

PCF AND INFINITE FREE
SUBSETS IN AN ALGEBRA
SH513

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ANNOTATED CONTENT

§1 Other variants of “G.C.H. holds almost always”

[We give another way to prove that for every $\lambda \geq \beth_\omega$ for every large enough regular $\kappa < \beth_\omega$ we have $\lambda^{[\kappa]} = \lambda$, dealing with sufficient conditions for replacing \beth_ω by \aleph_ω . This continues [Sh 460].]

§2 Large $\text{pcf}(\mathfrak{a})$ implies the existence of free sets

[A nice example of the implication stated in the title is that if $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every algebra M of cardinality \aleph_ω with countably many functions (or just $< \aleph_\omega$ many), for some $a_n \in M$ (for $n < \omega$) we have $a_n \notin \text{cl}_M(\{a_\ell : \ell \neq n, \ell < \omega\})$. Generally if $\text{pcf}(\mathfrak{a})$ is not just of cardinality $> |\mathfrak{a}|$, but $\langle J_{<\theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ has large rank (as defined below) then a relevant instance of IND connected to $\text{sup}(\mathfrak{a})$ holds.]

§3 Existence of free subsets implies restrictions on pcf

[We have results of forms complementary to those of §2 (though not close enough). So if $\text{IND}(\kappa, \sigma)$ (in every algebra with universe λ and $\leq \sigma$ functions there is an infinite independent subset) then for no distinct regular $\lambda_i \in \text{Reg} \setminus \kappa^+$ (for $i < \kappa$) does $\prod_{i < \kappa} \lambda_i / [\kappa]^{\leq \sigma}$ have true cofinality. We also look at

$\text{IND}(\langle J_{\kappa_n}^{\text{bd}} : n < \omega \rangle), J_n$ an ideal on κ_n (we ask for $\bar{\alpha} \in \prod_{n < \omega} \kappa_n$ such that $n < \omega \Rightarrow \alpha_n > \text{sup}(\text{cl}_M\{\alpha_\ell : \ell \in (n, \omega)\})$ for a model M with universe $\cup\{\kappa_n : n < \omega\}$ and more general version, and from assumptions as in §2 get results even for the non stationary ideal.]

§4 Sticks and Boolean Algebras

[We deal with some other measurements of $[\lambda]^{\leq \theta}$. We also give an application by a construction of a Boolean Algebra: one into which the free algebra with λ generators is embedded but no homomorphic image is the free Boolean Algebra with κ generators; with even weakened demand we can replace free by finite/cofinite Boolean Algebras.]

§5 More on free subsets and pcf

§6 Odds and ends

[In 6.1 we deal with a replacement for Δ -system lemma; with $> 2^\kappa$ sequences of ordinals of length κ . In 6.3 we look at how we can divide $F \subseteq \Pi \mathfrak{a}$ to few bounded sets. In 6.4 we relook at the characterization of a property of [GHS], generalizing the questions somewhat. We then deal with freeness properties for $F \subseteq {}^\delta \text{Ord}$ (modulo an ideal) and we give a correct version of [Sh:g, Ch.IX,3.5] on characterizing $\text{cov}(\lambda, \lambda, \theta, \sigma)$ when $\sigma > \aleph_0$ concerning the obtainment of the pp version. We shall continue in [Sh 589].]

§1 OTHER VARIANTS OF “G.C.H. HOLDS ALMOST ALWAYS”

We essentially redo the proof of [Sh 460], §2 in another more general way.

- 1.1 Notation.** 1) $\mathfrak{F}_\kappa(A)$ is the family of κ -complete filters \mathfrak{D} on $\mathcal{P}(A)$ so $\mathfrak{D} \subseteq \mathcal{P}(\mathcal{P}(A))$; so the points are subsets of A , and the members of \mathfrak{D} (which are $\subseteq \mathcal{P}(A)$) which we shall be most interested in are ideals and their compliments.
 2) We say $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$ has σ -complete character if for any $Y \subseteq \mathcal{P}(A)$ we have: $Y \in \mathfrak{D}$ iff $\text{id}_\sigma(Y) \in \mathfrak{D}$ where $\text{id}_\sigma(Y)$ is the σ -complete ideal on A generated by Y .
 3) For an ideal I on X let $I^+ = \mathcal{P}(X) \setminus I$, similarly for a filter.

- 1.2 Definition.** 1) For $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$, cardinals $\mu < \lambda$ and σ such that $|A| < \mu < \lambda$, we say that λ is $(\mathfrak{D}, \mu, \sigma)$ -inaccessible when: if $\mathfrak{a}_t \subseteq (\mu, \lambda) \cap \text{Reg}$ for $t \in A$, $|\mathfrak{a}_t| < \sigma$ then $\{B \subseteq A : \text{pcf}_{\sigma\text{-complete}}(\cup\{\mathfrak{a}_t : t \in B\}) \subseteq \lambda\} \in \mathfrak{D}$.
 2) If J is an ideal on A , we say λ is (J, μ, σ) inaccessible if $\mu < \lambda$ and for no $\theta_x \in (\mu, \lambda) \cap \text{Reg}$ for $x \in A$ and σ -complete ideal J_1 on A extending A is $\prod_x \theta_x / J, \lambda$ -directed.
 3) If we omit μ we mean: for $\mu = (|A| + \sigma)^+$.

1.3 Theorem. Suppose $\langle \kappa_n : n < \omega \rangle$ is a strictly increasing sequence of regular cardinals $> \aleph_2$. Stipulate $\kappa_{-1} = \aleph_1$ and assume $\mathfrak{D}_n \in \mathfrak{F}_{\kappa_{n-1}}(\kappa_n)$ for $n < \omega$ and $\kappa = \sum_{n < \omega} \kappa_n$ satisfies:

- ⊗ if $n < \omega, \aleph_0 < \theta = \text{cf}(\theta) < \kappa_n, h : \kappa_{n+1} \rightarrow \theta$ and $Y \in \mathfrak{D}_{n+1}^+$ (so $Y \subseteq \mathcal{P}(\kappa_{n+1})$) then for some $\zeta < \theta$ we have
 (*) $_{Y, \zeta}$ $\{B \in Y : \text{sup Rang}(h \upharpoonright B) < \zeta\} \in \mathfrak{D}_{n+1}^+$.

If $\lambda > \kappa$ then for every $n < \omega$ large enough, λ is $(\mathfrak{D}_n, \kappa, \aleph_1)$ -inaccessible.

- 1.4 Remark.** 1) We can replace ω, \aleph_1 by θ, θ^+ or $< \theta, \theta$ (when θ is regular uncountable) respectively (so $\kappa = \sum_{i < \theta} \kappa_i$, etc.) (why? repeat the proof or force by $\text{Levy}(\aleph_0, < \sigma)$). Of course, we can replace $\mathfrak{F}(\kappa_n)$ by $\mathfrak{F}(A)$ if $|A| = \kappa_n$.
 2) Note that the set defined in 1.1(2) is always an ideal on A .
 3) We can assume that every \mathfrak{D}_n has σ -complete character because: λ is $(\mathfrak{D}, \mu, \sigma)$ -inaccessible iff λ is $(\mathfrak{D}', \mu, \sigma)$ whenever $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$ and $\mathfrak{D}' = \{Y \subseteq \mathcal{P}(A) : \text{id}_\sigma(Y) \in \mathfrak{D}\}$.

Proof. We prove this by induction on λ . If $\lambda = \kappa^+$ this is an empty statement (as $(\kappa, \lambda) = \emptyset$). Also if $\lambda < \kappa^{+\omega_1}$ this is trivial, as $\bigcup_{t \in A} \mathfrak{a}_t$ is countable ($\subseteq \{\kappa^{+(\alpha+1)} : \alpha < \omega_1\} \cap \lambda$) hence $\text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{t \in A} \mathfrak{a}_t) = \bigcup_{t \in A} \mathfrak{a}_t \subseteq \{\kappa^{+(\alpha+1)} : \alpha < \omega_1\} \cap \lambda$. Also if this

holds for λ it holds for λ^+ because $\text{pcf}(\mathfrak{a} \cup \{\lambda\}) \subseteq \text{pcf}(\mathfrak{a}) \cup \{\lambda\}$. So we can assume that λ is a limit cardinal. If the conclusion fails then for some infinite $W \subseteq \omega$, for each $n \in W$ there is a sequence $\langle \mathfrak{a}_\alpha^n : \alpha < \kappa_n \rangle$ (where $\mathfrak{a}_\alpha^n \subseteq (\kappa, \lambda) \cap \text{Reg}$) which is a counterexample, i.e. $Y_n =: \{B \subseteq \kappa_n : \lambda \not\subseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}_\alpha^n)\} \in \mathfrak{D}_n^+$. If

$\text{cf}(\lambda) > \kappa$ then $\bigcup \{\mathfrak{a}_\alpha^n : n < \omega, \alpha < \kappa_n\}$ is a subset of λ of cardinality $\leq \kappa$, hence is bounded by some $\lambda' < \lambda$, so apply the induction hypothesis on λ' . If $\aleph_0 < \text{cf}(\lambda) < \kappa$ let $\lambda = \sum_{\zeta < \xi} \{\lambda_\zeta : \zeta < \text{cf}(\lambda)\}$, $\bigwedge_{\zeta < \xi} \lambda_\zeta < \lambda_\xi < \lambda$. Now as $\text{cf}(\kappa) = \aleph_0$ and $\kappa = \sum_{n < \omega} \kappa_n$,

for some $n(*) < \omega$ we have $\text{cf}(\lambda) < \kappa_{n(*)}$. For every $n \in W \setminus (n(*) + 2)$, we define a function $h_n : \kappa_n \rightarrow \text{cf}(\lambda)$ by $h_n(\alpha) = \text{Min}\{\zeta < \kappa_{n(*)} : \mathfrak{a}_\alpha \subseteq \lambda_\zeta\}$. Note that $h_n(\alpha)$ is well defined as $\text{cf}(\lambda) \geq \aleph_1 = \sigma$. Hence by the assumption \otimes , as $\text{cf}(\lambda) < \kappa_{n(*)} < \kappa_{n-1}$, for some $\zeta_n < \text{cf}(\lambda)$ we have $\{B \subseteq \kappa_n : \bigwedge_{\alpha \in B} \mathfrak{a}_\alpha^n \subseteq \lambda_{\zeta_n}\} \in \mathfrak{D}_n^+$. Now

we can contradict the induction hypothesis for $\lambda' = \sup\{\lambda_{\zeta_n} : n \in W \setminus (n(*) + 2)\}$. We are left with the case $\text{cf}(\lambda) = \aleph_0$ so let $\lambda = \sum_{n < \omega} \lambda_n, \lambda_0 = \kappa^+, \lambda_n < \lambda_{n+1}$. For

each $n, k < \omega$ define $Y_n^k = \{B \subseteq \kappa_n : \lambda \not\subseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}_\alpha^n \cap [\lambda_k, \lambda_{k+1}])\}$.

So $Y_n = \bigcup_{k < \omega} Y_n^k$, but $Y_n \in \mathfrak{D}_n^+$, and \mathfrak{D}_n^+ is \aleph_1 -complete hence for some $k_n < \omega, Y_n^{k_n} \in \mathfrak{D}_n^+$, so possibly shrinking W we get $\langle k_n : n \in W \rangle$ is constant or strictly increasing, the former contradicts the induction hypothesis on $\lambda_{k_{\text{Min}(W)+1}}$. Now renaming the λ_k 's we get $k_n = n$ and we can replace \mathfrak{a}_α^n by $\mathfrak{a}_\alpha^n \cap [\lambda_n, \lambda_{n+1})$. So without loss of generality $\text{Min}(W) > 4$ and for $n \in W$ we have $\bigcup_{\alpha} \mathfrak{a}_\alpha^n \subseteq [\lambda_n, \lambda_{n+1})$

and $\lambda_n < \lambda$, of course.

Let $n(*) = \text{min}(W)$. We try to define by induction on $k < \omega, \langle \theta_t : t \in w_k \rangle, w_k = \bigcup_{i < \kappa_{n(*)}} w_{k,i}, J_k$ and h_{k-1} if $k > 0$ such that:

- (a) $\theta_t \in \text{Reg} \cap \lambda \setminus \kappa$ for $t \in w_k$
- (b) $w_k = \bigcup_{i < \kappa_{n(*)}} w_{k,i}$ is disjoint to $\bigcup_{\ell < k} w_\ell$
- (c) $\langle w_{k,i} : i < \kappa_{n(*)} \rangle$ is a sequence of pairwise disjoint, countable sets

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- (d) $w_{0,i} = \{i\} \times \mathfrak{a}_i^{n(*)}$ and $\theta_{(i,\tau)} = \tau$ for $\tau \in \mathfrak{a}_i^{n(*)}$
- (e) h_k is a function from w_{k+1} to w_k mapping $w_{k+1,i}$ into $w_{k,i}$
- (f) $J_k = \{w \subseteq w_k : \lambda \supseteq \text{pcf}_{\aleph_2\text{-complete}}(\{\theta_t : t \in w\})\}$ is a proper ideal
- (g) if $w \in J_k^+$ then $\{t \in w_{k+1} : h_k(t) \in w\} \in J_{k+1}^+$
- (h) $t \in w_{k+1} \Rightarrow \theta_t < \theta_{h_k(t)}$.

During the induction, h_k is defined in the k -th step.

If we succeed, we shortly get a contradiction (by observation [Sh 460, 2.2]).

For $k = 0$ define $w_{0,i}, \theta_\tau$ for $\tau \in w_0 = \bigcup_{i < \kappa_{n(*)}} w_{0,i}$ by clause (d) and the clause (f)

holds as otherwise $\{\theta_t : t \in w_0\}$ can be represented as $\bigcup_{\varepsilon < \omega_1} \mathfrak{b}_\varepsilon$ with $\max \text{pcf}(\mathfrak{b}_\varepsilon) < \lambda$,

let $h : \kappa_{n(*)} \rightarrow \omega_1$ be $h(i) = \sup\{\min\{\varepsilon : \tau \in \mathfrak{b}_\varepsilon\} : \tau \in \mathfrak{a}_i^{n(*)}\}$ and apply \otimes from the hypothesis to get $Y \subseteq Y_{n(*)}$ and $\zeta < \omega_1$ such that $Y \in \mathfrak{D}_{n(*)}^+$ and $\{B \in Y : \sup \text{Rang}(h \upharpoonright B) < \zeta\} \in \mathfrak{D}_{n(*)}^+$, but $B \in Y$ implies $\lambda \not\subseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}_\alpha^{n(*)})$

because $B \in Y_{n(*)}$ and

$$\text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}_\alpha^{n(*)}) \subseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\varepsilon < \zeta} \mathfrak{b}_\varepsilon) \subseteq \bigcup_{\varepsilon < \zeta} \text{pcf}_{\aleph_1\text{-complete}}(\mathfrak{b}_\varepsilon) \subseteq \lambda;$$

contradiction.

So assume $w_k, \langle w_{k,i} : i < \kappa_{n(*)} \rangle, J_k, \langle \theta_t : t \in w_k \rangle$ are as required and we shall define $w_{k+1}, \langle w_{k+1,i} : i < \kappa_{n(*)} \rangle, J_{k+1}, \langle \theta_t : t \in w_{k+1} \rangle, h_k$.

Now for any $t \in w_k$ by the induction hypothesis for some $g(t) < \omega$ we have

(*)₁ if $m \in [g(t), \omega)$ and $\mathfrak{b}_i \subseteq \text{Reg} \cap \theta_t \setminus \kappa$ is countable for $i < \kappa_m$ then

$$\{B \subseteq \kappa_m : \text{pcf}_{\aleph_1\text{-complete}}(\cup\{\mathfrak{b}_i : i \in B\}) \subseteq \theta_t\} \in \mathfrak{D}_m$$

(*)₂ $g(t) > n(*) + 1$.

Let $u_{k,m} = \{t \in w_k : g(t) = m \text{ and } \theta_t > \kappa^+\}$. We shall prove

$\boxtimes_{k,m}$ if $u_{k,m} \notin J_k$, then we can find $\langle \mathfrak{c}_t : t \in u_{k,m} \rangle, \mathfrak{c}_t \subseteq \text{Reg} \cap \theta_t \setminus \kappa^+$ countable such that:

$u \subseteq u_{k,m}, u \in J_k^+$ implies $\text{pcf}_{\aleph_2\text{-complete}}(\bigcup_{t \in u} \mathfrak{c}_t) \not\subseteq \lambda$.

As J_k is \aleph_1 -complete (by its definition; even more) this suffices for carrying the induction.

[Why? Let $\langle \mathfrak{c}_t^m : t \in u_{k,m} \rangle$ for $m < \omega$ such that $u_{k,m} \notin J_k$ be as above, let

$$w_{k+1,i} = \bigcup \{ \{t\} \times \mathfrak{c}_t^m : \text{for some } m, u_{k,m} \notin J_k \text{ and } t \in u_{k,m} \cap w_{k,i} \},$$

$$\theta_{(t,\tau)} = \tau, w_{k+1} = \bigcup_{i < \kappa_{n(*)}} w_{k+1,i} \text{ and we define the function } h_{k+1} : w_{k+1} \rightarrow w_k$$

by $h_{k+1}((t,\sigma)) = t$ (note: every t belongs to at most one $u_{k,m}$ ($m < \omega$) and $w_k \setminus \cup \{u_{k,m} : u_{k,m} \notin J_k\} = \emptyset \text{ mod } J_k$.)

Proof of $\boxtimes_{k,m}$. For each $\tau \in \mathfrak{a}_m$, we apply [Sh:g, Ch.I,1.6] or [Sh:g, Ch.IX,4.1] on $\langle \theta_t : t \in u_{k,m} \rangle, J = J_k \upharpoonright u_{k,m}$ and τ (possible as $|u_{k,m}| < \kappa < \min\{\theta_t : t \in u_{k,m}\}$), each θ_t (for $t \in u_{k,m}$) is regular and $\prod_{t \in u_{k,m}} \theta_t / J_k$ is λ^+ -directed, $\tau < \lambda^+$ (the cases

$\tau < \theta_t$ can be ignored for several reasons, e.g. $\mathfrak{a}_m =: \bigcup_{\alpha} \mathfrak{a}_\alpha^m \subseteq [\lambda_m, \lambda_{m+1})$). So we can find $\langle \theta_{t,\tau} : t \in u_{k,m} \text{ and } \tau \in \mathfrak{a}_m \rangle$ such that:

- (α) $\theta_{t,\tau}$ is regular and $\kappa^+ \leq \theta_{t,\tau} < \theta_t$
- (β) $\prod_{t \in u_{k,m}} \theta_{t,\tau} / J_k$ has true cofinality τ

(note that $t \in u_{k,m} \Rightarrow \kappa^+ < \theta_t$, so we can assume $\theta_{t,\tau} \geq \kappa^+$).

Now for each $t \in u_{k,m}, \theta_{t,\tau} \in \text{Reg} \cap \theta_t \setminus \kappa$ for $\tau \in \mathfrak{a}_m$, but $g(t) = m$ (as $t \in u_{k,m}$), hence by the definition of g , letting $\mathfrak{a}_i^{m,t} = \{\theta_{t,\tau} : \tau \in \mathfrak{a}_i^m\}$, for $i < \kappa_m$, we have

$$\Gamma_t^m =: \{B \subseteq \kappa_m : \theta_t \supseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{i \in B} \mathfrak{a}_i^{m,t})\} \in \mathfrak{D}_m.$$

But \mathfrak{D}_m is $\kappa_{n(*)+1}$ -complete (as $m = g(t) > n(*) + 1$) and $|u_{k,m}| \leq \kappa_{n(*)} < \kappa_{n(*)+1}$ hence $\Gamma^* = \bigcap_{t \in u_{k,m}} \Gamma_t^m \in \mathfrak{D}_m$. On the other hand (by the choice of \mathfrak{a}_m)

$$\Gamma_m =: \{B \subseteq \kappa_m : \lambda \supseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{i \in B} \mathfrak{a}_i^m)\} \notin \mathfrak{D}_m.$$

So there is $B \in \Gamma_m \cap \Gamma^* = \Gamma_m \cap \bigcap_{t \in u_{k,m}} \Gamma_t^m$.

Let

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$$\mathfrak{a}^* = \{\theta_t : t \in u_{k,m}\} \cup \{\theta_{t,\tau} : t \in u_{k,m}, \tau \in \mathfrak{a}_m\} \cup \mathfrak{a}_m,$$

and we can for simplicity assume $|\lambda \cap \text{pcf}(\mathfrak{a}^*)| < \min(\mathfrak{a}^*)$, if $2^\mu < \lambda$ possible (by retroactive changes).

[Why? By [Sh 460, Th.2.5].] Hence there is a smooth close generating sequence $\langle \mathfrak{b}_\varrho[\mathfrak{a}^*] : \varrho \in \lambda \cap \text{pcf}(\mathfrak{a}^*) \rangle$ (see e.g. [Sh 430, 6.7], if not use [Sh 430, 6.7, 6.7F]). Clearly $B \neq \emptyset$. For each $t \in u_{k,m}$ we know $B \in \Gamma_t^m$ hence $\theta_t \supseteq \text{pcf}_{\aleph_1\text{-complete}}(\bigcup_{i \in B} \mathfrak{a}_i^{m,t})$. So

we can find countable $\mathfrak{c}_t \subseteq \theta_t \cap \text{pcf}(\bigcup_{i \in B} \mathfrak{a}_i^{m,t})$ such that

$$\bigcup_{i \in B} \mathfrak{a}_i^{m,t} \subseteq \bigcup_{\sigma \in \mathfrak{c}_t} \mathfrak{b}_\sigma[\mathfrak{a}^*].$$

The pcf calculus verifies clause (g) (as in [Sh 460, §2]).

□_{1.3}

It is now natural to look for suitable filters \mathfrak{D} , the simplest ones are:

1.5 Definition. For $\sigma < \theta < \kappa$ and $\mu \leq \kappa$ (always θ regular) let $\mathfrak{D} = \mathfrak{D}_{\sigma,\theta,\kappa,\mu}$ be the following filter on $\mathcal{P}(\kappa) : Y \in \mathfrak{D}$ iff there are functions $f_\alpha : \kappa \rightarrow \theta_\alpha$ for $\alpha < \alpha(*) < \theta$ where $\theta_\alpha \in [\sigma, \theta) \cap \text{Reg}$ such that $Y \supseteq \{a \subseteq \kappa : |a| \geq \mu \text{ and for every } \alpha < \alpha(*) \text{ for some } \zeta < \theta_\alpha \text{ we have } \text{Rang}(f_\alpha \upharpoonright a) \subseteq \zeta\}$. If $\mu = \theta$ we may omit it.

- 1.6 Observation.* 1) If $\sigma < \theta < \kappa_1 \leq \kappa_2$ and $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa_1,\mu}$ then $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa_2,\mu}$.
 2) $\mathfrak{D}_{\sigma,\theta,\kappa}$ is a θ -complete filter.
 3) If $2^{<\theta} < \kappa$ then $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa}$ and this is preserved by σ -c.c. forcing.

Proof. Straight.

1.7 Conclusion. Let μ be a limit singular cardinal of cofinality $< \sigma = \text{cf}(\sigma) < \mu$ and:

- ⊗ for every $\theta \in (\sigma, \mu) \cap \text{Reg}$ for some $\kappa \in (\theta, \mu)$ we have: $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa}$.

Then for every $\lambda > \mu$, for some $\theta = \theta_\mu \in (\sigma, \mu) \cap \text{Reg}$ for every $\kappa \in (\theta, \mu)$ we have

- (*) if $\lambda_i \in (\mu, \lambda) \cap \text{Reg}$ for $i < \kappa$ then

$$\{a \subseteq \kappa : \text{pcf}_{\sigma\text{-complete}}\{\lambda_i : i \in a\} \subseteq \lambda\} \in \mathfrak{D}_{\sigma,\theta,\kappa}.$$

Proof. Assume λ is a counterexample. Without loss of generality σ is regular, choose by induction on $\zeta < \sigma$, $\kappa_\zeta \in (\sigma, \mu) \cap \text{Reg}$ as follows

$\kappa_0 \in (\sigma, \mu) \cap \text{Reg}$ arbitrary;

$\kappa_\zeta \in (\bigcup_{\varepsilon < \zeta} \kappa_\varepsilon^+, \mu) \cap \text{Reg}$ is minimal κ which is a witness to \otimes for

$\theta_\zeta = (\bigcup_{\varepsilon < \zeta} \{\kappa_\varepsilon^+ : \varepsilon < \zeta\})^+$ (in particular $\emptyset \notin \mathfrak{D}_{\sigma, \theta_\zeta, \kappa_\zeta}$, so $\kappa_\zeta < \mu$).

Let $\kappa = \bigcup_{\zeta < \sigma} \kappa_\zeta$ and apply 1.3 for $\mathfrak{D}_{\sigma, \theta_\zeta, \kappa_\zeta}$ from Definition 1.5, more exactly the variant with σ instead \aleph_1 (see 1.4); alternatively use only $\langle \kappa_n : n < \omega \rangle$. $\square_{1.7}$

1.8 Remark. 1) In the proof of 1.3 we can change the universe during the proof, so weaken the demand \otimes .

2) The problematic example is: $T \subseteq {}^{\omega_1}\omega$ a Kurepa tree, say $T \cap {}^\alpha 2 = \{\gamma_\alpha(n) : m < \omega\}$, $\eta_j \in \lim_{\omega_1}(T)$ for $j < j^*$ and $\{a \subseteq \omega_1 \cup j^* : \delta =: a \cap \omega_1 \text{ a limit ordinal and for every } n < \omega \text{ for some } j \in j^* \cap a \text{ we have } \gamma_\alpha(n) = \eta_j(\alpha)\} \in \mathfrak{D}_{< \aleph_1}(\omega_1 \cup j^*)$.

3) We can replace in 1.7 above $\text{cf}(\mu) < \sigma$ by $\text{cf}(\mu) \neq \sigma$. We can replace $\emptyset \notin \mathfrak{D}_{\sigma, \theta, \kappa}$ by $\emptyset \notin \mathfrak{D}_{\sigma, \theta, \kappa; \Upsilon}$ for any fix $\Upsilon \in [\sigma, \mu)$. Note that the case $\Upsilon < \sigma$ is not interesting.

4) Note that the meaning of $\emptyset \in \mathfrak{D}_{\sigma, \theta, \kappa; \Upsilon}$ is that there are $\alpha(*) < \theta$ and functions $f_\alpha : \kappa \rightarrow \theta_\alpha$ where $\theta_\alpha \in \text{Reg} \cap [\sigma, \theta)$ such that for no $u \in [\kappa]^\Upsilon$ do we have $\alpha < \alpha(*) \Rightarrow \sup \text{Rang}(f_\alpha \upharpoonright u) < \theta_\alpha$. Recall if Υ is omitted it means $\Upsilon = \theta$.

1.9 Definition. Assume $\mathbf{J} \subseteq \text{Id}(\kappa)$ (= the family of ideals on κ).

1) We say (λ, μ) is \mathbf{J} -inaccessible if $\kappa \leq \mu < \lambda$ and there are no $\lambda_i \in \text{Reg} \cap (\mu, \lambda)$ for $i < \kappa$ and $J \in \mathbf{J}$ such that $\prod_{i < \kappa} \lambda_i / \mathbf{J}$ is λ -directed (equivalently, for some such

λ_i 's, $\prod_{i < \kappa} \lambda_i / J$ has true cofinality and it is $\geq \lambda$).

2) $(\lambda, *)$ means (λ, μ) for some $\mu \in [\kappa, \lambda)$, λ means (λ, κ) .

3) \mathbf{J} is σ -indecomposable when: if $J \in \mathbf{J}$ and $h : \text{Dom}(J) \rightarrow \sigma$ then for some $\zeta < \sigma$ and $I \in \mathbf{J}$ we have $J \upharpoonright h^{-1}\{\varepsilon : \varepsilon < \zeta\} \leq^* I$ (see below).

4) For ideals I_ℓ , on A_ℓ ($\ell = 0, 1$) $I_0 \leq^* I_1$ if there is $B_0 \in I_0^+$ and $B_1 \in I_1^+$ and one-to-one function g from B_0 into B_1 such that

$$Y \cap B_0 \in I_0 \Rightarrow g''(Y \cap B_0) \in I_1$$

5) \mathbf{J} is $[\sigma, \kappa)$ -indecomposable if it is θ -indecomposable for every $\theta \in [\sigma, \kappa) \cap \text{Reg}$.

1.10 Claim. *Assume J is a σ -complete ideal on κ and $X \in I^+ \Rightarrow I \upharpoonright X \cong I$ (e.g. $I = J_\kappa^{\text{bd}}$, cf $\kappa \geq \sigma$). Then (λ, μ) is $\{J\}$ -inaccessible if λ is (J, μ, σ) -inaccessible.*

Proof. Compare Definition 1.9(1) and 1.2(2).

§2 LARGE PCF(\mathfrak{a}) IMPLIES THE EXISTENCE OF FREE SETS

2.1 Definition. 1) Let $\bar{A} = \langle A_\alpha : \alpha < \alpha^* \rangle$ be a sequence of subsets of κ , no A_α in the ideal generated by $\{A_\beta : \beta < \alpha\}$. We define functions $\text{rk} = \text{rk}_{\bar{A}}$, $\text{rk}' = \text{rk}'_{\bar{A}}$ from $\mathcal{P}(\kappa)$ to the ordinals by:

$\text{rk}(A) \geq \zeta$ iff for every $\xi < \zeta$ for some α , $A \neq A \cap A_\alpha$ and $\text{rk}(A \cap A_\alpha) \geq \xi$
 $\text{rk}'(A) \geq \zeta$ iff for every $\xi < \zeta$ for some α , we have $\text{rk}'(A \cap A_\alpha) \geq \xi$ and $A \setminus A_\alpha, A \cap A_\alpha$ are not in $\text{id}_{\bar{A} \upharpoonright \alpha} =$ the ideal generated by $\{A_\beta : \beta < \alpha\}$.

2) Let $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ be a sequence of pairs of ideals on κ such that $[\alpha < \beta \Rightarrow J_\alpha \subseteq J'_\alpha \subseteq J_\beta \subseteq J'_\beta]$ and for some $A_\alpha \in J_\alpha^+, J'_\alpha = J_\alpha + A_\alpha$ we define $\text{rk}'_{\bar{J}}(A)$ for $A \subseteq \kappa$ by:

$\text{rk}'_{\bar{J}}(A) \geq \zeta$ iff for every $\xi < \zeta$ for some $\alpha < \alpha^*$ we have: $\text{rk}'_{\bar{J}}(A \cap A_\alpha) \geq \xi$ and $A \setminus A_\alpha, A \cap A_\alpha$ are not in J_α .

3) We identify \bar{A} with $\langle (\text{id}_{\bar{A} \upharpoonright \alpha}, \text{id}_{\bar{A} \upharpoonright (\alpha+1)}) : \alpha < \alpha^* \rangle$ (see 2.2(3) below). If $\bar{J} = \langle (J_\alpha, J_{\alpha+1}) : \alpha < \alpha^* \rangle$ is as required in (2) we may write $\langle J_\alpha : \alpha < \alpha^* \rangle$ instead \bar{J} . We can replace κ by any other set. We may write $\text{rk}^{(j)}(A, \bar{A})$ or $\text{rk}'(A, \bar{J})$ instead $\text{rk}_{\bar{A}}^{(j)}(A)$ or $\text{rk}'_{\bar{J}}(A)$.

2.2 Claim. 1) $\text{rk}_{\bar{A}}, \text{rk}'_{\bar{A}}$ are well defined (values: ordinals or ∞) and nondecreasing in A (under \subseteq).

2) $\text{rk}_{\bar{A}}(A) \geq \text{rk}'_{\bar{A}}(A)$.

3) $\text{rk}'_{\bar{A}}$ depend just on $\langle \text{id}_{\bar{A} \upharpoonright \alpha} : \alpha \leq \alpha^* \rangle$ and for $A \subseteq \kappa$, we have $\text{rk}'_{\bar{A}}(A) = \text{rk}'_{\bar{J}}(A)$ where $\bar{J} = \langle (\text{id}_{\bar{A} \upharpoonright \alpha}, \text{id}_{\bar{A} \upharpoonright (\alpha+1)}) : \alpha \leq \alpha^* \rangle$ (so we may write $\text{rk}'_{\langle J_\alpha : \alpha \leq \alpha^* \rangle}(A)$ with $J_\alpha = \text{id}_{\bar{A} \upharpoonright \alpha}$).

4) If $\text{rk}'_{\bar{A}}(\kappa) = \zeta$, then we can find $\bar{Y} = \langle Y_\varepsilon : \varepsilon < \zeta \rangle$, an increasing sequence of subsets of α^* , $\bigcup_{\varepsilon < \zeta} Y_\varepsilon = \alpha^*$ and for each $\varepsilon < \zeta$, the sets in the family $\{A_\alpha : \alpha \in$

$Y_\varepsilon \setminus \bigcup_{\xi < \varepsilon} Y_\xi\}$ are almost disjoint and positive modulo the ideal generated by $\{A_\alpha :$

$\alpha \in \bigcup_{\xi < \varepsilon} Y_\xi\}$.

5) If $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ is as in 2.1(2), where J_α, J'_α are ideals on κ , $J'_\alpha = J_\alpha + A_\alpha$ and $\bar{A} = \langle A_\alpha : \alpha < \alpha^* \rangle$ then for every $B \subseteq \kappa$ we have $\text{rk}_{\bar{A}}(B) \geq \text{rk}'_{\bar{A}}(B) \geq \text{rk}'_{\bar{J}}(B)$.

Proof. Straight: e.g. for the fourth part use $Y_\varepsilon = \{\alpha : \text{rk}'_{\bar{A}}(A_\alpha) < \varepsilon\}$. □_{2.2}

2.3 Claim. 1) If $\text{rk}_{\bar{A}}(\kappa) \geq \kappa^+$ then for some $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ we have $\alpha_n < \alpha_{n+1} < \kappa$ and for every $\ell < k < \omega$ for some $\alpha < \alpha^*$ we have $A_\alpha \cap \{\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_k\} = \{\alpha_{\ell+1}, \dots, \alpha_k\}$.

2) If $\text{rk}'_{\bar{J}}(\kappa) \geq \beta$ and $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ as in 2.1(2) then for some $\Gamma \subseteq \alpha^*$, $|\Gamma| \leq |\beta|$ we have $\text{rk}'_{\bar{J} \upharpoonright \Gamma}(\kappa) \geq \beta$.

3) If $\text{rk}_{\bar{J}}(B) \geq \beta$, $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ as in 2.1(2) and $J'_\alpha = J_\alpha + A_\alpha$ then we can find $\Gamma \subseteq \alpha^*$ such that

(*) $|\Gamma| \leq |\beta| + \aleph_0$ (even $|\Gamma| < |\beta|^+ + \aleph_0$) and if $A_\alpha \subseteq A'_\alpha \in J'_\alpha$ then $\text{rk}_{\langle A'_\alpha : \alpha \in \Gamma \rangle}(B) \geq \beta$.

Proof. 1) Let $\text{rk} = \text{rk}_{\bar{A}}$; choose by induction on n an ordinal $\alpha_n < \kappa$ and for every $\zeta < \kappa^+$ a decreasing sequence $\langle B_{\zeta,0}^n, \dots, B_{\zeta,n}^n \rangle$ of sets such that

(α) $(\forall \ell \leq n)(\forall m \leq n)[\alpha_\ell \in B_{\zeta,m}^n \Leftrightarrow \ell > m]$,

(β) $\text{rk}(B_{\zeta,n}^n) \geq \zeta$

(γ) each $B_{\zeta,n}^m$ is the intersection of finitely many A_α 's

For $n = 0$, for every $\zeta < \kappa^+$ there is $\alpha_\zeta < \alpha^*$ such that $\kappa \cap A_{\alpha_\zeta} \neq \kappa$, $\text{rk}(\kappa \cap A_{\alpha_\zeta}) \geq \zeta$, and choose $\alpha_\zeta^0 \in \kappa \setminus A_{\alpha_\zeta}$. So for some $\alpha_0 < \kappa$ we have $\kappa^+ = \sup\{\zeta < \kappa^+ : \alpha_\zeta^0 = \alpha_0\}$ and let $B_{\zeta,0}^0 = A_{\alpha_{\xi(\zeta)}}$ where $\xi(\zeta) < \kappa^+$ is the minimal $\xi \geq \zeta$ such that $\alpha_\xi^0 = \alpha_0$, as in demand (β) we ask “ $\geq \zeta$ ” not “ $= \zeta$ ”, we succeed. If we have defined for n , for each $\zeta < \kappa^+$, as $\text{rk}(B_{\zeta+1,n}^n) \geq \zeta + 1$, there is $\beta(\zeta, n) < \text{lg}(\bar{A})$ such that $\neg[B_{\zeta+1,n}^n \subseteq A_{\beta(\zeta,n)}]$ but $\text{rk}_{\bar{A}}(B_{\zeta+1,n}^n \cap A_{\beta(\zeta,n)}) \geq \zeta$, and choose $\gamma(\zeta, n) \in B_{\zeta+1,n}^n \setminus A_{\beta(\zeta,n)}$ so for some $\alpha_{n+1} < \kappa$, the set $S_n = \{\zeta < \kappa^+ : \gamma(\zeta, n) = \alpha_{n+1}\}$ is unbounded in κ^+ . For every $\zeta < \kappa^+$ let $\xi(\zeta, n) = \min\{\xi : \xi \in S_n, \xi > \zeta\}$, let $B_{\zeta,\ell}^{n+1}$ be $B_{\xi(\zeta,n),\ell}^n$ if $\ell \leq n$ and $B_{\xi(\zeta,n)+1,n}^n \cap A_{\beta(\zeta,n)}$ if $\ell = n + 1$. In the end we know that for $\ell < k < \omega$, for every $\zeta < \kappa^+$ we have $B_{\zeta,\ell}^k \cap \{\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_k\} = \{\alpha_{\ell+1}, \dots, \alpha_k\}$; also $B_{\zeta,\ell}^k$ has the form $\bigcap_{m < m(*)} A_{\alpha(m)}$ for some $m(*)$ and $\alpha(m) < \text{lg}(\bar{A})$ for $m < m(*)$, so

for some m we have $\alpha_\ell \notin A_{\alpha(m)}$, but $\{\alpha_{\ell+1}, \dots, \alpha_k\} \subseteq B_{\zeta,\ell}^k \subseteq A_{\alpha(m)}$ so $\alpha(m)$ is as required in 2.3(1) for our $\ell < k < \omega$. Lastly $\bigwedge_{n < m} \alpha_n \neq \alpha_m$ hence by Ramsey

theorem without loss of generality $\alpha_n < \alpha_{n+1}$ and we are done.

2) By induction on β or by part (3).

3) We can find $\langle (B_\eta, j_\eta) : \eta \in ds(\beta) \rangle$ where $ds(\beta) = \{\eta : \eta \text{ is a (strictly) decreasing sequence of cardinals } < \beta\}$, $B_{<} = B, \text{rk}'_{\bar{J}}(B_\eta) \geq \min(\{\beta\} \cup \{\eta(\ell) : \ell < \text{lg}(\eta)\})$ and $j_\eta < \alpha^*$ and if $\nu = \eta \hat{\ } \langle \gamma \rangle \in ds(\beta)$ then $B_\nu \neq B_\eta, B_\nu = B_\eta \cap A_{j_\eta} \notin J_{j_\eta}$ and $B_\eta \setminus B_\nu \notin J_{j_\eta}$. Let $\Gamma = \{j_\eta : \eta \in ds(\beta)\}$, now if $A_\alpha \subseteq A'_\alpha \in J_\alpha$ for $\alpha \in \Gamma$

then we can prove by induction on $\gamma < \beta$ that: $\eta \in ds(\beta) \Rightarrow \text{rk}'_{\langle A'_\alpha : \alpha \in \Gamma \rangle}(B_\eta) \geq \max(\{\beta\} \cup \{\eta(\ell) : \ell < \ell g(\eta)\})$. □_{2.3}

2.4 Definition. 1) For $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ strictly increasing, let $\text{IND}(\bar{\lambda})$ mean:

- (*) $_{\bar{\lambda}}$ for every algebra M with universe $\bigcup_{n < \omega} \lambda_n$ and \aleph_0 functions (all finitary) there is $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ such that:
- (a) $\alpha_n < \lambda_n$
 - (b) α_n is not in the M -closure of

$$\{\alpha_\ell : \ell \in (n, \omega)\} \cup \{i : \bigvee_{m < n} i < \lambda_m\}.$$

2) $\text{IND}(\lambda)$ means that $\lambda > \text{cf}(\lambda) = \aleph_0$ and for every (equivalently some, see below) $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ strictly increasing with limit λ we have

- (*)' $_{\bar{\lambda}}$ for every algebra M with universe λ and countably many functions there is a sequence $\langle \alpha_n : n \in \omega \rangle$ such that:
- (a) $w \subseteq \omega$ is infinite
 - (b) $\alpha_n < \lambda_n$ for $n \in w$
 - (c) for $n \in w$, α_n is not in the M -closure of

$$\{\alpha_\ell : \ell \in w, \ell > n\} \cup \{i : \bigvee_{\ell \in n \cap w} i < \lambda_\ell\}$$

3) $\text{IND}(\lambda, \kappa) = \text{IND}^0(\lambda, \kappa)$ means: if M is a model with universe λ and κ functions we can find $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ such that

$$\alpha_n < \lambda, \alpha_n \notin \text{cl}_M\{\alpha_\ell : \ell < \omega, \ell > n\}.$$

4) $\text{IND}^1(\lambda, \kappa)$ is defined similarly but demanding

$$\alpha_n \notin \text{cl}_M\{\alpha_\ell : \ell < \omega, \ell \neq n\}.$$

2.5 Observation. 1) In 2.4(2), if (*)' $_{\bar{\lambda}}$ holds for one strictly increasing $\bar{\lambda}$ with limit λ , then it holds for every strictly increasing $\bar{\lambda}' = \langle \lambda'_n : n < \omega \rangle$ with limit λ .

2) If λ is uncountable with cofinality \aleph_0 , \mathbb{P} a forcing notion of cardinality $\leq \mu < \lambda$ or satisfying the μ^+ -c.c. for some $\mu < \lambda$, or λ -complete then: $\text{IND}(\lambda) \Leftrightarrow \Vdash_{\mathbb{P}} \text{“IND}(\lambda)\text{”}$ and if $\kappa \in [\mu, \lambda)$, μ as above then $\text{IND}(\lambda, \kappa) \Leftrightarrow \Vdash_{\mathbb{P}} \text{“IND}(\lambda, \kappa)\text{”}$ and if in addition $\mu < \lambda_n < \lambda_{n+1}$, then $\text{IND}(\langle \lambda_n : n < \omega \rangle) \Leftrightarrow \Vdash_{\mathbb{P}} \text{“IND}(\langle \lambda_n : n < \omega \rangle)\text{”}$.

3) $\text{IND}(\bar{\lambda}) \Rightarrow \text{IND}(\lambda) \Rightarrow \text{IND}^1(\lambda, \kappa) \Rightarrow \text{IND}^0(\lambda, \kappa)$ if $\lambda = \bigcup_{n < \omega} \lambda_n$ and $\lambda_n < \lambda_{n+1}$ and $\lambda_0 > \kappa$. If $\kappa < \lambda \leq \lambda'$ and $i \in \{0, 1\}$ then

$$\text{IND}^i(\lambda, \kappa) \Rightarrow \text{IND}^i(\lambda', \kappa).$$

4) If ($i \in \{0, 1\}$ and) $\text{IND}^i(\lambda, \kappa)$, λ minimal for this κ then

(a) $\kappa \leq \kappa_1 < \lambda \Rightarrow \text{IND}^i(\lambda, \kappa_1)$

(b) $\text{cf}(\lambda) = \aleph_0$ and $\text{IND}^1(\lambda, \kappa)$ or λ is inaccessible

(c) if $\lambda = \sum_{n < \omega} \lambda_n$ and $\lambda_n < \lambda_{n+1}$ then not only $(*)'_{\bar{\lambda}}$ (from 2.4(2)), where $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$) but if \mathbb{P} is a c.c.c. forcing adding a dominating real then in $V^{\mathbb{P}}$ for some infinite $w \subseteq \omega$ we have $(*)_{\bar{\lambda} \upharpoonright w}$ from 2.4(1) holds.

5) $\text{IND}^1(\lambda, \kappa)$ is equivalent to $\text{IND}^0(\lambda, \kappa)$.

Proof. 1, 2), 3) Check.

4) Clause (a) of (4): Assume not and first let $i = 0$. Let $\chi = \beth_3(\lambda)^+$ and let M be the model with universe λ and the functions (n -place from λ to λ for some n) definable in $(\mathcal{H}(\chi), \in, <^*_\chi)$ with the additional individual constants $\lambda, \kappa, \kappa_1$. Clearly $(M, \alpha)_{\alpha < \kappa_1}$ exemplifies $\neg \text{IND}^i(\lambda, \kappa_1)$. Let F_n^-, F_n^+ be such that for $\bar{\beta} = \langle \beta_\ell : \ell < n \rangle, \beta_\ell < \lambda$ we have: $F_n^+(-, \bar{\beta})$ is a one-to-one function from $\text{cl}_M(\{\beta_0, \dots, \beta_{n-1}\} \cup \kappa_1)$ onto κ_1 and $F_n^-(-, \bar{\beta})$ is its inverse. We can apply the assumption $\text{IND}^i(\lambda, \kappa)$ to the model $(M, F_n^+, F_n^-, \beta)_{n < \omega, \beta < \kappa}$, so there are $\alpha_n (n < \omega)$ as in 2.4(3). By the assumption for no infinite $w \subseteq \omega$ is $\{\alpha_n : n \in w\}$ as required in 2.4(3) for $(M, \beta)_{\beta < \kappa_1}$. We claim

(*) for some infinite $w \subseteq \omega, \bigwedge_{n \in w} \alpha_n \in \text{cl}_M(\{\alpha_\ell : n < \ell \in w\} \cup \kappa_1)$.

[Why? Try to choose by induction on $\ell < \omega, u_\ell, n_\ell$ such that: $\bigwedge_{m < \ell} n_m < n_\ell < \omega, u_\ell \subseteq (n_\ell, \omega)$ is infinite, $u_{\ell+1} \subseteq u_\ell \subseteq w$ and $\alpha_{n_\ell} \notin \text{cl}_M(\{\alpha_n : n \in u_\ell\})$, we cannot succeed so w or some u_ℓ is as required.]

By renaming $w = \omega$, so for every n for some $k_n \in (n+1, \omega)$ we have $\alpha_n \in \text{cl}_M(\{\alpha_{n+1}, \dots, \alpha_{k_n}\} \cup \kappa_1)$; as we can increase k_n , without loss of generality $k_n <$

k_{n+1} hence $m \leq n \Rightarrow \alpha_m \in \text{cl}_M(\{\alpha_{n+1}, \dots, \alpha_{k_n}\} \cup \kappa_1)$ (just prove this by induction on n), so $\gamma_{n,m} =: F_{k_n-n}^+(\alpha_m, \alpha_{n+1}, \dots, \alpha_{k_n}) < \kappa_1$ and for each n the sequence $\langle \gamma_{n,m} : m \leq n \rangle$ is with no repetitions. Choose by induction on $\ell, m_\ell \in [\ell^2, (\ell+1)^2)$ such that $\gamma_{(\ell+1)^2, m_\ell} \notin \{\gamma_{(q+1)^2, m_q} : q < \ell\}$. But as $\neg \text{IND}^i(\kappa_1, \kappa)$ (because $\lambda > \kappa_1$ was minimal such that ...) for some $\ell < p < \omega$ we have $\gamma_{(\ell+1)^2, m_\ell} \in \text{cl}_M(\{\gamma_{(q+1)^2, m_q} : \ell < q < p\} \cup \kappa)$ and using some $F_{p^2-(\ell+1)^2}^-$ we have $\alpha_{m_\ell} \in \text{cl}_M(\{\alpha_q : q \text{ is } \geq (\ell+1)^2 \text{ but } \leq k_{(p^2)}\} \cup \kappa)$.

[Why? First note that $\gamma_{(q+1)^2, m_q}$ belong to this model for $q = \ell+1, \dots, p-1$, (using $F_{k_{(q+1)^2}}$) hence also $\gamma_{(\ell+1)^2, m_\ell}$ belongs to this model by the choice of ℓ and p ; a contradiction.] So we have proved clause (a) for $i = 0$.

If $i = 1$ the proof is similar: choose, by induction on $\ell, k_\ell, m_\ell, m_\ell$ such that $k_\ell < m_\ell < k_{\ell+1}$ and $\alpha_{m_\ell} \in \text{cl}_M(\{\alpha_n : n \in [k_\ell, m_\ell] \text{ or } n \in (m_\ell, k_{\ell+1})\} \cup \kappa_1)$, this is possible as otherwise $\{\alpha_n : n \in [k_\ell, \omega)\}$ contradict “ M exemplifies $\neg \text{IND}^1(\lambda, \kappa_1)$ ”. Let

$$\gamma_\ell = F_{k_{\ell+1}-k_\ell-1}^+(\alpha_{m_\ell}; \alpha_{k_\ell}, \alpha_{k_\ell+1}, \dots, \alpha_{m_\ell-1}, \alpha_{m_\ell+1}, \dots, \alpha_{k_{\ell+1}-1}) < \kappa_1.$$

For some $\ell < \ell(*) < \omega$ we have $\gamma_\ell \in \text{cl}_M(\{\gamma_0, \dots, \gamma_{\ell-1}, \gamma_{\ell+1}, \dots, \gamma_{\ell(*)-1}\} \cup \kappa)$ (because $\neg \text{IND}^1(\kappa_1, \kappa)$ as λ is first and the choice of M), hence $\alpha_\ell \in \text{cl}_M(\{\alpha_n : n < \omega, n \leq \ell\} \cup \kappa)$; a contradiction.

Clause (b) of (4): By the definition easily $\aleph_0 < \text{cf}(\lambda) \leq \kappa$ is impossible.

[Why? Let $\lambda = \sum_{i < \kappa} \lambda_i$, with $\lambda_i < \lambda$, and by the minimality of λ let M_i be a model with universe λ_i and $\leq \kappa$ functions exemplifying $\neg \text{IND}(\lambda_i, \kappa)$ and lastly let M be the model with universe λ and the functions of all the M_i ; check that M exemplifies $\neg \text{IND}(\lambda, \kappa)$.]

By 2.5(4) clause (a) it follows that

$$[\text{cf}(\lambda) > \aleph_0 \ \& \ \kappa_1 < \lambda \Rightarrow \kappa_1 < \text{cf}(\lambda)].$$

So if $\text{cf}(\lambda) > \aleph_0$ then λ is regular, it is inaccessible as it is not a successor as trivially $\neg \text{IND}^i(\mu^+, \mu)$ so by clause (a) we have

$$\neg \text{IND}^i(\mu, \kappa) \Rightarrow \neg \text{IND}^i(\mu^+, \kappa).$$

We still have to prove $\text{IND}^1(\lambda, \aleph_0)$ when $\text{cf}(\lambda) = \aleph_0$; if $i = 1$ this is trivial so assume $i = 0$. So assume $\text{cf}(\lambda) = \aleph_0, \bar{\lambda} = \langle \lambda_n : n < \omega \rangle, \lambda_n < \lambda_{n+1}$ and $\lambda = \sum_{n < \omega} \lambda_n$. We should prove $\text{IND}^1(\lambda, \kappa)$, it follows from part (5).

Clause (c) of (4): Left to the reader.

5) By 2.5(3),

$$\text{IND}^1(\lambda, \kappa) \Rightarrow \text{IND}^0(\lambda, \kappa) \text{ and } \lambda \leq \lambda' \ \& \ \text{IND}^i(\lambda, \kappa) \Rightarrow \text{IND}^i(\lambda', \kappa).$$

Hence it suffices to prove: if λ is minimal such that $\text{IND}^0(\lambda, \kappa)$ then $\text{IND}^1(\lambda, \kappa)$. Let M be a model with universe λ and vocabulary of cardinality $\leq \kappa$ and we shall prove the conclusion of 2.4(4) (= the Definition of $\text{IND}^1(\lambda, \kappa)$). Let for $n < \omega$, F_n^+ , F_n^- be $(n+2)$ -place functions from λ to λ such that

- (*) if $\gamma < \lambda$ and $\bar{\beta} \in {}^n\lambda$ then $F_n^+(-, \bar{\beta}, \gamma)$ is a one-to-one function from $\text{cl}_M(\{\beta_\ell : \ell < \ell g(\bar{\beta})\} \cup \{i : i \leq \gamma \text{ or } i < \kappa\})$ onto $|\kappa + \gamma|$ and $F_n^-(-, \bar{\beta}, \gamma)$ be the inverse function.

Let $M^* = (M, F_n^+, F_n^-)_{n < \omega}$ [Saharon: addition!] $\text{IND}(\lambda, \kappa)$ we can apply Definition 2.4(3) and find a sequence $\langle \alpha_n : n < \omega \rangle$ satisfying $\alpha_n < \lambda$, $\alpha_n \notin \text{cl}_{M^+}(\{\alpha_\ell : \ell \in (n, \omega)\})$. Without loss of generality $\langle \alpha_n : n < \omega \rangle$ is strictly increasing; $\alpha_n > \kappa$ (as each $i \leq \kappa$ is an individual constant of M^+), clearly it suffices to prove

- (**) for any $n < \omega$ for some $m \in (n, \omega)$ we have
 $\alpha_m \notin \text{cl}_M(\{\alpha_\ell : \ell < n \text{ or } \ell > m\})$
 hence it suffices to prove
 (**)' for any $n < \omega$ for some $m \in (n, \omega)$ we have

$$\alpha_m \notin \text{cl}_M(\{\alpha_\ell : \ell > m\} \cup \{i : i \leq \alpha_m\}).$$

[Saharon: addition!!]

□_{2.5}

2.6 Claim. 1) Assume $\lambda > \text{cf}(\lambda)$, $|\mathfrak{a}|^+ < \min(\mathfrak{a})$ and $\text{sup}(\mathfrak{a}) = \lambda$ and $\text{rk}'_{\langle J_{<\theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle}(\mathfrak{a}) \geq |\mathfrak{a}|^+$. Then $\text{IND}(\lambda, |\mathfrak{a}|)$.

2) Moreover, for any model M with universe λ and $|\mathfrak{a}|$ functions and \mathfrak{c} such that $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{pcf}(\mathfrak{a})$, $|\mathfrak{c}| < \min(\mathfrak{a})$ and $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ a generating sequence we can find $\bar{\alpha} = \langle \alpha_\theta : \theta \in \mathfrak{a} \rangle \in \Pi \mathfrak{a}$ such that, defining for $\mathfrak{b} \subseteq \mathfrak{a}$:

$$\text{cl}_{M, \bar{\alpha}}(\mathfrak{b}) = \mathfrak{b} \cup \{\theta \in \mathfrak{a} : \alpha_\theta \in \text{cl}_M(\{\alpha_\mu : \mu \in \mathfrak{b}\})\};$$

we have

$$\otimes_1 [\theta \in \mathfrak{c} \Rightarrow \text{cl}_{M, \bar{\alpha}}(\mathfrak{b}_\theta[\mathfrak{a}]) \in J_{\leq \theta}[\mathfrak{a}]];$$

\otimes_2 $cl_{M,\bar{\alpha}}(-)$ is a closure operation on \mathbf{a} , i.e.

$$\mathbf{b}_1 \subseteq \mathbf{b}_2 \Rightarrow cl_{M,\bar{\alpha}}(\mathbf{b}_1) \subseteq cl_{M,\bar{\alpha}}(\mathbf{b}_2),$$

$$\mathbf{b} \subseteq cl_{M,\bar{\alpha}}(\mathbf{b}),$$

$$cl_{M,\bar{\alpha}}(cl_{M,\bar{\alpha}}(\mathbf{b})) = cl_{M,\bar{\alpha}}(\mathbf{b}).$$

Remark. See 3.17 - 3.20 for more.

Proof. 1) Let us define $\bar{J} = \langle (J_{<\theta}[\mathbf{a}], J_{\leq\theta}[\mathbf{a}]) : \theta \in \text{pcf}(\mathbf{a}) \rangle$. We prove part (1) assuming part (2). Choose $\mathbf{c} \subseteq \text{pcf}(\mathbf{a})$, $|\mathbf{c}| = |\mathbf{a}|^+$ such that $\text{rk}'_{\bar{J} \upharpoonright \mathbf{c}}(\mathbf{a}) \geq |\mathbf{a}|^+$ (this is possible by 2.3(2)) and without loss of generality $\mathbf{a} \subseteq \mathbf{c}$ and let $\langle \mathbf{b}_\theta[\mathbf{a}] : \theta \in \text{pcf}(\mathbf{a}) \rangle$ be a generating sequence for \mathbf{a} (exists by [Sh 371, 2.6]). For proving $\text{IND}(\lambda, |\mathbf{a}|)$ let M be a model with universe λ and $\leq |\mathbf{a}|$ functions, by part (2) there is a sequence $\bar{\lambda} = \langle \alpha_\tau : \tau \in \mathbf{a} \rangle$ as there. Let for $\theta \in \mathbf{c}$, $\mathfrak{d}_\theta =: cl_{M,\bar{\alpha}}(\mathbf{b}_\theta[\mathbf{a}]) \subseteq \mathbf{a}$ (as defined in part (2)) so $\mathbf{b}_\theta[\mathbf{a}] \subseteq \mathfrak{d}_\theta \in J_{\leq\theta}[\mathbf{a}]$ hence $J_{<\theta}[\mathbf{a}] + \mathfrak{d}_\theta = J_{\leq\theta}[\mathbf{a}]$. So by 2.2(5) we know $\text{rk}_{\langle \mathfrak{d}_\theta : \theta \in \mathbf{c} \rangle}(\mathbf{a}) \geq \text{rk}'_{\langle \mathfrak{d}_\theta : \theta \in \mathbf{c} \rangle}(\mathbf{a}) \geq \text{rk}'_{\bar{J} \upharpoonright \mathbf{c}}(\mathbf{a}) \geq \kappa^+$ (by the choice of \mathbf{c} above). Now by 2.3(1) we can find $\tau_n \in \mathbf{a}$ for $n < \omega$, pairwise distinct and strictly increasing with n such that for every $n < m < \omega$ for some $\theta_{n,m} \in \mathbf{c}$ we have $\{\tau_n, \tau_{n+1}, \dots, \tau_m\} \cap \mathfrak{d}_{\theta_{n,m}} = \{\tau_{n+1}, \dots, \tau_m\}$, note: as $\tau_m \in \mathfrak{d}_{\theta_{n,m}}$ necessarily $\theta_{n,m} \geq \tau_m$. So by the choice of $\bar{\alpha}$ and the \mathfrak{d} 's, we have $\alpha_{\tau_n} \notin cl_M(\{\alpha_{\tau_{n+1}}, \alpha_{\tau_{n+2}}, \dots, \alpha_{\tau_m}\})$. So $\langle \alpha_{\tau_n} : n < \omega \rangle$ are as required in the definition of $\text{IND}(\lambda, |\mathbf{a}|)$.

2) Let $\bar{\mathbf{b}} = \langle \mathbf{b}_\theta[\mathbf{a}] : \theta \in \text{pcf}(\mathbf{a}) \rangle$ be the generating sequence for \mathbf{a} ; without loss of generality $\max \text{pcf}(\mathbf{a}) \in \mathbf{c}$ and $\theta \in \mathbf{c} \setminus \{\max \text{pcf}(\mathbf{a})\} \Rightarrow \min(\text{pcf}(\mathbf{a}) \setminus \theta^+) \in \mathbf{c}$. We know that

(*)_a there is $\bar{F} = \langle F_\theta : \theta \in \text{pcf}(\mathbf{a}) \rangle$ such that:

- (a) $F_\theta \subseteq \Pi \mathbf{a}$ and $|F_\theta| \leq \theta$ and F_θ is $(< \theta)$ -directed
- (b) $\{f \upharpoonright \mathbf{b} : f \in F_\theta\}$ is cofinal in $\Pi \mathbf{b}$ for every $\mathbf{b} \in J_{\leq\theta}[\mathbf{a}]$
- (c) F_θ includes $\cup \{F_\tau : \tau \in \theta \cap \text{pcf}(\mathbf{a})\}$ and is closed under some natural operations
- (d) if $\tau \in \theta \cap \mathbf{c}$ then $f \in F_\theta \Rightarrow (\exists g \in F_\tau)(f \upharpoonright \mathbf{b}_\tau[\mathbf{a}] = g \upharpoonright \mathbf{b}_\tau[\mathbf{a}])$.

[Why? E.g. the proof of [Sh 355, 3.5].]

Let M be a model with universe λ and vocabulary of cardinality $|\mathfrak{a}|$. For every $f \in \Pi\mathfrak{a}$ (e.g., $f \in F_\theta, \theta \in \text{pcf}(\mathfrak{a})$) we define $g_f \in \Pi\mathfrak{a}$ by $g_f(\tau) =: \sup[\tau \cap \text{cl}_M(\text{Rang } f)]$. For every $\theta \in \mathfrak{c} \subseteq \text{pcf}(\mathfrak{a}), \{g_f : f \in \bigcup_{\tau \in \theta \cap \mathfrak{c}} F_\tau\}$ is a subset of $\Pi\mathfrak{a}$ of cardinality $< \theta$ (here instead $|\mathfrak{c}| < \min(\mathfrak{a})$, just $\theta \in \mathfrak{c} \Rightarrow |\mathfrak{c} \cap \theta| < \theta$ suffice), so there is $g^\theta \in \Pi\mathfrak{a}$ such that:

$$(*)_1 \quad f \in \bigcup_{\tau \in \theta \cap \mathfrak{c}} F_\tau \Rightarrow g_f < g^\theta \text{ mod } J_{<\theta}[\mathfrak{a}].$$

Define $g^* \in \Pi\mathfrak{a}$ by $g^*(\tau) = \sup\{g^\theta(\tau) + 1 : \theta \in \mathfrak{c}\}$ (remember $|\mathfrak{c}| < \min(\mathfrak{a})$). So there is $h \in F_{\max \text{pcf}(\mathfrak{a})}$ such that $g^* < h$ (see $(*)_{\mathfrak{a}}(b)$); we shall show that:

$$(*)_2 \quad \text{for any such } h \text{ the sequence } \langle h(\tau) : \tau \in \mathfrak{a} \rangle \text{ is as required.}$$

Proof of $()_2$.* So let $\alpha_\tau = h(\tau)$. Note that \otimes_2 from 2.6(2) is trivial so we shall prove \otimes_1 . Assume $\theta \in \mathfrak{c}$ and let $\mathfrak{b} = \mathfrak{b}_\theta[\mathfrak{a}] \in J_{\leq\theta}[\mathfrak{a}]$ so by clause (d) of $(*)_{\mathfrak{a}}$ for some $f_1 \in F_\theta$ we have

$$\oplus_1 \quad h \upharpoonright \mathfrak{b} = f_1 \upharpoonright \mathfrak{b};$$

we can assume $\theta < \max \text{pcf}(\mathfrak{a})$ (otherwise conclusion is trivial) and let $\sigma = \min(\text{pcf}(\mathfrak{a}) \setminus \theta^+)$, by an assumption made in the beginning of the proof $\sigma \in \mathfrak{c}$ and so as $f_1 \in F_\theta$ by the choice of g^σ we have:

$$g_{f_1} < g^\sigma \text{ mod } J_{<\sigma}[\mathfrak{a}]$$

but by the choice of g^*

$$g^\sigma < g^*$$

and by the demand of h

$$g^* < h$$

together

$$g_{f_1} < h \text{ mod } J_{<\sigma}[\mathfrak{a}]$$

so for some $\mathfrak{d} \in J_{<\sigma}[\mathfrak{a}] = J_{\leq\theta}[\mathfrak{a}]$ we have:

$$\oplus_2 \quad g_{f_1} \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}) < h \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}).$$

Now for any $\tau \in \mathfrak{a}$

$$\begin{aligned} \oplus_3 \quad \tau \in \text{cl}_{M, \bar{\alpha}}(\mathfrak{b}) &\Rightarrow \alpha_\tau \in \text{cl}_M[\{\alpha_\kappa : \kappa \in \mathfrak{b}\}] \Rightarrow \alpha_\tau \in \text{cl}_M[\text{Rang}(h \upharpoonright \mathfrak{b})] \Rightarrow \alpha_\tau \in \\ &\text{cl}_M[(\text{Rang}(f_1 \upharpoonright \mathfrak{b}))] \Rightarrow \alpha_\tau \in \text{cl}_M(\text{Rang}(f_1)) \Rightarrow \alpha_\tau \leq g_{f_1}(\tau) \Rightarrow h(\tau) = \alpha_\tau \leq \\ &g_{f_1}(\tau). \end{aligned}$$

By $\oplus_2 + \oplus_3$ we have $\text{cl}_{M, \bar{\alpha}}(\mathfrak{b}_\theta) \subseteq \mathfrak{d}$ so the required conclusion follows. $\square_{2.6}$

E.g.

2.7 Claim. *If $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then $\text{IND}(\aleph_\omega)$.*

Proof. Let $\text{pp}(\aleph_\omega) = \aleph_{\alpha^*}$ (so $\alpha^* < \omega_4, \alpha^* = \beta^* + 1$, see [Sh:g, Ch.IX,2.1]) let $\mathfrak{a} = \{\aleph_{i+1} : 5 \leq i < \omega\}$ so we know $\text{pcf}(\mathfrak{a}) = \{\aleph_{i+1} : 5 \leq i < \alpha^*\}$ and let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \text{pcf}(\mathfrak{a}) \rangle$ be a normal generating sequence for $\text{pcf}(\mathfrak{a})$ (not $\mathfrak{a}!$, exists as $|\text{pcf}(\mathfrak{a})| < \min(\text{pcf}(\mathfrak{a}))$; without loss of generality $\mathfrak{b}_{\aleph_{\alpha^*}} = \text{pcf}(\mathfrak{a})$). Let $\bar{\mathfrak{c}} = \langle \mathfrak{b}_\lambda \cap \mathfrak{a} : \lambda \in \text{pcf}(\mathfrak{a}) \rangle$. Now by localization ([Sh:g, Ch.VIII,2.6] we know that for some club E of $\omega_1 : \delta \in E \Rightarrow \text{pp}(\aleph_\delta) > \aleph_{\omega_1}$ (so we can assume $\omega \in E$).

Let $E = \{\beta_\zeta : \zeta < \omega_1\}$ (increasing in ζ). Hence by [Sh:g, Ch.II,3.4,pg.337 + II,2.1,pg.55] we have

$$(*)_0 \text{ if } \delta < \omega_1 \text{ is limit, } \delta < \beta < \omega_1 \text{ and } \delta \in E, \text{ then for some unbounded } \mathfrak{d} \in \aleph_\delta \cap \text{Reg}, \text{ we have } \aleph_{\beta+1} \in \text{pcf}_{J^{\text{bd}}}(\mathfrak{d}).$$

Hence we can prove by induction on $\varepsilon < \omega_1$ that:

$$(*) \text{ if } \beta_\varepsilon \leq \zeta < \omega_1, \mathfrak{b} \subseteq \mathfrak{a} \text{ and } \mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}_{\aleph_{\zeta+1}} \text{ mod } J_{< \aleph_{\zeta+1}}[\mathfrak{a}] \text{ then } \text{rk}'_{\bar{\mathfrak{c}}}(\mathfrak{b}) \geq \varepsilon.$$

[Why? For $\varepsilon = 0$ this is trivial and also for ε limit. If $\varepsilon = \xi + 1$ we can find $\mathfrak{d} \in \aleph_{\beta_\varepsilon} \cap \text{Reg} \setminus \aleph_{\beta_\xi}$ such that $\aleph_{\zeta+1} \in \text{pcf}_{J^{\text{bd}}}(\mathfrak{d})$ so without loss of generality $\aleph_{\zeta+1} = \max \text{pcf}(\mathfrak{d})$. By [Sh:g, Ch.I,1.12] we know that the set \mathfrak{d}' of $\theta \in \mathfrak{d}$ such that $\mathfrak{b}_\theta \cap \mathfrak{a} \subseteq \mathfrak{b} \text{ mod } J_{< \theta}[\mathfrak{a}]$ is $= \mathfrak{d} \text{ mod } J_{< \aleph_{\zeta+1}}[\mathfrak{d}]$ hence is not bounded in \mathfrak{d} , hence is not bounded in $\aleph_{\beta_\varepsilon}$. But $\mathfrak{d}' \subseteq \mathfrak{d} \subseteq \aleph_{\beta_\varepsilon} \cap \text{Reg} \setminus \aleph_{\beta_\varepsilon}$, hence by the induction hypothesis $\theta \in \mathfrak{d}' \Rightarrow \text{rk}'_{\bar{\mathfrak{c}}}(\mathfrak{b}_\theta \cap \mathfrak{b}) \geq \xi$, but of course $\mathfrak{b} \setminus \mathfrak{b}_\theta \neq \emptyset$.]

Now apply 2.6. $\square_{2.7}$

2.8 Claim. *If $|\mathfrak{a}| < \min(\mathfrak{a}), \lambda = \sup(\mathfrak{a})$ is singular and $\text{pcf}_{J^{\text{bd}}}(\mathfrak{a})$ contains an interval of Reg of cardinality $|\mathfrak{a}|^+$ then $\text{IND}(\lambda)$.*

Proof. Similar to the proof of 2.7 and even can weaken the assumption as in [Sh 410].

Saharon: more in §3.

2.9 Discussion. We can also prove e.g.: if $\lambda = \text{tcf}(\prod_{\varepsilon < \kappa} \lambda_\varepsilon / [\kappa]^{< \aleph_0})$, satisfies $\lambda > \lambda_\varepsilon = \text{cf}(\lambda_\varepsilon) > \kappa$, then for every algebra M on $\sum_{\varepsilon < \kappa} \lambda_\varepsilon$ with $< \min\{\lambda_\varepsilon : \varepsilon < \kappa\}$ functions there are $\alpha_\varepsilon < \lambda_\varepsilon (\varepsilon < \kappa)$ such that: for finite $u \subseteq \kappa$ we have $\{\zeta : \alpha_\zeta \in \text{cl}_M(\{\alpha_\varepsilon : \varepsilon \in u\})\}$ is finite (and more). Not clear how interesting is this statement and where it leads.

2.10 Claim. Assume $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{pcf}(\mathfrak{a})$ and $e|\mathfrak{c}| < \min(\mathfrak{a})$ (or $\mathfrak{c} \in J_*[\text{pcf}(\mathfrak{a})]$, see [Sh:g, Ch.VIII, §3] and [Sh:E11]), so $\text{pcf}(\mathfrak{a}) = \text{pcf}(\mathfrak{c})$. Then

$$\text{rk}'_{\langle J_{< \theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle}(\mathfrak{a}) = \text{rk}'_{\langle J_{< \theta}[\mathfrak{c}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle}(\mathfrak{c}).$$

Proof. Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ be a generating sequence for \mathfrak{a} , hence we know that letting $\mathfrak{b}_\theta[\mathfrak{c}] =: \mathfrak{c} \cap \text{pcf}(\mathfrak{b}_\theta[\mathfrak{a}])$, we have: $\bar{\mathfrak{b}}' = \langle \mathfrak{b}_\theta[\mathfrak{c}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ is a generating sequence for \mathfrak{c} . Without loss of generality $\mathfrak{b}_\theta[\mathfrak{c}] \cap \mathfrak{a} = \mathfrak{b}_\theta[\mathfrak{a}]$ and $\mathfrak{b}_{\max \text{pcf}(\mathfrak{a})}[\mathfrak{a}] = \mathfrak{a}$ hence $\mathfrak{b}_{\max \text{pcf}(\mathfrak{a})}[\mathfrak{c}] = \mathfrak{c}$. So we can prove easily:

$$\mathfrak{d} \subseteq \mathfrak{a} \Rightarrow \text{rk}'_{\langle J_{< \theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle}(\mathfrak{d}) \leq \text{rk}'_{\langle J_{< \theta}[\mathfrak{c}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle}(\mathfrak{d})$$

(as $J_{< \theta}[\mathfrak{a}] = J_{< \theta}[\mathfrak{c}] \upharpoonright \mathfrak{a}$).

For the other direction we prove

(*) if $n < \omega$, $\{\theta_1, \dots, \theta_n\} \subseteq \text{pcf}(\mathfrak{a})$ then

$$\text{rk}'_{\langle J_{< \theta}[\mathfrak{c}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle} \left(\bigcap_{\ell=1}^n \mathfrak{b}_{\theta_\ell}[\mathfrak{c}] \right) \leq \text{rk}'_{\langle J_{< \theta}[\mathfrak{c}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle} \left(\bigcap_{\ell=1}^n \mathfrak{b}_{\theta_\ell}[\mathfrak{a}] \right).$$

[Saharon: details]

□_{2.10}

§3 EXISTENCE OF FREE SETS IMPLIES RESTRICTIONS ON PCF

3.1 Definition. Suppose $\bar{\mathcal{J}} = \langle (\kappa_n, I_n) : n < \omega \rangle$ is such that: I_n is an ideal on κ_n .

1) We define $J_n = J_n^{\bar{\mathcal{J}}}$ an ideal on $\prod_{\ell < n} \kappa_\ell$:

(*)₀ $J_0 =$ the empty ideal on $\{\langle \rangle\}$

(*)₁ $J_{n+1} = \{A \subseteq \prod_{\ell \leq n} \kappa_\ell : \{\alpha < \kappa_n : \{\eta \in \prod_{\ell < n} \kappa_\ell : \eta \hat{\ } \langle \alpha \rangle \in A\} \notin J_n\} \in I_n\}$

we let $\bar{J}^{\bar{\mathcal{J}}} = \langle J_n^{\bar{\mathcal{J}}} : n < \omega \rangle$.

2) We say $\langle J_n : n < \omega \rangle$ is a candidate (for $\bar{\mathcal{J}}$) if in (*)₁ we weaken “ $J_{n+1} = \dots$ ” to “ $J_{n+1} \subseteq \dots$ ”. [So there may be many candidates for a given $\bar{\mathcal{J}}$.]

3.2 Fact. 1) In definition 3.1(1) above, each J_n is an ideal on $\prod_{\ell < n} \kappa_\ell$.

2) If I_0, \dots, I_{n-1} are σ -complete then so is J_n .

3) If $\langle J'_n : n < \omega \rangle$ is a candidate for $\bar{\mathcal{J}}$, then $\bigwedge_{n < \omega} J'_n \subseteq J_n^{\bar{\mathcal{J}}}$.

3.3 Claim. For $\bar{\mathcal{J}}, \bar{J}^{\bar{\mathcal{J}}}$ as in 3.1(1) we have $A \in J_n$ iff for some functions f_0, \dots, f_{n-1} we have:

$$\text{Dom}(f_\ell) = \prod_{m=\ell+1}^{n-1} \kappa_m$$

and

$$\text{Rang}(f_\ell) \subseteq I_\ell,$$

and $A \subseteq \bigcup_{\ell < n} A_\ell^n(f_\ell)$ where

$$A_\ell^n(f_\ell) = \{\eta \in \prod_{m < n} \kappa_m : \eta(m) \in f_m(\eta \upharpoonright (m, n))\}.$$

Proof. By induction on n .

3.4 Theorem. Let $\bar{\mathcal{F}} = \langle (\kappa_n, I_n) : n < \omega \rangle$ be as in 3.1, $\kappa = \sum_{n < \omega} \kappa_n \leq \mu_0 = cf(\mu_0) < \mu_1 < \mu = cf(\mu)$. Then $\otimes_1 \Rightarrow \otimes_2$ where

- \otimes_1 for each n there is $\langle \lambda_i^n : i < \kappa_n \rangle$ such that:
 $\lambda_i^n \in [\mu_0, \mu_1) \cap \text{Reg}$ and $\prod_{i < \kappa_n} \lambda_i^n / I_n$ is μ -directed
- \otimes_2 for some $\bar{\mathcal{F}}$ -candidate, $\bar{J} = \langle J_n : n < \omega \rangle$ (except for clause (δ) , $J_n = J_n^{\bar{\mathcal{F}}}$ is O.K.) we have:
- $\otimes_2^{\bar{J}}$ there are $\bar{\lambda}^n = \langle \lambda_\eta : \eta \in \prod_{\ell < n} \kappa_\ell \rangle$ for $n \in (0, \omega)$ such that for each n :
- (α) $(\prod \{ \lambda_\eta : \eta \in \prod_{\ell < n} \kappa_\ell \} / J_n)$ has true cofinality μ
- (β) $\mu_0 \leq \lambda_\eta = cf(\lambda_\eta) < \mu_1$ (note that by clause (α) we have $\{ \eta \in \prod_{\ell < n} \kappa_\ell : \lambda_\eta = \mu_0 \} \in J_n$)
- (γ) if $0 < n < \omega$, $\alpha < \kappa_n$ and $\eta \in \prod_{\ell < n} \kappa_\ell$ then $\lambda_\eta > \mu_0 \Rightarrow \lambda_\eta > \lambda_{\eta \hat{\ } \langle \alpha \rangle}$ so $\{ \eta \in \prod_{\ell < n} \kappa_\ell : \lambda_{\eta \hat{\ } \langle \alpha \rangle} \not\leq \lambda_\eta \} \in J_n$ hence
- (γ') $\{ \eta \in \prod_{\ell \leq n} \kappa_\ell : \lambda_\eta \not\leq \lambda_{\eta \upharpoonright n} \} \in J_{n+1}$
- (δ) $J_n = \{ A \subseteq \prod_{\ell < n} \kappa_\ell : \max \text{pcf} \{ \lambda_\eta : \eta \in A \} < \mu \}$.

Question: Can we prepare the ground to 3.8 with IND^+ instead IND ?

Proof. We choose $\bar{\lambda}^n$ by induction on n . For $n = 1$ apply [Sh:g, Ch.II,1.5A] to the sequence $\langle \lambda_i^1 : i < \kappa_0 \rangle$, the ideal $\{ A \subseteq \kappa_1 : \max \text{pcf} \{ \lambda_i^1 : i < \kappa_0 \} < \mu_0 \}$ and the cardinal μ and get $\langle \lambda_{\langle i \rangle} : i < \kappa_0 \rangle$. For $n + 1$ for each $i < \lambda_n$ we apply [Sh:g, Ch.II,1.5A] to $\langle \lambda_\eta : \eta \in \prod_{\ell < n} \kappa_\ell \rangle$, the ideal $\{ A \subseteq \prod_{\ell < n} \kappa_\ell : \max \text{pcf} \{ \lambda_\eta : \eta \in A \} < \mu \}$ and the cardinal λ_i^n and we get $\langle \lambda_{\eta \hat{\ } \langle i \rangle} : \eta \in \prod_{\ell < n} \kappa_\ell \rangle$. $\square_{3.4}$

3.5 Claim. In 3.4, from \otimes_2 we can deduce

$$\otimes^3 \text{ there are functions } f_{\ell,n} : \prod_{m=\ell+1}^n \kappa_m \rightarrow I_\ell \text{ (for } \ell < n < \omega) \text{ such that for every } \eta \in \prod_{m < \omega} \kappa_m \text{ for some } \ell < n < \omega \text{ we have } \eta(\ell) \in f_{\ell,n}(\eta \upharpoonright (\ell, n)).$$

Proof. Otherwise $\langle \lambda_{\eta \upharpoonright n} : n < \omega \rangle$ is a strictly decreasing sequence of cardinals.

□_{3.5}

3.6 Definition. 1) $\text{IND}(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ (note that $|\text{Dom}(J_\varepsilon)|$ is not necessarily increasing with ε) means that each J_ε is an ideal on $\text{Dom}(J_\varepsilon)$, say κ_ε and

- (*) for every sequence $\langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \setminus (\varepsilon + 1) \text{ finite} \rangle$ such that¹ $f_{\varepsilon,u}$ a function from $\prod_{\zeta \in u} \kappa_\zeta$ to J_ε there is an increasing sequence $\varepsilon_0 < \varepsilon_1 < \dots < \varepsilon_n < \dots < \varepsilon^*$ (for $n < \omega$) and $\alpha_\ell \in \kappa_{\varepsilon_\ell}$ (for $\ell < \omega$) such that:
- (**) for $\ell < n < \omega$ we have $\alpha_\ell \notin f_{\varepsilon_\ell, u}(\langle \alpha_{\varepsilon_{\ell+1}}, \dots, \alpha_{\varepsilon_n} \rangle)$ for $u = \{\alpha_{\varepsilon_{\ell+1}}, \dots, \alpha_{\varepsilon_n}\}$.

2) $\text{IND}^+(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ means J_ε is an ideal (on $\text{Dom}(J_\varepsilon)$ which is say κ_ε) such that:

- (*) for every sequence $\langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \setminus (\varepsilon + 1) \text{ finite} \rangle$ such that $f_{\varepsilon,u} : \prod_{\zeta \in u} \kappa_\zeta \rightarrow J_\varepsilon$ there is $\langle \alpha_\varepsilon : \varepsilon < \varepsilon^* \rangle \in \prod_{\varepsilon < \varepsilon^*} \kappa_\varepsilon$ such that
- (**) for $\varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \setminus (\varepsilon + 1)$ finite we have: $\alpha_\varepsilon \notin f_{\varepsilon,u}(\dots, \alpha_\zeta, \dots)_{\zeta \in u}$.

3) Let function $\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ be the set of \bar{f} as in (*) of part (1), i.e. $\bar{f} = \langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*$ and $u \subseteq \varepsilon^* \setminus (\varepsilon + 1)$ is finite \rangle where $f_{\varepsilon,u}$ is a function with domain $\prod_{\zeta \in u} \kappa_\zeta$ and range $\subseteq J_\varepsilon$. We say for $\bar{f} \in$ function $\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$, that $\bar{\varepsilon}$ is candidate if $\bar{\varepsilon}$ is an increasing sequence of length ω of ordinals $< \varepsilon^*$. In this case we say that $\bar{\alpha}$ is $(\bar{f}, \bar{\varepsilon})$ -free if $\bar{\alpha} \in \prod_{n < \omega} \text{Dom}(J_{\varepsilon_n})$ and the statement (**) of part (1) holds.

4) Above if $J_\varepsilon = J_{\lambda_\varepsilon}^{\text{bd}}$ for $\varepsilon < \varepsilon^*$ then we may write $\langle \lambda_\varepsilon : \varepsilon < \varepsilon^* \rangle$ instead of $\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$.

¹of course, without loss of generality $v \subseteq u \Rightarrow f_{\varepsilon,v}(\langle \alpha_\zeta : \zeta \in v \rangle) \subseteq f_{\varepsilon,u}(\langle \alpha_\zeta : \zeta \in u \rangle)$

- 3.7 *Observation.* 1) If $\text{IND}(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ is as in 3.6, each J_ε is $(|\varepsilon^*|^{\aleph_0})^+$ -complete, then for some $\varepsilon_0 < \varepsilon_1 < \dots < \varepsilon^*$ we have $\text{IND}^+(\langle J_{\varepsilon_n} : n < \omega \rangle)$.
 2) If $\text{IND}(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ as in 3.6, each J_ε is $\text{cov}(|\varepsilon^*|, \mu, \aleph_1, 2)^+$ -complete then for some infinite $u \in \mathcal{S}_{<\mu}(\varepsilon^*)$ we have $\text{IND}(\bar{J} \upharpoonright u)$.
 3) Definition 3.6(4) and Definition 2.4(1) are compatible. [Saharon: explain!]
 4) $\text{IND}(\langle \lambda_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ is equivalent to $\text{IND}(\langle [\lambda_\varepsilon]^{\leq \mu_\varepsilon} : \varepsilon < \varepsilon^* \rangle)$, similarly with IND^+ .

Before we prove 3.7

3.8 *Conjecture.* if $\text{IND}(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$, $\varepsilon^* < \omega_1$ and each J_ε is \aleph_1 -complete then for some c.c.c. forcing \mathbb{P} we have:

$$\Vdash_{\mathbb{P}} \text{“for some } \varepsilon_0 < \dots < \varepsilon_n < \varepsilon_{n+1} < \dots < \varepsilon^*, \text{IND}(\langle J_{\varepsilon_n} : n < \omega \rangle)\text{”}.$$

3.9 *Remark.* In the proof of 3.7(2) it is enough to demand on \mathcal{P} :

- (*) if $\varepsilon_0 < \varepsilon_1 < \dots < \varepsilon_n < \dots < \varepsilon^*$ (for $n < \omega$) then for some $b \in \mathcal{P}$,
 $(\exists^\infty n)\varepsilon_n \in b$

this seems to weaken $\text{cov}(\dots)$ but does not.

Proof of 3.7. 1) Similar to the proof of part (2), as $\text{cov}(\lambda, \aleph_1, \aleph_2, 2) \leq |\varepsilon^*|^{\aleph_0}$.
 2) Let $\mu =: \text{cov}(\varepsilon^*, \mu, \aleph_1, 2)$ and let $\mathcal{P} \subseteq [\varepsilon^*]^{<\mu}$ be of cardinality μ exemplifying its definition i.e. $(\forall a)[a \subseteq \varepsilon^* \ \& \ |a| = \aleph_0 \Rightarrow (\exists b \in \mathcal{P})[a \subseteq b]]$.

If for some $b \in \mathcal{P}$, $\text{IND}(\bar{J} \upharpoonright u)$ hold then we are done. Otherwise for each $b \in \mathcal{P}$, we can find $\bar{f}^b = \langle f_{\varepsilon, u}^b : \varepsilon \in b \text{ and } b \subseteq u \setminus (\varepsilon + 1) \text{ is finite} \rangle \in \text{function}(J \upharpoonright u)$ such that for no $\bar{\varepsilon} = \langle \varepsilon_n : n < \omega \rangle$ strictly increasing sequence of ordinals from b and $\alpha_n \in \text{Dom}(J_{\varepsilon_n})$ (for $n < \omega$) do we have $n < \omega \ \& \ u \subseteq \{\varepsilon_{n+1}, \varepsilon_{n+2}, \dots\} \Rightarrow \alpha_n \notin f_{\varepsilon, u}^b(\{\alpha_m : m \in u\})$. Let us define for $\varepsilon < \varepsilon^*$ and $u \subseteq \varepsilon^* \setminus (\varepsilon + 1)$ finite a function $f_{\varepsilon, u}$ from $\prod_{\zeta \in u} \text{Dom}(J_\zeta)$ to J_ε by:

- (*) if $\alpha_\zeta \in \text{Dom}(J_\zeta)$ for $\zeta \in u$ then $f_{\varepsilon, u}(\dots, \alpha_\zeta, \dots)_{\zeta \in u} = \bigcup \{f_{\varepsilon, u}^b(\dots, \alpha_\zeta, \dots)_{\zeta \in u} : \varepsilon \notin b \text{ and } u \subseteq b \text{ and } b \in \mathcal{P}\}$.

(As each J_ζ is $|\mathcal{P}|^+$ -complete (by assumption) $\text{Rang}(f_{\zeta, u}) \subseteq J_\zeta$. As $\text{IND}(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ necessarily there is a strictly increasing $\langle \varepsilon_n : n < \omega \rangle$, $\varepsilon_n < \varepsilon^*$, and $\alpha_n \in \text{Dom}(J_{\varepsilon_n})$ (for $n < \omega$) such that:

- (**) if $n < \omega$, $u \subseteq \{\varepsilon_{n+1}, \varepsilon_{n+2}, \dots\}$ finite then $\alpha_n \notin f_{\varepsilon_n, u}(\dots, \alpha_m, \dots)_{\varepsilon_m \in u}$.

By the choice of \mathcal{P} for some $b \in \mathcal{P}$ we have $\{\varepsilon_n : n < \omega\} \subseteq b$, but then $\langle (\varepsilon_n, \alpha_n) : n < \omega \rangle$ contradict the choice of $\bar{f}^b = \langle f_{\zeta, u}^b : \zeta \in b, u \subseteq b \setminus (\zeta + 1) \text{ finite} \rangle$.

3),4) easy. □_{3.7}

3.10 Conclusion. 1) If $\text{IND}^+(\langle I_n : n < \omega \rangle)$ and $\text{Dom}(I_n) = \kappa_n$ then

(a) the conclusion of 3.5 (i.e. \otimes^3 there) fails, hence \otimes^2 of 3.4 fails hence \otimes^1 of 3.4 fails

(b) if $\lambda > \sum_{n < \omega} \kappa_n$ and $\kappa_n < \text{cf}(\kappa_{n+1})$, then for every n large enough for no

$\lambda_i \in (\sum_{n < \omega} \kappa_n, \lambda) \cap \text{Reg}$ (for $i < \kappa_n$) is $\prod_{i < \kappa_n} \lambda_i / I_n$ λ -directed.

2) If we weaken the assumption to $\text{IND}(\langle I_n : n < \omega \rangle)$ then in (b) we have just for arbitrarily large $n < \omega$.

3) If in addition $X \in I_n^+ \Rightarrow I_n \upharpoonright X \cong I_n$ (e.g. $I_n = J_{\kappa_n}^{\text{bd}}$) then in clause (b) for n large enough λ is (I_n, \aleph_0) -inaccessible.

Proof. 1) [Saharon: details?]

(a) straight

(b) our problem is to get μ_1 , which is not serious.

2), 3) Similarly. □_{3.10}

3.11 Conclusion. 1) Assume $\langle \kappa_\varepsilon : \varepsilon < \delta \rangle$ is strictly increasing, $|\delta| \leq \sigma < \kappa_0, \kappa = \sum_{i < \delta} \kappa_i$ and $\text{IND}(\kappa, \sigma)$. If $\lambda > \kappa$ then for every large enough $\varepsilon < \delta$, there are no

$\lambda_\alpha \in (\kappa, \lambda) \cap \text{Reg}$ for $\alpha < \kappa_\varepsilon$ such that $\prod_{\alpha < \kappa_\varepsilon} \lambda_\alpha / [\kappa_\varepsilon]^{\leq \sigma}$ is λ -directed recalling

$[\kappa_\varepsilon]^{\leq \sigma} = \{a \subseteq \kappa_\varepsilon : |a| \leq \sigma\}$.

2) If $\text{IND}(\kappa), \text{cf}(\kappa) = \aleph_0 = \sigma, \kappa = \sum_{n < \omega} \kappa_n, \kappa_n < \kappa_{n+1}$ then the conclusion of part

(1) holds.

3) If $\text{IND}(\kappa, \sigma), \delta = \omega, \kappa_\varepsilon = \kappa$, then the conclusion of (1) holds.

Proof. Check.

3.12 Discussion: Let $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ and assume $\text{IND}(\bar{J})$.

1) Note that if \mathbb{P} is a θ -c.c. forcing notion, each J_ε is θ -complete then for any $\bar{f} = \langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \in [\varepsilon^* \setminus (\varepsilon + 1)]^{<\aleph_0} \rangle \in \mathbf{V}^{\mathbb{P}}$ as in Definition 3.6 we can find $\bar{f}' = \langle f'_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \in [\varepsilon^* \setminus (\varepsilon + 1)]^{<\aleph_0} \rangle \in \mathbf{V}$ such that for every $\bar{\alpha} \in \prod_{\zeta \in u} \text{Dom}(J_\zeta)$

we have $f_{\varepsilon,u}(\bar{\alpha}) \subseteq f'_{\varepsilon,u}(\bar{\alpha})$, so we can consider only $\bar{f} \in V$. For each such \bar{f} let $A_{\bar{f}} = \{v : \text{for some strictly increasing sequence } \bar{\varepsilon} = \langle \varepsilon_n : n < \omega \rangle \text{ of ordinals } < \varepsilon^* \text{ and } \bar{\alpha} \in \prod_{n < \omega} \text{Dom}(J_{\varepsilon_n}) \text{ the conclusion } (**) \text{ of Definition 3.6 holds and } v = \{\varepsilon_\ell : \ell < m\} \text{ for some } m < \omega\}$.

For $\bar{g}, \bar{f} \in \text{function}(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$, let $\bar{g} \leq \bar{f}$ iff for every $\varepsilon < \varepsilon^*$ and $u \in [\varepsilon^* \setminus (\varepsilon + 1)]^{<\aleph_0}$ we have $\bar{\alpha} \in \prod_{\varepsilon \in u} \text{Dom}(J_\varepsilon) \Rightarrow g_{\varepsilon,u}(\bar{\alpha}) \subseteq f_{\varepsilon,u}(\bar{\alpha})$. Clearly

$$\bar{g} \leq \bar{f} \Rightarrow A_{\bar{f}} \subseteq A_{\bar{g}}.$$

2) In \mathbf{V} we can define a filter D on $[\bigcup_{\varepsilon < \varepsilon^*} \text{Dom}(J_\varepsilon)]^{<\aleph_0}$:

$$A \in D \text{ iff for some } \bar{f} \in \text{function}(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle) \text{ we have } A_{\bar{f}} \subseteq A.$$

Now $D \subseteq \mathcal{P}([\bigcup_{\varepsilon < \varepsilon^*} \text{Dom}(J_\varepsilon)]^{<\aleph_0})$ and D is upward closed trivially. Also D is closed

under intersection of countable many members if each J_n is \aleph_1 -closed (similarly σ -closed) because if $A_n \in D$ let $\bar{f}^n \in \text{function}(\bar{J})$ be such that $A_{\bar{f}^n} \subseteq A_n$. Now for some $g \in \text{function}(J)$ we have $[n < \omega \Rightarrow \bar{f}^n \subseteq g]$, hence $A_g \subseteq A_{\bar{f}^n}$, so $A_g \subseteq A_n$ for $n < \omega$ and obviously $A_g \in D$. Lastly $\emptyset \notin D$ as $\text{IND}(\bar{J})$ holds.

3.13 Claim. Suppose for $\alpha < \alpha^*$, $\bar{I}^\alpha = \langle I_n^\alpha : n < \omega \rangle$, $\kappa = \sup\{|\text{Dom}(I_n^\alpha)| : \alpha < \alpha^*\}$, $\text{IND}^+(\langle I_n^\alpha : n < \omega \rangle)$, and:

(*) if $\alpha < \alpha^*$, $f_n : \text{Dom}(I_n^\alpha) \rightarrow \text{Ord}$ then for some $n(*) < \omega$, $\beta < \alpha^*$ and ordinal γ

$$I_{1+n}^\beta \cong I_{n(*)+1+n}^\alpha \upharpoonright \{x \in \text{Dom}(I_{n(*)+1, n(*)+1+n}^\alpha) : f_{1+n}(x) > \gamma\}$$

$$I_0^\beta = I_{n(*)}^\alpha \upharpoonright \{x \in \text{Dom}(I_{n(*)}^\alpha) : f_0(x) < \gamma\}.$$

[Saharon: think].

Then for no $\lambda > \kappa$ and $\alpha < \alpha^*$ do we have for every $n < \omega$:

$$x \in \text{Dom}(I_n^\alpha) \Rightarrow \lambda_x^n \in (\kappa, \lambda) \cap \text{Reg}$$

and

$$\prod_{x \in \text{Dom}(I_n^\alpha)} \lambda_x^n / I_n^\alpha \text{ is } \lambda^+ \text{-directed.}$$

Proof. No new point.

3.14 Remark. 1) This claim is used in the proof of 5.2.

2) If in 3.13, $\alpha^* = 1$ and I_{n+1}^α is λ_n -complete, $\lambda_n > |\text{Dom}(I_\ell^\alpha)|$ for $\ell < n$ then (*) there holds.

3.15 Question: If $\text{IND}(\lambda, \sigma)$, $\text{cf}(\lambda) = \aleph_0$ do we have $\text{IND}(\langle J_{\lambda_n}^{\text{bd}} : n < \omega \rangle)$ for some $\lambda_n = \text{cf}(\lambda_n) < \lambda, \sigma < \lambda_n$?

3.16 Conclusion. 1) Assume $\text{IND}(\langle J_{\kappa_n}^{\text{bd}} : n < \omega \rangle)$ and $\kappa_n < \kappa_{n+1}, \kappa = \Sigma\{\kappa_n : n < \omega\}$. For any $\lambda > \kappa$ for infinitely many $n < \omega$, λ is $J_{\kappa_n}^{\text{bd}}$ -inaccessible.

2) If moreover $\text{IND}^+(\langle J_{\kappa_n}^{\text{bd}} : n < \omega \rangle)$, then the conclusion holds for every n large enough.

3.17 Theorem. If $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$ and $\text{rk}'_{\langle J_{<\theta[\mathfrak{a}]:\theta \in \text{pcf}(\mathfrak{a})} \rangle}(\mathfrak{a}) \geq |\mathfrak{a}|^+$, then $\text{IND}(\langle J_\theta^{\text{bd}} : \theta \in \mathfrak{a} \rangle)$.

Proof. Reread the proof of 2.6: let $f_{u,\varepsilon}$ be as in Definition 3.6, so without loss of generality $\text{Rang}(f_{\varepsilon,u}) \subseteq \kappa_\varepsilon$ (that is any value of $f_{\varepsilon,u}$ is an ordinal $< \kappa_\varepsilon$, remembering $\alpha = \{\beta : \beta < \alpha\}$) and $M = (\kappa, \kappa_\varepsilon, f_{u,\varepsilon})_{\varepsilon,u}$. Now repeat the proof of 2.6 or see below. $\square_{3.17}$

Next we improve the ideals from “bounded” to “nonstationary”

3.18 Theorem. 1) Assume $\lambda > \text{cf}(\lambda)$, $|\mathfrak{a}| < \min(\mathfrak{a})$ and $\lambda = \sup(\mathfrak{a})$ and $\text{rk}'_{\langle J_{<\theta[\mathfrak{a}]:\theta \in \text{pcf}(\mathfrak{a})} \rangle}(\mathfrak{a}) \geq |\mathfrak{a}|^+$ and $\sigma^* \in (|\mathfrak{a}|, \min(\mathfrak{a})) \cap \text{Reg}$, and

$$I_\theta =: \{S : S \subseteq \theta \text{ and } \{\delta \in S : \text{cf}(\delta) = \sigma^*\} \text{ is not stationary}\}$$

then $\text{IND}(\langle I_\theta : \theta \in \mathfrak{a} \rangle)$.

2) Moreover for any sequence $\bar{H} = \langle H_\theta : \theta \in \mathfrak{a} \rangle$, where $\mathfrak{a} \subseteq \lambda$, H_θ a function from $[\lambda]^{<\aleph_0}$ to I_θ and $\mathfrak{c}, \mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{pcf}(\mathfrak{a})$, $|\mathfrak{c}| < \min(\mathfrak{a})$ we can find $\bar{\alpha} = \langle \alpha_\tau : \tau \in \mathfrak{a} \rangle \in \Pi \mathfrak{a}$ such that defining for $\mathfrak{b} \subseteq \mathfrak{a}$

$$cl_{\bar{H}, \bar{\alpha}}(\mathbf{b}) = \{\tau \in \mathbf{a} : \alpha_\tau \in H_\tau(\bar{\alpha} \upharpoonright \mathfrak{c}) \text{ for some finite } \mathfrak{c} \subseteq \mathbf{b}\}$$

we have

$$(*) \theta \in \mathfrak{c} \ \& \ \mathbf{b} \in J_{<\theta}[\mathbf{a}] \Rightarrow cl_{\bar{H}, \bar{\alpha}}(\mathbf{b}) \in J_{<\theta}[\mathbf{a}].$$

Proof. 1) We can prove it from part 2) exactly as in the proof of 2.6(1).

2) Let $\bar{\mathbf{b}} = \langle \mathbf{b}_\theta[\mathbf{a}] : \theta \in \text{pcf}(\mathbf{a}) \rangle$ be a generating sequence for \mathbf{a} , without loss of generality $|\mathfrak{c}|^+ < \min(\mathbf{a})$ and $\max \text{pcf}(\mathbf{a}) \in \mathfrak{c}$ and

$$[\theta \in \mathfrak{c} \ \& \ \theta \neq \max \text{pcf}(\mathbf{a}) \Rightarrow \min(\text{pcf}(\mathbf{a}) \setminus \theta^+) \in \mathfrak{c}].$$

Before we continue, recall that we know:

3.19 Fact If $|\mathbf{a}| \leq |\mathfrak{a}| < \min(\mathbf{a})$, $\mathfrak{a} \subseteq \text{Reg}$, $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{pcf}(\mathbf{a})$ and $\langle \mathbf{b}_\sigma[\mathbf{a}] : \sigma \in \text{pcf}(\mathbf{a}) \rangle$ a generating sequence for \mathbf{a} then

(*) _{\mathfrak{a}} there is $\langle \bar{f}^\theta : \theta \in \text{pcf}(\mathbf{a}) \rangle$ such that

- (a) $\bar{f}^\theta = \langle f_\alpha^\theta : \alpha < \theta \rangle$ is $<_{J_{<\theta}[\mathbf{a}]}$ -increasing
- (b) \bar{f}^θ is cofinal in $(\Pi \mathfrak{a}, <_{J_\theta[\mathbf{a}]})$ where $J_\theta[\mathbf{a}] =: J_{<\theta}[\mathbf{a}] + (\mathfrak{a} \setminus \mathbf{b}_\theta[\mathbf{a}])$
- (c) if $\sigma \in \theta \cap \mathfrak{c}$, $\sigma < \theta$, $\delta < \theta$ and $\mathbf{b} = \mathbf{b}_\sigma[\mathbf{a}]$ then for some $n < \omega$ and $\alpha_\ell < \theta_\ell \leq \sigma$ (for $\ell \leq n$) we have $f_\delta^\theta \upharpoonright \mathbf{b} = (\max\{f_{\alpha_\ell}^{\theta_\ell} : \ell < n\}) \upharpoonright \mathbf{b}$ (the max is pointwise)
- (d) if $\delta < \theta \in \text{pcf}(\mathbf{a})$, $\text{cf}(\delta) \in (|\mathfrak{a}|, \min(\mathbf{a}))$ then for every $\tau \in \mathfrak{a}$

$$f_\delta^\theta(\tau) = \min\{\cup\{f_\alpha^\theta(\tau) : \alpha \in C\} : C \text{ a club of } \delta\}$$

provided that the function defined satisfies condition (c) above

(exist by [Sh:g, Ch.VIII,§1] or we choose by induction on θ).

Let

$$S_\theta^{\text{gd}}, S_{\bar{f}^\theta}^{\text{gd}} =: \{\delta < \theta : \text{cf}(\delta) \in (|\mathfrak{a}|, \min(\mathbf{a})), f_\delta^\theta \text{ is a } <_{J_{<\theta}^{\text{bd}}[\mathbf{a}]}\text{-eub of } \bar{f}^\theta \upharpoonright \delta \text{ and} \\ \{\tau \in \mathfrak{a} : \text{cf}(f_\alpha^\theta(\tau)) = \text{cf}(\delta)\} = \mathbf{b}_\theta[\mathbf{a}] \text{ mod } J_{<\theta}[\mathbf{a}]\}$$

(alternatively use simultaneous witnesses for $I[\theta]$ as in [Sh 420, §1].

Note:

(*)_a (e) if E_τ is a club of τ for $\tau \in \mathfrak{a}$ and $\theta \in \text{pcf}(\mathfrak{a})$ then for some club E of θ :

$$\delta \in E \cap S_\theta^{\text{gd}} \Rightarrow \{\tau \in \mathfrak{a} : f_\delta^\theta(\tau) \in E_\tau\} = \mathfrak{a} \text{ mod } J_\theta[\mathfrak{a}]$$

(f) if $\sigma^* \in (|\mathfrak{c}|, \text{Min}(\mathfrak{a}))$ and $N_i \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ for $i \leq \delta^*$ is increasing continuous, $\langle N_j : j \leq i \rangle \in N_{i+1}$, $\|N_i\| = \sigma^*$, $\sigma^* + 1 \subseteq N_i$, $\langle \bar{f}^\theta : \theta \in \text{pcf}(\mathfrak{a}) \rangle \in N_i$, $\delta^* < (\sigma^*)^+$ has cofinality $> |\mathfrak{a}|$ and $\{\mathfrak{a}, \mathfrak{c}\} \in N_i$ then:
 $\theta \in \mathfrak{c}$ implies $\sup(N_\delta \cap \theta) \in S_\theta^{\text{gd}}$ and
 $\{\sigma \in \mathfrak{a} : f_{\sup(N_\delta \cap \theta)}(\sigma) = \sup(N_\delta \cap \sigma)\} = \mathfrak{a} \text{ mod } J_\theta[\mathfrak{a}]$.

Why? See [Sh:g, ChVIII,1.2,1.4].

Let $\bar{H} = \langle H_\tau : \tau \in \mathfrak{a} \rangle$ be as in the claim, so H_τ is a function from $[\lambda]^{<\aleph_0}$ to I_τ . Now for every $\theta \in \text{pcf}(\mathfrak{a})$ and $\alpha < \theta$ and $\tau \in \mathfrak{a}$ we define $A_\alpha^{\theta, \tau} = A^\tau(f_\alpha^\theta) \in I_\tau$ as

$$\bigcup \{H_\tau(u) : u \subseteq \text{Rang}(f_\alpha^\theta) \text{ is finite}\}$$

(as I_τ is τ -complete, $\tau > |\mathfrak{a}| = |\{u : u \subseteq \text{Rang}(f_\alpha^\theta) \text{ finite}\}|$, really $A_\alpha^\theta \in I_\tau$).

Now for each $\alpha < \theta \in \text{pcf}(\mathfrak{a})$ and $\sigma \in \text{pcf}(\mathfrak{a})$ by (*)_a(e) applied with σ , $\langle A_\alpha^{\theta, \tau} : \tau \in \mathfrak{a} \rangle$ here standing for θ , $\langle E_\tau : \tau \in \mathfrak{a} \rangle$ there, we get a club $C_{\theta, \sigma, \alpha}$ of σ .

For each $\sigma \in \text{pcf}(\mathfrak{a})$ let $C_\sigma = \bigcap_{\theta \in \mathfrak{c} \cap \sigma, \alpha < \theta} C_{\theta, \sigma, \alpha}$ (note: $|\mathfrak{c} \cap \sigma| < \sigma$), so C_σ is a club of σ . Lastly let $\sigma^* = \text{cf}(\sigma^*) \in (|\mathfrak{c}|, \text{min}(\mathfrak{a}))$ and $\langle N_i : i \leq \sigma^* \rangle$ be as in clause (f) of (*)_a, so

$$\oplus_1 N_i \prec (\mathcal{H}(\chi), \in, <_\chi^*) \text{ is increasing continuous, for } i \leq \sigma^* \text{ we have } \|N_i\| = \sigma^*, \langle N_j : j \leq i \rangle \in N_{i+1}.$$

Let $\alpha_\theta = \sup(N_{\sigma^*} \cap \theta)$ for $\theta \in \mathfrak{c}$; so $\alpha_\theta \in S_\theta^{\text{gd}}$ for every $\theta \in \mathfrak{c}$, and we shall show that $\bar{\alpha} = \langle \alpha_\tau : \tau \in \mathfrak{a} \rangle$ is as required. [Saharon: details on clubs?]

For each $\theta \in (N_{\sigma^*} \cap \text{pcf}(\mathfrak{a}))$ and $\sigma \in \mathfrak{c}$, $\sigma < \theta$ we have: $\alpha_\theta \in C_\theta$ hence $\alpha_\theta \in C_{\sigma, \theta, \alpha_\sigma}$ hence

$$\{\tau \in \mathfrak{a} : f_{\alpha_\sigma}^\sigma(\tau) \in \tau \setminus A_{\alpha_\sigma}^{\sigma, \tau}\} = \mathfrak{b}_\tau[\mathfrak{a}] \text{ mod } J_\sigma[\mathfrak{a}].$$

The rest should be clear. □_{3.18}

We next point out another connection; if the rank is small and $|\text{pcf}(\mathfrak{a})|$ is large, then we have a case of “ $\Pi\mathfrak{d}/\mathcal{S}_{\leq \lambda}(\mathfrak{d})$ has large true cofinality”.

3.20 Claim. *If $\zeta > \text{rk}'(\mathfrak{a}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle)$ and $\bar{\lambda} = \langle \lambda_\varepsilon : \varepsilon \leq \zeta \rangle$ is strictly increasing and $|\text{pcf}(\mathfrak{a})| \geq \lambda_\zeta$ then for some $\varepsilon < \zeta$ and $\mathfrak{c} \subseteq \mathfrak{a}$ we have $|\text{pcf}(\mathfrak{c})| \geq \lambda_{\varepsilon+1}$ and $\mathfrak{d} \in J_*[\text{pcf}(\mathfrak{c})] \Rightarrow \Pi \mathfrak{d} / [\mathfrak{d}]^{\leq \lambda_\varepsilon}$ has the true cofinality which is $\max \text{pcf}(\mathfrak{c})$.*

Proof. If not, prove by induction on $\varepsilon \leq \zeta$ that

(*) if $\mathfrak{c} \subseteq \mathfrak{a}, \varepsilon = \text{rk}'(\mathfrak{c}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle)$ then $|\text{pcf}(\mathfrak{c})| < \lambda_{\varepsilon+1}$.

Let $J_0 = \{\mathfrak{d} : \mathfrak{d} \subseteq \mathfrak{c}, \varepsilon > \text{rk}'(\mathfrak{d}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle)\}$. By the induction hypothesis $[\mathfrak{d} \in J_0 \Rightarrow |\text{pcf}(\mathfrak{d})| \leq \lambda_\varepsilon]$. Let J be the ideal on \mathfrak{c} which J_0 generates. We have: $[\mathfrak{d} \in J \Rightarrow |\text{pcf}(\mathfrak{d})| \leq \lambda_\varepsilon]$, so by the assumption toward contradiction $\mathfrak{c} \notin J$. Let $\theta = \max \text{pcf}(\mathfrak{c})$. So by the definition of rk' we have $J_{<\theta}[\mathfrak{c}] \subseteq J$. Hence (see [Sh:g, Ch.VIII,§3] or [Sh:E11]) for some $\mathfrak{d} \subseteq \text{pcf}(\mathfrak{c}), \mathfrak{d} \in J_*[\text{pcf}(\mathfrak{a})]$ we have $\Pi \mathfrak{d} / J_{<\theta}^*[\text{pcf}(\mathfrak{c})]$ has the true cofinality θ and $J_*[\text{pcf}(\mathfrak{a})]$ is generated by $\{\text{pcf}(\mathfrak{b}) : \mathfrak{b} \in J_{<\theta}[\mathfrak{c}]\}$. So the conclusion holds. □_{3.20}

Remark. There is a model of small cardinality [Saharon].

§4 STICKS AND BA'S

4.1 Lemma. *Assume $\theta \leq \mu < \lambda \leq \lambda^*$, J an ideal on θ and assume*

$$\otimes_{\theta, \mu, \lambda, \lambda^*}^J \text{ if } n < \omega, \mathfrak{a}_i \in [\text{Reg} \cap \lambda^+ \setminus \mu^+]^n \text{ for } i < \theta \text{ then}$$

$$\{a \in J : \max \text{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*\} \text{ is generated by } \leq \mu \text{ sets.}^2$$

Then there is a set H such that

- (a) H a set of partial functions from θ to $[\lambda]^{\leq \mu}$
- (b) $|H| \leq \lambda^*$
- (c) for every function $g : \theta \rightarrow \lambda$ we can find h and $\bar{a} = \langle \mathfrak{a}_i : i < \theta \rangle$ such that
 - (i) \mathfrak{a}_i is a finite set of regular cardinals from $(\mu, \lambda]$
 - (ii) h is a function from θ to $[\lambda]^{\leq \mu}$ such that $i < \mu \Rightarrow g(i) \in h(i)$
 - (iii) for any $n < \omega$ and $a \in J$:
 if $(\forall i \in a)[|\mathfrak{a}_i| \leq n]$ and $\max \text{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*$ then for some b satisfying $a \subseteq b \subseteq \theta$ we have $h \upharpoonright b \in H$.

Proof. Like [Sh 430, §2]; [Saharon: details?].

4.2 Remark. 1) But we can then change the bound (in clause (c)(ii)) to $h(i) \in [\lambda]^{< \mu}$.

Then $\otimes_{\theta, \mu, \lambda, \lambda^*}$ is changed to

$$\otimes'_{J, \theta, \mu, \lambda, \lambda^*} \text{ if } n < \omega, \mathfrak{a}_i \in [\text{Reg} \cap \lambda^+]^n \text{ for } i < \theta \text{ then}$$

$$\{a \in J : \text{for some } \mu_0 < \mu, \max \text{pcf}(\bigcup_{i \in a} \mathfrak{a}_i \setminus \mu_0^+) \leq \lambda^*\}$$

is generated by $< \mu$ sets.

2) We can weaken \otimes to

$$\otimes^-_{J, \theta, \mu, \lambda, \lambda^*} \text{ if } n < \omega, \mathfrak{a}_i \in [\text{Reg} \cap \lambda^+]^n \text{ for } i < \theta \text{ then}$$

$$\{a \in J : \max \text{pcf}(\bigcup_{i < a} \mathfrak{a}_i) \leq \lambda^*\}$$

is generated (as an ideal) by some $\mathcal{P} \subseteq J$ such that

$$\kappa \in \bigcup_{i < \theta} \mathfrak{a}_i \Rightarrow \kappa > |\{a \in \mathcal{P} : \bigvee_{i \in a} \kappa \in \mathfrak{a}_i\}|.$$

²see 4.2(3); we can use \mathfrak{a}_i which are singletons

3) Instead $\otimes_{\theta, \mu, \lambda, \lambda^*}^J$ in 4.1 we can define a game

$\otimes_{\theta, \mu, \lambda, \lambda^*} [\mathfrak{D}]$ First player has no winning strategy in the game defined below

$GM'_{\theta, \mu, \lambda, \lambda^*} [\mathfrak{D}]$ The play lasts ω moves, in the n -th move:

first player chooses $\bar{\lambda}^n = \langle \lambda_i^n : i \in A_n \rangle$, $A_0 = \theta$, $\bigwedge_{m < n} A_m \subseteq A_n$, $\bigwedge_{m < n} \bigwedge_{i \in A_n} \lambda_i^n <$

$\lambda_i^m, \lambda_i^n = \text{cf}(\lambda_i^n) \in (\mu, \lambda]$ and

second player chooses an ideal J_n on A_n , $J_n \subseteq \{a \subseteq A_n : \max \text{pcf}\{\lambda_i^n : i \in a\} \leq \lambda^*\}$, J_n generated by $\leq \mu$ sets.

In the end (clearly $\bigcap_{n < \omega} A_n = \emptyset$) they produce the ideal J , the one generated

by $\{a \subseteq \theta : \text{for some } n, a \subseteq A_n \setminus A_{n+1} \text{ and } a \in J_n\}$.

Second player wins if $J \in \mathfrak{D}$.

4.3 Definition. Assume $\bar{J} = \langle J_\ell : \ell < 3 \rangle$, where $J_0 \subseteq J_1 \subseteq J_2 \subseteq \mathcal{P}(\theta)$, each J_ℓ is downward closed (usually is an ideal); we let $J_\ell^+ =: \mathcal{P}(\theta) \setminus J_\ell$.

1) $\text{def}_{\bar{J}}(\lambda, < \mu) = \min \left\{ |\mathcal{F}| : \mathcal{F} \text{ is a family of functions each with domain from}$

J_1^+ and range included in $[\lambda]^{<1+\mu}$ such that:

(*) $_{\mathcal{F}}$ for every $b \in J_2^+$ and $f \in {}^b \lambda$ for some $a \in J_1^+$

and $g \in \mathcal{F} \cap {}^a \lambda$ we have $(\forall^{J_0} i \in a)(i \in b \ \& \ g(i) \in f(i))$

i.e. $\{i : i \in a \text{ and } i \notin b \vee g(i) \notin f(i)\} \in J_0 \}$

2) $\text{ecf}_{\bar{J}}(\lambda, < \mu) = \min \left\{ |\mathcal{F}| : \mathcal{F} \text{ is a family of functions each with domain from}$

J_1^+ and range included in $[\lambda]^{<\mu}$ such that:

(*) $_{\mathcal{F}}$ for every $b \in J_2^+$ and $f \in {}^b \lambda$ for some $a \in J_1^+$

and $g \in \mathcal{F}$ we have

$(\exists^{J_0} i \in a)(i \in b \ \& \ g(i) \in f(i))$

i.e. $\{i : i \in a, i \in b \text{ and } g(i) \in f(i)\} \notin J_0 \}$

3) Let x be d or e . Now $\text{xcf}_{\bar{J}}(\lambda, < \mu^+)$ is written $\text{xcf}_{\bar{J}}(\lambda, \leq \mu)$ or $\text{xcf}_{\bar{J}}(\lambda, \mu)$. Also $\text{xcf}_{\bar{J}}(\lambda, 1)$ is written $\text{xcf}_{\bar{J}}(\lambda)$. Also $\text{xcf}_{\langle [\lambda]^{<\sigma}, [\lambda]^{<\theta}, [\lambda]^{<x} \rangle}(\lambda, < \mu)$ is written $\text{xcf}_{\chi, \theta, \sigma}(\lambda, < \mu)$, etc.

Remark. We may consider replacing J_0 by $J_{0,b}$; i.e, it depends on b .

4.4 Definition. $\mathfrak{I}_{\lambda,\mu,\theta} =: \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\theta \text{ is such that for every } A \in [\lambda]^\mu \text{ for some } a \in \mathcal{P} \text{ we have } a \subseteq A\}$. If $\mu = \lambda$ we may omit it. Let \mathfrak{I}_λ mean $\mathfrak{I}_{\lambda,\aleph_0}$ and $\mathfrak{I} = \mathfrak{I}_{\aleph_1}$.

4.5 Claim. 1) $\mathfrak{I}_{\lambda,\mu,\theta} = \text{dcf}_{\langle\{\emptyset\}, [\lambda]^{<\theta}, [\lambda]^{<\mu}\rangle}(\lambda)$ when $\theta \leq \mu \leq \lambda$.

2) $\text{ecf}_{\mu,\theta,\sigma}(\lambda, \theta) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\theta \text{ and for every } A \in [\lambda]^\mu \text{ for some } a \in \mathcal{P} \text{ we have } |A \cap a| \geq \sigma\}$.

Proof. Read the definitions. □_{4.5}

Now we can phrase the analog of 4.1 for dcf.

4.6 Lemma. Assume $\theta < \mu < \lambda \leq \lambda^*$, $\bar{J} = \langle J_0, J_1, J_2 \rangle$ is an increasing sequence of ideals on θ and assume

$\otimes_{\theta,\mu,\lambda,\lambda^*}^J$ if $n < \omega$, $\mathfrak{a}_i \in [\text{Reg} \cap \lambda^+ \setminus \mu^+]^n$ for $i < \theta$ then $\{a/J_\kappa : a \in J_2 \text{ and } \max \text{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*\} \subseteq \mathcal{P}(\theta)/J_2$ has a dense subset of cardinality $\leq \mu$.³

Then there is a set H such that

- (a) H a st of partial functions from θ to $[\lambda]^{\leq \mu}$
- (b) $|H| \leq \lambda^*$
- (c) for every function $g : \theta \rightarrow \lambda$ we can find h and $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i < \theta \rangle$ such that
 - (i) \mathfrak{a}_i is a finite set of regular cardinals from $(\mu, \lambda]$
 - (ii) h is a function from θ to $[\lambda]^{\leq \mu}$ such that $i < \theta \Rightarrow g(i) \in h(i)$
 - (iii) for any $n < \omega$ and $a \in J_2^+$:
if $(\forall i \in a)[\mathfrak{a}_i| \leq n]$ and $\max \text{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*$ then for some $b \in J_1^+$
 such that $a \subseteq b \text{ mod } J$ we have $h \upharpoonright b \in H$.

³see 4.2(3)

4.7 Claim. *Assume:*

- (*)₀ $\aleph_0 < \aleph_{\alpha(*)} \leq \uparrow$
- (*)₁ $\aleph_{\alpha(*)} < \aleph_{\omega_2}$ or at least

$$\text{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \uparrow$$

- (*)₂ $\mathfrak{a} \subseteq \text{Reg} \cap \uparrow \setminus \aleph_{\alpha(*)+1}$ & $|\mathfrak{a}| \leq \aleph_0 \Rightarrow |\text{pcf}(\mathfrak{a})| \leq \aleph_{\alpha(*)}$
- (*)₃ if $\lambda_i \in (\aleph_1, \uparrow) \cap \text{Reg}$ for $i < \omega_1$ then for some $a \in [\omega_1]^{\aleph_1}$, for every $b \in [a]^{\aleph_0}$ we have $\max \text{pcf}(\{\lambda_i : i \in b\}) \leq \uparrow$.

Then $\uparrow \uparrow = \uparrow$.

4.8 Remark. 1) This means that the conclusion holds except when some dubious statements on pcf holds; that is, ones which have high consistency strength (or are inconsistent) and \uparrow is somewhat large.

2) There are obvious monotonicity properties and $\text{ecf}_J(\lambda, < \mu) \leq \text{dcf}_J(\lambda, < \mu)$.

Proof. Let $\theta = \aleph_1, \mu = \aleph_{\alpha(*)}, \lambda = \uparrow, \lambda^* = \uparrow, J = [\omega_1]^{\leq \aleph_0}$. Apply 4.1. The assumption $\otimes_{\theta, \mu, \lambda, \lambda}^J = \otimes_{\aleph_1, \aleph_{\alpha(*)}, \uparrow, \uparrow}^J$ holds by (*)₂. So let $H \subseteq \{h : h \text{ is a function from } \aleph_1 \text{ to } [\uparrow]^\mu, |H| = \lambda^* = \lambda \text{ be as in the conclusion there. Let } \chi = \beth_7^+, \mathfrak{B} \prec (\mathcal{H}(\chi), \in, <_\chi^*), |\mathfrak{B}| = \lambda, \lambda + 1 \subseteq \mathfrak{B}, H \in \mathfrak{B}. \text{ We want to show } \mathcal{P} = \mathfrak{B} \cap [\lambda]^{\aleph_0} \text{ exemplifies } \uparrow_\lambda = \lambda. \text{ So assume } g : \theta \rightarrow \lambda, \text{ hence there are } h, \langle \langle \lambda_i^n : n < n_i \rangle : i < \omega_1 \rangle, \text{ as there. Let } n^* \text{ be such that } B_0 =: \{i < \omega_1 : n_i = n^*\} \text{ is uncountable. By using } n \text{ times } (*)_3 \text{ we can find an uncountable } B \subseteq B_0 (\subseteq \omega_1) \text{ such that}$

$$(*) \ a \subseteq B \ \& \ a \in J \Rightarrow \max \text{pcf}\{\lambda_i^n : n < n^*, i \in a\} \leq \lambda.$$

So for every $a \in [B]^{\aleph_0}$, for some $b \in [\omega_1]^{\aleph_0}$ we have $a \subseteq b$ and $h \upharpoonright b \in H \subseteq \mathfrak{B}$.

Let for a set $b \in \mathcal{H}(\chi)$ of ordinals, f_b be the $<_\chi^*$ -first one-to-one function from $|b|$ onto b . Let $g'(i) = f_{h(i)}^{-1}(g(i))$, so g' is a function from $\theta = \aleph_1$ to $\mu = \aleph_{\alpha(*)}$ (as $|h(i)| \leq \mu$). Now $\text{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \uparrow$ so $\text{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \lambda$ (the only property of $\alpha(*)$ we use) so there is $\mathcal{P}' \subseteq [\aleph_{\alpha(*)}]^{\leq \aleph_1}, |\mathcal{P}'| \leq \lambda$ exemplifying this and without loss of generality $\mathcal{P}' \subseteq \mathfrak{B}$ & $\mathcal{P}' \in \mathfrak{B}$. So the set $\{g'(i) : i \in B\}$ is included in a countable union of members of \mathcal{P}' , so for some $Y \in \mathcal{P}'$ (so $Y \in [\aleph_{\alpha(*)}]^{\leq \aleph_1}$ and $Y \in \mathfrak{B}$) we have $B^* =: \{i \in B : g'(i) \in Y\}$ is uncountable.

Define h' :

$$\text{Dom}(h') = \omega_1, h'(i) = \{\alpha \in h(i) : f_{h(i)}(\alpha) \in Y\}.$$

So h' is a function from ω_1 to $[\lambda]^{\leq \aleph_1}$ (as $|Y| \leq \aleph_1$) and $i \in B^* \Rightarrow g(i) \in h'(i)$; remember $a \in [B^*]^{\leq \aleph_0} \Rightarrow (\exists b)[a \subseteq b \subseteq \omega_1 \ \& \ h \upharpoonright b \in \mathfrak{B}]$.

Let $Z =: \{(i, f_{h'(i)}^{-1}(g(i))) : i \in B\}$, it is a subset of $\omega_1 \times \omega_1$ of cardinality λ , but $\upharpoonright = \lambda, \lambda + 1 \subseteq \mathfrak{B}$, so for some infinite $z \in \mathfrak{B}$ we have $z \subseteq Z$. Let $z_0 = \{i < \omega_1 : \bigvee_j (i, j) \in z\}$, so $z_0 \in \mathfrak{B}, z_0 \in [\omega_1]^{\aleph_0}$ and even $z_0 \subseteq B^*$, hence $h' \upharpoonright z_0 \in \mathfrak{B}$.

So as $h' \upharpoonright z_0 \in \mathfrak{B}$ and $\{(i, f_{h'(i)}^{-1}(g(i))) : i \in z_0\} = z \in \mathfrak{B}$ also $g \upharpoonright z_0 \in \mathfrak{B}$, so $\text{Rang}(g \upharpoonright z_0) \in B$, so $\text{Rang}(g \upharpoonright z_0) \in \mathcal{P}$ and we are done. $\square_{4.7}$

4.9 Definition.

$$St_{\lambda, \kappa}^3 = \min\{|\mathcal{P}| : (a) \ \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) \ (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \text{ infinite}) \\ (c) \ \mathcal{P} \text{ is AD which means } a \neq b \in \mathcal{P} \Rightarrow a \cap b \text{ finite}\}$$

(the main case $\kappa = \aleph_1$).

4.10 Definition.

$$St_{\lambda, \kappa}^4 = \min\{|\mathcal{P}| : (a) \ \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) \ (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \text{ infinite}) \\ (c) \ \sup\{\text{otp}(a) : a \in \mathcal{P}\} < \omega_1\}.$$

4.11 Definition.

$$St_{\lambda, \kappa}^5 = \min\{\mathcal{P} : (a) \ \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) \ (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \text{ infinite}) \\ (c) \ \text{the BA of subsets of } \lambda \text{ which } \mathcal{P} \text{ and the singletons} \\ \text{generate is superatomic of rank } < \omega_1\}.$$

4.12 Fact: $\text{dcf}_{\kappa, \aleph_0, \aleph_0}(\lambda) \leq St_{\lambda, \kappa}^\ell$.

* * *

4.13 Claim. 1) Given $\lambda \geq \kappa = \text{cf}(\kappa) > \aleph_0$, the following cardinals are equal for $k < \omega, k > 0$:

(a) $\text{ecf}_{\kappa, \aleph_0, \aleph_0}(\lambda)$

(b)_k $\min\{|\mathcal{F}|\}$:

(i) \mathcal{F} is a family of partial functions f from λ to k

(ii) $f \in \mathcal{F} \Rightarrow |\text{Dom}(f)| = \aleph_0$,

(iii) $f \in \mathcal{F}, \ell < k \Rightarrow f^{-1}(\{\ell\})$ is infinite

(iv) if $\langle A_0, \dots, A_{k-1} \rangle$ are pairwise disjoint subsets of λ each of cardinality κ then for some $f \in \mathcal{F}$ we have $\ell < k \Rightarrow f^{-1}(\{\ell\}) \cap A_\ell$ is infinite}

(c)_k like (b)_k replacing (iii), (iv) by

(iii)⁺ if $\langle \alpha_{\varepsilon, \ell} : \varepsilon < \kappa \text{ and } \ell < k \rangle$ is a sequence of ordinals, with no repetitions and $f \in \mathcal{F}$ then for infinitely many $\varepsilon < \kappa$, for each $\ell < k, f(\alpha_{\varepsilon, \ell}) = \ell$ (so $\alpha_{\varepsilon, \ell} \in \text{Dom}(f)$).

Proof. Let λ_k^b, λ_k^c be the cardinal from (b)_k, (c)_k respectively and λ^* the cardinal from (a). Clearly $\lambda_k^b \leq \lambda_{k+1}^b, \lambda_k^c \leq \lambda_{k+1}^c, \lambda_k^b \leq \lambda_k^c, \lambda_1^c = \lambda_1^b = \lambda^*$. So it suffices to prove $\lambda_k^c \leq \lambda^*$, assume \mathcal{P} exemplifies $\lambda^* = \text{ecf}_{\kappa, \aleph_0, \aleph_0}(\lambda)$ as phrased in 4.5 (2). If $\lambda^* \geq 2^{\aleph_0}$ let $\mathcal{F}^* = \{f : \text{for some } a \in \mathcal{P}, f \text{ is a function from } a \text{ to } k \text{ such that } \ell < k \Rightarrow |f^{-1}(\{\ell\})| = \aleph_0\}$ clearly it exemplifies $\lambda_k^c \leq |\mathcal{F}^*| = 2^{\aleph_0} \times \lambda^* = \lambda^*$. So assume $\lambda^* < 2^{\aleph_0}$ and let $\bar{\eta} = \langle \eta_i : i < \lambda^* \rangle$ be a sequence of pairwise distinct members of ${}^\omega 2$. Let $\bar{g} = \langle g_\ell : \ell < k \rangle$ be such that: $g_\ell : \lambda \rightarrow \lambda$ and $(\forall \alpha_0, \dots, \alpha_{k-1} < \lambda)(\exists \beta < \lambda)[\bigwedge_{\ell < k} g_\ell(\beta) = \alpha_\ell]$. Let $\mathcal{P}' = \{a^{\bar{g}} : a \in \mathcal{P}\}$ where $a^{\bar{g}} = \{g_\ell(x) : \ell < k, x \in a\}$ and let $\mathcal{F} = \{f_{b,h} : b \in \mathcal{P}' \text{ and for some } n \text{ we have } h \in ({}^{n2})k \text{ and } \ell < k \Rightarrow |f_{b,h}^{-1}(\{\ell\})| = \aleph_0\}$ where $f_{b,h}$ is the function with domain $b^{\bar{g}}$ and $f_{b,h}(i) = h(\eta_i \upharpoonright n)$ where $h \in ({}^{n2})k$; note that $b^{\bar{g}} \in \mathcal{P}'$.

Clearly \mathcal{F} has the right cardinality and form. Let us show that it satisfies the main requirement: let $\langle A_0, \dots, A_{k-1} \rangle$ be a sequence of pairwise disjoint subsets of λ each of cardinality κ . Let $A_\ell = \{\gamma_{\ell, \varepsilon} : \varepsilon < \kappa\}$ (no repetition). Let $\gamma_\varepsilon < \lambda$ be such that $\bigwedge_{\ell < k} g_\ell(\gamma_\varepsilon) = \gamma_{\ell, \varepsilon}$. For each ε for some $n(\varepsilon)$ we have: $\langle \eta_{\gamma_{\ell, \varepsilon}} \upharpoonright n(\varepsilon) : \ell < k \rangle$ is with no repetitions. As $\kappa = \text{cf}(\kappa) > \aleph_0$ and as we can replace each A_ℓ by any subset of cardinality κ without loss of generality for some $\bar{\nu} = \langle \nu_\ell : \ell < k \rangle$ and $n(*)$ we have $\varepsilon < \kappa \Rightarrow \eta_{\gamma_{\ell, \varepsilon}} \upharpoonright n(*) = \nu_\ell$ and $\varepsilon < \kappa \Rightarrow n(\varepsilon) = n(*)$. Now by the choice of

\mathcal{P} for some $a \in \mathcal{P}$, $W = \{\varepsilon : \gamma_\varepsilon \in a\}$ is infinite. Let $b = a^{\bar{g}}$, let $h : {}^{n(*)}2 \rightarrow k$ be such that $h(\nu_\ell) = \ell$, now $f_{b,h} \in \mathcal{F}$ is as required. □_{4.13}

4.14 Claim. *Assume*

$$(*)_1 \quad \biguparrow_{\lambda, \kappa, \aleph_0} = \lambda \text{ equivalently } \lambda = \Sigma \lambda_n \text{ where } \lambda_{n+1} \geq \text{dcf}_{\kappa, \aleph_0, \aleph_0}(\lambda_n, \aleph_0) \text{ for } n < \omega, \lambda_0 \geq \kappa \text{ and } \kappa = \text{cf}(\kappa) > \aleph_0.$$

Then there is a Boolean Algebra B of cardinality λ into which the free Boolean algebra generated by λ elements can be embedded but such that there is no homomorphism from B onto the free Boolean Algebra generated by κ elements.

4.15 Remark. 1) So we can find quite many pairs $\lambda = \sum_{n < \omega} \lambda_n$ and κ as required in 4.14 (or 4.15). E.g. any $\lambda = \lambda_n > \beth_\omega$ and $\kappa \in [\beth_\omega, \lambda] \cap \text{Reg}$ is large enough $\kappa = \text{cf}(\kappa) \leq \beth_\omega$ is as required by [Sh 460].
 2) On the problem see Fuchino, Shelah, Soukup [FShS 543], the proof is similar.
 3) From the proof we can strengthen the last phrase in the conclusion to “no homomorphism from B into $Fr(\kappa)$ with range of cardinality κ ”. Similarly in 4.17.

Proof. Let $\mathcal{F}_n = \{f_\alpha^n : \alpha < \lambda_1\}$ be as guaranteed in clause $(b_z)_2$ of 4.13, exists by $(*)_1$. Without loss of generality $\alpha < \lambda_{n+1} \wedge \ell < 2 \Rightarrow (\exists^{\aleph_0} i)(f_\alpha^n(i) = \ell)$. So let $\text{Dom}(f_\alpha^n) = \{j_{\alpha, k, \ell}^n, \ell < 2, k < \omega\}$ with no repetitions such that $f_\alpha^n(j_{\alpha, k, \ell}^n) = \ell$.

Remember the variety of Boolean rings has the operations $x \cup y, x \cap y, x - y$ and constant 0 (but no 1 and no $-x$), so any ideal of a Boolean algebra is a Boolean ring and if the ideal is maximal, the Boolean algebra is definable in the Boolean ring.

Let B_0 be the Boolean ring freely generated by $\{x_i^0 : i < \lambda\}$. Let B_1 be the Boolean ring generated by $B_0 \cup \{x_i^1 : i < \lambda\}$ freely except:

- (a) the equations which holds in B_0
- (b) $x_\alpha^1 \cap x_{j_{\alpha, k, 0}^n}^0 - \bigcup_{m < k} x_{j_{\alpha, m, 1}^n}^0 = 0$
- (c) $x_\alpha^1 \cap x_{j_{\alpha, k, 1}^n}^0 - \bigcup_{m < k} x_{j_{\alpha, m, 0}^n}^0 = x_{j_{\alpha, k, 1}^n}^0 - \bigcup_{m < k} x_{j_{\alpha, m, 0}^n}^0$.

Similarly let B_{n+1} be the Boolean ring generated by $B_n \cup \{x_\alpha^{n+1} : \alpha < \lambda\}$ freely except

- (a) the equations which holds in B_n

$$(b) \ x_\alpha^{n+1} \cap \sigma_{\alpha,k}^{n,1} \text{ where } \sigma_{\alpha,k}^n = x_{j_{\alpha,k,0}}^n - \bigcup_{m < k} x_{j_{\beta,m,1}}^n = 0$$

$$(c) \ x_\alpha^{n+1} \cap \sigma_{\alpha,k}^{n,2} = \sigma_{\alpha,k}^{n,2} \text{ where } \sigma_{\alpha,k}^{n,2} = x_{j_{\alpha,k,1}}^n - \bigcup_{m < k} x_{j_{\alpha,m,0}}^n.$$

Now we can prove by induction on n that

$\boxtimes_n(i)$ $\langle x_\alpha^n : \alpha < \lambda \rangle$ is a sequence of independent members of B_{n+1} , with no repetitions

(ii) $B_n \subseteq B_{n+1}$.

Now $B_\omega = \bigcup_{n < \omega} B_n$ is a Boolean ring and let B be the Boolean algebra for which

B_ω is a maximal ideal. Assume f is a homomorphism from B onto $Fr(\kappa)$, the Boolean algebra freely generated say by $\{z_i : i < \kappa\}$. Now B is generated by $\{x_\alpha^n : n < \omega, \alpha < \lambda_n\}$. So as f is onto and $\text{cf}(\kappa) > \aleph_0$, for some n , for every $\zeta < \kappa$ for some α , $f(x_\alpha^n) \notin \langle z_\varepsilon : \varepsilon < \zeta \rangle_{Fr(\kappa)}$. By the Δ -system lemma, we can find a stationary $S \subseteq \kappa$, Boolean term $\sigma_1 = \sigma_1(x_0, \dots, x_{n(*)-1})$, $m(*) < n(*)$, ordinals $\varepsilon(0) < \dots < \varepsilon(m(*)-1) < \min(S)$, and for each $\zeta \in S$, ordinals $\varepsilon(m(*), \zeta) < \dots < \varepsilon(n(*)-1, \zeta)$ all in the interval $[\zeta, \min(S \setminus (\zeta + 1))]$ and $\alpha_\zeta < \lambda_n$ such that $f(x_{\alpha_\zeta}^n) = \sigma_1(z_{\varepsilon(0)}, \dots, z_{\varepsilon(m(*)-1)}, z_{\varepsilon(m(*), \zeta)}, \dots, z_{\varepsilon(n(*)-1, \zeta)})$, where all the $n(*)$ variables are needed in the term σ_1 .

Let $S = \{\zeta(i) : i < \kappa\}$ with $\zeta(i)$ increasing in i , let h be a homomorphism from $Fr(\kappa)$ to the two element Boolean Algebra such that $h(f(x_{\alpha_{2\zeta+\ell}}^n)) = \ell$ for $\ell = 0, 1$ (exist as $\langle f(x_{\alpha_\zeta}^n) : \zeta < \omega \rangle$ is independent). Let $g = h \circ f$ and f_α^n as guaranteed by the choice of \mathcal{F}_n for g and $A_\ell = \{\alpha_{2\zeta+\ell} : \zeta < \kappa\}$ for $\ell = 0, 1$. So for $\ell = 0, 1$ the sets $W_\ell =: \{\alpha_{\zeta(2i+\ell)} : \alpha_{\zeta(2i+\ell)} \in \text{Dom}(f_\alpha^n)\}$ are infinite and $f_\alpha^n \upharpoonright W_\ell$ is constantly ℓ . So $z^* = f(x_\alpha^{n+1}) \in Fr(\kappa)$ satisfies

$$(*)_0 \ \alpha_\zeta \in W_0 \Rightarrow Fr(\kappa) \models z^* \cap f(\sigma_{\alpha,k}^{n,1}) = 0$$

$$(*)_1 \ \alpha_\zeta \in W_1 \Rightarrow Fr(\kappa) \models z^* \cap f(\sigma_{\alpha,k}^{n,2}) = f(\sigma_{\alpha,k}^{n,k}).$$

But for some finite $u \subseteq \kappa$, $z^* \in \langle z_\gamma : \gamma \in u \rangle_{Fr(\kappa)}$, so there is $\alpha_{\zeta(2i_0)} \in W_0$, such that u is disjoint to $\{\varepsilon(m(*), \zeta(2i_0)), \dots, \varepsilon(n(*)-1, \zeta(2i_0+1))\}$ and there is $\alpha_{\zeta(2i_1+1)} \in W_1$ such that u is disjoint to $\{\varepsilon(m(*), \zeta(2i_1+1)), \dots, \varepsilon(n(*)-1, \zeta(2i_1+1))\}$. For them $(*)_0, (*)_1$ gives a contradiction as $\langle z_i : i < \kappa \rangle$ generate $Fr(\kappa)$ freely.

So B is a Boolean algebra, of cardinality $\leq \lambda$ (as it is generated by $\{x_\alpha^n : \alpha < \lambda_n, n < \omega\}$), we can embed into it the free Boolean algebra with λ generators $\{x_\alpha^0 : \alpha < \lambda\}$ and B is with no homomorphism onto the free Boolean algebra with κ generators. $\square_{4.14}$

4.16 Definition. Let B_μ^{fcf} be the Boolean Algebra of finite and cofinite subsets of μ .

4.17 Claim. Assume $\lambda_{n+1} \geq \text{ecf}_{\kappa, \aleph_0, \aleph_0}(\lambda_n, \aleph_0)$, $\kappa = \text{cf}(\kappa) > \aleph_0$ and $\lambda = \sum_n \lambda_n$.
Then there is a Boolean Algebra B of cardinality λ into which B_λ^{fcf} can be embedded but such that there is no homomorphism from B onto B_κ^{fcf} .

Proof. Let

$$\mathcal{P}_n = \{a_\alpha^n : \alpha < \lambda_{n+1}\} \subseteq [\lambda_n]^{\aleph_0}$$

exemplifies $\lambda_{n+1} \geq \text{ecf}_{\kappa, \aleph_0, \aleph_0}(\lambda_n, \aleph_0)$ by 4.5(2). We define by induction on n a countable subset x_α^n of $(n+1) \times \lambda$ for each $\alpha < \lambda_n$. For $n = 0$ let $x_\alpha^0 = \{(0, \alpha)\}$. For $n + 1$ let

$$x_{2\alpha}^{n+1} = \{(n+1, 2\alpha)\} \cup \bigcup_{\beta \in a_\alpha^n} x_\beta^n \text{ and } x_{2\alpha+1}^{n+1} = \{(n+1, 2\alpha+1)\}.$$

Let B be the Boolean Algebra of subsets of $\omega \times \lambda$ generated by $\{x_\alpha^n : \alpha < \lambda_n, n < \omega\}$.

Clearly $|B| \leq \sum_n \lambda_n = \lambda$, also $\{x_{2\alpha+1}^n : \alpha < \lambda, n < \omega\}$ generate a subalgebra isomorphic to B_λ^{fcf} hence $|B| \geq \lambda$ (so $|B| = \lambda$) and B_λ^{fcf} can be embedded into B .

Lastly, suppose g is a homomorphism from B onto B_κ^{fcf} . Let $z_\zeta = \{\zeta\} \in B_\kappa^{\text{fcf}}$. For each $\zeta < \kappa$ for some $n(\zeta) < \omega, \alpha(\zeta) < \lambda_{n(\zeta)}$ we have $g(x_{\alpha(\zeta)}^{n(\zeta)}) \notin \langle z_\zeta : \varepsilon < \zeta \rangle_{B_\kappa^{\text{fcf}}}$, so for some stationary $S \subseteq \kappa, [\zeta \in S \Rightarrow n(\zeta) = n^*]$. If $g(x_\alpha^n) \in B_\kappa^{\text{fcf}}$ is infinite then $\{g(x) : x \in B \text{ and } g(x) \leq g(x_\alpha^n)\} = \{g(x \cap x_\alpha^n) : x \in B\}$ is countable so g is not onto B , a contradiction. So possibly shrinking S without loss of generality $\langle g(x_{\alpha(\zeta)}^{n(\zeta)}) : \zeta \in S \rangle$ is a Δ -system of finite subsets of κ with heart called w .

For some $\beta < \lambda_{n+1}$ the set $u = \{\zeta \in S : \alpha(\zeta) \in a_\beta^{n+1}\}$ is infinite, clearly $\zeta \in u \Rightarrow x_{\alpha(\zeta)}^n \leq x_{2\beta}^{n+1}$ hence $g(x_{\alpha(\zeta)}^{n(\zeta)}) \leq g(a_{2\beta}^{n+1})$ hence $g(x_{2\beta}^{n+1})$ is infinite hence it is co-finite, contradicting an earlier statement. $\square_{4.17}$

4.18 Definition.

$St_{\lambda, \kappa}^6 = \min\{|\mathcal{P}| : \text{there is } \mathcal{P} \subseteq [\lambda]^{\aleph_0} \text{ such that}$

- (i) \mathcal{P} is \aleph_2 -free i.e. if $a_i \in \mathcal{P}$ for $i < i^* < \aleph_2$ are disjoint then for some finite $b_i \subseteq a_i$ the sets $\langle a_i \setminus b_i : i < i^* \rangle$ are pairwise disjoint
- (ii) for every $f : \kappa \rightarrow \lambda$ for some $a \in \mathcal{P}$
 $(\exists^\infty \alpha < \kappa)(f(\alpha) \in a)$.

4.19 Claim. *Assume $\lambda_{n+1} \leq St_{\lambda_n, \kappa}^6$ for $n < \omega$, $\lambda = \sum_{n < \omega} \lambda_n$. Then there is a Boolean Algebra B as in the previous claim which is superatomic.*

Proof. Like the previous claim. [Saharon: details?]

§5 MORE ON FREE SUBSETS AND PCF

5.1 Claim. Assume $IND(\langle J_{\lambda_\varepsilon}^{\text{bd}} : \varepsilon < \varepsilon(*) \rangle)$ with λ_ε increasing and $\lambda = \sum_{\varepsilon < \varepsilon(*)} \lambda_\varepsilon$. If $\mu > \lambda$ and $\theta_i \in \text{Reg} \cap \mu \setminus \lambda$ for $i < \lambda$, then for some ε we can find $\mathfrak{c} \subseteq \mu \cap \text{pcf}\{\theta_i : i < \lambda\}$ and $\mathfrak{b}_\tau \in J_{\leq \tau}[\{\theta_i : i < \lambda\}]$ for $\tau \in \mathfrak{c}$ such that

(*) there is no $a \in [\lambda]^{\lambda_\varepsilon}$ such that $[\tau \in \mathfrak{c} \Rightarrow (\forall^{\lambda_\varepsilon} i \in a)\theta_i \notin \mathfrak{b}_\tau]$.

Proof. Easy from the definition.

5.2 Claim. 1) For every $\mathfrak{a} \subseteq \text{Reg}$, $\lambda \leq |\mathfrak{a}| < \min(\mathfrak{a})$ for some $\mathfrak{b} \subseteq \mathfrak{a}$, $\kappa < |\mathfrak{b}| \leq \lambda$, $\Pi \mathfrak{b}/[\mathfrak{b}]^{\leq \kappa}$ has true cofinality (so if λ is minimal for this κ , this holds for any $\kappa' < \lambda$).

2) If $IND(\langle J_{\lambda_n}^{\text{bd}} : n < \omega \rangle)$, $\lambda_n < \lambda_{n+1}$, λ_n regular, $|\mathfrak{a}| < \lambda_0$, $|\mathfrak{a}| < \min(\mathfrak{a})$ and

$$\mu = \sup_{n < \omega} \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda_n]^{|\mathfrak{a}|^+} \text{ and } (\forall A \in [\lambda_n]^{\lambda_n})(\exists B \in \mathcal{P})[B \subseteq A]\}$$

then $|\text{pcf}(\mathfrak{a})| \leq \mu$.

3) Assume $\sigma < \theta < \lambda_n$, (for $n < \omega$) and $IND(\langle J_n : n < \omega \rangle)$, J_n an ideal on λ_n and μ satisfies: $\bigwedge_n \mu > \lambda_n$ and (we can guess filters which are $(< \theta)$ -based).

(*) $_{\mu, J_n}$ there is a set \mathcal{E} , $|\mathcal{E}| \leq \mu$, each member of \mathcal{E} is an ideal on some bounded subset of κ_n such that:

⊗ if $Y \in J_n^+$ (so $Y \subseteq \lambda_n$), and I is a $(< \theta)$ -based σ -complete ideal on Y generated by $\leq \mu$ sets then for some $I' \in \mathcal{E}$, we have $(\text{Dom}(I') \cap Y \in I^+$ and $\text{Dom}(I') \setminus Y \in I'$ and $I' \upharpoonright (Y \cap \text{Dom}(I')) \supseteq I \upharpoonright (Y \cap \text{Dom}(I'))$).

If $|\mathfrak{a}| \leq \theta$ and $\Sigma\{\lambda_n : n < \omega\} < \text{Min}(\mathfrak{a})$ and $\mathfrak{a} \subseteq \text{Reg}$ then $(\text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})) \leq \mu$.

Proof. 1) Straight, e.g. let $\langle \mathfrak{b}_\theta : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ be a generating sequence for $\text{pcf}(\mathfrak{a})$, let $\mathfrak{b} = \mathfrak{b}_\theta$, θ minimal such that $|\mathfrak{b}_\theta| > \kappa$.

2) Suppose not so $|\text{pcf}(\mathfrak{a})| \geq \mu^+$. Clearly $n < \omega \Rightarrow \lambda_n \leq \mu$. Let $\langle \tau_{i+1} : i < \mu^+ \rangle$ be the first μ^+ members of $\text{pcf}(\mathfrak{a})$ listed in increasing order. Let $\tau_\delta = \bigcup_{i < \delta} \tau_{1+i}$ for limit

$\delta \leq \mu^+$. For each limit $\delta < \mu^+$ for some $n = n_\delta < \omega$, τ_δ is $\{J_{\lambda_n}^{bd}\}$ -inaccessible (by 3.16). So for some $n(*) < \omega$, $\{\delta < \mu^+ : n_\delta = n(*)\}$ is stationary, hence

(*) for no $\theta_\alpha \in \text{Reg} \cap \tau_{\mu^+}$ for $\alpha < \lambda_{n(*)}$, do we have

$$\prod_{\alpha < \lambda_{n(*)}} \theta_\alpha / J_{\lambda_{n(*)}}^{bd} \text{ is } \tau_{\mu^+}\text{-directed.}$$

By [Sh 420, §1] we can find $\langle C_\alpha : \alpha \in S \rangle$, $S \subseteq \mu^+$, $C_\alpha \subseteq \alpha$, $\text{otp}(C_\alpha) \leq \lambda_{n(*)}$, $[\beta \in C_\alpha \Rightarrow \beta \in S \ \& \ C_\beta = C_\alpha \cap \beta]$ and $\text{otp}(C_\alpha) = \lambda_{n(*)} \Rightarrow \alpha = \sup(C_\alpha)$ and $\{\alpha \in S : \text{otp}(C_\alpha) = \lambda_{n(*)}\}$ is stationary. Now we imitate [Sh 400, §2]. [Saharon: or use [Sh 410]? check.]

3) Similar to (2). [Saharon: more?]

□_{5.2}

§6 ODDS AND ENDS

As in [Sh 430, §6] this section is dedicated to things I forgot to say. We repeat and elaborate older things from [Sh 430, 6.6D,6.6E,6.6F], [Sh 410, 3.7].

6.1 Claim. *Suppose D is a σ -complete filter on $\theta = \text{cf}(\theta)$ such that $[\alpha < \theta \Rightarrow \theta \setminus \alpha \in D]$, σ is regular $> \kappa^+ + |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar{\beta} = \langle \beta_\epsilon^\alpha : \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \neq \emptyset \text{ mod } D$ there is $\langle \beta_\epsilon^* : \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:*

- (a) $\epsilon \in \kappa \setminus w \Rightarrow \sigma \leq \text{cf}(\beta_\epsilon^*) \leq \theta$,
- (b) $B =: \{\alpha \in X : \text{if } \epsilon \in w \text{ then } \beta_\epsilon^\alpha = \beta_\epsilon^* \text{ and: if } \epsilon \in \kappa \setminus w \text{ then } \beta_\epsilon^\alpha \text{ is } < \beta_\epsilon^* \text{ but } \beta_\epsilon^\alpha > \sup\{\beta_\zeta^* : \zeta < \kappa, \beta_\zeta^* < \beta_\epsilon^*\}\}$ is $\neq \emptyset \text{ mod } D$
- (c) if $\beta'_\epsilon < \beta_\epsilon^*$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B : \text{if } \epsilon \in \kappa \setminus w \text{ then } B'_\epsilon < \beta_\epsilon^\alpha\} \neq \emptyset \text{ mod } D$.

6.2 Remark. 1) Of course, we can replace κ by any set of this cardinality as the index set for the $\bar{\beta}$'s.

2) May look at [Sh 620], §7, there more is said concerning 6.1.

Proof. Let $f_\alpha : \kappa \rightarrow \text{Ord}$ be $f_\alpha(i) = \beta_i^\alpha$.

Let χ be large enough. We choose by induction on $i < \sigma$, a model N_i such that:

- $N_i \prec (\mathcal{H}(\chi), \in, <_\chi^*)$;
- $\|N_i\| \leq 2^\kappa + |i|^\kappa$;
- $2^\kappa \subseteq N_0$;
- $\kappa, \sigma, \theta, X \in N_0, \langle f_\alpha : \alpha < \theta \rangle \in N_0$;
- $i < j \Rightarrow N_i \prec N_j$;
- $\langle N_j : j \leq i \rangle \in N_{i+1}$;
- N_i increasing continuous.

Let $\delta_i =: \min(X \cap Y_i)$ where $Y_i = \cap\{B : B \in N_i \cap D\}$, now δ_i is well defined (as D is σ -complete and $\sigma > 2^\kappa + |i|^\kappa \geq \|N_i\|$, hence the intersection is in D). Also $\delta_i \geq \sup(\theta \cap N_i)$ as $\alpha \in \theta \cap N_i \Rightarrow \theta \setminus \alpha \in D \cap N_i$. As this and as $\delta_i \in N_{i+1}$ (as $\{N_i, D, \theta\} \in N_{i+1}$) clearly $\langle \delta_i : i < \sigma \rangle$ is strictly increasing. We define for $i < \sigma$, a function $g_i \in {}^\kappa \chi$ by

$$g_i(\zeta) = \min(N_i \cap \chi \setminus f_{\delta_i}(\zeta))$$

(it is well defined as $f_{\delta_i}(\zeta) < \bigcup_{\alpha < \theta} (f_\alpha(i) + 1) \in N_0 \prec N_i$). Clearly $E =: \{\alpha < \sigma : N_\alpha \cap \sigma = \alpha\}$ is a club of σ , and as $(\forall \alpha < \sigma)[|\alpha|^\kappa < \sigma]$ clearly $\alpha < \beta \in E$ &

$a \subseteq N_\alpha$ & $|a| \leq \kappa \Rightarrow a \in N_\beta$. Now $i \in E$, $\text{cf}(i) = \kappa^+$ implies $N_i = \bigcup_{j < i} N_j$ and $\text{Rang}(g_i) \subseteq \bigcup_{j < i} N_j$ hence $\bigvee_{j < i} [\text{Rang}(g_i) \subseteq N_j]$; but by the previous sentence every subset of N_j of cardinality $\leq \kappa$ belongs to N_i , hence $g_i \in \bigcup_{j < i} N_j$. So by

Fodor Lemma for some stationary subset S of $\{i \in E : \text{cf}(i) = \kappa^+\}$ and some $g^* : \kappa \rightarrow \text{Ord}$ and some $u \subseteq \kappa$ and some $i(*) < \sigma$ we have: $[i \in S \Rightarrow g_i = g^*]$, $(\forall i \in S)(\forall \zeta < \kappa)[f_{\delta_i}(\zeta) = g^*(\zeta) \Leftrightarrow \zeta \notin u]$ and $g^* \in N_{i(*)}$; note $u \in N_0 \subseteq N_{i(*)}$ as $u \subseteq \kappa$ and we can assume $i(*) < \min(S)$ and $i(*) \in E$.

Let $w =: \kappa \setminus u$ and $\beta_i^* =: g^*(i)$ for $i < \kappa$ now we shall show that $w, \langle \beta_i^* : i < \kappa \rangle$ are as required.

Clause (b):

The set B is defined from: $\langle \beta_i^* : i < \kappa \rangle$ and w and $\bar{f} = \langle f_\alpha : \alpha < \theta \rangle$. As all of them belong to $N_{i(*)}$ clearly $B \in N_{i(*)}$, so if $B = \emptyset \bmod D$ then $(\theta \setminus B) \in D \cap N_{i(*)}$ hence $\zeta \in S \Rightarrow \delta_\zeta \in \theta \setminus B \Rightarrow \delta_\zeta \notin B$; but $\delta_\zeta \in B$ by the definition of B, g^*, g_ζ, S .

Clause (a):

If $\epsilon \in \kappa \setminus w$, $\text{cf}(\beta_\epsilon^*) > \theta$ then $\gamma_\epsilon^* =: \sup\{f_\alpha(\epsilon) : \alpha < \theta \text{ and } f_\alpha(\epsilon) < \beta_\epsilon^*\}$ is $< \beta_\epsilon^*$ and it belongs to $N_{i(*)}$ (as $\epsilon, \langle f_\alpha : \alpha < \lambda \rangle$ and β_ϵ^* belongs to $N_{i(*)}$) and for any $\zeta \in S$ we get a contradiction to $g_\zeta(\epsilon) = \beta_\zeta^*$.

Clearly $\zeta \in \kappa \setminus w \Rightarrow \text{cf}[g^*(\zeta)] \geq \sigma$ as otherwise $g^*(\zeta) = \sup(N_i \cap g^*(\zeta))$ (as $N_i \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ and $N_{i(*)} \cap \sigma = i(*)$ because $i(*) \in E$) and easy contradiction.

Clause (c):

If there is $\bar{\beta}' = \langle \beta'_\epsilon \in u \rangle$ contradicting clause (c), then there is such a sequence defined from $B, \langle f_\alpha : \alpha < \theta \rangle, u, w, \langle \beta_i^* : i < \kappa \rangle$, just use the $<_\chi^*$ -first one, hence without loss of generality $\bar{\beta}' \in N_{i(*)}$, so for any $\zeta \in S$ we get a contradiction. $\square_{6.1}$

6.3 Observation. If $|\mathbf{a}| < \min(\mathbf{a}), H \subseteq \Pi \mathbf{a}, |H| = \theta = \text{cf}(\theta) \notin \text{pcf}(\mathbf{a})$ and also $\theta > \sup(\theta \cap \text{pcf}(\mathbf{a}))$ then for some $g \in \Pi \mathbf{a}$, the set $H_g =: \{f \in H : f < g\}$ has cardinality θ ; in fact H is the union of $\leq \sup(\theta \cap \text{pcf}(\mathbf{a}))$ sets of the form H_g .

Proof. Why? This is as $\Pi \mathbf{a} / J_{< \theta}[\mathbf{a}]$ is $\min(\text{pcf}(\mathbf{a})) \setminus \theta$ -directed and the ideal $J_{< \theta}[\mathbf{a}]$ is generated by $< \theta$ sets.

In details, let $\langle \mathbf{b}_\sigma[\mathbf{a}] : \sigma \in \text{pcf}(\mathbf{a}) \rangle$ be a generating sequence for \mathbf{a} (exists by [Sh:g, Ch.VIII,2.6]).

For $\sigma \in \text{pcf}(\mathbf{a})$ let $f_\alpha^\sigma \in \Pi \mathbf{a}$ for $\alpha < \sigma$ be such that $\langle f_\alpha^\sigma : \alpha < \sigma \rangle$ is $<_{J_\theta[\mathbf{a}]}$ -increasing and cofinal and moreover $\{f_\alpha^\sigma \upharpoonright \mathbf{b}_\sigma[\mathbf{a}] : \alpha < \sigma\}$ is cofinal in $\Pi \mathbf{b}_\sigma[\mathbf{a}]$ (where $J_\sigma[\mathbf{a}] = J_{<\sigma}[\mathbf{a}] + \mathbf{b}_\sigma[\mathbf{a}]$), exists as $(\Pi \mathbf{b}_\sigma, [\mathbf{a}], <)$ has cofinality σ by [Sh:g, Ch.II,3.1].

Now as $\Pi \mathbf{a} / J_{<\theta}[\mathbf{a}]$ is $\min(\text{pcf}(\mathbf{a}) \setminus \theta^+)$ -directed (as $J_{<\theta}[\mathbf{a}] = J_{<\theta^+}[\mathbf{a}] = J_{<\min(\text{pcf}(\mathbf{a}) \setminus \theta^+)}[\mathbf{a}]$) and by [Sh:g, Ch.I,1.5] there is $g \in \Pi \mathbf{a}$ such that: $h \in H \Rightarrow h > g \bmod J_{<\theta}[\mathbf{a}]$; hence for each $h \in H$ for some finite $\Theta(h) \subseteq \theta \cap \text{pcf}(\mathbf{a})$ we have

$$\{\sigma \in \mathbf{a} : h(\sigma) \geq g(\sigma)\} \subseteq \bigcup \{\mathbf{b}_\sigma[\mathbf{a}] : \sigma \in \Theta(h)\}.$$

Also for every $\sigma \in \text{pcf}(\mathbf{a})$ we can find $\alpha_\sigma(h)$ (for $\sigma \in \text{pcf}(\mathbf{a}), h \in H$) such that $h \upharpoonright \mathbf{b}_\sigma[\mathbf{a}] < f_{\alpha_\sigma(h)}^\sigma \upharpoonright \mathbf{b}_\sigma[\mathbf{a}]$. So $h < \max(\{g, f_{\alpha_\sigma(h)}^\sigma : \sigma \in \Theta(h)\})$. Let

$$G =: \left\{ \max\{g, f_{\alpha_1}^{\sigma_1}, \dots, f_{\alpha_n}^{\sigma_n}\} : n < \omega, \{\sigma_1, \dots, \sigma_n\} \subseteq \theta \cap \text{pcf}(\mathbf{a}) \right. \\ \left. \text{and } \alpha_1 < \sigma_1, \dots, \alpha_n < \sigma_n \right\}$$

it has cardinality $\leq \aleph_0 + \sup(\theta \cap \text{pcf}(\mathbf{a})) < \theta$ and $\theta = \text{cf}(\theta) = |H|$ and $\forall h \in H \exists g' \in G (h < g')$.

So for some $g^* \in G$ the set $\{h \in H : h < g^*\}$ has cardinality θ as required. $\square_{6.3}$

We comment to [Sh 410, 3.7] (which solve a problem from Gerlits Hajnal Szentmiklossy [GHS]).

6.4 Claim. 1) Suppose γ^* and $i^* = i(*)$ are ordinals and $\bar{\chi} = \langle \chi_i : i < i^* \rangle$ is a sequence of infinite cardinals; so of course

(*) we can find $n, \bar{w} = \langle w_\ell : \ell < n \rangle, \bar{\kappa} = \langle \kappa_\ell : \ell < n \rangle, \bar{\sigma} = \langle \sigma_\ell : \ell < n \rangle$ such that

(**) \bar{w} is a partition of i^* , $|w_\ell| = \kappa_\ell$, $\bigwedge_{i \in w_\ell} \chi_i \leq \sigma_\ell$, and

$$(\forall \chi < \sigma_\ell)(\exists^{\kappa_\ell} i)(i \in w_\ell \ \& \ \chi < \chi_i \leq \sigma_\ell)$$

and $\bar{\kappa}$ is strictly increasing, $\bar{\sigma}$ is strictly decreasing, in fact $\bar{w}, \bar{\kappa}, \bar{\sigma}$ are unique for our given $i^*, \bar{\chi}$.

Then the following are equivalent

(A) $_{\bar{\chi}, \gamma^*}$ there are $f_\alpha \in \prod_{i < i^*} \chi_i$ for $\alpha < \gamma^*$ satisfying $\alpha < \beta \Rightarrow (\exists i < i^*) [f_\alpha(i) < f_\beta(i)]$

(B) $_{\bar{\chi}, \gamma^*}$ for some $\ell < n$ we have:
 $2^{\kappa_\ell} \geq |\gamma^*|$ or for every regular $\mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+$ for some singular $\lambda^* \leq \sigma_\ell$ we have

(*) $\text{cf}(\lambda^*) \leq \kappa_\ell, \lambda^* > 2^{\kappa_\ell}$ and $\text{pp}^+(\lambda^*) > \mu_1$

2) If

$$\otimes (\forall \mu_1)(\mu_1 = \text{cf}(\mu_1) \leq |\gamma^*| \rightarrow (\exists \mu_2)[\mu_2 = \text{cf}(\mu_2) \ \& \ \mu_1 \leq \mu_2 \leq |\gamma^*| \ \& \ (\forall \alpha < \mu_2)|\alpha|^{\aleph_0} < \mu_2]),$$

then in part (1), $(B)_{\bar{\chi}, \gamma^*}$ the demand $(*)$ on λ^* can be replaced by

$$(*)' \text{ cf}(\lambda^*) \leq \kappa_\ell, \lambda^* > 2^{\kappa_\ell} \text{ and } (\forall \mu < \lambda^*)(\mu^{\kappa_\ell} < \lambda^*) \text{ and } (\lambda^*)^{\kappa_\ell} \geq \mu_1.$$

Now we call it $(B)'_{\bar{\chi}, \gamma^*}$ (so if \otimes then $(A)_{\bar{\chi}, \gamma^*} \Leftrightarrow (B)_{\bar{\chi}, \gamma^*} \Leftrightarrow (B)'_{\bar{\chi}, \gamma^*}$).

3) If \otimes from part (2) holds, then also $(A)_{\bar{\chi}, \gamma^*} \Leftrightarrow (B)''_{\bar{\chi}, \gamma^*} \Leftrightarrow (B)^+_{\bar{\chi}, \gamma^*}$ where

$$(B)''_{\bar{\chi}, \gamma^*} \quad |\gamma^*| \leq \max\{(\sigma_\ell)^{\kappa_\ell} : \ell < n\}$$

$(B)^+$ for some $\ell < n$ we have:

$$2^{\kappa_\ell} \geq |\gamma^*| \text{ or for every regular } \mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+ \text{ for some singular } \lambda^* \leq \sigma_\ell \text{ we have}$$

$$(*)^+ \text{ cf}(\lambda^*) \leq \kappa_\ell, \lambda^* > 2^{\kappa_\ell} \text{ and } (\forall \mu < \lambda^*)(\mu^{\kappa_\ell} < \lambda^*) \text{ and } \text{pp}^+(\lambda^*) > \mu_1.$$

4) In part (2), (3) instead \otimes we may let $\lambda_0 = \max\{2^{\kappa_\ell} : \ell < n\}$, $\lambda_1 = |\gamma^*|$, and demand

$$\oplus_{\lambda_0, \lambda_1} \text{ if } \lambda_0 < \mu \leq \lambda_1, \text{cf}(\mu) = \aleph_0 \text{ and } (\forall \lambda < \mu)(|\lambda|^{\aleph_0} < \mu) \text{ then } \text{pp}(\mu) =^+ \mu^{\aleph_0}.$$

Remark. 1) On $\oplus_{\lambda_0, \lambda_1}$ from 6.4(4), see [Sh 430], §1.

2) Note that we could in 6.4(1) demand

$$(\forall \chi < \sigma_\ell)(\exists^{\kappa_\ell} i)(i \in w_\ell \ \& \ \chi \leq \chi_1 < \sigma_\ell)$$

and can allow χ_i (hence σ_i) to be any ordinal, and even let κ_i be ordinal so the demand is $(\forall \alpha < \sigma_\ell)[\kappa_\ell = \text{otp}\{i \in w_\ell : \alpha \leq \chi_i < \sigma_\ell\}]$. This causes no serious change. [Saharon: re-read]

Proof. 1) $(B)_{\bar{\chi}, \gamma^*} \Rightarrow (A)_{\bar{\chi}, \gamma^*}$.

Let $\ell < n$ exemplifies $(B)_{\bar{\chi}, \gamma^*}$ so there are $f'_\alpha \in {}^{(\kappa_\ell)}(\sigma_\ell)$ for $\alpha < \gamma^*$ such that $\alpha < \beta < \gamma^* \Rightarrow (\exists j < \kappa_\ell)[f'_\alpha(j) > f'_\beta(j)]$.

[Why? If $2^{\kappa_\ell} \geq |\gamma^*|$ then by the use of [EK] and if $2^{\kappa_\ell} \leq \gamma^*$ prove it by induction

on γ^* : if $|\text{gamma}^*|$ is regular call it μ_1 and use the definition of $\text{pp}^+(\lambda^*)$ and if not manipulate previous examples.]

Let $h : w_\ell \rightarrow \kappa_\ell$ be such that:

$$(\forall j < \kappa_\ell)(\forall \sigma < \sigma_\ell)(\exists^{\kappa_\ell} i)[i \in w_\ell \ \& \ \sigma < \chi_i \leq \sigma_\ell \ \& \ h(i) = j].$$

Let $f_\alpha \in \prod_{i < i^*} \chi_i$ be: $f_\alpha(i) = f'_\alpha(h(i))$ if $i \in w_\ell$ & $f'_\alpha(h(i)) < \chi_i$ and $f_\alpha(i) = 0$

otherwise. So if $\alpha < \beta < \gamma^*$ then for some $j < \kappa_\ell$, $f'_\alpha(j) < f'_\beta(j) < \sigma_\ell$ so for some $i \in w_\ell$ we have: $h(i) = j$ & $\chi_i > f'_\beta(j)$. So $f_\alpha(i) \leq f'_\alpha(j) = f'_\beta(j) < f_\beta(i)$ as required.

$$(A)_{\bar{\chi}, \gamma^*} \Rightarrow (B)_{\bar{\chi}, \gamma^*}$$

Assume this fails, note that clearly $\gamma_1 < \gamma_2$ & $(A)_{\bar{\chi}, \gamma_2} \Rightarrow (A)_{\bar{\chi}, \gamma_1}$ and $\gamma_1 < \gamma_2$ & $(B)_{\bar{\chi}, \gamma_2} \Rightarrow (B)_{\bar{\chi}, \gamma_1}$. Without loss of generality γ^* is minimal (for our $\bar{\chi}$) for which the implication fails; so γ^* is minimal such that $(B)_{\bar{\chi}, \gamma^*}$ fails. Inspecting $(B)_{\bar{\chi}, \gamma^*}$, as n is finite, clearly γ^* is a regular cardinal, call it θ . By renaming we can assume that i^* is a cardinal. Now let $\langle f_\alpha : \alpha < \theta \rangle$ exemplifies $(A)_{\bar{\chi}, \theta}$; now apply 6.1 above with i^* , $\langle f_\alpha : \alpha < \theta \rangle$, D_θ^{cb} , the filter of cobounded subsets of θ and $\max_{\ell < n} (2^{\kappa_\ell})^+ = (2^{i^*})^+$ here standing for $\kappa, \langle \langle \beta_i^\alpha : i < \kappa \rangle : \alpha < \theta \rangle, D, \sigma$ there.

So we get $w, \langle \beta_i^* : i < i^* \rangle, B$ as there, and let $\mathfrak{a} = \{\text{cf}(\beta_i^*) : i \in i^* \setminus w\}$, so $\mathfrak{a} \subseteq \text{Reg} \cap (\max_{\ell < n} \sigma_\ell) \setminus (2^{i^*})^+$ (see clause (a) of 6.1) and $|\mathfrak{a}| \leq |i^*|$. Now if $\theta \leq$

$\max \text{pcf}(\mathfrak{a})$ then for some $\ell, \theta \leq \max \text{pcf}\{\text{cf}(\beta_i^*) : i \in w_\ell \setminus w\}$, and so $(B)_{\bar{\chi}, \theta}$ does not fail, contradicting an earlier assumption. So $\theta > \max \text{pcf}(\mathfrak{a})$, so there

is a cofinality $H \subseteq \prod_{i \in (i^* \setminus w)} \beta_i^*$ of cardinality $< \theta$, so there are $h_\alpha \in H$ such that

$f_\alpha \upharpoonright (i^* \setminus w) < h_\alpha$ but $|B| = \theta > |H|$ (by the choice of D_θ^{cb} as D) so for some $h^* \in H$ the set $B_1 = \{\alpha \in B : h_\alpha = h^*\}$ is unbounded in θ . By clause (c) of the conclusion of 6.1 for some $\alpha \in B$ we have $i \in i^* \setminus w \Rightarrow h^*(i) < f_\alpha(i)$. Choose $\beta \in B_1 \setminus (\alpha + 1)$, so $\alpha < \beta$ are in B , hence $f_\alpha \upharpoonright w = f_\beta \upharpoonright w$ and $i \in i^* \setminus w \Rightarrow f_\beta(i) < h^*(i) < f_\alpha(i)$, so $\bigwedge_i f_\beta(i) \leq f_\alpha(i)$, a contradiction to “ $\langle f_\gamma : \gamma < \theta \rangle$ exemplifies $(A)_{\bar{\chi}, \theta}$ ”.

2), 3) Clearly $(B)_{\bar{\chi}, \gamma^*}^+ \Rightarrow (B)_{\bar{\chi}, \gamma^*} \Rightarrow (B)_{\bar{\chi}, \gamma^*}''$ by checking. So we should just prove the following two implications:

$$(B)_{\bar{\chi}, \gamma^*}'' \Rightarrow (B)_{\bar{\chi}, \gamma^*}'$$

We can assume $|\gamma^*| > 2^{\kappa_\ell}$ for $\ell < n$ (otherwise the conclusion is trivial). We know by $(B)_{\bar{\chi}, \gamma^*}''$ that for some $\ell, (\sigma_\ell)^{\kappa_\ell} \geq |\gamma^*|$. Let us check now $(B)_{\bar{\chi}, \gamma^*}'$ so let a regular $\mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+$ be given. So $(\sigma_\ell)^{\kappa_\ell} \geq |\gamma^*| \geq \mu_1$ hence $\lambda =: \min\{\lambda : \lambda^{\kappa_\ell} \geq \mu_1\}$ is

$\leq \sigma_\ell$ but is $> 2^{\kappa_\ell}$, hence it is singular, $\text{cf}(\lambda) \leq \kappa_\ell$ and $(\forall \alpha < \lambda)(|\alpha|^{\kappa_\ell} < \lambda)$, i.e. as required in $(*)'$ which means that $(B)'_{\bar{\chi}, \gamma^*}$ holds.

$$(B)'_{\bar{\chi}, \gamma^*} \Rightarrow (B)^+_{\bar{\chi}, \gamma^*}$$

Again we can assume $|\gamma^*| > 2^{\kappa_\ell}$ for $\ell < n$. Let us check $(B)_{\bar{\chi}, \gamma^*}$, so let a regular $\mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+$ be given. As we are assuming \otimes from 6.4(2) there is $\mu_2 \in \text{Reg} \cap (\gamma^* + 1) \setminus \mu_1$ such that $(\forall \alpha < \mu_2)(|\alpha|^{\aleph_0} < \mu_2)$. Apply $(B)'_{\bar{\chi}, \gamma^*}$ for μ_2 and get λ_2^* as in $(*)$ for μ_2 instead of μ_1 . Clearly $(\lambda_2^*)^{\kappa_\ell} \geq \mu_2$ and let $\lambda^* = \min\{\lambda : \lambda^{\kappa_\ell} \geq \mu_2\}$, so $\lambda^* \leq \lambda_2^* \leq \sigma_\ell$, $\lambda^* > 2^{\kappa_\ell}$ and $(\forall \mu < \lambda^*)[\mu^{\kappa_\ell} < \lambda^*]$, and clearly $\text{cf}(\lambda^*) \leq \kappa_\ell$, so $(\lambda^*)^{\kappa_\ell} = (\lambda^*)^{\text{cf}(\lambda^*)}$.

By the choice of μ_2 necessarily $\text{cf}(\lambda^*) > \aleph_0$ (otherwise $\mu_2 \leq (\lambda^*)^{\kappa_\ell} = (\lambda^*)^{\aleph_0} < \mu_2$). By [Sh:g], (see 6.5 below), $\text{pp}(\lambda^*) =^+ (\lambda^*)^{\text{cf}(\lambda^*)} = (\lambda^*)^{\kappa_\ell}$ as required in $(*)^+$.

4) The only place we use the assumption \otimes in the proof of part (2), (3) was in choosing μ_2 in the proof of $(B)'_{\bar{\chi}, \gamma^*} \Rightarrow (B)^+_{\bar{\chi}, \gamma^*}$ and the use of its property is to show $\text{cf}(\lambda^*) > \aleph_0$ (to be able to use [Sh:g, Ch.VIII], §1) but we can use instead $\oplus_{\lambda_0, \lambda_1}$. □_{6.4}

Remember that by [Sh:g]:

6.5 Observation. If μ is singular, $\text{cf}(\mu) > \aleph_0$ and $\alpha < \mu \Rightarrow |\alpha|^{\text{cf}(\mu)} < \mu$ then $\mu^{\text{cf}(\mu)} =^+ \text{pp}(\mu)$.

Proof. By [Sh 430, 3.5], [Sh:g, CH.VIII,1.8], [Sh:g, Ch.II,5.6]. □_{6.5}

6.6 Definition. 1) For $F \subseteq {}^\delta \text{Ord}$, we say F is free ^{ℓ} when we can find $\zeta_f < \delta$ for $f \in F$ such that:

(a) if $\ell = 1$ then

$$f \neq g \in F \ \& \ \zeta = \max\{\zeta_f, \zeta_g\} \Rightarrow f \upharpoonright \zeta \neq g \upharpoonright \zeta$$

(b) if $\ell = 2$ then

$$f \neq g \in F \ \& \ \delta > \zeta \geq \max\{\zeta_f, \zeta_g\} \Rightarrow f(\zeta) \neq g(\zeta)$$

(c) if $\ell = 3$ then

$$f \neq g \in F \ \& \ \delta > \zeta > \varepsilon = \max\{\zeta_f, \zeta_g\} \ \& \ f(\varepsilon) \leq g(\varepsilon) \Rightarrow f(\zeta) \leq g(\zeta)$$

(d) if $\ell = 4$ then for $f, g \in F, f \leq g$ (i.e.

$$\zeta < \delta \Rightarrow f(\zeta) \leq g(\zeta)) \text{ or } g \leq f$$

(e) if $\ell = 5$ then for some ζ and h we have

$$f \in F \Rightarrow f \upharpoonright \zeta = h$$

$\{f(\zeta) : f \in F\}$ is with no repetition.

2) Let $\text{free}^{\ell,m}$ means free^ℓ and free^m , similarly $\text{free}^{k,\ell,m}$, we may write $\text{free}^{\{\ell,m\}}$. For J an ideal on δ we write $J\text{-free}^\ell$ if $\zeta_f < \delta$ is replaced by $s_f \in J$, that is there are $s_f \in J$ for $f \in F$ such that:

(a) if $\ell = 1$ then

$$f \neq g \in F \ \& \ s = s_f \cup s_g \Rightarrow f \upharpoonright s \neq g \upharpoonright s$$

(b) if $\ell = 2$ then

$$f \neq g \in F \ \& \ \zeta \in \delta \setminus s_f \setminus s_g \Rightarrow f(\zeta) \neq g(\zeta)$$

(c) if $\ell = 3$ then

$$f \neq g \in F \ \& \ \{\xi, \zeta\} \subseteq \delta \setminus s_f \setminus s_g \Rightarrow [f(\xi) \leq g(\xi) \equiv f(\zeta) \leq g(\zeta)]$$

(d) if $\ell = 4$ then

$$f, g \in F \Rightarrow [f \leq g \vee g \leq f].$$

6.7 Definition. For $F \subseteq {}^\delta \text{Ord}$:

- 1) We say F is $\mu\text{-free}^x$ if every $F' \in [F]^\mu$ is free^x .
- 2) We say F is $(\mu, \kappa)\text{-free}^x$ if every $F' \in [F]^\mu$ there is $F'' \in [F']^\kappa$ which is free^x .

6.8 Fact: 1) “ F is free^ℓ implies F is free^m ” when (ℓ, m) is one of $(2,1)$, $(4,3)$, $(5,1)$.
2) Similarly for $\mu\text{-free}^x$ and $(\mu, \kappa)\text{-free}^x$.

Proof. Straight.

On 6.9 see [Sh:g, Ch.II],§1, [Sh:g, Ch.II],§3, [Sh 282], [Sh:g, Ch.II,4.10], Shelah Zapletal [ShZa 561].

6.9 Claim. 1) If $J_\delta^{\text{bd}} \subseteq J, J$ an ideal on $\delta, \prod_{i < \delta} \lambda_i / J$ is λ^+ -directed, $\langle \lambda_i : i < \delta \rangle$ an increasing sequence of regulars $> \delta$ with limit $\mu, \mu < \lambda = \text{cf}(\lambda)$, then there are regulars $\lambda'_i < \lambda_i$ with $\mu = \text{tlim}_J \langle \lambda'_i : i < \delta \rangle$ and $f_\alpha \in \prod_{i < \delta} \lambda'_i$ for $\alpha < \lambda$ such that $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$ -increasing cofinal in $\prod_{i < \delta} \lambda'_i$ and $\{f_\alpha : \alpha < \lambda\}$ is $\mu^+ - J$ -free^{1,2,3}.

2) Assume \mathfrak{a} is a set of regular $> |\mathfrak{a}|$ with no last element, J an ideal on \mathfrak{a} extending $J_\mathfrak{a}^{\text{bd}}$ and $\mathfrak{c} = \{\theta \in \mathfrak{a} : \theta > \max \text{pcf}(\theta \cap \mathfrak{a})\}$ and $\lambda = \max \text{pcf}(\mathfrak{a})$. Then there is $\langle f_\alpha : \alpha < \lambda \rangle$ cofinal in $\Pi \mathfrak{a}, <_J$ -increasing, such that:

(*) if $\theta \in \mathfrak{c}$ then $\{f_\alpha \upharpoonright (\theta \cap \mathfrak{a}) : \alpha < \lambda\}$ has cardinality $< \theta$.

3) If $\mu > \mu_0 \geq \kappa \geq \text{cf}(\mu), \lambda = \mu^+$ (or just $\mu < \lambda = \text{cf}(\lambda) < \text{pp}_\kappa^+(\mu)$) then for some $\mathfrak{a} \subseteq (\mu_0, \mu) \cap \text{Reg}$ we have $|\mathfrak{a}| \leq \kappa, [\theta \in \mathfrak{a} \Rightarrow \max \text{pcf}(\theta \cap \mathfrak{a}) < \theta]$ and $\lambda = \max \text{pcf}(\mathfrak{a})$ (if $[\alpha < \mu \Rightarrow |\alpha|^\kappa < \mu]$ we can have $\mu = \text{sup}(\mathfrak{a}), \text{otp}(\mathfrak{a}) = \text{cf}(\mu)$ (so part (2) is not empty)).

4) If J is an ideal on $\mathfrak{a}, \langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$ -increasing $<_J$ -cofinal in $\Pi \mathfrak{a}$ and J is generated by $< \min(\mathfrak{a})$ sets (as an ideal) then for every $A \in [\lambda]^\lambda$ for some $\mathfrak{d} \in J$ we have: for every $g \in \Pi \mathfrak{a}$ for λ ordinals $\alpha \in A, g \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}) < f_\alpha \upharpoonright (\mathfrak{a} \setminus \mathfrak{d})$. Hence \bar{f} is $(\lambda, \min(\mathfrak{a})) - J$ -free^{2,3}.

5) Assuming $|\mathfrak{a}| < \min(\mathfrak{a}), \lambda = \text{tcf}(\Pi \mathfrak{a}, <_{J_\mathfrak{a}^{\text{bd}}}), \mathfrak{c} = \{\theta \in \mathfrak{a} : \theta > \max \text{pcf}(\theta \cap \mathfrak{a})\}$ is unbounded in \mathfrak{a} and $\langle f_\alpha : \alpha < \lambda \rangle$ is as in part (2). Then not only for each $\theta \in \mathfrak{a}$ the family $\{f_\alpha : \alpha < \lambda\}$ is (λ, θ) -free^{2,3}, but for any $A \in [\lambda]^\lambda$ for every large enough $\theta \in \mathfrak{c}$ there is $B \in [A]^\theta$ such that $\langle f_\alpha \upharpoonright (\theta \cap \mathfrak{a}) : \alpha \in B \rangle$ is constant and $\langle f_\alpha \upharpoonright (\mathfrak{a} \setminus \theta) : \alpha \in B \rangle$ is strictly increasing.

6) Assume $\lambda = \text{tcf}(\Pi \mathfrak{a}, <_J), J_\mathfrak{a}^{\text{bd}} \subseteq J, \lambda > \mu = \text{sup}(\mathfrak{a})$ and $\langle f_\alpha : \alpha \in \lambda \rangle$ is $<_J$ -increasing and $<_J$ -cofinal in $\Pi \mathfrak{a}$. Then

(a) if $\kappa \leq \mu^+$ and⁴ $\{\delta < \lambda : \text{cf}(\delta) < \kappa\} \in I[\lambda]$, then for some $A \in [\lambda]^\lambda$ we have $\{f_\alpha : \alpha \in A\}$ is $(< \kappa)$ -free^{2,3}, also we can find $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ as above with $A = \lambda$, such that \bar{f} is ^bcontinuous (see [Sh:g, Ch.I, §3])

(b) if $\kappa = \text{cf}(\kappa) < \mu^+$ and $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ belongs to $I[\lambda]$ then \bar{f} is (κ, κ) -free^{2,3}.

7) Assume $\langle \mu_i : i \leq \kappa \rangle$ is an increasing continuous sequence of singulars $> \kappa, \kappa = \text{cf}(\kappa) > \aleph_0, \theta \in \text{Reg} \cap \mu_0 \setminus \kappa^+$ and⁵ $\{\delta < \mu_i^+ : \text{cf}(\delta) = \theta\} \in I[\mu_i^+]$ for $i = \kappa$. Then some $\bar{f} = \langle f_\alpha : \alpha < \mu_\kappa^+ \rangle$ is $<_{J_\mathfrak{C}^{\text{bd}}}$ -increasing and cofinal in $\prod_{i < \kappa} \mu_i^+$ and is

⁴note: if λ is a successor of regular and $\lambda > \kappa^+$ then this holds

⁵note: if μ_i is regular $> \theta$ then this holds

(θ, θ) -free^{2,3}. If we demand in addition $\{\delta < \mu_\kappa^+ : \text{cf}(\delta) \leq \theta\} \in I[\mu_\kappa^+]$ then \bar{f} is $< \theta^+$ -free^{2,3}.

Proof. 1) By [Sh:g, Ch.II,§1].

2) By [Sh:g, Ch.II,3.5].

3) By [Sh:g, Ch.II,§3].

4) Can prove as in [Sh 282]. Or as in 6.1, as below.

Let $A \in [\lambda]^\lambda$ and let $\{\mathfrak{d}_\zeta : \zeta < \zeta^*\} \subseteq J$ be a family generating J closed under finite union such that $\zeta^* < \lambda_0$. We shall prove that for some $\zeta < \zeta^*$ we have $(\forall g \in \Pi\mathfrak{a})(\exists^\lambda \alpha \in A)(g \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}_\zeta) < f_\alpha \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}_\zeta))$. If not, for each $\zeta < \zeta^*$ some $g_\zeta \in \Pi\mathfrak{a}$ and $\alpha_\zeta < \lambda$ exemplifies it, i.e. $\alpha \in A \setminus \alpha_\zeta \Rightarrow \neg(g \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}_\zeta) < f_\alpha \upharpoonright (\mathfrak{a} \setminus \mathfrak{d}_\zeta))$. So defined the function $g \in \Pi\mathfrak{a}$ by $g(\theta) =: \sup[\{g_\zeta(\theta) : \zeta < \zeta^*\} \cup \{f_{\alpha^*}(\theta) + 1\}]$ is well defined where we let $\alpha^* = \sup\{\alpha_\zeta : \zeta < \zeta^*\} < \lambda$. Now $\{f_\alpha : \alpha < \lambda\}$ is $<_J$ -increasing and $<_J$ -cofinal in $\Pi\mathfrak{a}$ so for some $\alpha \in A \setminus (\alpha^* + 1)$ we have $g <_J f_\alpha$, so for some $\zeta < \zeta^*$ we have $\{\theta \in \mathfrak{a} : \neg(g(\theta) < f_\alpha(\theta))\} \subseteq \mathfrak{d}_\zeta$, so for \mathfrak{d}_ζ , the pair (g_ζ, α_ζ) is not as required, a contradiction.

5), 6, 7) Left to the reader. □_{6.9}

6.10 Definition. We say $(\Pi\mathfrak{a}, <_J)$ is x -free ^{y} if there is a $<_J$ -increasing $<_J$ -cofinal $\langle f_\alpha : \alpha < \lambda \rangle$ from $\Pi\mathfrak{a}$ which is x -free ^{y} .

6.11 Fact: If $y \in \{1, 2, 3, \{2, 3\}\}$, $x \in \{\mu - J, (\mu, \theta) - J\}$ and $(\Pi\mathfrak{a}, <_J)$ is x -free ^{y} and $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$ -increasing $<_J$ -cofinal in $\Pi\mathfrak{a}$ then for some $A \in [\lambda]^\lambda$ the set $\{f_\alpha : \alpha \in A\}$ is x -free ^{y} .

Proof. Straight.

6.12 Question: Let $\mu > \kappa \geq \text{cf}(\mu)$. For how many $\mathfrak{a} \subseteq \text{Reg} \cap \mu$ such that $\mu = \sup(\mathfrak{a}), J_\mathfrak{a}^{\text{bd}} \subseteq J, J$ an ideal on $\mathfrak{a}, \lambda = \text{tcf}(\Pi\mathfrak{a}, <_J)$, is $(\Pi\mathfrak{a}, <_J)$ not μ -free?

The proof of [Sh:g, Ch.IX,3.5] has a gap (in the reference to [Sh:g, Ch.IX,3.3A]). What we know is only

6.13 Lemma. 1) Assume $\lambda > \theta > \text{cf}(\lambda) \geq \sigma = \text{cf}(\sigma) > \aleph_0$. Then $\text{cov}(\lambda, \lambda, \theta, \sigma) =^+ \sup(\cup\{\text{pcf}_{\Gamma(\theta, \sigma), J_\mathfrak{a}^{\text{bd}}}(\mathfrak{a}) : \mathfrak{a} \subseteq \text{Reg} \cap \lambda, \lambda = \sup(\mathfrak{a}), |\mathfrak{a}| < \min(\mathfrak{a})\})$ where

$\text{pcf}_{\Gamma(\theta, \sigma), J}(\mathfrak{a}) = \{\text{tcf}(\Pi\mathfrak{a}, <_I) : I \text{ an ideal on } \mathfrak{a} \text{ extending } J, \text{tcf}(\Pi\mathfrak{a}, <_I) \text{ well defined, } I \text{ is } \sigma\text{-complete and for some } \mathfrak{b} \in I, |\mathfrak{a} \setminus \mathfrak{b}| < \theta\}$

(the $=^+$ means that if the left side is regular then the supremum in the right side is obtained).

2) If in addition $(\forall \mu < \lambda)(\text{cov}(\mu, \theta, \theta, \sigma) < \lambda)$, (θ, σ) regular) then

$$\text{cov}(\lambda, \lambda, \theta, \sigma) =^+ \text{pp}_{\Gamma(\theta, \sigma)}(\lambda).$$

3) So for $\mathcal{Y} = \mathcal{Y}_\mu$, $Eq = Eq_\mu$ be as in [Sh 420, §3-§5], $\text{cf}(\mu) = \sigma > \aleph_0$ for simplicity, $\mu > \theta > \text{cf}(\mu)$.

If $\mathfrak{a} \subseteq \text{Reg} \setminus \mu$, $|\mathfrak{a}| < \mu$, $|\mathfrak{a}| < \min(\mathfrak{a})$, λ inaccessible, $J = J_{\mathfrak{a}}^{\text{bd}}$, $\lambda = \sup \text{pcf}_{\Gamma(\theta, \sigma)}(\mathfrak{a})$, $J^{\text{bd}}(\mathfrak{a})$, then we can find $e \in Eq$, $\bar{\lambda} = \langle \lambda_x : x \in \mathcal{Y}_\mu/e \rangle$ and $D \in \text{FIL}(\mathcal{Y}_\mu)$ such that:

$$\lambda = \text{tcf}(\Pi \bar{\lambda}/D)$$

$$\lim_D(\bar{\lambda}) = \mu$$

$$\lambda_x = \text{cf}(\lambda_x).$$

Proof. As in [Sh 410], §1 (replacing normal by σ -complete) or make the following changes in the proof of [Sh:g, Ch.IX, 3.5]: $\|N_k^x\| = \mu_k, \mu_k + 1 \subseteq N_k^x$. [Saharon: write more?]

6.14 Claim. 1) Assume

- (i) λ is inaccessible
- (ii) $|\mathfrak{a}|^+ < \min(\mathfrak{a})$,
- (iii) $\mu =: \sup(\mathfrak{a}) < \lambda$
- (iv) $R \subseteq \lambda \cap \text{Reg} \setminus \mu$, $|R| = \lambda$,
- (v) for $\tau \in R$, $\mathfrak{b}_\tau \subseteq \mathfrak{a}$, $\sup(\mathfrak{b}_\tau) = \mu$, J_τ an ideal on \mathfrak{b}_τ including $J_{\mathfrak{b}_\tau}^{\text{bd}}$, $\tau = \text{tcf}(\Pi \mathfrak{b}_\tau, <_{J_\tau})$
- (vi) $\lambda \notin \text{pcf}(\mathfrak{a})$

Then for some $\langle \lambda_\theta : \theta \in \mathfrak{a} \rangle$ we have

- (a) $\lambda_\theta = \text{cf}(\lambda_\theta) < \theta$, $\lambda_\theta > |\mathfrak{a}|$
- (b) $\lim_{J_\tau} \langle \lambda_\theta : \theta \in \mathfrak{b}_\theta \rangle = \mu$
- (c) $\lambda = \text{tcf}(\prod_{\theta \in \mathfrak{a}} \lambda_\theta, <_{J_{< \lambda}[\mathfrak{a}]})$ so $\lambda = \max \text{pcf}\{\lambda_\theta : \theta \in \mathfrak{a}\}$
- (d) $R' =: \{\min \text{pcf}_{J_\tau}(\prod_{\theta \in \mathfrak{b}_\tau} \lambda_\theta) : \tau \in R\}$ is⁶ unbounded in λ .

⁶recall $\text{pcf}_J(\Pi\{\lambda_x : x \in X\})$, J an ideal on X is $\{\text{tcf}(\Pi \lambda_*/D) : D \text{ an ultrafilter on } X \text{ disjoint to the ideal } J\}$.

So $\mathfrak{a}' =: \{\lambda_\theta : \theta \in \mathfrak{a}\}$, R', λ, μ satisfy clauses (i), (ii), (iv), (v) and

- (iii)⁻ $|\mathfrak{a}'|^+ \leq \min(\mathfrak{a}')$
- (vi)^{*} $\max \text{pcf}(\mathfrak{a}') = \lambda$.

2) Assume (i), (ii), (iii)⁻, (iv), (v), (vi)^{*} are satisfied by $\mathfrak{a}, R, \mathfrak{b}_\tau, J_\tau$ (for $\tau \in R$). Then for some $f_\tau \in \text{Pb}_\tau$ for $\tau \in R$ we have:

(*) for every $g \in \text{Pb}_\tau$ for some τ we have $g \upharpoonright \mathfrak{b}_\tau <_{J_\tau} f_\tau$.

Proof. 1) By [Sh:g, Ch.II,1.5A] we can find $\langle \lambda_\theta : \theta \in \mathfrak{a} \rangle$ satisfying (a), (b), (c). If (d) fails, choose for each $\tau \in R, \mathfrak{b}'_\tau \in J_\tau^+$ such that $\chi_\tau = \text{tcf}(\prod_{\theta \in \mathfrak{b}'_\tau} \lambda_\theta, <_{J_\tau \upharpoonright \mathfrak{b}'_\tau})$

is well defined and equal to $\min \text{pcf}_{J_\tau}(\prod_{\theta \in \mathfrak{b}'_\tau} \lambda_\theta, <_{J_\tau})$. So $\{\chi_\tau : \tau \in R\}$ is bounded

in λ hence for some $\chi, R' = \{\tau \in R : \chi_\tau = \chi\}$ is unbounded in λ . Hence we can find $\zeta^* < |\mathfrak{a}|^+$ and $\tau_\zeta \in R'$ for $\zeta < \zeta^*$ such that $\lambda \leq \max \text{pcf}\{\tau_\zeta : \zeta < \zeta^*\}$ (why? choose by induction on $\zeta < |\mathfrak{a}|^+, \tau_\zeta \in R', \tau_\zeta > \max \text{pcf}\{\tau_\varepsilon : \varepsilon < \zeta\}$ and use localization). Let D_ζ be an ultrafilter on \mathfrak{a} such that $\mathfrak{b}'_{\tau_\zeta} \in D_\zeta, J_{\tau_\zeta} \cap D_\zeta = \emptyset$, so $\text{tcf}(\text{Pb}_\tau, <_{D_\zeta}) = \tau_\zeta$, and let E be an ultrafilter on $\{\tau_\zeta : \zeta < \zeta^*\}$ such that $\text{tcf}(\prod_{\zeta} \tau_\zeta, <_E) \geq \lambda$. Let $D = \{\mathfrak{c} \subseteq \mathfrak{a} : \{\zeta : \mathfrak{c} \in D_\zeta\} \in E\}$. By [Sh:g, Ch.I,1.10,1.11]

we conclude $\text{tcf}(\text{Pb}_\tau, <_D) \geq \lambda$ hence $D \cap J_{<\lambda}[\mathfrak{a}] = \emptyset$. By clause (c) clearly $\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D$

has true cofinality λ . Also letting $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \chi \notin \text{pcf}\{\lambda_\theta : \theta \in \mathfrak{b}\}\}$, it is an ideal such that for any ultrafilter D' on \mathfrak{a} we have $\chi = \text{tcf}(\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D') = \chi \Leftrightarrow D \cap J = \emptyset$,

hence in particular $\zeta < \zeta^* \Rightarrow D_\zeta \cap J = \emptyset$ hence by the choice of D we have $D \cap J = \emptyset$ hence $\text{tcf}(\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D) = \chi < \lambda$ contradicting an earlier sentence.

2) Let $\langle f_\alpha^* : \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\mathfrak{a}]}$ -increasing and cofinal. Let $f_\alpha = f_\alpha^* \upharpoonright \mathfrak{b}_\alpha$. □_{6.14}

Concerning [Sh 430, 3.1] we comment

6.15 Theorem. 1) Assume $\lambda > \theta \geq \kappa > \aleph_0$ are regular and

- (*) _{θ, κ} if $\mathfrak{a} \subseteq \text{Reg} \cap \lambda \setminus \theta$ and $|\mathfrak{a}| < \theta$ then there are $\zeta^* < \lambda$ and $\mathfrak{b}_\zeta \in J_{<\lambda}[\mathfrak{a}]$ for $\zeta < \zeta^*$ such that for every $\mathfrak{b} \in [\mathfrak{a}]^{<\kappa}$ for some $\zeta < \zeta^*, \mathfrak{b} \subseteq \mathfrak{b}_\zeta$.

Then the following conditions are equivalent:

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- (A) = $(A)_{\lambda, \theta, \kappa}$ for every $\mu < \lambda$ we have $\text{cov}(\mu, \theta, \kappa, 2) < \lambda$
- (B) = $(B)_{\lambda, \theta, \kappa}$ if $\mu < \lambda$ and $a_\alpha \in [\mu]^{<\kappa}$ for $\alpha < \lambda$ then for some $W \subseteq \lambda$ of cardinality λ we have $|\bigcup_{\alpha \in W} a_\alpha| < \theta$
- (C) = $(C)_{\lambda, \theta, \kappa}$ if a_α is a set of cardinality $< \kappa$ for $\alpha < \lambda$ and $W_0 \subseteq \{\delta < \lambda : \text{cf}(\delta) \geq \kappa\}$ then for some stationary $W \subseteq W_0$ and set b of cardinality $< \theta$ we have $\langle a_\alpha \setminus b : \alpha \in W \rangle$ is a sequence of pairwise disjoint sets.

2) If $\lambda > \theta_1 > \theta_2 \geq \kappa > \aleph_0$ where λ, κ are regular, $(A)_{\lambda, \theta, \kappa} \Leftrightarrow (B)_{\lambda, \theta_1, \kappa}$ and $\text{cov}(\theta_1, \theta_2, \kappa, 2) < \lambda$ then $(A)_{\lambda, \theta_2, \kappa} \Leftrightarrow (B)_{\lambda, \theta_2, \kappa}$ (so if for some $\theta_1, (*_{\theta_1, \kappa}, \lambda > \theta_1 > \theta_2 = \text{cf}(\theta) \geq \kappa$ and $\text{cov}(\theta_1, \theta, \kappa, 2) < \lambda$ then the conclusion holds).

Proof. 1) Read the proof of [Sh 430, 3.1] (which was written in a way appropriate to this generalization), but defining the $M_n, \langle N_\zeta^n : \zeta < \theta \rangle$, we omit clause (d), that is, $N_\zeta^n \in \mathfrak{A}_{\delta(*)}$ and instead demand

- (d)' for each n we can find $\mathcal{P}_n \subseteq [\theta]^{<\theta}$ such that $(\forall a \in [\theta]^{<\kappa})(\exists b \in \mathcal{P}_n)(a \subseteq b)$ and $\langle N_b^n : b \in \mathcal{P}_n \rangle$ such that $N_b^n \prec \mathfrak{B}_{\delta(*)}, N_\zeta^n = \bigcup \{N_b^n : b \in \mathcal{P}_n, b \subseteq \zeta\}$ and $b_1 \subseteq b_2 \Rightarrow N_{b_1}^n \prec N_{b_2}^n$ and $f_n \upharpoonright (\text{Reg} \cap N_b^n) \in \mathfrak{B}$ for $b \in \mathcal{P}_n$.

2) Left to the reader.

6.16 Question: Can we in [Sh 430, 4.2](1) weaken clause (β) in the conclusion to “ $\lambda_x > \mu_0$ for D -almost all $x \in \mathcal{Y}/e$ ” then we can weaken the hypothesis [Sh 420, 6.1C] (was stated in [Sh 430], earlier version clear).

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