

# VIVE LA DIFFÉRENCE III

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ABSTRACT. We show that, consistently, there is an ultrafilter  $\mathcal{F}$  on  $\omega$  such that if  $N_n^\ell = (P_n^\ell \cup Q_n^\ell, P_n^\ell, Q_n^\ell, R_n^\ell)$  (for  $\ell = 1, 2, n < \omega$ ),  $P_n^\ell \cup Q_n^\ell \subseteq \omega$ , and  $\prod_{n < \omega} N_n^1/\mathcal{F} \equiv \prod_{n < \omega} N_n^2/\mathcal{F}$  are models of the canonical theory  $t^{\text{ind}}$  of the strong independence property, then every isomorphism from  $\prod_{n < \omega} N_n^1/\mathcal{F}$  onto  $\prod_{n < \omega} N_n^2/\mathcal{F}$  is a product isomorphism.

## 0. INTRODUCTION

In a previous paper [Sh 326] we gave two constructions of models of set theory in which the following isomorphism principle fails in various strong respects:

**(Iso 1):** If  $M, N$  are countable elementarily equivalent structures and  $\mathcal{F}$  is a non-principal ultrafilter on  $\omega$ , then the ultrapowers  $M^*, N^*$  of  $M, N$  with respect to  $\mathcal{F}$  are isomorphic.

As is well known, this principle is a consequence of the Continuum Hypothesis. Recall that Keisler celebrated theorem (from [Ke67]) says that, if  $2^\lambda = \lambda^+$  then two models,  $M, N$  of cardinality at most  $\lambda^+$  (and vocabulary of cardinality  $\leq \lambda$ ) are elementarily equivalent iff for some ultrafilter  $\mathcal{F}$  on  $\lambda$ , the ultrapowers  $M^\lambda/\mathcal{F}, N^\lambda/\mathcal{F}$  are isomorphic. This has given an algebraic characterization of elementary equivalence.

In [Sh 405] our aim originally was to give a related example in connection with the well-known isomorphism theorem of Ax and Kochen. In its general formulation, that result states that a fairly broad class of Henselian fields of characteristic zero satisfying a completeness (or saturation) condition are classified up to isomorphism by the structure of their residue fields and their value groups. That is, the statement that interest us in the second paper in this series [Sh 405], was:

**(Iso 2):** If  $\mathcal{F}$  is a non-principal ultrafilter on  $\omega$ , then the ultraproducts  $\prod_p \mathbb{Z}_p/\mathcal{F}$  and  $\prod_p \mathbb{F}_p[[t]]/\mathcal{F}$  are isomorphic.

The answer we got was, more generally:

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**Theorem 0.1** (See [Sh 405]). *It is consistent with the axioms of set theory that there is a non-principal ultrafilter  $\mathcal{F}$  on  $\omega$  such that for any two sequences of discrete rank 1 valuation rings  $(R_n^i)_{n=1,2,\dots}$  ( $i = 1, 2$ ) having countable residue fields, any isomorphism  $F : \prod_n R_n^1/\mathcal{F} \rightarrow \prod_n R_n^2/\mathcal{F}$  is an ultraproduct of isomorphisms  $F_n : R_n^1 \rightarrow R_n^2$  (for a set of  $n$ 's contained in  $\mathcal{F}$ ). In particular, for  $\mathcal{F}$ -majority of the  $n$ , the valuation rings  $R_n^1, R_n^2$  are isomorphic.*

In the case of the rings  $\mathbb{F}_p[[t]]$  and  $\mathbb{Z}_p$ , we see that (Iso 2) fails. For this our main work was to show the following statement which actually from model theoretic point of view is more basic and interesting.

**Theorem 0.2** (See [Sh 405]). *It is consistent with the axioms of set theory that there is a non-principal ultrafilter  $\mathcal{F}$  on  $\omega$  such that for any two sequences of countable trees  $(T_n^i)_{n=1,2,\dots}$  for  $i = 1, 2$ , with each tree  $T_n^i$  countable with  $\omega$  levels, and with each node having at least two immediate successors, if  $\mathcal{T}^i = \prod_n T_n^i/\mathcal{F}$ , then for any isomorphism  $F : \mathcal{T}^1 \xrightarrow{\simeq} \mathcal{T}^2$  there is an element  $a \in \mathcal{T}^1$  such that the restriction of  $F$  to the cone above  $a$  is the restriction of an ultraproduct of maps  $F_n : T_n^1 \rightarrow T_n^2$ .*

From a model theoretic point of view this still is not the right level of generality for a problem of this type. There are two natural ways to pose the problem. From now on

**Convention 0.3.** In the rest of §0 and §2, §3 models are countable with countable vocabulary if not said otherwise, and we use  $M, N$  to denote models. If we say a model may be uncountable we still assume its vocabulary is countable if not said otherwise.

**Problem 1.** Characterize the pairs of countable models  $M, N$  which are pseudo-isomorphic, where

**Definition 0.4.** We say that the countable models  $M, N$  are pseudo-isomorphic if:

- (a) if  $\mathcal{F}$  is a non principal ultrafilter over  $\omega$  then  $M^\omega/\mathcal{F}, N^\omega/\mathcal{F}$  are isomorphic, and
- (b) clause (a) continue to hold after forcing by any (set) forcing .

Of course this is not isomorphism (see below on models of a stable theory). A related problem is

**Problem 2.** Characterize the pairs of countable models  $M, N$  with non-isomorphic ultrapowers modulo any non-principal ultrafilter  $\mathcal{F}$ ,  $M^\omega/\mathcal{F}, N^\omega/\mathcal{F}$  in some forcing extension. (I.e., the negation is: such that for every forcing extension there is a non-principal ultrafilter  $\mathcal{F}$  on  $\omega$  we have  $M^\omega/\mathcal{F} \simeq N^\omega/\mathcal{F}$ .)

There are two variants of the second problem: the ultrapowers may be formed either using one ultrafilter twice (called 2(A)), or may consider using

any two ultrafilters (called 2(B)), but see below. As when the continuum hypothesis holds is too easy ask:

**Problem 3.** Characterize the pairs  $M, N$  of countable models such that in some forcing extension failing in continuum hypothesis, for every non-principal ultrafilter  $\mathcal{F}$  on  $\omega$ ,  $M^\omega/\mathcal{F} \cong N^\omega/\mathcal{F}$

**Problem 4.** Let us write  $M \leq N$  whenever in every forcing extension, if  $\mathcal{F}$  is an ultrafilter on  $\omega$  such that  $N^\omega/\mathcal{F}$  is saturated, then  $M^\omega/\mathcal{F}$  is also saturated. Characterize this relation.

This is related to the Keisler order (see Keisler [Ke67], or [Sh:a], or [Sh:c, Chapter VI]), but does not depend on the fact that the ultrafilter is regular, so some of the results there apply to Problem 4, this in turn implies results on Problem 2(A). By [Sh:c, VI] we know the following. Let  $\mathcal{D}$  be a non-principle ultrafilter on  $\omega$ , and  $M$  (countable) model (with countable vocabulary). If  $\text{Th}(M)$  is stable then  $M^\omega/\mathcal{D}$  is saturated. We can replace  $\aleph_0$  here by any cardinal  $\kappa$  satisfying  $\kappa^{<\kappa} = \kappa$  using regular ultrafilter on  $\kappa$ .

Now, by [Sh 13], there is an ultrafilter  $\mathcal{D}$  on  $2^{\aleph_0}$  such that for countable models (with countable vocabulary)  $M, N$

$$M \equiv N \quad \Rightarrow \quad M^{2^{\aleph_0}}/\mathcal{D} \cong N^{2^{\aleph_0}}/\mathcal{D}.$$

and we can add “ $M^\omega/\mathcal{D}$  is  $\kappa$ -saturated” for every  $\kappa$  such that  $2^{<\kappa} = 2^{2^{\aleph_0}}$ . Also, if  $2^{\aleph_0} = \aleph_1$ ,  $\mathcal{F}$  is a non-principal ultrafilter on  $\omega$  and  $M_1 \equiv M_2$  are countable, then  $M_1^\omega/\mathcal{F} \cong M_2^\omega/\mathcal{F}$  (as they are saturated); similarly if  $M_n^\ell$  are countable models (for  $\ell = 1, 2, n < \omega$ ),  $M_\ell = \prod_{n < \omega} M_n^\ell/\mathcal{F}_\ell$ , and  $\mathcal{F}_\ell$  are non-

principal ultrafilters on  $\omega$ , then  $M_1 \equiv M_2 \Rightarrow M_1 \cong M_2$ . On the other hand, if  $2^{\aleph_0} > \aleph_1$ , then by [Sh:c, Ch VI] for every regular cardinal  $\theta$ ,  $\aleph_1 \leq \theta < 2^{\aleph_0}$  there is a non-principal ultrafilter  $\mathcal{F}_\theta$  on  $\omega$  such that the downward cofinality of  $(\omega, <)^\omega/\mathcal{F}_\theta$  above  $\omega$  is  $\theta$  so  $\theta_1 \neq \theta_2 \Rightarrow (\omega, <)^\omega/\mathcal{F}_{\theta_1} \not\cong (\omega, <)^\omega/\mathcal{F}_{\theta_2}$ . This gives negative results on Problem 2(B) above. If  $\text{Th}(M)$  is unstable then some such  $\mathcal{D}$ ,  $M^\omega/\mathcal{D}$  is not  $\aleph_2$ -saturated. Why? We can choose  $\varphi(\bar{x}, \bar{y})$  which has the order property,  $\text{lg}(\bar{x}) = m$  choose  $\bar{a}_{n,i} \in {}^m M (i < n < \omega)$  be such that  $M \models \varphi[\bar{a}_{n,i}, \bar{a}_{n,j}]$  iff  $i < j < n$ . Let  $P_n = \{\bar{a}_{n,i} : i < \omega\}$ ,  $<_n = \{(\bar{a}_{n,i}, \bar{a}_{n,j}) : i < j < n\}$ . Consider  $(N, P) := \prod_{n < \omega} (M, P_n, <_n) \setminus D$ , now use a “cut” of  $\prod_{n < \omega} (P_n, <_n)/\mathcal{D}$  with cofinality  $(\aleph_0, \aleph_1)$ . So for Problem 4, the stable theories are minimal.

More general problem is

**Problem 5.** For which quadruples  $(M_1, N_1, M_2, N_2)$  of countable models, in some forcing extension for some ultrafilter  $\mathcal{F}$  on  $\omega$ ,  $M_1^\omega/\mathcal{F} \cong N_1^\omega/\mathcal{F}$  but  $M_2^\omega/\mathcal{F} \not\cong N_2^\omega/\mathcal{F}$ ? (and other variants as above).

We can also replace above the countable model  $M$  by the first order theory  $\text{Th}(M)$  e.g. we can define:  $T_1 \leq T_2$  iff  $(T_1, T_2)$  are countable theories such that) for every countable model  $M_1$  of  $T_1$  there is a countable model  $M_2$  of

$T_2$  such that  $M_1 \leq M_2$ . The present paper is dedicated to shedding some further light.

**Problem 6.** We may be more interested in the ultrafilter, so define the order on the family of ultrafilters on  $\omega$  but here our focus is on model theory. More specifically, we may ask to investigate  $\leq_{uf}$  where  $\mathcal{F}_1 \leq_{uf} \mathcal{F}_2$  iff  $\mathcal{F}_1, \mathcal{F}_2$  are non-principal ultrafilter on  $\omega$  such that for every countable model  $M$ , if  $M^\omega/\mathcal{F}_1$  is saturated then  $M^\omega/\mathcal{F}_2$  is saturated.

Working on [Sh 405] we had hoped to continue it sometime. However, we actually began only when Jarden asked:

- (\*) Suppose that  $F_n^\ell$  are finite fields (for  $n < \omega$ ,  $\ell = 1, 2$ ) satisfying  $F_n^1 \not\cong F_n^2$ . Can we have (a universe and) an ultrafilter  $\mathcal{F}$  on  $\omega$  such that  $\prod_{n < \omega} F_n^1/\mathcal{F}$  and  $\prod_{n < \omega} F_n^2/\mathcal{F}$  are elementarily equivalent but not isomorphic?

That was not an arbitrary question: he knew that many such pairs of ultraproducts are elementarily equivalent, because the first order theory of a field  $F$  which is isomorphic to an ultraproduct of finite fields is determined by its characteristic and its subfield of algebraic elements. Hence we can find an equivalence relation  $E_k$  on the family of finite fields for  $k < \omega$  each with finitely many equivalence classes of the form: an equation from  $\Delta_n$  has a solution in one iff it has a solution in the other with  $\Delta_n$  finite, and such that if  $F_n^1, F_n^2$  are finite fields for  $n < \omega$  and  $\mathcal{F}$  is a non-principal ultrafilter on  $\omega$  and for each  $k$  the set  $\{n < \omega : (F_n^1)E_k(F_n^2)\}$  belongs to  $\mathcal{F}$  then the respective ultraproducts are elementarily equivalent.

When Jarden asked me, I inquired whether it has the strong independence property and told him what it is, he said yes. Cherlin gave me the reference to the strong independence property for finite fields: Duret [Du80, pp. 136–157].

Here we continue [Sh 326, §3], [Sh 405, §1]. To give an affirmative answer to (\*), we show that after adding  $\aleph_3$  Cohen reals to a suitable ground model, one gets a universe with an ultrafilter  $\mathcal{F}$  on  $\omega$  and a sequence of models  $\langle M_n : n < \omega \rangle$  on  $\omega$  such that

- (\*\*) if  $N_n^\ell = (P_n^\ell \cup Q_n^\ell, P_n^\ell, Q_n^\ell, R_n^\ell)$  (for  $\ell = 1, 2$ ,  $n < \omega$ ),  $P_n^\ell \cup Q_n^\ell \subseteq \omega$ , and  $\prod_{n < \omega} N_n^1/\mathcal{F} \equiv \prod_{n < \omega} N_n^2/\mathcal{F}$  are models of the canonical theory  $t^{\text{ind}}$

of the strong independence property (see Definition 1.5 below), then:

□ every isomorphism from  $\prod_{n < \omega} N_n^1/\mathcal{F}$  onto  $\prod_{n < \omega} N_n^2/\mathcal{F}$  is (first order)

definable in  $\prod_{n < \omega} M_n/\mathcal{F}$  for some models  $M_n$  with universe  $\omega$

or what is equivalent but hopefully more transparent

□' if  $F$  is an isomorphism from  $N^1 = \prod_{n < \omega} N_n^1/\mathcal{F}$  onto  $N^2 = \prod_{n < \omega} N_n^2/\mathcal{F}$

then we can find unary functions  $F_n$  from  $N_n^1$  into  $N_n^2$  for every  $n < \omega$

such that the set of  $n$  for which  $F_n$  is an isomorphism from  $N_n^1$  onto  $N_n^2$  belongs to the ultrafilter and  $\prod_{n < \omega} (N_n^1, N_n^2, F_n) / \mathcal{F}$  is  $(N^1, N^2, F)$ .

Our forcing is adding  $\aleph_3$  Cohen reals, but we need that our model of set theory, i.e. the universe over which we force, satisfies some conditions. There are two ways to get a “suitable” ground model. The first way involves taking any ground model which satisfies a relevant portion of the GCH, and extending it by an appropriate preliminary forcing, which generically adds the *name* for an ultrafilter which will appear after addition of the Cohen reals. The alternative approach, which we consider more model-theoretic, is to start with an  $\mathbf{L}$ -like ground model and use instances of diamond (or related weaker principles) to prove that a sufficiently generic name already exists in the ground model. We will fully present the first approach - the second one should be then an easy modification of the arguments presented in [Sh 405, §1].

Our presentation is somewhat more general than needed for (\*\*). By allowing more what we call “bigness” properties to be involved in the definition of  $\text{App}$ , we leave room for getting analogs of (\*\*) for more classes of models (getting the conclusion for all of them at once, or possibly only for some) - as long as the respective bigness notions are as in Definition 1.4. This, we hope, would be helpful in connection with the Problems above (particularly problem 2 and 5). For the problem on fields only the case associated with the strong independence property is needed; general bigness notions appear for possible general treatment.

Let us comment on our general point of view. In this paper we try to advance in Problems 1+2(A) and for this, it seemed, we can take the maximal  $\Gamma$ , i.e., allow all  $\aleph_0$ -bigness notions. However, concerning Problem 4 (investigating the partial order  $\leq$  on models), for showing  $M \not\leq N$ , the construction causes  $N^\omega / \mathcal{F}$  to be almost always non  $\aleph_3$ -saturated. We need finer tools for them e.g. using some bigness notion but not others.

The two previous papers benefited from Gregory Cherlin, the present one benefited from Andrzej Rosłanowski, thank you!

We continue those investigations in [Sh 800].

*Notation 0.5.* Our notation is standard and compatible with that of classical textbooks (like Hodges [Ho93] Chang and Keisler [?] and Jech [J]). In forcing we keep the older convention that *a stronger condition is the larger one*.

- (1) We will use two forcing notions denoted by  $\mathbb{C}_{\aleph_3}$  and  $\text{App}$  (see Definitions 2.1 and 2.4, respectively). Conditions in these forcing notions will be called  $p, q, r$  (with possible sub/super-scripts). Note that the product  $\text{App} \times \mathbb{C}_{\aleph_3}$  is a dense subset of the composition  $\text{App} * \mathbb{C}_{\aleph_3}$
- (2) All names for objects in forcing extensions will be denoted with a tilde below (e.g.,  $\underline{a}, \underline{p}$ ).
- (3) The letter  $\tau$  (with possible sub/super-scripts) stands for a vocabulary of a first order language; we may also write  $\tau(M), \tau(T)$  for a

model  $M$  or theory  $T$  with the obvious meaning. We will use the letters  $\mathfrak{p}, \mathfrak{q}$  (with sub/super-scripts) to denote types.

- (4) The universe of a model  $M$  will be denoted  $|M|$ , but we will often abuse this notation and write, e.g.,  $a \in M$ . The cardinality of a set  $A$  will be denoted  $\|A\|$ , and, for a model  $M$ ,  $\|M\|$  will stand for the cardinality of its universe.

Comment: Why the  $\aleph_3$ ? We like to have a preliminary forcing notion  $\mathbb{A}pp$  which for some  $\kappa$ , is  $\kappa$ -complete,  $\kappa^+$ -c.c.,  $\kappa^{<\kappa} = \kappa$ ; so that every cardinal is preserved. But for  $\kappa = \aleph_1$ ,  $A \subseteq \kappa^+$  countable the number of conditions with this domains (i.e. the number of names of ultrafilters on  $\omega$  as above) is more than  $\kappa$  hence in the natural choice the  $\kappa$ -c.c may fail, we may remedy this but it is easier to use a cardinal  $\kappa$  such that  $\mu < \kappa \Rightarrow \mu^{\aleph_0} < \kappa$ .

## 1. BIGNESS NOTIONS

In this section we will quote relevant definitions and results from [Sh:e, Chapters X, XI] (= [Sh 384], [Sh 482]), but we somewhat restrict ourselves here. The reader interested in the field case only and/or finding Definition 1.1 obscure, may jump directly to Definition 1.5.

**Definition 1.1** (See [Sh:e, Chapter XI, §1]). Let  $T$  be a complete first order theory (in a vocabulary  $\tau$ ), and  $\mathcal{K} = \mathcal{K}_T$  be a class of models of  $T$  (normally: all models of  $T$ ) partially ordered by the relation  $\prec$  of being elementary submodel. Also let  $t$  be a first order theory with a countable vocabulary  $\tau(t)$  (including equality, treating function symbols as predicates).

- (1) We say that  $\mathcal{K}'$  is an  $A$ -place in  $\mathcal{K}$  if
- (a)  $\mathcal{K}' \subseteq \mathcal{K}$ ,
  - (b) if  $M \in \mathcal{K}'$ , then  $A \subseteq M$ ,
  - (c) if  $M \prec N$  are from  $\mathcal{K}$  and  $A \subseteq M$ , then  $(M \in \mathcal{K}') \Leftrightarrow (N \in \mathcal{K}')$ ,
  - (d) if  $M \in \mathcal{K}$  and  $A \subseteq N \in \mathcal{K}$  and  $M, N$  are isomorphic over  $A$ , then  $M \in \mathcal{K}' \Leftrightarrow N \in \mathcal{K}'$ .
- (1A) A place is an  $A$ -place for some  $A$  (alternatively use only  $M \prec \mathfrak{C}$  of cardinality  $< \bar{\kappa}$ , where  $\mathfrak{C}$  is  $\bar{\kappa}$ -saturated model of  $T$ , as in [Sh:c]).
- (2) For  $A \subseteq M \in \mathcal{K}$  we let  $\mathcal{K}' = \mathcal{K}_{A,M}$  be the class

$$\{N \in \mathcal{K} : A \subseteq N \text{ and } \bar{a} \in {}^{\omega}A \Rightarrow \text{tp}(\bar{a}, \emptyset, M) = \text{tp}(\bar{a}, \emptyset, N) \}.$$

We call it *the  $(A, M)$ -place*.

- (1) A *local bigness notion*  $\Gamma$  for  $\mathcal{K}$  (without parameters, in one variable  $x$ ) is a function with domain  $\mathcal{K}$  which for every model  $M \in \mathcal{K}$  gives

$$\begin{aligned} \Gamma_M^- &= \Gamma^-(M) \subseteq \{\varphi(x, \bar{a}) : \varphi \in \mathcal{L}(\tau) \ \& \ \bar{a} \subseteq M\}, \\ \Gamma_M^+ &= \Gamma^+(M) = \{\varphi(x, \bar{a}) : \varphi \in \mathcal{L}(\tau) \ \& \ \bar{a} \subseteq M\} \setminus \Gamma_M^- \end{aligned}$$

such that

- (a)  $\Gamma_M^-$  is preserved by automorphisms of  $M$ ,
- (b)  $\Gamma_M^-$  is a proper ideal, i.e.,  $\Gamma_M^+ \neq \emptyset$  and

- ( $\alpha$ ) if  $M \models (\forall x)(\varphi(x, \bar{a}) \rightarrow \psi(x, \bar{b}))$  and  $\psi(x, \bar{b}) \in \Gamma_M^-$ , then  $\varphi(x, \bar{a}) \in \Gamma_M^-$ ,
- ( $\beta$ ) if  $\varphi_1(x, \bar{a}_1), \varphi_2(x, \bar{a}_2) \in \Gamma_M^-$ , then  $\varphi_1(x, \bar{a}_1) \vee \varphi_2(x, \bar{a}_2) \in \Gamma_M^-$ .

Elements of  $\Gamma_M^-$  are called  $\Gamma$ -small in  $M$ , members of  $\Gamma_M^+$  are  $\Gamma$ -big.

A local bigness notion  $\Gamma$  for  $\mathcal{K}$  with parameters<sup>1</sup> from  $A$  is defined similarly but  $\text{Dom}(\Gamma)$  is an  $A$ -place  $\mathcal{K}'$  in  $\mathcal{K}$  and in clause (a) the automorphisms are over  $A$ .

- (2) We say that a local bigness notion  $\Gamma$  is *invariant* for  $\mathcal{K}$  (or for an  $A$ -place  $\mathcal{K}'$ ) if for  $M \prec N$  from  $\mathcal{K}$  (or from the  $A$ -place  $\mathcal{K}'$ ) we have  $\Gamma_M^- \subseteq \Gamma_N^-$  and  $\Gamma_M^+ \subseteq \Gamma_N^+$ .
- (3) A  $\Gamma$ -big type  $\mathbf{p}(x)$  in  $M$  is a set of formulas  $\psi(x, \bar{a})$  all of whose finite conjunctions are  $\Gamma$ -big in  $M$ .
- (4) A pre  $t$ -bigness notion scheme  $\Omega$  is a sentence  $\psi_\Omega$  (in possibly infinitary logic) in the vocabulary  $\tau(t) \cup \{P^*\}$ , where  $P^*$  is a unary predicate, we may say “using  $P^*$ ”.
- (5) An interpretation with parameters of  $t$  in a model  $M \in \mathcal{K}$  is  $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$ , where  $\varphi_R \in \mathcal{L}(\tau)$  and  $\bar{a}_R$  is a sequence of appropriate length of elements of  $M$ . So a predicate  $R$  from  $\tau(t)$  is interpreted as

$$\{\bar{b} : M \models \varphi_R(\bar{b}, \bar{a}_R), \text{lg}(\bar{b}) = \text{lg}(\bar{y}_R) (= \text{the arity of } R) \}.$$

The interpreted model is called  $M[\bar{\varphi}]$  or  $M^{[\bar{\varphi}]}$  and we demand that it is a model of  $t$ ; so in particular  $M[\bar{\varphi}]$  is a  $\tau(t)$ -model and its universe is  $\{b \in M : M \models \varphi_=(b, b, \bar{a}_=)\}$  defined by  $\varphi_=(x, y, \bar{a}_=)$  which we demand is an equivalence relation; here usually equality on its domains, so we may write just  $\varphi_=(x, \bar{a}_=)$  or just  $\varphi(x, \bar{a})$ ; of course we could use  $k$ -tuples for elements and then  $\text{lg}(\bar{y}_R) = kn$  for  $R$  an  $n$ -place predicate from  $\tau(t)$

- (6) For a pre  $t$ -bigness notion scheme  $\Omega = \psi_\Omega$  and an interpretation  $\bar{\varphi}$  of  $t$  in  $M \in \mathcal{K}$  with parameters from  $A \subseteq M$ , we define the  $\bar{\varphi}$ -derived local pre-bigness notion  $\Gamma = \Gamma_{\psi, \bar{\varphi}} = \Gamma_\psi[\bar{\varphi}]$  with parameters from  $A \subseteq M$  (in the  $A$ -place  $\mathcal{K}_{A, M}$ ) as follows:

Given  $M' \in \mathcal{K}_{A, M}$ , a formula  $\vartheta(x, \bar{b})$  in  $\mathcal{L}(\tau)$  (with parameters from  $M'$  of course) is  $\Gamma_\psi[\bar{\varphi}]$ -big in  $M'$  if for any quite saturated  $N^*$ ,  $M' \prec N^*$ , letting

$$P^* = \{a \in N^*[\bar{\varphi}] : N^* \models \vartheta[a, \bar{b}]\}$$

we have  $(N^*[\bar{\varphi}], P^*) \models \psi$ .

In full we may write  $\Gamma = \Gamma_{(\psi, t, \bar{\varphi})}$  and even  $\Gamma = \Gamma_{(\psi, t, \bar{\varphi}, M, A)}$ .

- (7) We say  $\psi$  is a  $t$ -bigness notion (for  $T$ ) if for every interpretation  $\bar{\varphi}$  of  $t$  in some  $A$ -place  $\mathcal{K}' \subseteq \mathcal{K}$ ,  $\Gamma_{t, \psi, \bar{\varphi}}$  is an invariant<sup>2</sup> local bigness notion

<sup>1</sup>Alternatively use the monster model.

<sup>2</sup>the “invariant” really follows

for our fixed  $\mathcal{K}$ . If there is no  $T$  mentioned or understood we mean “for every  $T$ ”. So it is enough in (6) above if we define  $\Gamma_{M'}$  when  $M \prec M'$ .

- Proposition 1.2.** (1) *If  $\Gamma$  is a local bigness notion for  $\mathcal{K}$  with parameters in  $A$ ,  $M \in \mathcal{K}_{A,M'}$  and  $\mathfrak{p}(x)$  is a  $\Gamma$ -big type in  $M$ , then it can be extended to  $\Gamma$ -big type  $\mathfrak{q}$  in  $M$  which is a complete type over  $M$ .*
- (2) *Assume  $t, \psi, \bar{\varphi}, M, A$  are as in Definition 1.1(6). The truth value of “ $\vartheta(y, \bar{a})$  is  $\Gamma_{(t, \psi, \bar{\varphi})}$ -big” depends just on  $(M \upharpoonright \tau', \bar{a}, c)_{c \in A}$  whenever the formulas in  $\bar{\varphi}$  and  $\vartheta$  belong to  $\mathbb{L}(\tau')$ .*

**Proposition 1.3.** *For  $T, \mathcal{K} = \mathcal{K}_T$  and  $t$  as in 1.1,*

- ( $\boxtimes$ ) *if  $N \prec M$  are from  $\mathcal{K}$ , and  $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$  is an interpretation of  $t$  in  $N$ , then  $\bar{\varphi}$  is an interpretation of  $t$  in  $M$  (i.e.,  $M[\bar{\varphi}] \models t$ ) and moreover  $N[\bar{\varphi}] \prec M[\bar{\varphi}]$ .*

The following definition is crucial in our application, the proofs give some amount of definability, “a local version” and we need to deduce from it a global one. This is a good property criterion for closing the gap which have in fact been used for  $t^{\text{ind}}$ , see more systematically in [Sh 800].

**Definition 1.4.** Let  $t$  be a first order theory in a vocabulary  $\tau(t)$ . Suppose that  $\psi$  is a  $t$ -bigness notion scheme, using  $P \in \tau(t)$ , a unary predicate, and  $\vartheta(y, x)$  is a  $\tau(t)$ -formula. We say that  $\psi$  is  $(\aleph_2, \aleph_1)$ - $(P, \vartheta)$ -separative whenever the following condition  $(\otimes)_{\Gamma}^{P, \vartheta}$  holds and for simplicity we assume  $\varphi_{=} (x, y, \bar{a}_{=})$  is equality on its domain<sup>3</sup>.

$(\otimes)_{\Gamma}^{P, \vartheta}$  For every  $\aleph_2$ -compact<sup>4</sup>  $\tau$ -model  $M$  and every interpretation  $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(t) \rangle$  of  $t$  in  $M$  and a set  $X \subseteq |M|$  of cardinality at most  $\aleph_1$ , including all parameters of  $\bar{\varphi}$  we have:

if  $N \prec M$ ,  $X \subseteq |N|$ ,  $\|N\| \leq \aleph_1$ , and  $\mathfrak{p}(x)$  is a  $\Gamma_{\psi}[\bar{\varphi}]$ -big type over  $N$ ,  $\|\mathfrak{p}(x)\| \leq \aleph_1$ , and  $a_1, a_2$  are distinct members of  $|M| \setminus |N|$  with (recalling 1.1(5))

$$M \models \varphi_P[a_1, \bar{a}_P] \wedge \varphi_P[a_2, \bar{a}_P]$$

then the type  $\mathfrak{p}(x) \cup \{\vartheta(a_1, x) \leftrightarrow \neg\vartheta(a_2, x)\}$  is  $\Gamma_{\psi}[\bar{\varphi}]$ -big.

We now define the main bigness notion used

**Definition 1.5** (See [Sh:e, Def. 3.4, 3.5, Chapter XI]). (1)  $t^{\text{ind}} = t_0^{\text{ind}}$  is the first order theory in vocabulary  $\tau(t^{\text{ind}}) = \{P, Q, R\}$ , where  $P, Q$  are unary predicates and  $R$  is a binary predicate, including sentences

$$\begin{aligned} &(\forall x)(\forall y)(x R y \rightarrow P(x) \wedge Q(y)), \quad \text{and} \\ &(\forall x)(P(x) \vee Q(x)) \end{aligned}$$

<sup>3</sup>Otherwise we should inside  $(\otimes)_{\Gamma}^{P, \vartheta}$ , demands further that for any  $c \in N$  we have  $M \models \neg\varphi_{=}(c, a_1, \bar{a}_{=}) \wedge \neg\varphi_{=}(c, a_2, \bar{a}_{=}) \wedge \neg\varphi_{=}(a_1, a_2, \bar{a}_{=})$ .

<sup>4</sup>A model  $M$  is called  $\kappa$ -compact if every type over it of cardinality  $< \kappa$  is realized; if we omit  $\kappa$  we mean the cardinality of the model



and saying that for each  $n < \omega$  and any pairwise distinct elements  $a_1, \dots, a_{2n} \in P$ , there is  $c \in Q$  such that

$$a_i R c \quad \text{if and only if} \quad i \leq n.$$

$t_1^{\text{ind}}$  is  $t_0^{\text{ind}}$  plus

$$(\forall x)(\forall y)(\exists z)(Q(x) \wedge Q(y) \wedge x \neq y \rightarrow P(z) \wedge (z R x \equiv \neg z R y)).$$

- (2) We define a pre  $t^{\text{ind}}$ -bigness notion scheme  $\Gamma^{\text{ind}}$  as follows. The sentence  $\psi^{\text{ind}}$  says that  $P^* \subseteq Q$  and  $(P, Q, R, P^*)$  satisfies:  
 for every  $n < \omega$ , there is a finite set  $A \subseteq P$  such that  
 for every distinct  $a_1, \dots, a_{2n} \in P \setminus A$  there is  $c \in P^*$   
 satisfying

$$a_\ell R c \quad \text{for } \ell \leq n, \quad \text{and} \quad \neg(a_\ell R c) \quad \text{for } n < \ell \leq 2n.$$

(So  $\psi^{\text{ind}}$  is not first order.)

- (3) We say that a first order theory  $T$  has the strong independence property if some<sup>5</sup> formula  $\vartheta(x, y)$  defines a two place relation which is a model of  $t_1^{\text{ind}}$  with  $P, Q$  chosen as the whole model i.e. for  $M \models T$  define the  $\tau_{t_1^{\text{ind}}}$ -model  $M', |M'| = |M| = P^{M'} = Q^{M'}, R^{M'} = \{(a, b) : M \models \vartheta(a, b)\}$

In such case we may also say “ $\vartheta(x, y)$  has the strong independence properties (for  $\tau$ )”

Plainly,

- Proposition 1.6.** (1) For a model  $M$  of  $t_1^{\text{ind}}$ , an automorphism  $\pi$  of  $M$  is determined by  $\pi \upharpoonright P^M$  (i.e., if  $\pi_1, \pi_2 \in \text{Aut}(M)$  are such that  $\pi_1 \upharpoonright P^M = \pi_2 \upharpoonright P^M$ , then  $\pi_1 = \pi_2$ ).
- (2) Moreover, if  $\bar{\varphi}$  is an interpretation of  $t_1^{\text{ind}}$  in  $M^*$ ,  $M = M^*[\bar{\varphi}]$ ,  $\pi \in \text{Aut}(M)$  and  $\pi \upharpoonright P^M$  is definable in  $M^*$  (with parameters in  $M^*$ ), then so is  $\pi$ .

**Proposition 1.7.** (See [Sh:e, Chapter XI, §3] and [Sh 107])  $\psi^{\text{ind}}$  is a  $t^{\text{ind}}$ -bigness notion scheme. It is  $(\aleph_2, \aleph_1)$ - $(P, \vartheta)$ -separative where  $P \in \tau(t_0^{\text{ind}})$  is given and we choose  $\vartheta(y, x) := y R x$ .

**Definition 1.8.** A mapping  $F : N^1 \rightarrow N^2$  is a  $\Delta$ -embedding from  $N^1$  to  $N^2$  whenever  $\Delta$  is a set of formulas in  $\mathbb{L}_{\omega, \omega}(\tau(N^1) \cap \tau(N^2))$  and

$$\begin{aligned} &\text{if } \varphi \in \Delta \text{ and } N^1 \models \varphi[a_1, \dots, a_n], \\ &\text{then } N^2 \models \varphi[F(a_1), \dots, F(a_n)]. \end{aligned}$$

[of course, if  $\Delta$  is closed under negation, then we have “if and only if”.]

<sup>5</sup>of course  $\vartheta(\bar{x}, \bar{y}), \text{lg}(\bar{x}) = m = \text{lg}(\bar{y})$  can serve as well

2. THE FORCING NOTION  $\mathbb{A}pp$ 

As explained in the introduction, we work in a Cohen generic extension of a suitable ground model. In this section we present how that suitable ground model can be obtained: we start with  $\mathbf{V} \models \text{GCH}$  and we force with the forcing notion  $\mathbb{A}pp$  defined in 2.4 below, the  $\mathbb{A}pp$  comes for approximations, as the members are approximations to a name for an ultrafilter as we desire.

- Definition 2.1.** (1) The Cohen forcing of adding  $\aleph_3$  Cohen reals is denoted by  $\mathbb{C}_{\aleph_3}$ . Thus a condition  $p$  in  $\mathbb{C}_{\aleph_3}$  is a finite partial function from  $\aleph_3 \times \omega$  to  $\omega$ , and the order of  $\mathbb{C}_{\aleph_3}$  is the natural one. The canonical  $\mathbb{C}_{\aleph_3}$ -name for  $\beta^{\text{th}}$  Cohen real will be called  $x_\beta$ .
- (2) Let  $\mathbf{A} \subseteq \aleph_3$ . For a condition  $p \in \mathbb{C}_{\aleph_3}$ , its restriction to  $\mathbf{A} \times \omega$  is called  $p \upharpoonright \mathbf{A}$ , and we let  $\mathbb{C}_{\aleph_3} \upharpoonright \mathbf{A} = \mathbb{C}_{\mathbf{A}} = \{p \upharpoonright \mathbf{A} : p \in \mathbb{C}_{\aleph_3}\}$ . Also, we let  $\omega_{\mathbf{A}}^* = (\omega_\omega)^{\mathbf{V}^{\mathbb{C}_{\aleph_3} \upharpoonright \mathbf{A}}}$ .
- (3) For a sequence  $\langle A_n : n < \omega \rangle$  of non-empty sets (and  $\mathbf{A} \subseteq \aleph_3$ ), we define

$$\prod_{n < \omega}^{\mathbf{A}} A_n = \{f \in \mathbf{V}^{\mathbb{C}_{\aleph_3} \upharpoonright \mathbf{A}} : f \text{ is a function with domain } \omega, \\ \text{and such that } f(n) \in A_n \text{ for every } n \},$$

and similarly for models.

- (4) For  $\mathbf{A} \subseteq \aleph_3$  and  $m < \omega$ , let  $I_{\mathbf{A}}^m$  be the set of all  $\omega$ -sequences of canonical  $\mathbb{C}_{\mathbf{A}}$ -names for subsets of  ${}^m\omega$ . Let  $Q_{\bar{s}}$  (for  $\bar{s} \in I_{\mathbf{A}}^m$ ,  $m < \omega$ ) be an  $m$ -ary predicate,  $Q_{\bar{s}_0} \neq Q_{\bar{s}_1}$  whenever  $\bar{s}_0 \neq \bar{s}_1$  i.e. even when they are forced to be equal they may be different as sequences of names, and let

$$\tau_{\mathbf{A}} = \{Q_{\bar{s}} : \bar{s} \in I_{\mathbf{A}}^m \ \& \ m < \omega\}$$

(so because of the demand “canonical”,  $\|\tau_{\mathbf{A}}\| = \aleph_1 \cdot \|\mathbf{A}\|$ ). Let  $M_{\mathbf{A}}^n$  be a  $\mathbb{C}_{\mathbf{A}}$ -name for the  $\tau_{\mathbf{A}}$ -model with universe  $\omega$  such that if  $\bar{s} = \langle s_n : n < \omega \rangle \in I_{\mathbf{A}}^m$ , then  $\Vdash_{\mathbb{C}_{\mathbf{A}}} (Q_{\bar{s}})^{M_{\mathbf{A}}^n} = s_n$ . So the vocabulary  $\tau_{\mathbf{A}}$  is an object in  $\mathbf{V}$ , not a name.

- (5) If  $\mathbf{A}_1 \subseteq \mathbf{A}_2$ , and for  $\ell = 1, 2$   $\mathcal{F}_\ell$  is a  $\mathbb{C}_{\mathbf{A}_\ell}$ -name of an ultrafilter on  $\omega$  then  $\mathcal{F}_1 = \mathcal{F}_2 \upharpoonright \mathbf{A}_1$  means that  $\Vdash_{\mathbb{C} \upharpoonright \mathbf{A}_2} \mathcal{F}_1 \subseteq \mathcal{F}_2$ , so  $\mathcal{F}_2 \upharpoonright \mathbf{A}_2$  is unique but not always well defined.<sup>6</sup>

In the definition below the reader can restrict himself to the case  $t = t^{\text{ind}}, \psi = \psi^{\text{ind}}$ , see Definition 1.5 (so we later in Definition 2.4 use only  $\Gamma = \Gamma^{\text{ind}}$ )

- Definition 2.2.** (1) A function  $\mathbf{G}$  is called an  $(\aleph_3, \aleph_2)$ -bigness guide if the domain  $\text{Dom}(\mathbf{G})$  of  $\mathbf{G}$  is

$$\{(\mathbf{A}, \mathcal{F}) : \mathbf{A} \subseteq \aleph_3, \|\mathbf{A}\| \leq \aleph_1, \text{ and} \\ \mathcal{F} \text{ is a } \mathbb{C}_{\mathbf{A}}\text{-name of a non principal ultrafilter on } \omega \},$$

<sup>6</sup>as for  $\mathbb{C}_{\mathbf{A}_1}$ -name  $A$  of a subset of  $\omega$ , the truth value of “ $A \in \mathcal{F}_2$ ” is an  $\mathbb{C}_{\mathbf{A}_2}$ -name but in general not a  $\mathbb{C}_{\mathbf{A}_1}$ -name.

and

- ( $\alpha$ )  $\mathbf{G}(\mathbf{A}, \mathcal{F})$  is a non-empty set of triples  $(t, \psi, \bar{\varphi})$ , where<sup>7</sup>  $t$  is a (countable) first order theory (or just a  $\mathbb{C}_{\mathbf{A}}$ -name of a (countable) first order theory),  $\psi$  is a  $\mathbb{C}_{\mathbf{A}}$ -name of  $t$ -bigness notion scheme, and  $\bar{\varphi}$  is (a  $\mathbb{C}_{\mathbf{A}}$ -name for) an interpretation of  $t$  in  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}}^n / \mathcal{F}$ , and  $\|\mathbf{G}(\mathbf{A}, \mathcal{F})\| \leq \aleph_2$ , and
- ( $\beta$ ) if  $(\mathbf{A}^\ell, \mathcal{F}^\ell) \in \text{Dom}(\mathbf{G})$  for  $\ell = 1, 2$ ,  $\mathbf{A}^1 \subseteq \mathbf{A}^2$  and  $\Vdash_{\mathbb{C}_{\mathbf{A}^2}} \mathcal{F}^1 \subseteq \mathcal{F}^2$ , then  $\mathbf{G}(\mathbf{A}^1, \mathcal{F}^1) \subseteq \mathbf{G}(\mathbf{A}^2, \mathcal{F}^2)$ .
- (2) An  $(\aleph_3, \aleph_2)$ -bigness guide  $\mathbf{G}$  is *ind-full* if
- ( $\gamma$ ) for every  $(\mathbf{A}, \mathcal{F}) \in \text{Dom}(\mathbf{G})$  and a canonical  $\mathbb{C}_{\mathbf{A}}$ -name  $\bar{\varphi}$  for an interpretation of  $t^{\text{ind}}$  in  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}}^n / \mathcal{F}$  we have  $(t^{\text{ind}}, \psi^{\text{ind}}, \bar{\varphi}) \in \mathbf{G}(\mathbf{A}, \mathcal{F})$ .
- (3) We say that  $\mathbf{G}$  is *full* whenever the following condition holds.
- ( $\boxplus$ ) Assume  $(\mathbf{A}, \mathcal{F}) \in \text{Dom}(\mathbf{G})$  and  $\underline{t}$  is a canonical  $\mathbb{C}_{\mathbf{A}}$ -name of a (countable) first order theory in the vocabulary  $\tau(\underline{t}) \in \mathcal{H}(\aleph_1)$ ,  $\psi$  is a canonical  $\mathbb{C}_{\mathbf{A}}$ -name for a pre  $\underline{t}$ -bigness notion scheme,  $\psi \in \mathbb{L}_{\aleph_1, \aleph_1}(\tau(\underline{t}) \cup \{P^*\})$ . Let  $\bar{\varphi}$  be a  $\mathbb{C}_{\mathbf{A}}$ -name for an interpretation of  $\underline{t}$  in  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}}^n / \mathcal{F}$ ; no need for parameters as all elements are interpretation of an individual constant. Suppose  $(\underline{t}, \psi, \bar{\varphi})$  is forced to define a bigness notion<sup>8</sup>  $\Gamma = \Gamma_{(\underline{t}, \psi, \bar{\varphi})}$ . Then  $(\underline{t}, \psi, \bar{\varphi}) \in \mathbf{G}(\mathbf{A}, \mathcal{F})$ .

The clause 2.2(2) is added for our particular application. It can be replaced by the use of a family of bigness notions relevant to your interest.

**Proposition 2.3.** (1) *There is a full  $(\aleph_3, \aleph_2)$ -bigness guide  $\mathbf{G}$ .*

(2) *If a bigness guide  $\mathbf{G}$  is full, then it is ind-full.*

(3) *Full and even just ind-full implies non-emptiness, i.e.  $\mathbf{G}(\mathbf{A}, \mathcal{F}) \neq \emptyset$  when defined.*

*Proof.* Trivial. □

**Definition 2.4.** Let  $\mathbf{G}$  be an  $(\aleph_3, \aleph_2)$ -bigness guide. We define the forcing notion  $\text{App} = \text{App}_{\mathbf{G}}$ . (When  $\mathbf{G}$  is fixed, as typically in the present paper, we may and usually will not mention it.)

(1) **A condition  $q$  in  $\text{App}$**  is a triple  $q = (\mathbf{A}, \mathcal{F}, \bar{\Gamma}) = (\mathbf{A}^q, \mathcal{F}^q, \bar{\Gamma}^q)$  such that:

(a)  $\mathbf{A}$  is a subset of  $\aleph_3$  of cardinality  $\leq \aleph_1$ ;

<sup>7</sup>note that our forcing  $\text{App}$  will add no real so as we are considering only countable  $t$ , it is OK to use only old ones. As we may consider names in the Cohen forcing, things are different so we allow such names

<sup>8</sup>We can fix a  $\mathbb{C}_{\aleph_3}$ -name of countable first order theory.

- (b)  $\mathcal{F}$  is a canonical  $\mathbb{C}_{\mathbf{A}}$ -name of a non-principal ultrafilter on  $\omega$ , such that for  $\beta < \aleph_3$  divisible by  $\aleph_2$ ,

$$\mathcal{F} \upharpoonright (\mathbf{A} \cap \beta) \stackrel{\text{def}}{=} \mathcal{F} \cap \{a : a \text{ is a } \mathbb{C}_{\mathbf{A} \cap \beta}\text{-name of a subset of } \omega \}$$

is a  $\mathbb{C}_{\mathbf{A} \cap \beta}$ -name (of an ultrafilter on  $\omega$ );

Why “canonical”? for the same reasons as in 2.1(4)

- (c)  $\bar{\Gamma} = \langle \bar{\Gamma}_\beta : \beta \in \mathbf{A} \ \& \ \text{cf}(\beta) = \aleph_2 \rangle$ , where each  $\bar{\Gamma}_\beta$  is a local bigness notion  $\Gamma_\psi[\bar{\varphi}]$  for some  $(t, \psi, \bar{\varphi}) \in \mathbf{G}(\mathbf{A} \cap \beta, \mathcal{F} \upharpoonright (\mathbf{A} \cap \beta))$ ;
- (d) If  $\text{cf}(\beta) = \aleph_2$ ,  $\beta \in \mathbf{A}$ , then it is forced (i.e.,  $\Vdash_{\mathbb{C}_{\aleph_3}}$  equivalently  $\Vdash_{\mathbb{C}_{\mathbf{A}}}$ ) that:

the type realized by the element  $x_\beta$  in the model  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A} \cap \beta}^n / \mathcal{F}$

over the model  $\prod_{n < \omega}^{\mathbf{A} \cap \beta} M_{\mathbf{A} \cap \beta}^n / (\mathcal{F} \upharpoonright (\mathbf{A} \cap \beta))$  (so it is a type in the vocabulary  $\tau_{\mathbf{A} \cap \beta}$ ) is  $\bar{\Gamma}_\beta$ -big and complete of course, and moreover this type is a  $\mathbb{C}_{\mathbf{A} \cap \beta}$ -name ; actually we should say “by the element  $x_\beta / (\mathcal{F} \upharpoonright \mathbf{A})$ ”. We call it “the type induced by  $x_\beta$  according to  $q$ ”.

- (2) **The order**  $\leq_{\text{App}} = \leq$  **of**  $\text{App} = \text{App}_{\mathbf{G}}$  is the natural one:  $q_1 \leq q_2$  if and only if  $\mathbf{A}^{q_1} \subseteq \mathbf{A}^{q_2}$ ,  $\Vdash_{\mathbb{C}_{\mathbf{A}^{q_2}}} \mathcal{F}^{q_1} \subseteq \mathcal{F}^{q_2}$ , and  $\bar{\Gamma}^{q_2} \upharpoonright \mathbf{A}^{q_1} = \bar{\Gamma}^{q_1}$ .
- (3) We say that  $q_2 \in \text{App}$  is an *end extension* of  $q_1 \in \text{App}$ , and we write  $q_1 \leq_{\text{end}} q_2$ , if  $q_1 \leq q_2$  and  $\text{sup}(\mathbf{A}^{q_1}) \leq \min(\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_1})$ .
- (4) For a condition  $q \in \text{App}$  and an ordinal  $\beta \in \aleph_3$  we define  $q \upharpoonright \beta = (\mathbf{A}^q \cap \beta, \mathcal{F}^q \upharpoonright (\mathbf{A}^q \cap \beta), \bar{\Gamma}^q \upharpoonright (\mathbf{A}^q \cap \beta))$ .
- (5) For  $\beta < \aleph_3$  we let  $\text{App} \upharpoonright \beta = \{q \in \text{App} : \mathbf{A}^q \subseteq \beta\}$  with inherited order. If  $G \subseteq \text{App}$  is generic over  $\mathbf{V}$ , then we let  $G \upharpoonright \beta = G \cap (\text{App} \upharpoonright \beta)$ .

One easily checks that

- Proposition 2.5.** (1) If  $q \in \text{App}$ ,  $\beta < \aleph_3$ , then  $q \upharpoonright \beta \in \text{App}$  and  $q \upharpoonright \beta \leq_{\text{end}} q$ .
- (2) Both  $\leq_{\text{App}}$  and  $\leq_{\text{end}}$  are partial orders, (pedantically quasi orders) on  $\text{App}$ .

**Lemma 2.6.** If  $\langle q_\zeta : \zeta < \xi \rangle$  is an increasing sequence of members of  $\text{App}$ ,  $\xi \leq \aleph_1$ , and  $q_{\zeta_1} \leq_{\text{end}} q_{\zeta_2}$  for  $\zeta_1 < \zeta_2$ , then there is  $q \in \text{App}$  such that  $\mathbf{A}^q = \bigcup_{\zeta < \xi} \mathbf{A}^{q_\zeta}$  and  $q_\zeta \leq_{\text{end}} q$  for all  $\zeta < \xi$ .

*Proof.* We may assume that  $\xi > 0$  is a limit ordinal. If  $\text{cf}(\xi) > \aleph_0$ , then we let  $\mathbf{A}^q = \bigcup_{\zeta < \xi} \mathbf{A}^{q_\zeta}$ ,  $\mathcal{F}^q = \bigcup_{\zeta < \xi} \mathcal{F}^{q_\zeta}$  and  $\bar{\Gamma}^q = \bigcup_{\zeta < \xi} \bar{\Gamma}^{q_\zeta}$ . If  $\text{cf}(\xi) = \aleph_0$ , then additionally we have to extend  $\bigcup_{\zeta < \xi} \mathcal{F}^{q_\zeta}$  to a  $\mathbb{C}_{\mathbf{A}^q}$ -name of an ultrafilter on  $\omega$ , which is no problem.  $\square$

**Lemma 2.7.** *Suppose that  $q \in \text{App}$ ,  $\mathbf{A}^q \subseteq \gamma \in \aleph_3$ , and  $\underline{\mathfrak{p}}$  is a  $\mathbb{C}_{\mathbf{A}^q}$ -name of a type over the model  $\prod_{n < \omega}^{\mathbf{A}^q} M_{\mathbf{A}^q}^n / \mathcal{F}^q$  (so the type  $\underline{\mathfrak{p}} = \underline{\mathfrak{p}}(x)$  is in the vocabulary  $\tau_{\mathbf{A}^q}$ , finitely satisfiable in  $\prod_{n < \omega}^{\mathbf{A}^q} M_{\mathbf{A}^q}^n / \mathcal{F}^q$ ). Then:*

- (1) *If  $\text{cf}(\gamma) < \aleph_2$ , then there is a condition  $r \in \text{App}$  stronger than  $q$  such that  $\mathbf{A}^r = \mathbf{A}^q \cup \{\gamma\}$ , and*

$$\Vdash_{\mathbb{C}_{\mathbf{A}^r}} \text{“} \underline{x}_\gamma / \mathcal{F}^r \text{ realizes } \underline{\mathfrak{p}} \text{ in } \prod_{n < \omega}^{\mathbf{A}^r} M_{\mathbf{A}^r}^n / \mathcal{F}^r \text{”}.$$

- (2) *If  $\text{cf}(\gamma) = \aleph_2$ ,  $(t, \psi, \bar{\varphi}) \in \mathbf{G}(\mathbf{A}^q, \mathcal{F}^q)$  and the type  $\underline{\mathfrak{p}}$  is (forced to be)  $\Gamma_\psi[\bar{\varphi}]$ -big, then there is a condition  $r \in \text{App}$  as in (1) and such that  $\Gamma_\psi^r[\bar{\varphi}] = \Gamma_\psi[\bar{\varphi}]$ .*

*Proof.* 1) Extend  $\mathcal{F}^q$  to  $\mathcal{F}^r$  so that  $\underline{x}_\gamma / \mathcal{F}^r$  realizes the required type, (using “ $\underline{x}_\gamma$  is Cohen over  $\mathbf{V}^{\mathbb{C}|\mathbf{A}}$ ”).

2) Note that every  $\Gamma_\psi[\bar{\varphi}]$ -big type can be extended to a complete  $\Gamma_\psi[\bar{\varphi}]$ -big one by 1.2.  $\square$

**Lemma 2.8.** (1) *Suppose  $q_0, q_1, q_2 \in \text{App}$ ,  $q_0 = q_2 \upharpoonright \beta$ ,  $q_0 \leq q_1$ ,  $\mathbf{A}^{q_1} \subseteq \beta$ . Suppose further that  $\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_0} = \{\beta\}$  and  $\text{cf}(\beta) = \aleph_2$ . Assume further that  $\underline{\mathfrak{p}}_1$  is a  $\mathbb{C}_{\mathbf{A}^{q_1}}$ -name for a complete  $\Gamma_\beta^{q_2}$ -big type over  $(\prod_{n < \omega}^{\mathbf{A}^{q_1}} M_{\mathbf{A}^{q_1}}^n / \mathcal{F}^{q_1})$  such that  $\underline{\mathfrak{p}}_1$  contains the type  $\underline{\mathfrak{p}}_0$  induced by  $\underline{x}_\beta$  according to  $q_2$  (such  $\underline{\mathfrak{p}}_1$  necessarily exists, by the properties of bigness). Then there is  $q_3 \geq q_1, q_2$  with  $\mathbf{A}^{q_3} = \mathbf{A}^{q_1} \cup \{\beta\}$ , such that  $\underline{x}_\beta$  induces  $\underline{\mathfrak{p}}_1$  on  $(\prod_{n < \omega}^{\mathbf{A}^{q_1}} M_{\mathbf{A}^{q_1}}^n / \mathcal{F}^{q_1})$  (according to  $q_3$ ).*

- (2) *Assume  $q_0, q_1, q_2 \in \text{App}$ ,  $q_0 = q_2 \upharpoonright \beta$ ,  $q_0 \leq q_1$  and  $\mathbf{A}^{q_1} \subseteq \beta$ . If  $\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_0} = \{\beta\}$  and  $\text{cf}(\beta) < \aleph_2$ , then there is  $q_3 \in \text{App}$ ,  $q_3 \geq q_1, q_2$  such that  $\mathbf{A}^{q_3} = \mathbf{A}^{q_1} \cup \mathbf{A}^{q_2}$ . This clause is like the first one except the cofinality.*

- (3) *Assume that  $\delta_1, \delta_2 < \aleph_2$ , and  $\langle \beta_j : j < \delta_2 \rangle$  is a non-decreasing sequence of ordinals below  $\aleph_3$ . Let  $\langle p_i : i < \delta_1 \rangle$  be an  $\leq$ -increasing sequence from  $\text{App}$ . Suppose that  $q_j \in \text{App} \upharpoonright \beta_j$  (for  $j < \delta_2$ ) are such that:*

$$p_i \upharpoonright \beta_j \leq q_j \text{ for } i < \delta_1, j < \delta_2,$$

$$q_j \leq_{\text{end}} q_{j'} \text{ for } j < j' < \delta_2.$$

Then there is an  $r \in \text{App}$  with  $p_i \leq r$  and  $q_j \leq_{\text{end}} r$  for all  $i < \delta_1$  and  $j < \delta_2$ .

- (4) *If  $\bar{p} = \langle p_i : i < \delta_1 \rangle$  an increasing sequence in  $\text{App}$ ,  $\delta_1 < \aleph_2$ , then  $\bar{p}$  has an upper bound in  $\text{App}$ . If  $\text{cf}(\delta_1) = \aleph_1$  we use the (naturally defined) union.*
- (5) *Assume*
- (a)  *$\gamma$  is a limit ordinal of cofinality  $\aleph_0$  divisible by  $\aleph_2$*

- (b)  $p \in \text{App}_\gamma$  and  $\underline{p}$  is as  $\mathbb{C}_{\mathbf{A}^p}$ -name of a finitely satisfiable set of formulas in one free variable  $x$  over  $\prod_{n < \omega}^{\mathbf{A}^p} \underline{M}_{\mathbf{A}^p}^n / \mathcal{F}^p$
- (c)  $\gamma_n \in \gamma \setminus \mathbf{A}^p, \gamma_n < \gamma_{n+1}$  and  $\gamma = \cup \{\gamma_n : n < \omega\}$
- Then there is  $q$  such that
- ( $\alpha$ )  $p \leq q \in \text{App}_\gamma$
- ( $\beta$ )  $\mathbf{A}^q = \mathbf{A}^p \cup \{\gamma_n : n < \omega\}$
- ( $\gamma$ )  $\Vdash_{\mathbb{C}_{\mathbf{A}^q}} \text{“}\underline{p} \text{ is realized in } \prod_{n < \omega}^{\mathbf{A}^q} \underline{M}_{\mathbf{A}^q}^n / \mathcal{F}^q\text{”}$

*Proof.* 1) Note that this is a strong form of the  $\aleph_2$ -c.c., see the proof of 2.9 below. Let  $\mathbf{A}_i = \mathbf{A}^{q_i}$  and let  $\mathcal{F}_i = \mathcal{F}^{q_i}$  for  $i < 3$ , and  $\mathbf{A}_3 = \mathbf{A}_1 \cup \mathbf{A}_2 = \mathbf{A}_1 \cup \{\beta\}$ . The only possibly not clear part is to show that, in  $\mathbf{V}^{\mathbb{C}_{\mathbf{A}_3}}$ , there is an ultrafilter extending  $\mathcal{F}_1 \cup \mathcal{F}_2$  which contains  $\mathcal{F}'$ , the family of all the sets

$$\{n < \omega : \underline{M}_{\mathbf{A}_3}^n \models \varphi[x_\beta(n), \bar{a}(n)]\}$$

for  $\varphi(x, \bar{y}) \in \underline{p}_1$ ,  $lg(\bar{y}) = m$ , and a  $\mathbb{C}_{\mathbf{A}_1}$ -name  $\bar{a}$  of an  $m$ -tuple from  $\omega_{\mathbf{A}_1}^*$  (and in our notation above  $\bar{a}(n)$  is a  $\mathbb{C}_{\mathbf{A}_1}$ -name for an  $m$ -tuple of elements of  $\omega$ , so pedantically we define  $\bar{a} = \langle a_\ell : \ell < m \rangle$ ,  $a_\ell = \langle a_\ell(n) : n < \omega \rangle$  where  $a_\ell(n)$  is a  $(\mathbb{C} \upharpoonright \mathbf{A})$ -name of a natural number and  $\bar{a}(n) = \langle a_\ell(n) : \ell < m \rangle$  and we should use below  $\langle a_\ell / \mathcal{F} : \ell < m \rangle$  instead  $\bar{a}$ ). As  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}'$  are (forced, i.e.,  $\Vdash_{\mathbb{C}_{\mathbf{A}_3}}$ ) to be closed under intersections (of two, and hence of finitely many), clearly if this fails, then (as  $\mathcal{F}_0$  is forced to be a non-principal ultrafilter on  $\omega$  so  $m < \omega$  implies  $\Vdash [m, \omega] \in \mathcal{F}_0$ ) there are a condition  $p \in \mathbb{C}_{\mathbf{A}_3}$ , a  $\mathbb{C}_{\mathbf{A}_1}$ -name  $\underline{a}$  of a member of  $\mathcal{F}_1$ , a  $\mathbb{C}_{\mathbf{A}_2}$ -name  $\underline{b}$  of a member of  $\mathcal{F}_2$ , a  $(\mathbb{C}_{\mathbf{A}_1}$ -name for a)  $\tau_{\mathbf{A}_1}$ -formula  $\varphi$  and a  $\mathbb{C}_{\mathbf{A}_1}$ -name for an  $m$ -tuple  $\bar{a}$  from  $\omega_{\mathbf{A}_1}^*$  such that

$$p \upharpoonright \mathbf{A}_1 \Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“}\varphi(x, \bar{a}) \in \underline{p}_1 \text{”} \quad \text{and} \quad p \Vdash_{\mathbb{C}_{\mathbf{A}_3}} \text{“}\underline{a} \cap \underline{b} \cap \underline{c} = \emptyset \text{”},$$

where

$$\underline{c} = \{n : \underline{M}_{\mathbf{A}_3}^n \models \varphi[x_\beta(n), \bar{a}(n)]\}.$$

We may easily eliminate parameters, so we may assume that we have  $\varphi[x_\beta(n)]$  only (remember the definition of  $\tau_{\mathbf{A}_1}$ ). Let  $p_i = p \upharpoonright \mathbf{A}_i$  for  $i = 0, 1, 2$ , and let  $H^0 \subseteq \mathbb{C}_{\mathbf{A}_0}$  be generic over  $\mathbf{V}$  such that  $p_0 \in H^0$ . For  $n < \omega$  let  $\underline{A}_n^*$  be a  $\mathbb{C}_{\mathbf{A}_0}$ -name such that

$$\begin{aligned} \underline{A}_n^*[H^0] = \{y \in \underline{M}_{\mathbf{A}_2}^n : & \text{there is } p'_2 \in \mathbb{C}_{\mathbf{A}_2} \text{ such that} \\ & p_2 \leq p'_2, p'_2 \upharpoonright \mathbf{A}_0 \in H^0 \text{ and} \\ & p'_2 \Vdash \text{“} x_\beta(n) = y \text{ and } n \in \underline{b} \text{”}\} \end{aligned}$$

(recall  $y \in \underline{M}_{\mathbf{A}_2}^n$  means  $y \in \omega$ ). Let  $\underline{A}^* = \prod_{n < \omega}^{\mathbf{A}_0} \underline{A}_n^* / \mathcal{F}_0$ . So  $\underline{A}^*[H^0]$  is (the interpretation of) an unary predicate from  $\tau_{\mathbf{A}_0}$ ; in fact  $Q_{\langle \underline{A}_n^* : n < \omega \rangle}$  is such a predicate, but we shall write  $\underline{A}^*(x)$  instead  $Q_{\langle \underline{A}_n^* : n < \omega \rangle}(x)$ . Thus, in  $\mathbf{V}[H^0]$ , either  $\underline{A}^*(x) \in \underline{p}_0$  or  $\neg \underline{A}^*(x) \in \underline{p}_0$ . The latter is impossible by the choice of  $\underline{A}^*$ , so necessarily  $\underline{A}^*(x) \in \underline{p}_0$ . As also  $p \upharpoonright \mathbf{A}_1 \Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“}\varphi(y) \in \underline{p}_1 \text{”}$ , clearly if

$H^1 \subseteq \mathbb{C}_{\mathbf{A}_1}$  is generic over  $\mathbf{V}$  and  $H^0 \cup \{p_1\} \subseteq H^1$ , then in  $\mathbf{V}[H^1]$  we have

$$\{n \in \omega : M_{\mathbf{A}_1}^n \models (\exists y)(A^*(y) \ \& \ \varphi(y))\} \in \mathcal{F}_1[H^1]$$

(remember  $\mathfrak{p}_1$  is a type over  $\prod_{n < \omega}^{A^{q_1}} M_{\mathbf{A}^{q_1}}^n / \mathcal{F}_1$  extending  $\mathfrak{p}_0$ ). Consequently, we may find a condition  $p'_1 \in H^1 \subseteq \mathbb{C}_{\mathbf{A}_1}$  stronger than  $p_1$ , an integer  $n < \omega$ , and an element  $y \in M_{\mathbf{A}_1}^n$  (so  $y \in \omega$ ) such that

$$p'_1 \upharpoonright \mathbf{A}_0 \in H^0, \quad \text{and} \quad p'_1 \Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“ } M_{\mathbf{A}_1}^n \models (A^*(y) \ \& \ \varphi(y)) \text{ and } n \in \mathfrak{a} \text{”}.$$

As  $A_n^*$  is a  $\mathbb{C}_{\mathbf{A}_0}$ -name, we really have  $y \in A_n^*[H^0]$ , and hence (by its definition) for some  $p'_2 \in \mathbb{C}_{\mathbf{A}_2}$  we have

$$p_2 \leq p'_2, \quad p'_2 \upharpoonright \mathbf{A}_0 \in H^0, \quad \text{and} \quad p'_2 \Vdash \text{“ } y = x_\beta(n) \text{ and } n \in \mathfrak{b} \text{”}.$$

Now for our  $n$  we can force  $n \in \mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c}$  by amalgamating the corresponding conditions  $p'_1, p'_2$ , getting a contradiction. As said above this finishes the proof of the existence of  $q_3$ .

2) The proof is essentially contained in the previous one (use the very trivial bigness notion:  $\varphi(x, \bar{a})$  is big in  $M$  if and only if  $M \models (\exists x)\varphi(x, \bar{a})$ , so we may use a  $\mathfrak{p}_1$ ). See also the end of the proof of (3).

3) We will prove by induction on  $\gamma \in \aleph_3$  that if all  $\beta_j \leq \gamma$  and all  $p_i$  belong to  $\text{App} \upharpoonright \gamma$ , then the assertion in (3) holds for some  $r \in \text{App} \upharpoonright \gamma$ .

We may assume that  $\delta_1 > 0$  (otherwise apply 2.6) and  $\delta_2 > 0$  (otherwise let  $\delta'_2 = 1$ ,  $\beta_0 = 0$ ,  $q'_0 \in \text{App} \upharpoonright 0$  be above  $p_i \upharpoonright 0$  for  $i < \delta_1$ ; so it just means  $\mathcal{F}^{q'_0}$  is an ultrafilter extending  $\mathcal{F}^{p_i \upharpoonright 0}$  for  $i < \delta_1$ ; now if  $\gamma = 0$ , then  $r = q'_0$  is as required and otherwise we have reduced the case  $\delta_2 = 0$  to the case  $\delta_2 = 1$ ).

We may assume that  $\beta_j = \sup\{\alpha + 1 : \alpha \in \mathbf{A}^{q_j}\}$  (for  $j < \delta_2$ ), and also that the sequence  $\langle \beta_j : j < \delta_2 \rangle$  is strictly increasing. Let  $\beta = \sup_{j < \delta_2} \beta_j$  and let  $q = (\bigcup_{j < \delta_2} \mathbf{A}^{q_j}, \bigcup_{j < \delta_2} \mathcal{F}^{q_j}, \bigcup_{j < \delta_2} \bar{\Gamma}^{q_j})$ , this triple is not necessarily a member of  $\text{App}$ .

We first deal with

CASE 1:  $\text{cf}(\gamma) \neq \aleph_0$ .

If  $\gamma = \beta$ , then  $q \in \text{App}$  and we may take  $r = q$ . So let us assume  $\beta < \gamma$ . If  $\delta_2$  is a successor ordinal, or a limit ordinal of uncountable cofinality, then we let  $q^* = q$  (clearly  $q^* \in \text{App} \upharpoonright \beta$ ). If  $\text{cf}(\delta_2) = \aleph_0$ , then we may first apply the inductive hypothesis to  $\langle p_i \upharpoonright \beta : i < \delta_1 \rangle$  (and  $\langle \beta_j, q_j : j < \delta_2 \rangle$ ) to get a condition  $q^* \in \text{App} \upharpoonright \beta$  which is stronger than all  $p_i \upharpoonright \beta$  and which end-extends all  $q_j$ . So in all these cases, we have a condition  $q^* \in \text{App} \upharpoonright \beta$  end extending all  $q_j$  for  $j < \delta_2$  and stronger than all  $p_i \upharpoonright \beta$  for  $i < \delta_1$  (and we are looking for an end-extension of it which is a bound to all  $p_i \upharpoonright \beta$ ). The following three subcases suffice as we have already dealt with the possibility  $\gamma = 0$ .

THE SUBCASE 1A:  $\gamma = \gamma_0 + 1$  IS A SUCCESSOR

In this case our inductive hypotheses applies to the  $p_i \upharpoonright \gamma_0, q^*$ , and  $\gamma_0$ , yielding  $r_0$  in  $\text{App} \upharpoonright \gamma_0$  with  $p_i \upharpoonright \gamma_0 \leq r_0$  for  $i < \delta_1$  and  $q^* \leq_{\text{end}} r_0$ . What remains to be done is an amalgamation of  $r_0$  with all of the  $p_i$ , where  $\mathbf{A}^{p_i} \subseteq \mathbf{A}^{r_0} \cup \{\gamma_0\}$ , and where one may as well suppose that  $\gamma_0$  is in  $\mathbf{A}^{p_i}$  for all  $i$ . This is a slight variation on (1) or (2). For instance, suppose  $\text{cf}(\gamma_0) = \aleph_2$ . We let

- $\mathbf{A}_2 = \bigcup_{i < \delta_1} \mathbf{A}^{p_i}$ ,  $\mathbf{A}_0 = \mathbf{A}_2 \setminus \{\gamma_0\}$ ,  $\mathbf{A}_1 = \mathbf{A}^{r_0}$ ,  $\mathbf{A}_3 = \mathbf{A}_2 \cup \mathbf{A}_1$ .
- $\mathcal{F}_1 = \mathcal{F}^{r_0}$ ,  $\mathcal{F}_2 = \bigcup_{i < \delta_1} \mathcal{F}^{p_i}$ . (The latter might be only a  $\mathbb{C}_{\mathbf{A}_2}$ -name of a filter).
- For  $i < \delta_1$  let  $\mathfrak{p}^i$  be the  $\mathbb{C}_{\mathbf{A}^{p_i} \cap \gamma_0}$ -name for the  $(\Gamma_{\gamma_0}^{p_i}$ -big) type induced by  $x_{\gamma_0}$  over the model  $\prod_{n < \omega}^{p_i \cap \gamma_0} M_{\mathbf{A}^{p_i} \cap \gamma_0}^n / \mathcal{F}^{p_i \upharpoonright \gamma_0}$ . Then let  $\mathfrak{p}_0 = \bigcup_{i < \delta_1} \mathfrak{p}^i$ , and note that it is a  $\mathbb{C}_{\mathbf{A}_0}$ -name for a  $\Gamma_{\gamma_0}^{p_i}$ -big type over the model  $\prod_{n < \omega}^{A_0} M_{\mathbf{A}_0}^n / \mathcal{F}_0$ .
- Let  $\mathfrak{p}_1$  be (a  $\mathbb{C}_{\mathbf{A}_1}$ -name for) a complete  $\Gamma_{\gamma_0}^{p_i}$ -big type over  $\prod_{n < \omega}^{A_1} M_{\mathbf{A}_1}^n / \mathcal{F}_0$  extending  $\mathfrak{p}_0$ . (Exists by 1.2; the role of  $\mathfrak{p}_1$  is to be the type which  $x_{\gamma_0}$  realizes over  $\prod_{n < \omega}^{A_1} M_{\mathbf{A}_1}^n / \mathcal{F}^{r_0}$  according to a condition  $r$  which we will choose below so necessarily it extends  $\bigcup_{i < \delta_1} \mathfrak{p}^i$ ).

Now, in  $\mathbf{V}^{\mathbf{C}_{\mathbf{A}_3}}$ , we would like to extend  $\mathcal{F}_1 \cup \mathcal{F}_2$  to an ultrafilter  $\mathcal{F}'$  containing the sets of the form  $\{n < \omega : M_{\mathbf{A}_3}^n \models \varphi[x_{\gamma_0}(n)]\}$  for all  $\varphi(x) \in \mathfrak{p}_1$ . If this fails, then as

$$\Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“} \langle \mathcal{F}^{p_i} : i < \delta_1 \rangle \text{ is increasing ”}$$

we find a condition  $p \in \mathbb{C}_{\mathbf{A}_3}$ , a  $\mathbb{C}_{\mathbf{A}_1}$ -name  $\mathfrak{a}$  of a member of  $\mathcal{F}_1$ , and  $i < \delta_1$ , and a  $\mathbb{C}_{\mathbf{A}_2}$ -name  $\mathfrak{b}$  for a member of  $\mathcal{F}_i$ , and  $\varphi$  such that

$$p \upharpoonright \mathbf{A}_1 \Vdash \text{“} \varphi(x) \in \mathfrak{p}^i \subseteq \mathfrak{p}_1 \text{”} \quad \text{and} \quad p \Vdash_{\mathbb{C}_{\mathbf{A}_3}} \text{“} \mathfrak{a} \cap \mathfrak{b} \cap \{n : M_{\mathbf{A}_3}^n \models \varphi[x_{\beta}(n)]\} = \emptyset \text{”}.$$

Next we continue exactly as in the proof of (1).

THE SUBCASE 1B:  $\gamma$  IS A LIMIT ORDINAL OF COFINALITY  $\aleph_2$

Since  $\delta_1 < \aleph_2$  there is some  $\gamma_0 < \gamma$  such that all  $p_i$  lie in  $\text{App} \upharpoonright \gamma_0$  and  $\beta < \gamma_0$ , and the induction hypothesis then yields the claim.

THE SUBCASE 1C:  $\gamma$  IS A LIMIT ORDINAL OF COFINALITY  $\aleph_1$

Choose a strictly increasing and continuous sequence  $\langle \gamma_j : j < \aleph_1 \rangle$  with supremum  $\gamma$ , starting with  $\gamma_0 = \beta$ . By induction on  $j$  choose  $r_j \in \text{App} \upharpoonright \gamma_j$  (for  $j < \aleph_1$ ) such that:

- $r_0 = q^*$ ;
- $r_j \leq_{\text{end}} r_{j'}$  for  $j < j' < \aleph_1$ ;
- $p_i \upharpoonright \gamma_j \leq r_j$  for  $i < \delta_1$  and  $j < \aleph_1$ .



[Thus, at a successor stage  $j + 1$ , the inductive hypothesis is applied to  $p_i \upharpoonright \gamma_{j+1}, r_j, \gamma_j$ , and  $\gamma_{j+1}$ . At a limit stage  $j$ , we apply the inductive hypothesis to  $p_i \upharpoonright \gamma_j$  for  $i < \delta_1$ ,  $r_{j'}$  for  $j' < j$ ,  $\gamma_{j'}$  for  $j' < j$ , and  $\gamma_j$ .] Finally, we let  $r = (\bigcup_{j < \aleph_1} \mathbf{A}^{r_j}, \bigcup_{j < \aleph_1} \mathcal{F}^{r_j}, \bigcup_{j < \aleph_1} \bar{\Gamma}^{r_j})$ . Clearly  $r \in \text{App}$  is as required.

Now we are going to consider the remaining case:

THE CASE 2:  $\gamma$  IS A LIMIT ORDINAL OF COFINALITY  $\aleph_0$

If  $\beta < \gamma$  (where  $\beta$  is as defined at the beginning of the proof), then we first pick a strictly increasing sequence  $\langle \gamma_j : j < \aleph_0 \rangle$  of ordinals such that  $\beta \leq \gamma_0$  and  $\sup_{j < \aleph_0} \gamma_j = \gamma$ . Then we apply repeatedly the inductive hypothesis

to build a sequence  $\langle q'_j : j < \aleph_0 \rangle$  such that  $q'_j \in \text{App} \upharpoonright \gamma_j$ ,  $q'_{j_0} \leq_{\text{end}} q'_{j_1}$  for  $j_0 < j_1$ ,  $q_j \leq_{\text{end}} q'_0$  (for all  $j < \delta_2$ ), and  $p_i \upharpoonright \gamma_j \leq q'_j$  (for all  $i < \delta_1, j < \aleph_0$ ). Thus we have reduced this sub-case to the only one remaining:  $\beta = \gamma$ . Now if for some  $j < \delta_2$  we have  $\beta_j = \gamma$ , then  $r = q_j$  is as required, so without loss of generality  $(\forall j < \delta_2)(\beta_j < \gamma)$ . Then necessarily  $\text{cf}(\delta_2) = \aleph_0$  and we may equally well assume that  $\delta_2 = \aleph_0$ .

We take  $q$  as defined earlier (so it is the “union” of all  $q_j$ ), but it does not have to be a condition in  $\text{App}$ : the filter  $\bigcup_{j < \aleph_0} \mathcal{F}^{q_j}$  does not have to be an ultrafilter, and we need to extend it to one that contains also  $\bigcup_{i < \delta_1} \mathcal{F}^{p_i}$ . Note

that  $\mathbf{A}^* \stackrel{\text{def}}{=} \bigcup_{i < \delta_1} \mathbf{A}^{p_i} \subseteq \bigcup_{j < \aleph_0} \mathbf{A}^{q_j} \stackrel{\text{def}}{=} \mathbf{A}^+$ , but there might be  $\mathbb{C}_{\mathbf{A}^*}$ -names for elements of  $\bigcup_{i < \delta_1} \mathcal{F}^{p_i}$  that are not  $\mathbb{C}_{\mathbf{A}^{q_j}}$ -names for any  $j < \aleph_0$ , so seemingly it could happen that one name like that is forced to be disjoint from some element of  $\mathcal{F}^{q_j}$ . Still, also here  $\bigcup_{j < \aleph_0} \mathcal{F}^{q_j}$  is closed under finite intersection and similarly  $\bigcup_{i < \delta_1} \mathcal{F}^{p_i}$ . So assume toward contradiction, that there are a condition  $p \in \mathbb{C}_{\mathbf{A}^+}$ , ordinals  $i < \delta_1$  and  $j < \aleph_0$ , a  $\mathbb{C}_{\mathbf{A}^{p_i}}$ -name  $\mathbf{a}$ , and a  $\mathbb{C}_{\mathbf{A}^{q_j}}$ -name  $\mathbf{b}$  such that

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^+}} \text{ “ } \mathbf{a} \in \mathcal{F}^{p_i} \ \& \ \mathbf{b} \in \mathcal{F}^{q_j} \ \& \ \mathbf{a} \cap \mathbf{b} = \emptyset \text{ ”}.$$

Increasing  $j$  if necessary, we may also assume that  $p \in \mathbb{C}_{\mathbf{A}^{q_j}}$  so  $\text{Dom}(p) \subseteq \beta_j \times \omega$ . Let  $H^0 \subseteq \mathbb{C}_{\mathbf{A}^{p_i \cap \beta_j}}$  be generic over  $\mathbf{V}$  such that  $p \upharpoonright \mathbf{A}^{p_i} \in H^0$ , and let

$$\mathbf{c} = \{n \in \omega : \text{ there is a condition } p' \in \mathbb{C}_{\mathbf{A}^{p_i}} \text{ stronger than } p \upharpoonright \mathbf{A}^{p_i} \text{ and such that } p' \upharpoonright (\mathbf{A}^{p_i} \cap \beta_j) \in H^0 \text{ and } p' \Vdash_{\mathbb{C}_{\mathbf{A}^{p_i}}} \text{ “ } n \in \mathbf{a} \text{ ”}\}.$$

Clearly,  $\mathbf{c} \in \mathbf{V}[H^0]$  is a set from  $(\mathcal{F}^{p_i} \upharpoonright (\mathbf{A}^{p_i} \cap \beta_j))[H^0]$ . Since  $p_i \upharpoonright \beta_j \leq q_j$ , we find a condition  $p'' \in \mathbb{C}_{\mathbf{A}^{q_j}}$  and  $n \in \mathbf{c}$  such that

$$p \leq p'' \ \& \ p'' \upharpoonright (\mathbf{A}^{p_i} \cap \beta_j) \in H^0 \ \& \ p'' \Vdash_{\mathbb{C}_{\mathbf{A}^{q_j}}} \text{ “ } n \in \mathbf{b} \text{ ”}.$$

For this  $n$  we find  $p' \in \mathbb{C}_{\mathbf{A}^{p_i}}$  witnessing that  $n \in \mathbf{c}$  (i.e.  $p' \upharpoonright (A^{p_i} \cap \beta_j) \in H^0$  and  $p' \Vdash_{\mathbb{C}_{\mathbf{A}^{p_i}}} "n \in \mathfrak{a}"$ ) and next we let  $p^* = p' \cup p''$ . Clearly  $p^* \Vdash n \in \mathfrak{a} \cap \mathfrak{b}$ , a contradiction.

4) Follows, i.e., it is the case  $\delta_2 = 0$  of part (3).

5) We choose  $q_n \in \text{App}_{\gamma_n}$  for  $n < \omega$  such that  $\mathbf{A}^{q_n} := \mathbf{A}^p \cup \{\gamma_\ell : \ell < n\}$ ,  $p \upharpoonright \gamma_n \leq q_n$  and  $q_n \leq_{\text{end}} q_{n+1}$  for  $n < \omega$  and let  $\mathbf{A} = \cup\{\mathbf{A}^{q_n} : n < \omega\}$

This is possible for  $n = 0$  let  $q_n = p \upharpoonright \gamma_{n+1}$ , for  $n = k + 1$ , let  $q'_n \in \text{App}$  be such that  $\mathbf{A}^{q'_n} = \mathbf{A}^{q_k} \cup \{\gamma_n\}$  and  $q_k \leq_{\text{end}} q'_n$ , exists by 2.7, and then  $q_n$  as required exists by 2.8(1).

Let  $x$  be the following  $\mathbb{C}_{\mathbf{A}}$ -name of an  $\omega$ -sequence:

$$x = \langle x_{\gamma_n}(n) : n < \omega \rangle.$$

Now we shall choose  $q$  such that  $\mathbf{A}^q = \mathbf{A} = \cup\{\mathbf{A}^{q_n} : n < \omega\} = \mathbf{A}^p \cup \{\gamma_n : n < \omega\}$ ,  $n < \omega \Rightarrow q_n \leq_{\text{end}} q$  and  $p \leq q$  and  $\Vdash_{\mathbb{C}_{\mathbf{A}}} "x \text{ realizes } \mathfrak{p}"$ .

Again the only problem is to find a  $\mathbb{C}_{\mathbf{A}}$ -name of an ultrafilter on  $\omega$  which include

$$\mathcal{F}^p \cup \bigcup\{\mathcal{F}^{q_n} : n < \omega\} \cup \{\{n : M_{\mathbf{A}^p}^n \models \varphi(x(n))\} : \varphi(x) \in \mathfrak{p}\}$$

As without loss of generality  $\mathfrak{p}$  is closed under conjunction it is enough to show that:

- ⊗ if  $\mathfrak{a}$  is a  $\mathbb{C}_{\mathbf{A}^p}$ -name of a member of  $\mathcal{F}^p$ ,  $n < \omega$ ,  
 $\mathfrak{b}$  is a  $\mathbb{C}_{\mathbf{A}^{q_n}}$ -name of a member of  $\mathcal{F}^{q_n}$   
 $\varphi(x)$  is a  $\mathbb{C}_{\mathbf{A}^p}$ -name of a formula from  $\mathfrak{p}$   
then  $\Vdash_{\mathbb{C}_{\mathbf{A}}} "\mathfrak{a} \cap \mathfrak{b} \cap \{n : M_{\mathbf{A}^p} \models \varphi(x(n))\} \neq \emptyset"$ . As in previous cases this is easy. □

**Lemma 2.9.** *Assume  $\mathbf{V} \models \text{GCH}$ . The forcing notion  $\text{App}$  satisfies the  $\aleph_3$ -chain condition, it is  $\aleph_2$ -complete,  $\|\text{App}\| = \aleph_3$  and  $\|\text{App} \upharpoonright \gamma\| \leq \aleph_2$  for every  $\gamma \in \aleph_3$ . Consequently, the forcing with  $\text{App}$  does not collapse cardinals nor changes cofinalities, and  $\Vdash_{\text{App}} \text{GCH}$ .*

*Proof.* The only perhaps unclear part is the chain condition. Suppose towards contradiction that we have an antichain  $\{q_\alpha : \alpha \in \aleph_3 \ \& \ \text{cf}(\alpha) = \aleph_2\} \subseteq \text{App}$  (the index  $\alpha$  is taken to vary over ordinals of cofinality  $\aleph_2$  just for convenience). An important point is that  $\mathbf{G}$  can “offer” at most  $\aleph_2$  candidates for the bigness notion at  $\delta < \aleph_3$ ,  $\text{cf}(\delta) = \aleph_2$ , hence for each  $\gamma \in \aleph_3$  the restricted forcing  $\text{App} \upharpoonright \gamma$  has cardinality  $\leq \aleph_2$ . Applying Fodor’s lemma twice, we find a stationary set  $S \subseteq \{\alpha \in \aleph_3 : \text{cf}(\alpha) = \aleph_2\}$  and a condition  $q^* \in \text{App}$  such that  $(\forall \alpha \in S)(q_\alpha \upharpoonright \alpha = q^*)$ . Pick  $\alpha_1, \alpha_2 \in S$  such that  $\text{sup}(\mathbf{A}^{q_{\alpha_1}}) < \alpha_2$ ; it follows from Lemma 2.8(3) that the conditions  $q_{\alpha_1}, q_{\alpha_2}$  are compatible, a contradiction. □

**Proposition 2.10.** (1) *For each  $p \in \text{App}$  and  $\alpha \in \aleph_3$ , there is a condition  $q \in \text{App}$  stronger than  $p$  and such that  $\alpha \in \mathbf{A}^q$ .*

- (2)  $\mathcal{F} \stackrel{\text{def}}{=} \bigcup \{ \mathcal{F}^r : r \in G_{\text{App}} \}$  is an App-name of a  $\mathbb{C}_{\aleph_3}$ -name for a non-principal ultrafilter on  $\omega$ . Also, for each  $r \in G_{\text{App}}$  we have:  $\mathcal{F} \cap \mathcal{P}(\omega)^{\mathbf{V}[G_{\text{App}}]}^{\mathbf{C}^{\mathbf{A}^r}} = \mathcal{F}^r$ .

*Proof.* Should be clear (for (1) use 2.7 + 2.8(3); then (2) follows).  $\square$

- Definition 2.11.** (1) Suppose  $G_{\text{App}} \subseteq \text{App}$  is generic over  $\mathbf{V}$ ,  $\mathbf{V}^* = \mathbf{V}[G_{\text{App}}]$ . For  $\alpha \leq \aleph_3$  we let  $G_\alpha = G_{\text{App}} \cap (\text{App} \upharpoonright \alpha)$ . It is a generic subset of  $\text{App} \upharpoonright \alpha$ ; let  $\mathcal{F}^\alpha$  be the  $(\text{App} \upharpoonright \alpha)$ -name of the  $\mathbb{C}_\alpha$ -name  $\bigcup \{ \mathcal{F}^q : q \in G_\alpha \}$ . Note:  $\mathcal{F}^q$  being a  $\mathbb{C}_{\mathbf{A}^q}$ -name is a  $\mathbb{C}_\alpha$ -name when  $\mathbf{A}^q \subseteq \alpha$ . So in  $\mathbf{V}^*$  the sequence  $\langle \mathcal{F}^\alpha : \alpha < \aleph_3 \rangle$  is forced (i.e.  $\Vdash_{\mathbb{C}}$ ) to be increasing, let  $\mathcal{F} = \mathcal{F}^{\aleph_3}$  so  $\mathcal{F}^\alpha$  is the  $\mathbb{C}_\alpha$ -name for the restriction  $\mathcal{F} \upharpoonright \alpha$  of the ultrafilter  $\mathcal{F}$  to the sets from the universe  $(\mathbf{V}^*)^{\mathbb{C}_\alpha}$ .
- (2) We define an App-name  $\Gamma_\delta$  of a  $\mathbb{C}_\delta$ -name as  $\Gamma_\delta^p$  for every  $p \in G_{\text{App}}$  such that  $\delta \in \mathbf{A}^p$ . (So it is an App \*  $\mathbb{C}_\delta$ -name.)

**Lemma 2.12.** (1) Suppose that  $G_{\text{App}} \subseteq \text{App}$  is generic over  $\mathbf{V}$ ,  $\mathbf{V}^* = \mathbf{V}[G_{\text{App}}]$ , and  $\delta < \aleph_3$ ,  $\text{cf}(\delta) = \aleph_2$ , and  $H^\delta \subseteq \mathbb{C}_\delta$  is generic over  $\mathbf{V}^*$ . Then, in  $\mathbf{V}[G_{\text{App}} \cap (\text{App} \upharpoonright \delta)][H^\delta]$ , we have<sup>9</sup>:

- $\prod_{n < \omega} M_\delta^n / \mathcal{F}^\delta[H^\delta]$  is  $\aleph_2$ -compact.
- (2) Also if  $H \subseteq \mathbb{C}_{\aleph_3}$  is generic over  $\mathbf{V}^*$ ,  $H \supseteq H^\delta$ , then in  $\mathbf{V}^*[H]$ :
- (a)  $\prod_{n < \omega} M_{\aleph_3}^n / \mathcal{F}[H]$  is  $\aleph_2$ -compact,
- (b)  $x_\delta[H] / \mathcal{F}[H] \in \prod_{n < \omega} M_{\aleph_3}^n / \mathcal{F}[H]$  realizes a  $\Gamma_\delta[G][H^\delta]$ -big type over  $\prod_{n < \omega} M_\delta^n / \mathcal{F}^\delta[H^\delta]$ .

*Proof.* By 2.7(1)+2.7(2). We can use some  $x_\beta$  with  $\beta$  of cofinality less than  $\aleph_2$  to realize each type.  $\square$

### 3. DEFINABILITY

*Hypothesis 3.1.* In this section we assume that  $\mathbf{G}$  is an  $(\aleph_3, \aleph_2)$ -bigness guide,  $\text{App} = \text{App}_{\mathbf{G}}$ ,  $G^* \subseteq \text{App}$  is a generic filter over  $\mathbf{V}$ , and  $\mathbf{V}^* = \mathbf{V}[G^*]$ . For an ordinal  $\alpha < \aleph_3$ , we let  $G_\alpha^* = G^* \cap (\text{App} \upharpoonright \alpha)$ . Also,  $\underline{H}$ ,  $\underline{H}^\alpha$  are the canonical  $\mathbb{C}_{\aleph_3}$ - and  $\mathbb{C}_\alpha$ -names of the generic subsets of  $\mathbb{C}_{\aleph_3}$  and  $\mathbb{C}_\alpha$ , respectively. We work mostly in  $\mathbf{V}^*$ .

[Note that, by Lemma 2.9,  $\mathbf{V}^* \models \text{GCH}$ .]

**Definition 3.2.** (1) We say that  $\mathbf{m}$  is an  $(\aleph_3, \aleph_2)$ -isomorphism candidate (or just an isomorphism candidate, in  $\mathbf{V}$  or in  $\mathbf{V}^*$ , see below) if;

- (i)  $\mathbf{m}$  consists of  $\mathbf{A}^* = \mathbf{A}^*[\mathbf{m}] \in [\aleph_3]^{<\aleph_2}$ ,  $p^* = p^*[\mathbf{m}]$ ,  $N_n^\ell = N_n^\ell[\mathbf{m}]$ ,  $t_n^\ell$  (for  $n < \omega$ ,  $\ell \in \{1, 2\}$ ),  $\underline{F} = \underline{F}[\mathbf{m}]$ ,  $\underline{\Gamma} = \underline{\Gamma}[\mathbf{m}]$  and  $(\underline{t}, \underline{\varphi}, \underline{\psi}, \underline{\Delta}) = (t[\mathbf{m}], \varphi[\mathbf{m}], \psi[\mathbf{m}], \Delta[\mathbf{m}])$ ,

<sup>9</sup>Note:  $M_\delta^n$  is  $M_{\mathbf{A}}^n$  for  $\mathbf{A} = \delta$

- (ii)  $t, \psi, \bar{\varphi}$  are  $\mathbb{C}_{\mathbf{A}^*}$ -names as in 2.2(3),  $\underline{\Delta} \subseteq \mathbb{L}(\tau(t))$  is a  $\mathbb{C}_{\mathbf{A}^*}$ -name, equality belongs to it, and  $\underline{\Gamma} = \underline{\Gamma}_{(t, \psi, \bar{\varphi})}$  is a bigness notion as there,  $\tau(t)$  is countable; we can assume  $\tau(t)$  is an object (not a name) by adding for each  $m, \aleph_0$  predicates with  $m$  places said (by  $t$ ) to be empty.
- (iii)  $N_n^\ell$ , for  $n < \omega$  and  $\ell \in \{1, 2\}$ , are  $\mathbb{C}_{\mathbf{A}^*}$ -names for countable models of a (countable) theory  $t_n^\ell$ , and the universes  $|N_n^\ell|$  are subsets of  $\omega$  and with vocabulary  $\tau(t)$ .

Also it is forced (i.e.,  $\Vdash_{\mathbb{C}_{\aleph_3}}$ ) that  $t \subseteq \text{Th}\left(\prod_{n < \omega} N_n^1/\mathcal{F}\right) = \text{Th}\left(\prod_{n < \omega} N_n^2/\mathcal{F}\right)$ ,

where the  $\prod$  is  $\prod_{n < \omega}^{\aleph_3}$ . Note that we cannot require that  $t_n^\ell = t$ ,

as  $t$  may be infinite, (e.g.  $t_0^{\text{ind}}$  is) and no  $N_n^\ell$  is a model of  $t$ .

- (iv) We have predicates  $Q_R^\ell \in \tau_{\mathbf{A}^*}$  (for  $R \in \tau(t)$ ) such that  $\bar{\varphi}^\ell = \langle Q_R^\ell : R \in \tau(t) \rangle$  is the interpretation of  $\tau(t)$  in  $\prod_{n < \omega}^{\mathbf{A}^*} M_{\mathbf{A}^*}^n/\mathcal{F}$  giving  $\prod_{n < \omega} N_n^\ell/\mathcal{F}$ . (Remember 2.1(4), 1.4(1); so by the choice of  $\tau_{\mathbf{A}^*}$  actually  $\bar{\varphi}^* = \bar{\varphi}^*$ .)
- (v)  $\underline{F}$  is a  $\mathbb{C}_{\aleph_3}$ -name (more accurately an  $\mathbb{A}\text{pp}$ -name of such name, but we sometimes write  $\underline{F}$  instead of  $\underline{F}[G^*]$  as when  $G^*$  is constant) and  $p^* \in \mathbb{C}_{\aleph_3}$  is a condition such that:

$$p^* \Vdash_{\mathbb{C}_{\aleph_3}} \text{“}\underline{F} \text{ is a map from } \prod_{n < \omega} N_n^1 \text{ into } \prod_{n < \omega} N_n^2\text{”}$$

$$p^* \Vdash_{\mathbb{C}_{\aleph_3}} \text{“}\underline{F} \text{ represents a } \underline{\Delta}\text{-embedding modulo } \mathcal{F}\text{”}.$$

[If  $\mathbf{m}$  is clear from the context we may omit it.]

*Remark 3.3.* (1) In  $\mathbf{m}$ , note that  $\underline{\Delta}$  tells us which first order formulas in the vocabulary  $\tau(t)$  does the function  $\underline{F}$  preserve. In our main case those are the atomic and negation of atomic formulas in  $\tau^{\text{ind}}$

- (2) Of course  $\mathbf{m}$  gives us two interpretations of  $t$  in the ultraproduct: one for  $\ell = 1$  and another for  $\ell = 2$ , and the interpreting formulas define  $N_n^\ell$  in the  $n$ -th coordinate. Without loss of generality the universe of  $N_n^\ell$  is non-empty for every  $n < \omega$  (and  $\ell = 1, 2$ ).

**Definition 3.4.** For  $\mathbf{m}$  as in 3.2 let

$$\mathbf{m}^- = \langle t, \psi, \bar{\varphi}, \underline{\Delta}, \langle N_n^\ell : n < \omega, \ell = 1, 2 \rangle \rangle,$$

those names involve countably many of the Cohens  $x_\beta$ . Also note that as  $\mathbb{A}\text{pp}$  is  $\aleph_2$ -complete, this forcing does not add new  $\mathbf{m}^-$ , i.e.,  $\mathbf{V}$  and  $\mathbf{V}^*$  have the same set of  $\mathbf{m}^-$ , though we have an  $\mathbb{A}\text{pp}$ -name  $\underline{\mathbf{m}}$  of such object.

**Observation 3.5.** Assume, in  $\mathbf{V}^*$ , that  $\mathbf{m}$  is an  $(\aleph_3, \aleph_2)$ -isomorphism candidate,  $\underline{\Gamma} = \underline{\Gamma}[\mathbf{m}] = \Gamma_{(t, \bar{\varphi}, \psi)}$ . Then there is a stationary set of ordinals  $\delta < \aleph_3$  such that:

- (a) $_\delta$   $\mathbf{A}^* = \mathbf{A}^*[\mathbf{m}] \subseteq \delta \cap \mathbf{A}^q$ ,  $\text{cf}(\delta) = \aleph_2$ , and  $p^* = p^*[\mathbf{m}] \in \mathbb{C}_\delta$ , and for some  $q \in G^*$  we have that  $\underline{\Gamma} = \underline{\Gamma}_\delta^q$  is  $\Gamma_\psi[\bar{\varphi}]$  (for  $(t, \psi, \bar{\varphi})$  from 2.2),

- (b) $_{\delta}$  for every  $\mathbb{C}_{\aleph_3} \upharpoonright \delta$ -name  $\underline{x}$  for an element of  $\prod_{n < \omega} N_n^1$ ,  $\underline{F}(\underline{x})$  is a  $(\mathbb{C}_{\aleph_3} \upharpoonright \delta)$ -name,  
 [recall  $\text{App}$  satisfies the  $\aleph_3$ -c.c.]  
 (c) $_{\delta}$  similarly for  $\underline{F}^{-1}$  and for “ $y \in \text{Rang}(\underline{F})$ ”,  
 (d) $_{\delta} \Vdash_{\mathbb{C}_{\aleph_3}}$  “ $\{n < \omega : \underline{x}_{\delta}(n) \in N_n^1\} \in \mathcal{F}$  (so  $\underline{x}_{\delta}/\mathcal{F} \in \prod_{n < \omega} N_n^1/\mathcal{F}$ )”.

For such  $\delta$ , we let  $\underline{y}^* = \underline{y}_{\delta}^* = \underline{y}_{\delta, \underline{F}}^* = \underline{y}_{\delta, \mathbf{m}}^*$  be  $\underline{F}(\underline{x}_{\delta}) \in \prod_{n < \omega} N_n^2$ .

Remark: Also notice that the clauses (b) $_{\delta}$ , (c) $_{\delta}$  of 3.5 above say that  $\underline{F}^{\delta}[G^*]$  is really a  $\mathbb{C}_{\delta}$ -name for a function from  $(\prod_{n < \omega} N_n^1)^{(\mathbf{V}^*)^{\mathbb{C}_{\delta}}}$  into  $(\prod_{n < \omega} N_n^2)^{(\mathbf{V}^*)^{\mathbb{C}_{\delta}}}$  preserving  $\Delta$ -formulas; in the main case it is “onto”.

**The Main Isomorphism Theorem 3.6.** *Assume that  $\mathbf{m}$  is an  $(\aleph_3, \aleph_2)$ -isomorphism candidate as in 3.2, and  $\delta < \aleph_3$  is as in Observation 3.5. Then there are  $q_{\delta}, \Gamma, \underline{y}$  such that*

- (a)  $q_{\delta} \in \text{App}$ , moreover  $q_{\delta} \in G^*$ , and  $\Gamma = \Gamma_{\delta}^{q_{\delta}}$  is  $\Gamma_{\psi}[\bar{\varphi}]$  for  $(\underline{t}, \underline{\psi}, \bar{\varphi})$  from 2.2 (the set of choices of  $q_{\delta}$  is dense and quite closed)
- (b)  $q_{\delta} \Vdash_{\text{App}} p^* \Vdash_{\mathbb{C}_{\aleph_3}}$  “ $\underline{F}(\underline{x}_{\delta}) = \underline{y}^*$ ”, where  $\underline{y}^*$  is a  $\mathbb{C}_{\mathbf{A}^{q_{\delta}}}$ -name of a member of  ${}^{\omega}\omega$ ,
- (c)  $\mathbf{A}^* \subseteq \mathbf{A}^{q_{\delta}}$ ,  $\mathbf{A}_{\delta} \stackrel{\text{def}}{=} \mathbf{A}^{q_{\delta}} \cap \delta$ ,
- (d) in  $\mathbf{V}[G_{\delta}^*][\underline{H}^{\delta}]$  we have:  
 (i)  $\mathcal{F}_{\delta} = \mathcal{F}_{\delta}[G_{\delta}^*][\underline{H}^{\delta}]$  is a non-principal ultrafilter on  $\omega$ .  
 (ii) The model  $M_{\delta} = \prod_{n < \omega}^{\delta} M_{\delta}^n/\mathcal{F}_{\delta}$  with the vocabulary  $\tau_{\delta}$  is  $\aleph_2$ -compact where  $M_{\delta}^n = M_{\delta}^n[G_{\delta}^*][\underline{H}^{\delta}]$  and  $N_n^{\ell} = N_n^{\ell}[G_{\delta}^*][\underline{H}^{\delta}]$ .  
 (iii) The vocabulary  $\tau_{\mathbf{A}_{\delta}} \subseteq \tau_{\delta}$  is of cardinality  $\leq \aleph_1$ .  
 (iv)  $M_{\mathbf{A}_{\delta}} = \prod_{n < \omega}^{\mathbf{A}_{\delta}} M_{\mathbf{A}_{\delta}}^n/\mathcal{F}^{q_{\delta} \upharpoonright \delta}[\underline{H}^{\delta}] \prec M_{\delta} \upharpoonright \tau_{\mathbf{A}_{\delta}}$ .  
 (v)  $p^* \Vdash_{\mathbb{C}_{\delta}}$  “ $\underline{F}_{\delta} = (\underline{F} \upharpoonright \delta)[\underline{H}^{\delta}] = ((\underline{F} \upharpoonright \delta)[G^* \cap (\text{App} \upharpoonright \delta)])[\underline{H}^{\delta}]$  is a  $\Delta$ -embedding from the model  $\prod_{n < \omega}^{\delta} N_n^1/\mathcal{F}_{\delta}$  into  $\prod_{n < \omega}^{\delta} N_n^2/\mathcal{F}_{\delta}$ ”, recalling  $p^* = p^*[\mathbf{m}]$
- (vi) Let  $\underline{p}_{\delta} = \underline{p}_{\delta}(x)$  be the  $(\mathbb{C}_{\mathbf{A}_{\delta}})$ -name of the 1-type in the vocabulary  $\tau_{\mathbf{A}_{\delta}}$  such that  $q_{\delta} \Vdash_{\text{App}} p^* \Vdash_{\mathbb{C}_{\delta}}$  “ $\underline{p}_{\delta}(x)$  is the type realized by  $\underline{x}_{\delta}$  over  $M_{\mathbf{A}_{\delta}}$  in  $\prod_{n < \omega} M_{\mathbf{A}_{\delta}}^n/\mathcal{F}^{q_{\delta}}$ ”. [Clearly it is a  $\mathbb{C}_{\mathbf{A}^{q_{\delta}}}$ -name, or an  $\text{App} * \mathbb{C}_{\mathbf{A}^{q_{\delta}}}$ -name; see clause (d) of Definition 2.4(1).] Clearly  $q_{\delta} \Vdash_{\text{App}} p^* \Vdash_{\mathbb{C}_{\delta}}$  “ $\underline{p}_{\delta}$  is  $\Gamma$ -big”.
- (vii) For  $\ell = 1, 2$  let  $N_{\delta}^{\ell} = \prod_{n < \omega}^{\delta} N_n^{\ell}/\mathcal{F}_{\delta}$  (they are in  $\mathbf{V}^*[\underline{H}^{\delta}]$ , even in  $\mathbf{V}[G_{\delta}^*][\underline{H}^{\delta}]$ ). We define  $R_{\delta, m} \subseteq (N_{\delta}^1)^m \times (N_{\delta}^2)^m$  for  $m < \omega$  so that they are  $(\text{App} \upharpoonright \delta) * \mathbb{C}_{\delta}$ -names and  $(q_{\delta} \upharpoonright \mathbf{A}_{\delta}, p^*)$  forces  $(\otimes)_1 R_{\delta, m}$  includes the graph of  $F_{\delta}$ , i.e., if  $\bar{a}$  is an  $m$ -tuple from  $N_{\delta}^1$ , then  $(\bar{a}, F_{\delta}(\bar{a})) \in R_{\delta, m}$ ,

- ( $\otimes$ )<sub>2</sub> the truth value of  $(\bar{a}, \bar{b}) \in R_{\delta, m}$  depends only on  $\mathbb{L}_{\omega, \omega}(\tau_{\mathbf{A}_\delta})$ -type realized by  $(\bar{a}, \bar{b})$  over  $M_{\mathbf{A}_\delta}$  in  $M_\delta$ ,
- ( $\otimes$ )<sub>3</sub>  $R_{\delta, m}$  is minimal such that ( $\otimes$ )<sub>1</sub> + ( $\otimes$ )<sub>2</sub> hold.
- (viii) The relations  $R_{\delta, m}$  mentioned above satisfy (i.e.  $(q_\delta \upharpoonright \mathbf{A}_\delta, p^*)$  forces):
- ( $\oplus$ )<sub>1</sub> if  $\bar{a}_1, \bar{a}_2$  are finite sequences of the same length  $m$  of members of  $N_\delta^1$ , and  $p_\delta \cup \{\vartheta^{N_\delta^1}(x, \bar{a}_1), \neg\vartheta^{N_\delta^1}(x, \bar{a}_2)\}$  is a  $\Gamma$ -big type over  $M_\delta$ , and  $\vartheta, \neg\vartheta \in \underline{\Delta}[\mathbf{m}]$ , where  $\vartheta^{N_\delta^1}$  is  $\vartheta$  as interpreted in the interpretation  $\bar{\varphi}^1$ ,  
then  $(\bar{a}_1, F_\delta(\bar{a}_2)) \notin R_{\delta, m}$ .
- ( $\oplus$ )<sub>2</sub> Above, we may replace  $\vartheta, \neg\vartheta$  by any pair  $\vartheta_0, \vartheta_1$  of contradictory formulas from  $\underline{\Delta}[\mathbf{m}]$ .
- (ix) Note that also
- (\*) <sub>$y^*, \delta$</sub>  <sup>$p^*$</sup>   $p^* \Vdash_{\mathbb{C}_{\aleph_3}}$  “the  $\underline{\Delta}$ -type which  $y^*$  realizes over  $\underline{N}_\delta^2 = (\prod_{n < \omega} \underline{N}_n^2 / \underline{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3} \upharpoonright \delta}}$  in the model  $\underline{N}^2 = (\prod_{n < \omega} \underline{N}_n^2 / \underline{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}}}$  includes the image under  $\underline{F}$  of the  $\underline{\Delta}$ -type which  $\underline{x}_\delta / \underline{\mathcal{F}}$  realizes over  $\underline{N}_\delta^1 = (\prod_{n < \omega} \underline{N}_n^1 / \underline{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3} \upharpoonright \delta}}$  in the model  $\underline{N}^1 = (\prod_{n < \omega} \underline{N}_n^1 / \underline{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}}}$ ”.

**The proof of the Main Isomorphism Theorem 3.6.** Note that we use the countability of  $\underline{t}$ .

Take a condition  $q_\delta \in G^*$  such that

- (A) <sup>$q_\delta$</sup>   $\mathbf{A}^* \subseteq \mathbf{A}^{q_\delta}$  recalling that  $\mathbf{m}$  determine  $\mathbf{A}^*$ ,  $\underline{x}_\delta, \underline{y}^*$  are  $\mathbb{C}_{\mathbf{A}^{q_\delta}}$ -names (so  $\delta \in \mathbf{A}^{q_\delta}$ ), and  $p^* \in \mathbb{C}_{\mathbf{A}^{q_\delta} \cap \delta}$ , and
- (B) <sup>$q_\delta$</sup>  the condition  $q_\delta$  forces (in  $\mathbb{A}pp$ ) that clauses (b) <sub>$\delta$</sub> , (c) <sub>$\delta$</sub>  and (d) <sub>$\delta$</sub>  from 3.5 hold true (so in particular  $q_\delta$  forces that  $\underline{x}_\delta / \underline{\mathcal{F}} \in \prod_{n < \omega} \underline{N}_n^1 / \underline{\mathcal{F}}$ ,  $\underline{y}^* \in \prod_{n < \omega} \underline{N}_n^2$  and (\*) <sub>$y^*, \delta$</sub>  <sup>$p^*$</sup>  from clause (ix) of 3.6 holds as  $\underline{F}$  is (forced to be) a  $\underline{\Delta}$ -embedding), and
- (C) <sup>$q_\delta$</sup>  if  $\underline{x}$  is a  $\mathbb{C}_{\mathbf{A}^{q_\delta}}$ -name for a member of  $\prod_{n < \omega} \underline{N}_n^1$  ( $\prod_{n < \omega} \underline{N}_n^2$ , respectively), **then**  $\underline{F}(\underline{x})$  ( $\underline{F}^{-1}(\underline{x})$ , respectively) is also a  $\mathbb{C}_{\mathbf{A}^{q_\delta}}$ -name.

Before we continue with the proof of 3.6, let us note the following.

**Lemma 3.7.** Let  $\delta < \aleph_3$ ,  $q_\delta \in \mathbb{A}pp$  and  $\underline{y}^*, p^*$  be as above. Suppose that

$$q_\delta \upharpoonright \delta = q \leq q' \in G^* \cap (\mathbb{A}pp \upharpoonright \delta).$$

Let  $\underline{y}^*$  be a  $\mathbb{C}_{\mathbf{A}^*}$ -name of a  $\tau(\underline{t})$ -formula. Assume further that  $\underline{x}', \underline{x}''$  and  $\underline{y}', \underline{y}''$  are  $\mathbb{C}_{\mathbf{A}^{q'}}$ -names, and  $p^* \leq p \in \mathbb{C}_{\mathbf{A}^{q'}}$ , and the condition  $p$  forces (in  $\mathbb{C}_{\mathbf{A}^{q'}}$ ) that

- ( $\alpha$ )  $\underline{x}', \underline{x}'' \in \prod_{n < \omega} \underline{N}_n^1$ , and  $\underline{y}', \underline{y}'' \in \prod_{n < \omega} \underline{N}_n^2$ , and

( $\beta$ ) the types of  $(x', y')$  and of  $(x'', y'')$  over  $\prod_{n < \omega}^{\mathbf{A}^q} M_{\mathbf{A}^q}^n / \mathcal{F}^q$  in the model

$\prod_{n < \omega}^{\mathbf{A}^{q'}} M_{\mathbf{A}^{q'}}^n / \mathcal{F}^{q'}$  (i.e., the vocabulary and the  $\omega$  structures are from  $\mathbf{V}[G_\delta^*][\underline{H} \cap \mathbb{C}_{\mathbf{A}^q}]$ , the ultraproduct is taken in  $\mathbf{V}[G_\delta^*][\underline{H} \cap \mathbb{C}_{\mathbf{A}^{q'}}]$ ) are equal.

Then the following conditions are equivalent.

(A) There is  $r^0 \in \text{App}$  such that  $q_\delta, q' \leq r^0$ ,  $r^0 \upharpoonright \delta \in G^* \cap (\text{App} \upharpoonright \delta)$ , and

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^0}}} \text{ “ } \prod_{n < \omega}^{\mathbf{A}^{r^0}} N_n^1 / \mathcal{F}^{r^0} \models \vartheta^*[x' / \mathcal{F}^{r^0}, x_\delta / \mathcal{F}^{r^0}] \text{ and } \\ \prod_{n < \omega}^{\mathbf{A}^{r^0}} N_n^2 / \mathcal{F}^{r^0} \models \neg \vartheta^*[y' / \mathcal{F}^{r^0}, y^* / \mathcal{F}^{r^0}] \text{ ” .}$$

(B) There is  $r^1 \in \text{App}$  such that  $q_\delta, q' \leq r^1$ ,  $r^1 \upharpoonright \delta \in G^* \cap (\text{App} \upharpoonright \delta)$  and

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^1}}} \text{ “ } \prod_{n < \omega}^{\mathbf{A}^{r^1}} N_n^1 / \mathcal{F}^{r^1} \models \vartheta^*[x'' / \mathcal{F}^{r^1}, x_\delta / \mathcal{F}^{r^1}] \text{ and } \\ \prod_{n < \omega}^{\mathbf{A}^{r^1}} N_n^2 / \mathcal{F}^{r^1} \models \neg \vartheta^*[y'' / \mathcal{F}^{r^1}, y^* / \mathcal{F}^{r^1}] \text{ ” .}$$

Remark: Note that  $y^*$  is not necessarily a  $\mathbb{C} \upharpoonright (\mathbf{A}^{q_\delta} \cap (\delta + 1))$ -name (though  $x_\delta$  is), this somewhat complicates the proof.

*Proof.* By symmetry it suffices to show that (A) implies (B). So suppose that  $r^0$  is as in (A). By 3.10+3.11 below we are done.  $\square$

*Proof.* CONTINUATION OF THE PROOF OF 3.6: We define some  $\mathbb{C}_\delta$ -names; recall  $\underline{H}^\delta \subseteq \mathbb{C}_{\aleph_3} \upharpoonright \delta$  is generic over  $\mathbf{V}^*$ ,  $\mathcal{F}_\delta[\underline{H}^\delta] = \bigcup \{ \mathcal{F}^{r'}[\underline{H}^\delta] : r' \in G_\delta \}$ , and

$$M_\delta^* = \prod_{n < \omega}^\delta M_\delta^n / \mathcal{F}_\delta, \quad \text{and} \quad N_\delta^\ell = \prod_{n < \omega}^\delta N_n^\ell / \mathcal{F}_\delta \quad (\text{for } \ell = 1, 2).$$

Let

$$\mathcal{Z}_\delta^1[\underline{H}^\delta] = \left\{ (x / \mathcal{F}_\delta, y / \mathcal{F}_\delta) \in N_\delta^1 \times N_\delta^2 : \text{there are a } \tau(t)\text{-formula } \vartheta \in \underline{\Delta} \text{ and} \right. \\ \text{conditions } p \in \mathbb{C}_{\aleph_3} \text{ and } r^0 \in \text{App} \text{ such that} \\ p^* \leq p, p \upharpoonright \delta \in \underline{H}^\delta, x, y \text{ are } \mathbb{C}_{\mathbf{A}^{r^0} \cap \delta}\text{-names, and} \\ q_\delta \leq r^0, r^0 \upharpoonright \delta \in G^* \cap (\text{App} \upharpoonright \delta), \text{ and} \\ p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^0}}} \text{ “ } \prod_{n < \omega}^{\mathbf{A}^{r^0}} N_n^1 / \mathcal{F}^{r^0} \models \vartheta[x / \mathcal{F}^{r^0}, x_\delta / \mathcal{F}^{r^0}] \text{ and} \\ \left. \prod_{n < \omega}^{\mathbf{A}^{r^0}} N_n^2 / \mathcal{F}^{r^0} \models \neg \vartheta[y / \mathcal{F}^{r^0}, y^* / \mathcal{F}^{r^0}] \text{ ” } \right\}, \\ \mathcal{Z}_\delta^0[\underline{H}^\delta] = (N_\delta^1 \times N_\delta^2) \setminus \mathcal{Z}_\delta^1.$$

Now, it follows from 3.7 (and 2.8) that

$(\square)_\delta$  in  $\mathbf{V}[G^* \cap (\text{App} \upharpoonright \delta)][\underline{H}^\delta]$ , if the types realized by  $(x' / \mathcal{F}_\delta, y' / \mathcal{F}_\delta)$  and  $(x'' / \mathcal{F}_\delta, y'' / \mathcal{F}_\delta)$  over the model  $\prod_{n < \omega}^{\mathbf{A}^{q_\delta \cap \delta}} M_{\mathbf{A}^{q_\delta \cap \delta}}^n / \mathcal{F}^{q_\delta \upharpoonright \delta}$  in the model

$\prod_{n < \omega}^{\delta} M_{\mathbf{A}^{q_{\delta} \cap \delta}}^n / \mathcal{F}_{\delta}$  are equal, **then**

$$(x' / \mathcal{F}_{\delta}, y' / \mathcal{F}_{\delta}) \in \mathcal{Z}_{\delta}^0 \quad \text{if and only if} \quad (x'' / \mathcal{F}_{\delta}, y'' / \mathcal{F}_{\delta}) \in \mathcal{Z}_{\delta}^0.$$

Now, most clauses of 3.6 should be clear; we say more on (d)(vii,viii), for notational simplicity for  $m = 1$ .

We let  $\mathcal{R}_{\delta,1} = \mathcal{Z}_{\delta}^0$ , so clause (d)(vii)( $\otimes$ )<sub>2</sub> holds.

Since  $\mathcal{F}$  is (an  $\text{App} * \mathbb{C}_{\aleph_3}$ -name for) a  $\Delta$ -embedding from  $\prod_{n < \omega} N_n^1 / \mathcal{F}$  onto  $\prod_{n < \omega} N_n^2 / \mathcal{F}$ , if  $x / \mathcal{F}_{\delta} \in N_{\delta}^1$ , then  $\Vdash_{\mathbb{C}_{\delta}} \text{“} (x / \mathcal{F}_{\delta}, \mathcal{F}(x) / \mathcal{F}_{\delta}) \in \mathcal{Z}_{\delta}^0 \text{”}$ . Hence clause (d)(viii)( $\oplus$ )<sub>1</sub> holds.

Thus the proof of 3.6 is completed.  $\square$

*Conclusion 3.8.* In  $\mathbf{V}[G^*][H^{\aleph_3}]$ , for each  $\mathbf{m}$ , there is a stationary set  $S \subseteq \{\delta < \aleph_3 : \text{cf}(\delta) = \aleph_2\}$  and conditions  $q, q_{\delta} \in \text{App}$  such that for each  $\delta \in S$ :

- clauses (a) <sub>$\delta$</sub> –(d) <sub>$\delta$</sub>  of 3.5 are satisfied,
- $q_{\delta} \in G^*$ ,  $q_{\delta} \upharpoonright \delta = q$ ,  $q_{\delta}, y_{\delta}$  as in 3.5,
- the conclusion of 3.6 holds,
- for every  $\delta_1, \delta_2 \in S$  there is a one-to-one order preserving function  $h : \mathbf{A}^{q_{\delta_1}} \xrightarrow{\text{onto}} \mathbf{A}^{q_{\delta_2}}$  (so it is the identity on  $\mathbf{A}^q$ ) which maps  $\delta_1, q_{\delta_1}, x_{\delta_1}, \mathcal{F}(x_{\delta_1}) = y_{\delta_1}$  onto  $\delta_2, q_{\delta_2}, x_{\delta_2}, \mathcal{F}(x_{\delta_2}) = y_{\delta_2}$ ,

*Proof.* Straightforward.  $\square$

We still have some debts as 3.11,3.10 were used in the proof of 3.6

**Definition 3.9.** (1) Let  $\otimes_{\beta,q,r,s,f}$  mean that

- (a)  $q, r, s \in \text{App}_{\beta}$
- (b)  $q \leq r$  and  $q \leq s$
- (c)  $\mathbf{A}^r = \mathbf{A}^s$  call it  $\mathbf{A}$
- (d)  $f$  is a  $\mathbb{C}_{\mathbf{A}}$ -name of a partial (one to one) elementary mapping from  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}^r$  into  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}^s$  over  $\prod_{n < \omega}^{\mathbf{A}^q} M_{\mathbf{A}^q}^n / \mathcal{F}^q$ ; i.e.
  - ( $\alpha$ )  $f$  is a subset of  $\{(a, b) : a, b \text{ are canonical } \mathbb{C}_{\mathbf{A}}\text{-names of } \omega\text{-sequences of natural numbers}\}$ ,
  - ( $\beta$ ) if  $G_{\mathbf{A}} \subseteq \mathbb{C}_{\mathbf{A}}$  is generic over  $\mathbf{V}$  then in  $\mathbf{V}[G_{\mathbf{A}}]$ , the set  $\{(a[G_{\mathbf{A}}], b[G_{\mathbf{A}}]) : (a, b) \in f\}$  is a function and
  - ( $\gamma$ ) if moreover in  $\mathbf{V}[G_{\mathbf{A}}]$  the first order formula  $\varphi(x_1, \dots, x_n)$  is in the vocabulary  $\tau_{\mathbf{A}^q}$  and  $(a_{\ell}, b_{\ell}) \in f$  for  $\ell = 1, \dots, n$  and we let  $\mathcal{F}_1 = \mathcal{F}^r[G_{\mathbf{A}}]$  and  $\mathcal{F}_2 = \mathcal{F}^s[G_{\mathbf{A}}]$  then

$$\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}_1 \models \varphi[(a_1[G_{\mathbf{A}}]) / \mathcal{F}_1, \dots, (a_n[G_{\mathbf{A}}]) / \mathcal{F}_1]$$

iff

$$\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}_2 \models \varphi[(b_1[G_{\mathbf{A}}]) / \mathcal{F}_2, \dots, (b_n[G_{\mathbf{A}}]) / \mathcal{F}_2].$$



- ( $\delta$ )  $\underline{f}$  include the identity map on  $\prod_{n < \omega}^{\mathbf{A}^q} M_{\mathbf{A}^q}^n / \mathcal{F}^q$
- (2) Let  $\otimes_{\beta, q, r, s, f}^+$  means that in part (1) we add:  $\underline{f}$  is in isomorphism from  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}^r$  onto  $\prod_{n < \omega}^{\mathbf{A}} M_{\mathbf{A}^q}^n / \mathcal{F}^s$  (i.e. this is  $\Vdash_{\mathbb{C}_{\mathbf{A}}}$ ).

**Observation 3.10.** Assume  $\otimes_{\beta, q, r, s, f}$

If  $\text{cf}(\beta) = \aleph_2$  or  $\beta$  is divisible by  $\aleph_2$  and has cofinality  $\aleph_0$  then we can find  $r', s', f'$  such that  $\otimes_{\beta, q, r', s', f'}^+$  and  $r \leq r', s \leq s'$  and  $\Vdash_{\mathbb{C}_{\mathbf{A}^{r'}}} \underline{f} \subseteq \underline{f}'$

*Proof.* By  $\aleph_1$  uses of 2.7(1) and 2.8(4) if  $\text{cf}(\beta) = \aleph_2$  and by  $\aleph_1$  uses of 2.8(5) and 2.8(4) if  $\text{cf}(\beta) = \aleph_0$ .  $\square$

**Lemma 3.11.** If  $\beta_1 < \beta_2 < \aleph_3$  are divisible by  $\aleph_2$ ,  $q_2 \in \text{App}_{\beta_2}$ ,  $q_0 = q_2 \upharpoonright \beta_1$ ,  $q_0 \leq r_0 \in \text{App}_{\beta_1}$ ,  $r_0 \leq r_1 \in \text{App}_{\beta_2}$ , and  $\otimes_{\beta_1, q_1, r_0, s, f}^+$  (see Definition 3.9) then we can find  $r_2, s_2$  and  $\underline{f}'$  such that:

- (i)  $\otimes_{\beta_2, q_2, r_2, s_2, f'}^+$
- (ii)  $r_1 \leq r_2$
- (iii)  $s \leq s_2$  and
- (iv)  $\Vdash_{\mathbb{C}_{\mathbf{A}^{r_2}}} \underline{f} \subseteq \underline{f}'$

Proof of 3.11: We prove this by induction on  $\beta_2$ .

Case 1:  $\beta_2 = 0$

Empty

Case 2:  $\beta_2 = \beta_1 + \aleph_2$  and  $\text{cf}(\beta_1) < \aleph_2$ .

It is enough to find  $s' \in \text{App}$  such that letting  $r' = r_1$  we have  $\mathbf{A}^{s'} = \mathbf{A}^{r'}$ ,  $p \leq s'$ ,  $s \leq s'$  and  $\otimes_{\beta_2, q_2, r', s', f}^+$  (i.e. without the +), this is enough by observation 3.10.

Let  $\underline{f} = \{(a_\epsilon, b_\epsilon) : \epsilon < \epsilon^*\}$ . So it suffices to find a  $\mathbb{C}_{\mathbf{A}^{r'}}$ -name of an ultrafilter which is forced to include the following families

- (a)  $\mathcal{F}^{q_2}$
- (b)  $\mathcal{F}^s$
- (c) the sets of the form  $\{n : M_{\mathbf{A}^{r'}}^n \models \varphi(b_{\epsilon_0}(n), \dots, b_{\epsilon_{k-1}}(n))\}$ : where  $\epsilon_0, \dots, \epsilon_{k-1} < \epsilon^*$  and  $\varphi(x_0, \dots, x_{k-1})$  is a  $\mathbb{C}_{\mathbf{A}^{q_2}}$ -name of a first order formula in the vocabulary  $\tau_{\mathbf{A}^{q_2}}$  such that

$$\prod_{n < \omega}^{\mathbf{A}^{r'}} M_{\mathbf{A}^{q_2}}^n / \mathcal{F}^{r'} \models \varphi[a_{\epsilon_0} / \mathcal{F}^{r'}, \dots, a_{\epsilon_{k-1}} / \mathcal{F}^{r'}].$$

So it suffices to prove that any finite intersection is not empty, but each of those families is closed under finite intersection, hence it suffices to prove the following

- $\otimes p \Vdash_{\mathbb{C}_{\mathbf{A}^{r'}}} \text{“}\mathbf{a} \cap \mathbf{b} \cap \mathbf{c} \neq \emptyset\text{”}$   
when
- (a)  $p \in \mathbb{C}_{\mathbf{A}^r}$

- (b)  $\underline{\mathfrak{a}}$  is a  $\mathbb{C}_{\mathbf{A}^{q_2}}$ -name such that  $p \upharpoonright \mathbf{A}^{q_2} \Vdash \underline{\mathfrak{a}} \in \mathcal{F}^{q_2}$
- (c)  $\underline{\mathfrak{b}}$  is a  $\mathbb{C}_{\mathbf{A}^s}$ -name such that  $p \upharpoonright \mathbf{A}^s \Vdash \underline{\mathfrak{b}} \in \mathcal{F}^s$
- (d)  $\underline{\mathfrak{c}} = \{n : M_{\mathbf{A}^{r'}}^n \models \varphi[b_{\epsilon_0}(n), \dots, b_{\epsilon_{k-1}}(n)]\}$  where  $\varphi$  is a  $\mathbb{C}_{\mathbf{A}^{q_2}}$ -name of a first order formula in the vocabulary  $\tau_{\mathbf{A}^{q_2}}$ , without loss of generality a predicate as an atomic formula, such that  $p \Vdash_{\mathbb{C}_{\mathbf{A}^{r'}}} \prod_{n < \omega} M_{\mathbf{A}^{q_2}}^n / \mathcal{F}^{r'} \models \varphi[a_{\epsilon_0} / \mathcal{F}^{r'}, \dots, a_{\epsilon_{k-1}} / \mathcal{F}^{r'}]$

Without loss of generality  $p$  forces that  $\varphi = Q_{\langle R_n^2 : n < \omega \rangle}$ .

Let  $H \subseteq \mathbb{C}_{\mathbf{A}^{q_0}}$  be generic over  $\mathbf{V}$  such that  $p \upharpoonright \mathbf{A}^{q_0} \in H$ . In  $\mathbf{V}[H]$  for each  $n$  we define a  $k$ -place relation  $R_n^0$  on  $\omega$

$R_n^0 = \{(m_0, \dots, m_{k-1}) : \text{there is } p', p \leq p' \in \mathbb{C}_{\mathbf{A}^{r'}}, p' \upharpoonright \mathbf{A}^{q_0} \in H \text{ such that } p' \Vdash n \in \underline{\mathfrak{a}} \text{ and } \langle m_0, \dots, m_{k-1} \rangle \in R_n^2\}$

Now

- (\*)<sub>0</sub>  $p \Vdash_{\mathbb{C}^{r'}} \prod_{n < \omega} M_{\mathbf{A}^{q_2}}^n / \mathcal{F}^r \models \varphi[a_{\epsilon_0} / \mathcal{F}^r, \dots]$   
hence
- (\*)<sub>1</sub>  $p \Vdash_{\mathbb{C}^r} \langle a_{\epsilon_0} / \mathcal{F}^r, \dots, a_{\epsilon_{k-1}} / \mathcal{F}^r \rangle \in Q_{\langle R_n^2 : n < \omega \rangle}$   
hence
- (\*)<sub>2</sub>  $p \upharpoonright \mathbf{A}^r \Vdash \langle a_{\epsilon_0} / \mathcal{F}^r, \dots, a_{\epsilon_{k-1}} / \mathcal{F}^r \rangle \in Q_{\langle R_n^0 : n < \omega \rangle}$   
hence
- (\*)<sub>3</sub>  $p \upharpoonright \mathbf{A}^r \Vdash \langle b_{\epsilon_0} / \mathcal{F}^s, \dots, b_{\epsilon_{k-1}} / \mathcal{F}^s \rangle$  satisfies  $Q_{\langle R_n^0 : n < \omega \rangle}$  in  $\prod_{n < \omega} M_{\mathbf{A}^{q_0}}^n / \mathcal{F}^s$   
so
- (\*)<sub>4</sub> in  $\mathbf{V}[H]$ , the set  $\mathbf{b}' \in \mathcal{F}^{q_0}$  where  $\mathbf{b}' = \{n : \text{for some } p', p \upharpoonright \mathbf{A}^{s_0} \leq p' \in \mathbb{C}_{\mathbf{A}^{s_0}} \text{ and } p' \upharpoonright \mathbf{A}^{q_0} \in H \text{ and } p' \Vdash_{\mathbb{C}_{\mathbf{A}^{s_0}}} \langle n \in \underline{\mathfrak{b}} \text{ and } \langle b_{\epsilon_0}(n), \dots, b_{\epsilon_{k-1}}(n) \rangle \in R_n^0[H]\}$

So clearly  $\mathbf{b}'$  is a non-empty set of natural numbers, so choose  $n \in \mathbf{b}'$ . So there is  $p_1 \in \mathbb{C}_{\mathbf{A}^s}, p \upharpoonright \mathbf{A}^s \leq p_1, p_1 \upharpoonright \mathbf{A}^{q_0} \in H, p_1 \Vdash \langle n \in \underline{\mathfrak{b}} \text{ and } \langle b_{\epsilon_0}(n), \dots, b_{\epsilon_{k-1}}(n) \rangle \in R_n^0[H]\}$ . Without loss of generality  $p_1$  forces values to  $b_{\epsilon_0}(n), \dots, b_{\epsilon_{k-1}}(n)$ , call them  $m_0, \dots, m_{k-1}$ . So  $\langle m_0, \dots, m_{k-1} \rangle \in R_n^0[H]$ , hence by its definition there is  $p_2$  such that  $p \upharpoonright \mathbf{A}^{q_2} \leq p_2 \in \mathbb{C}_{\mathbf{A}^{q_2}}, p_2 \Vdash \langle n \in \underline{\mathfrak{a}} \text{ and } \langle m_0, \dots, m_{k-1} \rangle \in R_n^2$ .

Now  $p^* =: p_1 \cup p_2 \in \mathbb{C}_{\mathbf{A}^{r'}}$  is above  $p, p^* \upharpoonright \mathbf{A}^{q_0} \in H$ , and it forces that  $n \in \underline{\mathfrak{a}} \cap \underline{\mathfrak{b}} \cap \underline{\mathfrak{c}}$ , which is enough.

Case 3:  $\beta_2 = \beta_1 + \aleph_2, \text{cf}(\beta_1) = \aleph_2$

First by 2.8 we can find  $r', s', f'$  such that

- (a)  $r_0 \leq r' \in \text{App} \upharpoonright \beta_1,$
- (b)  $s \leq s' \in \text{App} \upharpoonright \beta_1,$
- (c)  $q_2 \upharpoonright \beta_1 \leq s'$
- (d)  $\otimes_{\beta_1, q_2 \upharpoonright \beta_1, r', s', f'}^+$

Now we continue as in case 2, the  $\Gamma_{\beta_1}^{q_2}$ -bigness of  $x_{\beta_1}$  is automatic.

Case 4:  $(\forall \gamma < \beta_2)(\gamma + \aleph_2 < \beta_2)$

Let  $\langle \gamma_\epsilon : \epsilon < \text{cf}(\beta_2) \rangle$  be increasing continuous with limit  $\beta_2$  such that  $\gamma_0 = \beta_1$ ,  $\text{cf}(\gamma_\epsilon) < \aleph_2$  and each  $\gamma_\epsilon$  is divisible by  $\aleph_2$  and stipulate  $\gamma_{\text{cf}(\beta_2)} = \beta_2$ . We choose  $(r_\epsilon, s_\epsilon, \underline{f}_\epsilon)$  by induction on  $\epsilon \leq \text{cf}(\beta_2)$  such that

- ⊠ (a)  $\otimes_{\beta_{\gamma_\epsilon, q_2}^+ \upharpoonright \gamma_\epsilon, r_\epsilon, s_\epsilon, \underline{f}_\epsilon}$  holds.
- (b)  $(r_0, s_0, \underline{f}_0) = (r, s, \underline{f})$
- (c)  $r' \upharpoonright \gamma_\epsilon \leq r_\epsilon$
- (d) if  $\zeta < \epsilon$  then  $r_\zeta \leq r_\epsilon, s_\zeta \leq s_\epsilon$  and  $\Vdash_{\mathbb{C}_{\mathbf{A}^{r_\epsilon}}} \underline{f}_\zeta \subseteq \underline{f}_\epsilon$

Clearly if we succeed we are done with case 4.

For  $\epsilon = 0$  this is trivial.

For  $\epsilon = \zeta + 1$  first find  $r'_\zeta \in \text{App}_{\gamma_\zeta}$  such that  $r' \upharpoonright \gamma_\zeta \leq r'_\zeta$  and  $r'_\zeta \upharpoonright \gamma_\zeta = r_\zeta$ , possibly by 2.8(3). Second apply the induction hypothesis with  $(\beta_\zeta, \beta_\epsilon, q_2 \upharpoonright \beta_\zeta, q_2 \upharpoonright \beta_\epsilon, r_\zeta, s_\zeta, \underline{f}_\zeta, r'_\zeta)$  standing for  $(\beta_0, \beta_2, q_0, q_2, r, s, \underline{f}, r')$ .

For  $\epsilon$  limit of uncountable cofinality take the union (see 2.8(4)).

For  $\epsilon$  limit of countable cofinality, we first repeat the argument in case 2. Then use 2.8 and then 3.10.  $\square_{3.5A}$

#### 4. BACK TO MODEL THEORY

In this section we present just enough to solve the problem on finite fields.

**Definition 4.1.** Let  $M$  be a model. Assume  $N_1 = M^{\{\bar{\varphi}^1\}}$ ,  $N_2 = M^{\{\bar{\varphi}^2\}}$  are models of  $t_0$  interpreted in  $M$  by the sequences  $\bar{\varphi}^1, \bar{\varphi}^2$  of formulas with parameters from  $M$ , and they have the same vocabulary  $\tau^* = \tau(N_1) = \tau(N_2)$ . Furthermore, let  $\Gamma$  be an invariant bigness notion in  $M$  (over some set  $A_0$  of  $< \kappa$  parameters, more exactly in  $\mathcal{K}_{(M, A_0)}$ ), and  $\Delta \subseteq \mathbb{L}_{\omega, \omega}(\tau(N_1))$  and  $\kappa > \aleph_0$  (for simplicity) and for a formula  $\vartheta(\bar{x}) \in \Delta$  let  $\vartheta_{\bar{\varphi}^\ell}(\bar{x})$  be the result of substituting  $\bar{\varphi}^\ell$  in  $\vartheta$  so  $N^\ell \models \vartheta[\bar{a}]$  iff  $\bar{a} \in \text{lg}^{\bar{x}}(N^\ell)$  and  $M \models \vartheta_{\bar{\varphi}^\ell}[\bar{a}]$ .

- (1) We say that  $(N_1, N_2)$  is  $(\kappa, \Gamma, \Delta)$ -complicated in  $M$  when:  
for every  $\Delta$ -embedding  $F$  of  $N_1$  into  $N_2$ , and for every  $\Gamma$ -big type  $\mathfrak{p}_0(x)$  inside  $M$  of cardinality  $< \kappa$  such that  $\mathfrak{p}_0(M) \subseteq N_1$ , there is a  $\Gamma$ -big type  $\mathfrak{p}_1(x)$  inside  $M$  of cardinality  $< \kappa$  which includes  $\mathfrak{p}_0(x)$  and such that, letting  $\tau(\mathfrak{p}_1) \subseteq \tau(M)$  consist of those predicates and function symbols mentioned in  $\mathfrak{p}_1(x)$  (so  $|\tau(\mathfrak{p}_1)| < \kappa$ ) and  $A \subseteq M$  be the set of parameters of  $\mathfrak{p}_0$  union with  $A_0$  so  $|A| < \kappa$  and  $A_0 \subseteq A$ , we have

(\*) $_{\mathfrak{p}_1(x)}$  letting

$$R_m \stackrel{\text{def}}{=} \{(\bar{a}, \bar{b}) : \bar{a} \in {}^m(N_1), \bar{b} \in {}^m(N_2) \text{ and for some } \bar{c} \in {}^m(N_1) \text{ we have} \\ \text{tp}_{\mathbb{L}_{\omega, \omega}(\tau(\mathfrak{p}))}(\bar{a} \frown \bar{b}, A, M) = \text{tp}_{\mathbb{L}_{\omega, \omega}(\tau(\mathfrak{p}))}(\bar{c} \frown F(\bar{c}), A, M) \}$$

the parallel of 3.6(vii)+(viii) holds, so

- ( $\oplus$ ) $_1$  if  $\bar{a}_1, \bar{a}_2$  are finite sequences of the same length  $m$  of members of  $N^1$ , and  $\mathfrak{p}_1 \cup \{\vartheta_{\bar{\varphi}^1}^{N_1}(x, \bar{a}_1), \neg \vartheta_{\bar{\varphi}^1}^{N_1}(x, \bar{a}_2)\}$  is a  $\Gamma$ -big type over  $M$ , and  $\vartheta, \neg \vartheta \in \Delta$ , then  $(\bar{a}_1, F(\bar{a}_2)) \notin R_m$ .

- ( $\oplus$ )<sub>2</sub> Moreover, in  $\oplus_1$  we can replace  $\vartheta, \neg\vartheta$  by any pair  $\vartheta_0, \vartheta_1$  of contradictory formulas from  $\Delta$ .
- (2) In part (1):
- (i) We do not mention  $\Delta$  if it is the set of quantifier free formulas (of  $\mathbb{L}_{\omega, \omega}(\tau(N_1))$ ).
  - (ii) We replace  $\Gamma$  by  $(t, \psi)$  if we mean “for all bigness notions of the form  $\Gamma = \Gamma_{(t, \psi, \bar{\varphi})}$ , where  $\bar{\varphi}$  is an interpretation of  $t$  in  $M$  with  $< \kappa$  parameters and  $|t| < \kappa$ ,  $\psi \in \mathbb{L}_{\kappa, \omega}$ ” (i.e.,  $\psi \in \mathbb{L}_{\mu^+, \omega}$  for some  $\mu < \kappa$  and in the vocabulary  $\tau(t) \cup \{P^*\}$ ).
  - (iii) We omit  $\Gamma$  if we mean “for all  $\Gamma$ ’s as in (ii)”.
  - (iv) We say  $M$  is  $\kappa$ -complicated (or:  $(\kappa, \Gamma, \Delta)$ -complicated) and omit  $N_1, N_2$  if this holds for all  $N_1, N_2$  as in our assumptions, but with  $|\tau(N_1)| < \kappa$ .

*Remark 4.2.* More on the relation  $R_n$  etc., see [Sh 800].

**Theorem 4.3.** *Let  $\mathbf{G}$  be a full  $(\aleph_3, \aleph_2)$ -bigness guide (see 2.2; recall there is one by 2.3). Assume that  $G \subseteq \text{App}_{\mathbf{G}}$  is generic over  $\mathbf{V}$  and  $H \subseteq \mathbb{C}_{\aleph_3}$  is generic over  $\mathbf{V}[G]$  and  $\mathcal{F} = \mathcal{F}_{\aleph_3}[G][H]$ , and let  $\langle M_n = M_{\aleph_3}^n : n < \omega \rangle$  be a sequence of models as in 2.1(4), that is each with a countable universe being the set of natural numbers for simplicity, all with the same vocabulary such that for every  $k$  and a sequence  $\langle R_n : n < \omega \rangle$  with  $R_n$  being a  $k$ -place relation on  $M_n$  there is a  $k$ -place predicate in the common vocabulary satisfying  $R^{M_n} = R_n$  for each  $n$ . Then*

- (1) *in  $\mathbf{V}[G][H]$  the model  $M = \prod_{n < \omega} M_{\aleph_3}^n / \mathcal{F}$  is  $\aleph_2$ -complicated and  $\aleph_2$ -compact.*
- (2) *We can change the demands on  $\mathbf{G}$  accordingly to the version of  $\aleph_2$ -complicated we actually used (e.g. not all  $\Gamma$ -s, etc.), (so we are using a different  $\mathbf{G}$ ).*
- (3) *If  $N^1, N^2$  are models of  $t_1^{\text{ind}}$  (which is defined in from Definition 1.5), interpreted in  $M$ , then any isomorphism  $\pi$  from  $N^1$  onto  $N^2$  is definable in  $M$ .*
- (4) *If  $N^\ell = \prod_{n < \omega} N_n^\ell / \mathcal{F}$ , each  $N_n^\ell$  is countable, and  $N^\ell$  is a model of  $t_1^{\text{ind}}$  (for  $\ell = 1, 2$ ), and  $\pi$  is an isomorphism from  $N^1$  onto  $N^2$ , then there are  $A \in \mathcal{F}$  and isomorphisms  $\pi_n$  from  $N_n^1$  onto  $N_n^2$  (for  $n \in A$ ) such that  $\pi = \prod_{n < \omega} \pi_n / \mathcal{F}$ .*
- (5) *Above we may replace : “ $N^\ell$  is a model of  $t_1^{\text{ind}}$ ” by “some formula  $\phi(x, y)$  in the vocabulary of  $N^1$  which is equal to that of  $N^2$ , has the strong independence property” (in their common theory <sup>10</sup>, see Definition 1.5 on the strong independence property).*

<sup>10</sup>of course if the strong independence property holds when we restrict ourselves to say a predicate  $P$  we get less, but see [Sh 800]

(6) If  $N_n^\ell$  are finite fields (for  $\ell = 1, 2$  and  $n < \omega$ ), and  $\prod_{n < \omega} N_n^1/\mathcal{F}$  is isomorphic to  $\prod_{n < \omega} N_n^2/\mathcal{F}$ , then the set  $\{n < \omega : N_n^1 \simeq N_n^2\}$  belongs to  $\mathcal{F}$ .

*Proof.* (1) By 3.8.

(2) The same proof.

(3) By 4.4 below and 1.6(2).

(4) Without loss of generality, the universe of  $N_n^\ell$  is  $\alpha_n^\ell \leq \omega$ . Now, for  $\ell = 1, 2$ , we can find  $P_\ell \in \tau_M$  such that  $(P_\ell)^{M_{\aleph_3}^n} = |N_n^\ell|$  and for  $Q_\ell \in \tau(N_n^\ell)$  there is  $Q_\ell \in \tau_M$  with  $(Q_\ell)^{M_{\aleph_3}^n} = Q^{N_n^\ell}$  and  $R_\ell \in \tau_M$  with  $(R_\ell)^{M_{\aleph_3}^n} = R^{N_n^\ell}$ . Therefore,  $N^\ell = \prod_{n < \omega} N_n^\ell/\mathcal{F}$  can be viewed as an interpretation in  $M$  by  $\bar{\varphi}^\ell$ .

Now apply part (3) for  $\Gamma = \Gamma_{(t^{\text{ind}}, \psi^{\text{ind}}, \bar{\varphi}^1)}$ .

(5) This follows by part (4), as the vocabulary is finite, being an isomorphism is expressibly a first order sentence.

(6) This is a particular case of part (4). Of course without loss of generality the fields  $N^1, N^2$  are infinite. By part (5) it suffices for infinite ultraproducts  $N^\ell$  of finite fields to find a formula  $\vartheta(x, y)$  in the vocabulary of fields which has the strong independence property see Definition 1.5. First we deal with the case that the fields are of characteristic  $> 2$ . Consider the formula  $\vartheta(x, y)$  saying that  $x + y$  has a square root in the field.

We rely on a theorem of Duret, [Du80, p. 982, Lemma 10], for the value  $p = 2$  the hypothesis of this lemma holds as the field contains all  $p$ -th roots of the unit (that is  $1, -1$ ). The conclusion says that for  $n$  and any pairwise distinct elements  $a_1, \dots, a_n, b_1, \dots, b_n$  of the field there is an element  $c$  such that  $a_m + c$  has a square root and  $b_m + c$  does not have a square root for  $m = 1, \dots, n$ . So the formula  $\vartheta_p(x, y) = (\exists z)(z^p = x + y)$  is as required.

Of course, if the characteristic of the field is 2, then we naturally use the same theorem but choosing  $p = 3$ , so of course maybe the field fail to have all the  $p$ -th roots of the unit, however, as Duret does, in this case we consider an algebraic extension of  $N^\ell$  of order 3 by adding a root of  $x^3 - 1$  hence all of them getting a new field  $N_*^\ell$ . Now the set of elements of  $N_*^\ell$  can be represented as the set of triples of elements of  $N^\ell$ , and the operations of  $N_*^\ell$  are definable in  $N^\ell$ ; so our problem is almost notational. E.g. we can note that recalling  $N^\ell = \prod_{n < \omega} N_n^\ell/\mathcal{F}$  then  $N_*^\ell = \prod_{n < \omega} N_{*,n}^\ell/\mathcal{F}$  where  $N_{*,n}^\ell$  is

equal to  $N_n^\ell$  if  $N_n^\ell$  has three 3-th roots of the unit and an algebraic extension of  $N_n^\ell$  of order three which has this property otherwise. Again the first order theory of  $N_*^\ell$  has the strong independence property and for  $N_*^1, N_*^2$  (by asking on the existence of cubic roots) we get the desired conclusion; but any isomorphism from  $N^1$  onto  $N^2$  can be extended to an isomorphism from  $N_*^1$  onto  $N_*^2$  and we can easily finish. (We could have used the “strong independence property for  $m$ -types”.)  $\square$

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**Proposition 4.4.** *Assume that  $M$  is a  $\kappa$ -complicated  $\kappa$ -compact model. Let  $N_1, N_2$  be interpretations of  $t_1^{\text{ind}}$  in  $M$ . Then for any isomorphism  $\pi$  from  $N_1$  onto  $N_2$ , the function  $\pi$  is definable in  $M$  by a first order formula (with parameters).*

*Proof.* Let  $N_\ell = M^{\{\bar{\varphi}^\ell\}}$  (so  $\bar{\varphi}^\ell$  has parameters in  $M$ ) for  $\ell = 1, 2$  and let  $F$  be an isomorphism from  $N_1$  onto  $N_2$ .

Let  $\Gamma$  be the bigness notion  $\Gamma_{(t_1^{\text{ind}}, \psi^{\text{ind}}, \bar{\varphi}^1)}$  (so  $\psi^{\text{ind}} \in \mathbb{L}_{\omega_1, \omega}$ ). Let  $\mathfrak{p}_0(x)$  be the type just saying  $x \in Q^{N_1}$ , and let  $\mathfrak{p}_1$  be the type guaranteed to exist in Definition 4.1(1), without loss of generality closed under conjunctions. Let  $A \subseteq M$ ,  $|A| < \kappa$  and  $\tau^* \subseteq \tau_M$ ,  $|\tau^*| < \kappa$  be given by the definition of being  $\kappa$ -complicated (applied to  $F$ ). [Without loss of generality,  $A$  includes the parameters of  $\bar{\varphi}^1, \bar{\varphi}^2$  and is closed under  $F$  and  $F^{-1}$ , and for every  $n$  and for every formula  $\varphi(x) \in \mathfrak{p}_1$ ,  $A$  includes the finite set mentioned in 1.5(2).]

Let  $R_1$  be as in 4.1(1). Clearly, recalling Definition 1.5(2), there are no distinct  $a_1, a_2 \in P^{N_1} \setminus A$  and  $b \in N_2$  such that  $(a_1, b), (a_2, b) \in R_1$ , but  $a \in P^{N_1} \Rightarrow (a, F(a)) \in R_1$ . Hence

$$\{(b, a) : (a, b) \in R_1 \text{ and } a \in P^{N_1}\}$$

is the graph of a partial function from  $P^{N_2}$  into  $P^{N_1}$  which includes the graph of  $F^{-1} \upharpoonright P^{N_2}$ . But  $F$  is one-to-one and onto. Therefore,  $R_1 \upharpoonright (P^{N_1} \times P^{N_2})$  is the graph of  $F \upharpoonright P^{N_1}$ . But  $R_1 \upharpoonright P^{N_1}$  is definable in  $(M \upharpoonright \tau^*, c)_{c \in A}$  by a formula from  $\mathbb{L}_{\infty, \kappa}$ , so also  $F \upharpoonright P^{N_1}$  is, and thus if  $N_1, N_2$  are models of  $t_1^{\text{ind}}$  also  $F$  is (by 1.6). Applying [Sh 72, 1.9] (or [Sh:e, Ch XI]) we conclude that it is definable by a first order formula with parameters from  $M$ , as required.  $\square$

Similarly we can show the following.

**Proposition 4.5.** *Assume that  $\Gamma$  is a  $(\aleph_2, \aleph_1)$ - $(P, \vartheta)$ -separative bigness notion, see Definition 1.4. Suppose that  $N_1, N_2$  are interpretations of  $t$  in  $M$ , and  $M$  is  $\kappa$ -compact  $\kappa$ -complicated (or just  $\kappa$ -complicated for  $\Gamma$ ),  $\kappa > \aleph_0$ .*

- (1) *If  $F$  is an isomorphism from  $N_1$  onto  $N_2$ , then*
  - (\*)<sub>1</sub>  *$F \upharpoonright P^{N_1}$  is definable in  $(M \upharpoonright \tau^*, c)_{c \in A}$  by a formula from  $\mathbb{L}_{\infty, \kappa}$ , recalling  $\tau \subseteq \tau_M$ ,  $|\tau| < \kappa$ ,  $A \subseteq M$ ,  $|A| < \kappa$ .*
- (2) *If  $F$  is an embedding of  $N_1$  into  $N_2$ , then*
  - (\*)<sub>2</sub> *there is a partial function  $f$  from  $P^{N_2}$  into  $P^{N_1}$  which extends  $F^{-1}$  and is definable in  $(M \upharpoonright \tau^*, c)_{c \in A}$  by a formula from  $\mathbb{L}_{\infty, \kappa}$ , where  $\tau^*, A$  are as above.*

*Remark 4.6.* (1) The proposition 4.5 should be the beginning of an analysis of first order theories  $T$ . For more in this direction see [Sh 503], [Sh 800].

- (2) As stated in the introduction, we may avoid the preliminary forcing with  $\mathbb{A}$  and construct the name  $\mathcal{F}$  in the ground model  $\mathbf{V}$ , provided  $\mathbf{V}$  is somewhat  $\mathbf{L}$ -like. Assuming  $\diamond_{\{\delta < \aleph_3: \text{cf}(\delta) = \omega_2\}}$  is enough, but we

may also use the weaker principle from [HLS<sub>h</sub> 162] and [Sh 405, Appendix].

- (3) We may vary the cardinals, e.g., we may replace  $\aleph_2, \aleph_3$  by  $\kappa, \lambda$ , respectively, provided  $\lambda = \kappa^+$ ,  $\kappa = \kappa^{<\kappa}$  (so an approximation has size  $< \kappa$ ).

Moreover we can replace  $\aleph_0$  by  $\theta = \theta^{<\theta}$ , so in full let us assume that

$$\theta = \theta^{<\theta} < \kappa = \kappa^{<\kappa} < \lambda = \kappa^+.$$

- (a) For  $\mathbf{A} \subseteq \lambda$  let  $\mathbb{C}(\mathbf{A}) = \mathbb{C}_{\mathbf{A}} = \{p : p \text{ is a partial function from } \text{Dom}(p) \in [\mathbf{A}]^{<\theta} \text{ to } \theta^{>2} \}$  ordered by

$$p_1 \leq_{\mathbb{C}_{\mathbf{A}}} p_2 \quad \text{iff} \quad \text{Dom}(p_1) \subseteq \text{Dom}(p_2) \ \& \ (\forall \alpha \in \text{Dom}(p_1))(p_1(\alpha) \trianglelefteq p_2(\alpha)).$$

- (b) We define  $\text{App}_{\mathbf{G}}^-$  as the set of  $q = (\mathbf{A}^q, \mathcal{F}^q)$  where  $\mathbf{A}^q \in [\lambda]^{<\kappa}$  and  $\mathcal{F}^q$  is a  $\mathbb{C}_{\mathbf{A}^q}$ -name of a regular ultrafilter on  $\theta$  such that for each  $\alpha < \lambda$ ,  $\mathcal{F}^q \cap \mathcal{P}(\theta)^{\mathbf{V}^{\mathbb{C}(\mathbf{A}^q \cap \alpha)}}$  is a  $\mathbb{C}_{\mathbf{A}^q \cap \alpha}$ -name.

- (c) For  $\alpha \in \mathbf{A} \in [\lambda]^{<\kappa}$ ,  $\underline{x}_\alpha$  is the  $\mathbb{C}_{\mathbf{A}}$ -name  $\bigcup \{p(\alpha) : p \in G_{\mathbb{C}(\mathbf{A})}\}$  of a member of  ${}^\theta\theta$ .

- (d) We define  $M_{\mathbf{A}}^\varepsilon$  for  $\varepsilon < \theta$ ,  $\mathbf{A} \in [\lambda]^{<\kappa}$  as the following  $\mathbb{C}_{\mathbf{A}}$ -name: it is a model with universe  $\theta$ ,

$$\tau_{M_{\mathbf{A}}^\varepsilon} = \{P_{\bar{R}} : \bar{R} = \langle R_\varepsilon : \varepsilon < \theta \rangle, \text{ for some } m \text{ each } \bar{R}_\varepsilon \text{ is a } \mathbb{C}_{\mathbf{A}}\text{-name of an } m\text{-place relation on } \theta\},$$

$$(P_{\bar{R}})^{M_{\mathbf{A}}^\varepsilon} = \underline{R}_\varepsilon.$$

So we may think of  $\tau_{M_{\mathbf{A}}^\varepsilon}$  to be an old object whose members are indexed as  $P_{\bar{R}}$ , where each  $\bar{R}_\varepsilon$  is a  $\mathbb{C}_{\mathbf{A}}$ -name. Or we can consider  $\tau_{M_{\mathbf{A}}^\varepsilon}$  to be a name and interpret it in  $\mathbf{V}[G_{\mathbb{C}(\mathbf{A})}]$ .

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