equation

$$q'_{\psi(\zeta),m}a'_{\psi(\zeta),m+r+1} = a'_{\psi(\zeta),m+r} + \sum_{\ell<r} d'^{\ell}_{\psi(\zeta),m}a'_{\psi(\zeta),\ell} + \sum_{k=1}^{n} f'(\varphi'^k_{\psi(\zeta)}(m)) + c'_{\psi(\zeta)}(m)$$

is the desired equation

$$q_{\zeta,m}a_{\zeta,m+r+1} = a_{\zeta,m+r} + \sum_{\ell<r} d^{\ell}_{\zeta,m}a_{\zeta,\ell} + \sum_{k=1}^{n} f(\varphi^k_\zeta(m)) + c_\zeta(m).$$

□

5 (B+) IMPLIES (A+)

Theorem 11 *Let λ be a regular uncountable cardinal such that (B+) holds, i.e., there is a strong Whitehead λ-system. Then*
(A+) there are 2^λ strongly λ-free Whitehead groups of cardinality λ.

PROOF. Given a strong Whitehead λ-system $(S, \lambda_\eta, B_\eta : \eta \in S)$ together with $\varphi^k_\zeta, q_{\zeta,m}, d^\ell_{\zeta,m}$, we use them to define a group G in terms of generators and relations. Our group G will be the group F/K where F is the free abelian group with basis

$$\bigcup \mathcal{S} \cup \{z_{\zeta,j} : \zeta \in S_f,\, j \in \omega\}$$

and K is the subgroup of F generated by the elements $w_{\zeta,m} =$

$$q_{\zeta,m}z_{\zeta,m+r+1} - z_{\zeta,m+r} - \sum_{\ell<r} d^\ell_{\zeta,m}z_{\zeta,\ell} - \sum_{k=1}^{n} \varphi^k_\zeta(m)$$

for all $m \in \omega$, and $\zeta \in S_f$. Let us see first that G is a Whitehead group. (For this we need only (B).) It suffices to show that every group homomorphism $\psi : K \longrightarrow \mathbb{Z}$ extends to a homomorphism from F to $\mathbb{Z}$. (See, for example, [2, p.8].) Given ψ, define $c_\zeta(m) = \psi(w_{\zeta,m})$ for all $m \in \omega$, and $\zeta \in S_f$. Then by (B)(b), there are integers $a_{\zeta,j}$ ($\zeta \in S_f$, $j \in \omega$) and a function $f : \bigcup \mathcal{S} \to \mathbb{Z}$ such that for all $\zeta \in S_f$ and $m \in \omega$,

$$c_\zeta(m) = q_{\zeta,m}a_{\zeta,m+r+1} - a_{\zeta,m+r} - \sum_{\ell\ <r} d^\ell_{\zeta,m}a_{\zeta,\ell} - \sum_{k=1}^{n} f(\varphi^k_\zeta(m)).$$

Define $\theta : F \longrightarrow \mathbb{Z}$ by setting $\theta \restriction \bigcup \mathcal{S} = f$ and $\theta(z_{\zeta,j}) = a_{\zeta,j}$. We just need to check that θ extends ψ. But for all $\zeta \in S_f$ and $m \in \omega$, we have

$$\theta(w_{\zeta,m}) = q_{\zeta,m} a_{\zeta,m+r+1} - a_{\zeta,m+r} - \sum_{\ell<r} d^{\ell}_{\zeta,m} a_{\zeta,\ell} - \sum_{k=1}^{n} f(\varphi^k_\zeta(m))$$

by the definitions of θ and of $w_{\zeta,m}$. Thus $\theta(w_{\zeta,m}) = c_\zeta(m) = \psi(w_{\zeta,m})$, by (5).

Next let us show that G is not free. (Here again, we need only (B).) The proof is essentially the same as that of Lemma VII.3.9 of [2, pp. 205f], but we will give a somewhat different version of the proof here. The proof proceeds by induction on n where n is the height of our λ-system. For each $\alpha < \lambda$, let G_α be the subgroup of G generated by

$$\{z_{\zeta,j} : \zeta \in S_f, \zeta(0) < \alpha,\ j \in \omega\} \cup \bigcup \{s_\zeta : \zeta \in S_f, \zeta(0) < \alpha\}.$$

It suffices to prove that for all α in a stationary subset of λ, $G_{\alpha+1}/G_\alpha$ is not free (cf. [2, IV.1.7]). In fact, we will show that $G_{\alpha+1}/G_\alpha$ is not free when α is a limit point of $E_\emptyset$ and belongs to $C \cap E_\emptyset$, where C is the cub

> $\{\alpha < \lambda :$ whenever $\varphi^1_\zeta(m) \in \bigcup\{B_{\langle\beta\rangle} : \beta < \alpha\}$ then $\exists \sigma \in S_f$ with $\sigma(0) < \alpha$ and $\varphi^1_\zeta(m) \in \mathrm{rge}(\varphi^1_\sigma)\}$.

We begin with the case $n = 1$. Then for all $\alpha \in C \cap E_\emptyset$ such that α is a limit point of $E_\emptyset$, $G_{\alpha+1}/G_a$ is non-free because it is as described in the first part of Lemma 8 (with generators $\{z_{\langle\alpha\rangle,j} : j \in \omega\}$), since for all $m \in \omega$, $\varphi^1_{\langle\alpha\rangle}(m) \in B_{\langle\alpha\rangle} = \bigcup\{B_{\langle\beta\rangle} : \beta < \alpha\}$ by the definition of a λ-system (because α is a limit point of $E_\emptyset$) and hence $\varphi^1_{\langle\alpha\rangle}(m) \in G_\alpha$ since $\alpha \in C$.

Now suppose $n > 1$ and the result is proved for $n-1$. Again, let $\alpha \in C \cap E_\emptyset$ such that α is a limit point of $E_\emptyset$. Again we have that $\varphi^1_\zeta(m) \in G_\alpha$ for all $m \in \omega$ when $\zeta(0) = \alpha$. We will consider the $\lambda_{\langle\alpha\rangle}$-system $\Lambda^{\langle\alpha\rangle}$. (See Definition 6.) Note that $\Lambda^{\langle\alpha\rangle}$ has height $n-1$, and the group $G_{\alpha+1}/G_\alpha$ is defined as in (5) and (5) relative to this $\lambda_{\langle\alpha\rangle}$-system. Hence by induction $G_{\alpha+1}/G_\alpha$ is not free.

Finally, we will use the reshuffling property given by (B+) to prove that G is strongly λ-free. As in the proof of [2, VII.3.11], we will prove that for all $\alpha \in \lambda \cup \{-1\}$ and all $\beta > \alpha$, $G_\beta/G_{\alpha+1}$ is free. Let $I = \{\zeta \in S_f : \zeta(0) < \beta\}$, and let $<_I$ be the well-ordering given by the reshuffling property for I and α. Let s^k_ζ denote $\mathrm{rge}(\varphi^k_\zeta)$. We claim that there is a basis $\mathcal{Z}_{\beta,\alpha}$ of $G_\beta/G_{\alpha+1}$ consisting of the cosets of the members of the following two sets:

$$\{z_{\zeta,j} : \alpha < \zeta(0) < \beta, \text{ and either } j < r \text{ or } \exists k \text{ s.t. } \varphi^k_\zeta(j-r) \notin \bigcup\{s^k_\nu : \nu <_I \zeta\}\}$$

and

$$\{\varphi^k_\zeta(m) : \varphi^k_\zeta(m) \notin \bigcup\{s^k_\nu : \nu <_I \zeta\} \text{ and } \exists i < k[\varphi^i_\zeta(m) \notin \bigcup\{s^i_\nu : \nu <_I \zeta\}]\}.$$

To see that the elements of $\mathcal{Z}_{\beta,\alpha}$ generate $G_\beta/G_{\alpha+1}$, we proceed by induction with respect to $<_I$ to show that the coset of every $z_{\zeta,j}$ ($\zeta(0) < \beta, j \in \omega$) and the coset of every element of s_ζ ($\zeta(0) < \beta$) is a linear combination of the elements of $\mathcal{Z}_{\beta,\alpha}$. Since $s_\zeta \setminus \bigcup\{s_\nu : \nu <_I \zeta\}$ is infinite, for each $j \in \omega$ such that $z_{\zeta,j} + G_{a+1} \notin \mathcal{Z}_{\beta,\alpha}$, there is $t > j$ such that $z_{\zeta,t} + G_{\alpha+1}$ belongs to $\mathcal{Z}_{\beta,\alpha}$. Without loss of generality, $t = j + 1$. Then

$$z_{\zeta,j} = q_{\zeta,j-r} z_{\zeta,t} - \sum_{\ell<r} d^\ell_{\zeta,j-r} z_{\zeta,\ell} - \sum_{k=1}^{n} \varphi^k_\zeta(j-r)$$

by (5). By induction each $\varphi^k_\zeta(j-r) + G_{a+1}$ is a linear combination of members of $\mathcal{Z}_{\beta,\alpha}$ (because $\varphi^k_\zeta(j-r) \in \bigcup\{s^k_\nu : \nu <_I \zeta\}$ since $z_{\zeta,j} + G_{a+1} \notin \mathcal{Z}_{\beta,\alpha}$); hence $z_{\zeta,j} + G_{a+1}$ belongs to the subgroup generated by the members of $\mathcal{Z}_{\beta,\alpha}$.

For each $m, i \in \omega$, if $\varphi^i_\zeta(m) \in \bigcup\{s^k_\nu : \nu <_I \zeta\}$, then by induction $\varphi^i_\zeta(m) + G_{\alpha+1}$ is a linear combination of elements of $\mathcal{Z}_{\beta,\alpha}$. Otherwise, $\varphi^i_\zeta(m) + G_{\alpha+1}$ belongs to $\mathcal{Z}_{\beta,\alpha}$ unless i is minimal such that $\varphi^i_\zeta(m) \notin \bigcup\{s^k_\nu : \nu <_I \zeta\}$. But in the latter case,

$$\sum_{k=1}^{n} \varphi^k_\zeta(m) = q_{\zeta,m} z_{\zeta,m+r+1} - z_{\zeta,m+r} - \sum_{\ell<r} d^\ell_{\zeta,m} z_{\zeta,\ell}$$

so its coset is a linear combination of elements of $\mathcal{Z}_{\beta,\alpha}$; thus since $\varphi^k_\zeta(m) + G_{\alpha+1} \in \mathcal{Z}_{\beta,\alpha}$ for all $k \neq i$, $\varphi^i_\zeta(m) + G_{\alpha+1}$ belongs to the subgroup generated by the elements of $\mathcal{Z}_{\beta,\alpha}$. This completes the proof that $\mathcal{Z}_{\beta,\alpha}$ is a generating set. To see that the elements of $\mathcal{Z}_{\beta,\alpha}$ are independent, compare coefficients in F.

To construct not just one but 2^λ different strongly λ-free Whitehead groups, we use a standard trick: write $E_\emptyset$ as the disjoint union $\coprod_{\sigma<\lambda} X_\sigma$ of λ stationary sets; then for every non-empty subset W of λ, do the construction above for the generalized λ-system $\Lambda = (S_W, \lambda_\zeta, B_\zeta : \zeta \in S_W)$ with $E_\emptyset = \coprod_{\sigma\in W} X_\sigma$, i.e., where $S_W = \{\zeta \in S : \zeta(0) \in \coprod_{\sigma\in W} X_\sigma\}$. $\Box$

6 Appendix: Non-free Whitehead implies 2-uniformization

A *ladder system* on a stationary subset E of ω_1 is an indexed family of functions $\{\eta_\alpha : \alpha \in E\}$ such that each $\eta_\alpha : \omega \to \alpha$ is strictly increasing and $\sup(\mathrm{rge}(\eta_\alpha)) = \alpha$. If $\{\varphi_\alpha : \alpha \in I\}$ is an indexed family of functions each with domain ω, we say that it has the 2-*uniformization property* provided that for every family of functions $c_\alpha : \omega \to 2 = \{0,1\}$ ($\alpha \in I$), there exists a function H such that for all $\alpha \in I$, $H(\varphi_\alpha(n))$ is defined and equals $c_\alpha(n)$ for all but finitely many n. It is not hard, given a ladder system on E which has the 2-uniformization property, to construct, explicitly (by generators and relations), a non-free Whitehead group. (See [2, Prop. XII.3.6].) It is more difficult to go the other way: starting with an arbitrary non-free Whitehead group of cardinality $\aleph_1$ to show that there exists a ladder system on a stationary subset of ω_1 which has the 2-uniformization property. This was left to the reader in the original paper by the second author [9, Thm. 3.9, p. 277]. The only published proof is a rather complicated one in [2, §XII.3]; so considering the importance of this result, it seems to us worthwhile to give another proof which is conceptually and technically simpler than that one. The proof given here resembles the original proof found by the second author, which was also the basis of the proofs in [3] and in this paper.

Our goal is to prove the following.

Theorem 12 *If there is a non-free Whitehead group A of cardinality $\aleph_1$, then there is a ladder system $\{\eta_\alpha : \alpha \in E\}$ on a stationary set E which has the 2-uniformization property.*

We begin with an observation. It is sufficient to show that the hypothesis of Theorem 12 implies that there is a family $\{\varphi_\alpha : \alpha \in E\}$ of functions which has the 2-uniformization property and is *based on an ω_1-filtration*, that is, indexed by a stationary subset E of ω_1 and such that there is a continuous ascending chain $\{B_\nu : \nu \in \omega\}$ of countable sets such that for all $\alpha \in E$, $\varphi_\alpha : \omega \to B_\alpha$. (Note that what we are talking about, in the language of the preceding sections, is a family of countable sets based on an $\aleph_1$-system.) Indeed, by a suitable coding we can assume that $B_\alpha = \alpha$ and if the range of φ_α is not cofinal in α, we can choose a ladder η'_α on α, replace $\varphi_\alpha(n)$ by $\langle\varphi_\alpha(n), \eta'_\alpha(n)\rangle$, and re-code, to obtain a ladder system on $E \cap C$, (for some cub C) which has the 2-uniformization property.

Fromnow on, let A denote a non-free Whitehead group of cardinality $\aleph_1$. Then we can write A as the union, $A = \cup_{\nu<\omega_1} A_\nu$, of a continuous chain of countable free subgroups; since A is not free, we can assume that there is a stationary subset E of ω_1(consisting of limit ordinals) such that for all $\alpha \in E$ $A_{\alpha+1}/A_\alpha$ is not free. By Pontryagin's Criterion we can assume without loss of generality that $A_{\alpha+1}/A_\alpha$ is of finite rank and, in fact, that every subgroup of $A_{\alpha+1}/A_\alpha$ of smaller rank is free. Since

($\star$) *whenever $E = \cup_{n\in\omega} E_n$, at least one of the E_n is stationary*

(cf. [2, Cor. II.4.5]) we can also assume that all of the $A_{\alpha+1}/A_\alpha$ (for $\alpha \in E$) have the same rank $r+1$ ($r \geq 0$). In order to make clear the ideas involved in the proof of the Theorem, we will give the proof first in the special case when $r = 0$, i.e., $A_{\alpha+1}/A_\alpha$ is a rank one non-free group when $\alpha \in E$, and then describe how to handle the extension to the general case. In fact this special case divides into two subcases: using ($\star$) and replacing $A_{\alpha+1}$ by a subgroup if necessary, we can assume that either

1. for all $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ has a type all of whose entries are 0's or 1's [and there are infinitely many 1's]; or

2. there is a prime p such that for all $\alpha \in E$, the type of $A_{\alpha+1}/A_\alpha$ is $(0,0,...0,\infty,0,...)$ where the ∞ occurs in the pth place.

(See [5, pp. 107ff].) We next give the easy combinatorial lemmas needed for the first, and simplest, subcase.

Lemma 13 *Suppose Y and Y' are finite subsets of an abelian group G such that $|Y|^2 < |Y'|$. Then there exists $b \in Y'$ such that Y and $b+Y$ are disjoint. [Here $b+Y = \{b+y : y \in Y\}$.]*

PROOF. Choose $b \in Y' \setminus \{x-y : x,y \in Y\}$. □

Lemma 14 *For any positive integer $p > 1$ there are integers a_0 and a_1 and a function $F_p : \mathbb{Z}/p\mathbb{Z} \to 2 = \{0,1\}$ such that for all $m \in \mathbb{Z}$ with $(2|m|+1)^2 < p$, $F_p(m + a_\ell + p\mathbb{Z}) = \ell$, for $\ell = 0,1$.*

PROOF. Let $a_0 = 0$ and let $a_1 = b$ as in Lemma 13, where $G = \mathbb{Z}/p\mathbb{Z} = Y'$ and $Y = \{m + p\mathbb{Z} : (2|m|+1)^2 < p\}$. Then since $Y = a_0 + Y$ and $a_1 + Y$ are disjoint, we can define F_p. (Note that F_p is a set function, not a homomorphism.) □

PROOF OF THEOREM 12 (*in special subcase* (i)): For all $\alpha \in E$ there is an infinite set P_α of primes such that

$$A_{\alpha+1}/A_\alpha \cong \{\frac{m}{n} \in \mathbb{Q} : n \text{ is a product of } \textit{distinct} \text{ primes from } P_\alpha\}.$$

Then if $P_\alpha = \{p_{\alpha,n} : n \in \omega\}$, $A_{\alpha+1}$ is generated over A_α by a subset $\{y_{\alpha,n} : n \in \omega\}$ satisfying the relations (and only the relations)

$$(\dagger)\ \ p_{\alpha,n} y_{\alpha,n+1} = y_{\alpha,0} - g_{\alpha,n}$$

for some $g_{\alpha,n} \in A_\alpha$. We define $\varphi_\alpha(n) = \langle p_{\alpha,n}, g_{\alpha,n}\rangle$. Then $\{\varphi_\alpha : \alpha \in E\}$ is based on an ω_1-filtration, in fact on the chain $\{\mathbb{Z} \times A_\alpha : \alpha < \omega_1\}$.

Given functions $c_\alpha : \omega \to 2$, we are going to define a homomorphism $\pi : A' \to A$ with kernel $\mathbb{Z}e$ and then use the splitting $\rho : A \to A'$ to define the uniformizing function H.

We define $\pi_\nu : A'_\nu \to A_\nu$ inductively along with a set function ψ_ν: $A_\nu \to A'_\nu$ such that $\pi_\nu \circ \psi_\nu = 1_{A_\nu}$. The crucial case is when π_α and ψ_α have been defined and $\alpha \in E$. (When $\alpha \notin E$ we can use the fact that $\mathrm{Ext}(A_{\alpha+1}, \mathbb{Z}) \to \mathrm{Ext}(A_\alpha, \mathbb{Z})$ is onto.) We define $A'_{\alpha+1}$ by generators $\{y'_{\alpha,n} : n \in \omega\}$ over A'_α satisfying relations

$$(\dagger\dagger)\ \ p_{\alpha,n} y'_{\alpha,n+1} = y'_{\alpha,0} - \psi_\alpha(g_{\alpha,n}) + a_\ell e$$

where a_ℓ is as in Lemma 14 for $p = p_{\alpha,n}$ and $\ell = c_\alpha(n)$.

In the end we let $\pi = \cup_\nu \pi_\nu : A' = \cup_\nu A'_\nu \to A$ and $\psi = \cup_\nu \psi_\nu$. Then since A is a Whitehead group, there exists a homomorphism ρ such that $\pi \circ \rho = 1_A$. For any $g \in A$, $\psi(g) - \rho(g) \in \ker(\pi) = \mathbb{Z}e$; we will abuse notation and identify $\psi(g) - \rho(g)$ with the unique integer k such that $\psi(g) - \rho(g) = ke$. For any $w \in \cup_{\alpha \in E} \mathrm{rge}(\varphi_\alpha)$, if $w = \langle p, g \rangle$, let $H(w) = F_p(\psi(g) - \rho(g) + p\mathbb{Z})$.

Note that w may equal $\varphi_\alpha(n)$ $(= \langle p_{\alpha,n}, g_{\alpha,n} \rangle)$ for many pairs (α, n). To see that this definition of H works, fix $\alpha \in E$. For any $n \in \omega$, applying ρ to equation (†) and subtracting from equation (††), we have

$$p_{\alpha,n}(y'_{\alpha,n+1} - \rho(y_{\alpha,n+1})) = y'_{\alpha,0} - \rho(y_{\alpha,0}) - (\psi_\alpha(g_{\alpha,n}) - \rho(g_{\alpha,n})) + a_\ell$$

so that $\psi_\alpha(g_{\alpha,n}) - \rho(g_{\alpha,n})$ is congruent to $y'_{\alpha,0} - \rho(y_{\alpha,0}) + a_\ell$ mod $p_{\alpha,n}$. Then if n is large enough, $(2|\, y'_{\alpha,0} - \rho(y_{\alpha,0})| + 1)^2 < p_{\alpha,n}$ so by choice of $F_{p_{\alpha,n}}$ and a_ℓ,

$$H(\varphi_\alpha(n)) = F_{p_{\alpha,n}}(\psi_\alpha(g_{\alpha,n}) - \rho(g_{\alpha,n}) + p_{\alpha,n}\mathbb{Z}) =$$
$$F_{p_{\alpha,n}}(y'_{\alpha,0} - \rho(y_{\alpha,0}) + a_\ell + p_{\alpha,n}\mathbb{Z}) = \ell = c_\alpha(n).$$

This completes the proof in the first special subcase.

For the purposes of the second special subcase we need another combinatorial lemma.

Lemma 15 *Fix a positive integer $p > 1$. Define a strictly increasing sequence of positive integers t_i inductively, as follows. Let $t_0 = 0$. If t_{i-1} has been defined for some $i \geq 1$, let $t_i = t_{i-1} + d_i$ where d_i is the least positive integer such that $(2p^{t_{i-1}} + 1)^2 p^{2t_{i-1}} < p^{d_i}$. Then for every $i \geq 1$ there exists a function*

$$F_i : \mathbb{Z}/p^{t_i}\mathbb{Z} \to 2$$

and integers $a_n^\ell \in \{0, ..., p-1\}$ $(t_{i-1} \leq n < t_i, \ell = 0, 1)$ such that whenever $|m_0| \leq p^{t_{i-1}}$ and $a_j \in \{0, ..., p-1\}$ for $j < t_{i-1}$, then for $\ell = 0, 1$

$$F_i(m_0 + \sum_{j<t_{i-1}} p^j a_j + \sum_{n=t_{i-1}}^{t_i - 1} p^n a_n^\ell + p^{t_i}\mathbb{Z}) = \ell.$$

PROOF. We apply Lemma 13 to the sets $Y = \{m_0 + \sum_{j<t_{i-1}} p^j a_j + p^{t_i} Z : |m_0| \leq p^{t_{i-1}}, a_j \in \{0, ..., p-1\}\}$ (which has cardinality $\leq (2p^{t_{i-1}} + 1)p^{t_{i-1}}$) and $Y' = \{\sum_{n=t_{i-1}}^{t_i - 1} p^n x_n + p^{t_i} Z : x_n \in \{0, ..., p-1\}\}$ (which has cardinality p^{d_i}), to get $b \in Y'$. Then choose $a_n^0 = 0$ for all n, and a_n^1 so that $\sum_{n=t_{i-1}}^{t_i - 1} p^n a_n^1 = b$ and define F_i as in Lemma 14. □

PROOF OF THEOREM 12 (*in special subcase* (ii)): For all $\alpha \in E$

$$A_{\alpha+1}/A_\alpha \cong \{\frac{m}{n} \in \mathbb{Q} : n \text{ is a power of } p\}$$

Then $A_{\alpha+1}$ is generated over A_α by a subset $\{y_{\alpha,n} : n \in \omega\}$ satisfying the relations (and only the relations)

$$(\dagger) \quad py_{\alpha,n+1} = y_{\alpha,n} - g_{\alpha,n}$$

for some $g_{\alpha,n} \in A_\alpha$. Let $\varphi_\alpha(m) = \langle g_{\alpha,j} : j < t_{m+1}\rangle$ for all $m \in \omega$. Given functions $c_\alpha : \omega \to 2$, we define $\pi_\nu : A'_\nu \to A_\nu$ with kernel $\mathbb{Z}e$ inductively along with a set function ψ_ν: $A_\nu \to A'_\nu$ such that $\pi_\nu \circ \psi_\nu = 1_{A_\nu}$. The crucial case is when π_α and ψ_α have been defined and $\alpha \in E$. Then we define $A'_{\alpha+1}$ by generators $\{y'_{\alpha,n} : n \in \omega\}$ over A'_α satisfying relations

$$(\dagger\dagger) \quad py'_{\alpha,n+1} = y'_{\alpha,n} - \psi_\alpha(g_{\alpha,n}) + a_n^{\ell(n)}e$$

where $\ell(n)$ is taken to be $c_\alpha(i-1)$ when $t_{i-1} \leq n < t_i$. In the end we let $\pi = \cup_\nu \pi_\nu : A' = \cup_\nu A'_\nu \to A$ and $\psi = \cup_\nu \psi_\nu$. Then since A is a Whitehead group, there exists a homomorphism ρ such that $\pi \circ \rho = 1_A$. For any $w \in \cup_{\alpha\in E}\mathrm{rge}(\varphi_\alpha)$, if $w = \langle g_j : j < t_i\rangle$, let $H(w) = F_i(\sum_{n<t_i} p^n(\psi(g_n) - \rho(g_n)) + p^{t_i}\mathbb{Z})$. To see that this works, fix $\alpha \in E$ and for $i \geq 1$, consider $w_i = \varphi_\alpha(i-1) = \langle g_{\alpha,j} : j < t_i\rangle$. From the equations $(\dagger)$, for $n \leq t_i$ we obtain that

$$p^{t_i} y_{\alpha,t_i} = y_{\alpha,0} - \sum_{n<t_i} p^n g_{\alpha,n}$$

If we apply ρ to this and subtract from the corresponding equation derived from $(\dagger\dagger)$ we obtain that $\sum_{n<t_i} p^n(\psi(g_{\alpha,n}) - \rho(g_{\alpha,n}))$ is congruent to

$$(y'_{\alpha,0} - \rho(y_{\alpha,0})) + \sum_{n<t_i} p^n a_n^{\ell(n)}$$

mod p^{t_i}. So if $|y'_{\alpha,0} - \rho(y_{\alpha,0})| \leq p^{t_{i-1}}$, then by our choice of F_i and the $a_n^{\ell(n)}$ for $t_{i-1} \leq n < t_i$, $H(w_i)$ equals $c_\alpha(i-1)$.

This completes the proof of Theorem 12 when $r = 0$.

PROOF OF THEOREM 12 (*in the general case*): In the general case without loss of generality we have either

1. for all $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ has a free subgroup L_α/A_α of rank r such that $A_{\alpha+1}/L_\alpha$ has a type all of whose entries are 0's or 1's [and there are infinitely many 1's]; or
2. there is a prime p such that for all $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ has a free subgroup L_α/A_α of rank r such that the type of $A_{\alpha+1}/L_\alpha$ is $(0, 0, ...0, \infty, 0, ...)$ where the ∞ occurs in the pth place.

In other words, $A_{\alpha+1}$ is generated by A_α and a subset $\{z_{\alpha,k} : k = 1, ..., r\} \cup \{y_{\alpha,n} : n \in \omega\}$ modulo (only) the relations in A_α plus relations:

1. (†) $p_{\alpha,n} y_{\alpha,n+1} = y_{\alpha,0} + \sum_{k=1}^{r} \mu_{\alpha,k}(n) z_{\alpha,k} - g_{\alpha,n}$ for some family of distinct primes $p_{\alpha,n}$ and $\mu_{\alpha,k}(n) \in \mathbb{Z}$, $g_{\alpha,n} \in A_\alpha$; or
2. (†) $p y_{\alpha,n+1} = y_{\alpha,n} + \sum_{k=1}^{r} \mu_{\alpha,k}(n) z_{\alpha,k} - g_{\alpha,n}$ for some $\mu_{\alpha,k}(n) \in \mathbb{Z}$ and $g_{\alpha,n} \in A_\alpha$ for each $n \in \omega$.

For use in (the harder) subcase (ii), define a strictly increasing sequence of positive integers t_i inductively, as follows. Let $t_0 = 0$. If t_{i-1} has been defined for some $i \geq 1$, let $t_i = t_{i-1} + d_i$ where d_i is the least positive integer such that

$$(2p^{t_{i-1}} + 1)^{2r+2} p^{2t_{i-1}} < p^{d_i}.$$

Then we have the following generalization of Lemma 15. (Note that when $r = 0$ the sequence μ is empty.)

Lemma 16 *Fix $p > 1$ and $r \geq 0$. For every sequence of functions $\mu = \langle \mu_1, ..., \mu_r \rangle$, where $\mu_k : \omega \to \mathbb{Z}$ and every $i \geq 1$ there exists a function*

$$F_{i,\mu} : \mathbb{Z}/p^{t_i}\mathbb{Z} \to 2$$

and integers $a^\ell_{n,\mu} \in \{0, ..., p-1\}$ ($t_{i-1} \leq n < t_i, \ell = 0, 1$) such that $F_{i,\mu}$ and $a^\ell_{n,\mu}$ depend only on $\mu \restriction t_i$ ($= \langle \mu_1 \restriction t_i, ..., \mu_r \restriction t_i \rangle$) and are such that whenever $m_0, ..., m_r$ are integers with $|m_k| \leq p^{t_{i-1}}$ for all $k \leq r$ and $a_j \in \{0, ..., p-1\}$ for $j < t_{i-1}$, then

$$F_{i,\mu}(m_0 + \sum_{k=1}^{r} (\sum_{j<t_i} p^j \mu_k(j)) m_k + \sum_{j<t_{i-1}} p^j a_j + \sum_{n=t_{i-1}}^{t_i - 1} p^n a^\ell_{n,\mu} + p^{t_i}\mathbb{Z}) = \ell.$$

PROOF. We apply Lemma 13 with $G = \mathbb{Z}/p^{t_i}\mathbb{Z}$,

$$Y = \{m_0 + \textstyle\sum_{k=1}^{r} (\sum_{j<t_i} p^j \mu_k(j)) m_k + \sum_{j<t_{i-1}} p^j a_j + p^{t_i}\mathbb{Z} : \\ |m_k| \leq p^{t_i} \text{ , for all } k \leq r,\ a_j \in \{0, ..., p-1\}\}$$

and

$$Y' = \{\sum_{n=t_{i-1}}^{t_i - 1} p^n x_n + p^{t_i}\mathbb{Z} : x_n \in \{0, ..., p-1\}\}.$$

and proceed as in the proof of Lemma 15. □

Similarly we have the following generalization of Lemma 14 for use in subcase (i).

Lemma 17 *Given $p > 1$ and $r \geq 0$, and a sequence of integers $\mu = \langle \mu_1, ..., \mu_r \rangle$, let t_p be maximal such that $(2t_p + 1)^{2r+2} < p$. Then there exists a function*

$$F_{p,\mu} : \mathbb{Z}/p\mathbb{Z} \to 2$$

and integers $a^\ell_{p,\mu} \in \{0, ..., p-1\}$ ($\ell = 0, 1$) such that whenever $m_0, ..., m_r$ are integers such that $|m_k| \leq t_p$ for all $k \leq r$, then

$$F_{p,\mu}(m_0 + \sum_{k=1}^{r} \mu_k m_k + a^\ell_{p,\mu} + p\mathbb{Z}) = \ell.$$

□

Now define the function φ_α with domain ω by letting

1. $\varphi_\alpha(m) = \langle \langle \mu_{\alpha,k}(m) : k = 1, ..., r \rangle, p_{\alpha,m}, g_{\alpha,m} \rangle$; or
2. $\varphi_\alpha(m) = \langle \langle \mu_{\alpha,k}(n) : k = 1, ..., r \rangle, g_{\alpha,n} : n < t_{m+1} \rangle$.

Given functions $c_\alpha : \omega \to 2$, we define $\pi_\nu : A'_\nu \to A_\nu$ inductively along with a set function ψ_ν: $A_\nu \to A'_\nu$ such that $\pi_\nu \circ \psi_\nu = 1_{A_\nu}$. The crucial case is when π_α and ψ_α have been defined and $\alpha \in E$. Then we define $A'_{\alpha+1}$ by generators $\{z'_{\alpha,k} : k = 1, ..., r\} \cup \{y'_{\alpha,n} : n \in \omega\}$ over A'_α satisfying relations

1. (††) $p_{\alpha,n} y'_{\alpha,n+1} = y'_{\alpha,0} + \sum_{k=1}^{r} \mu_{\alpha,k}(n) z'_{\alpha,k} - \psi_\alpha(g_{\alpha,n}) + a^\ell_{p_{\alpha,n},\mu} e$; or
2. (††) $p y'_{\alpha,n+1} = y'_{\alpha,n} + \sum_{k=1}^{r} \mu_{\alpha,k}(n) z'_{\alpha,k} - \psi_\alpha(g_{\alpha,n}) + a^\ell_{\alpha,n,\mu} e$

where $a^\ell_{p_{\alpha,n},\mu}$ (respectively, $a^\ell_{\alpha,n,\mu}$) is as in Lemma 17 (respectively, Lemma 16) for $\ell = c_\alpha(n)$ (respectively, $\ell = c_\alpha(i-1)$ if $t_{i-1} \leq n < t_i$) (and the appropriate prime or primes are used).

In the end we use a splitting ρ of $\pi = \cup_\nu \pi_\nu : A' = \cup_\nu A'_\nu \to A$ to define $H(w)$ as follows:

1. if $w = \langle \langle \mu_k : k = 1, ..., r \rangle, p, g \rangle$, let $H(w) = F_{p,\mu}(\psi(g) - \rho(g) + p\mathbb{Z})$; or
2. if $w = \langle \langle \mu_k(n) : k = 1, ..., r \rangle, g_n : n < t_i \rangle$, let $H(w) =$ $F_{i,\mu}(\sum_{n<t_i} p^n (\psi(g_n) - \rho(g_n)) + p^{t_i}\mathbb{Z})$.

Then we check as before that this definition works. □

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