$Bext^2(G,T)$ CAN BE NONTRIVIAL, EVEN ASSUMING GCH

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Abstract. Using the consistency of some large cardinals we produce a model of Set Theory in which the generalized continuum hypothesis holds and for some torsion-free abelian group G of cardinality $\aleph_{\omega+1}$ and for some torsion group T

$$Bext^2(G,T) \neq 0.$$

Hence G.C.H. is not sufficient for getting the results of [10].

1. Introduction

All groups in this paper are abelian groups. For basic terminology about abelian groups in general we refer the reader to [9]. For terminology concerning Butler groups see [2, 1, 3, 10, 8]. It is commonly agreed that the three major questions concerning the infinite rank Butler groups are:

(1) Are B_1-groups necessarily B_2-groups?
(2) Does $Bext^2(G,T) = 0$ hold for all torsion-free groups G and torsion groups T?
(3) Which pure subgroups of B_2-groups are again B_2-groups? In particular: is a balanced subgroup of a B_2-group a B_2-group?

In [2] it is shown that the answer to all these questions is "Yes" for countable groups G. In the series of papers [1, 4, 3] it was shown that under the continuum hypothesis the answer is "Yes" to all three questions for groups G of cardinality $\leq \aleph_\omega$. In [5] it is shown that the answer to question 2 is "No" if the continuum hypothesis fails. In a more recent paper [10] it is shown that in the constructible universe, L the answer is "Yes" to all three questions for *arbitrary* groups G. Actually [10] used only the generalized continuum hypothesis and that the combinatorial principle $\square_\kappa$ holds for every singular cardinal κ whose cofinality is $\aleph_0$. Is the use made in [10] of the additional combinatorial principle really needed or does the affirmitive answer to our three questions follow simply from G.C.H.? Let us mention that a key tool used in [3, 10] was the representation of an arbitrary torsion-free group as the union of a chain of subgroups which are countable unions of balanced subgroups. In [7] it is shown that such a representation is equivalent to a weak version of $\square_\kappa$.

In this paper we show that at least for getting an affirmitive answer to questions 2 and 3, one needs some extra set theoretic assumptions in addition to G.C.H. We do it by producing a model of Set Theory, satisfying G.C.H., in which for some torsion-free G of cardinality $\aleph_{\omega+1}$ and some torsion T, $Bext^2(G,T) \neq 0$. Also in the

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same model there will be a balanced subgroup of a completely decomposable group which is not a B_2-group. Hence the answer to question 3 in this model is "No". The construction of the model requires the consistency of some large cardinals, which can not be avoided since getting a model in which $\square_\kappa$ fails for some singular κ requires assumptions stronger than the consistency of Set Theory. Let us stress that the status of question 1 is not known and it is possible (though unlikely) that the implication "every B_1-group is a B_2-group" is a theorem of Set Theory.

Since this paper is aimed at a mixed audience of set theorists and abelian group theorists it is divided into two sections with very different prerequisites. In the next section we describe the construction of the model of Set Theory with certain properties to be listed below. In the following section we shall describe how to use the listed properties to get a group G which will be the counterexample to $Bext^2(G,T) = 0$. A reader who is not familiar with standard set theoretical techniques, like forcing, can skip the set theoretic section and simply assume the properties of the model listed below. We do assume some basic Set Theory at the level introduced by [6].

We now describe the properties of the model which will be used in the construction of the counterexample to questions 2 and 3. The model will naturally satisfy G.C.H. Hence by standard cardinal arithmetic $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$. Therefore we can enumerate all the ω-sequences from $\aleph_\omega$ in a sequence of order type $\aleph_{\omega+1}$. Let $\langle f_\alpha | \alpha < \aleph_{\omega+1}\rangle$ be this enumeration. Let F_α be the range of f_α. The important property of the model is the following:

For some stationary subset S of $\aleph_{\omega+1}$ such that every point of S has cofinality $\aleph_1$, and for some choice of a cofinal set C_β in β of order type ω_1, for every $\beta \in S$ and for some fixed countable ordinal δ we have:

(1)
$$\bigcup_{\alpha \in D} F_\alpha$$
has order type δ for every $D \subseteq C_\beta$ which is cofinal subset of C_β and for every $\beta \in S$. In particular for $D = C_\beta$
$$E_\beta = \bigcup_{\alpha \in C_\beta} F_\alpha$$
has order type δ.

(2) *If $\beta \neq \gamma$ both in S, then $E_\beta \cap E_\gamma$ has order type less than δ.*

(3) *δ is an indecomposable ordinal, namely δ can not be represented as a finite sum of smaller ordinals. Or equivalently, δ is not the finite union of sets of ordinals of order type less than δ.*

Denote the conjunction of all the properties above by (*). The main theorem of Section 1 is

Theorem 1. *Assume the consistency of a supercompact cardinal. Then there is a model of Set Theory in which (*) holds. The model also satisfies the Generalized Continuum Hypothesis.*

The construction of the model is very close to the construction in [11]. The main tool that will be used to get in Section 3 an example of a group G satisfying $Bext^2(G,T) \neq 0$ is the notion of $\aleph_0$-prebalancedness (see [8]). We are rephrasing the original definition in a form which is clearly equivalent to the original definition.

Definition 1. *Let G be a pure subgroup of the group H. G is said to be $\aleph_0$-prebalanced in H if for every element $h \in H - G$ there are countably many elements $g_0, g_1, \ldots$ of G such that for every element g of G the type (in H) of $h-g$ is bounded by the the union of finitely many types of the form $lh - g_i$ for some natural number l. More explicitly for some $n, l \in \omega$*

$$\boldsymbol{t}(h-g) \leq \boldsymbol{t}(lh-g_0) \cup \ldots \cup \boldsymbol{t}(lh-g_n).$$

Also the group G is said to admit an $\aleph_0$-prebalanced chain if G can be represented as a continuous increasing union of pure $\aleph_0$-prebalanced subgroups where at the successor stages the factors are of rank 1.

We shall use the following fundamental result of Fuchs ([8]):

Theorem 2. *A torsion-free group G admits an $\aleph_0$-prebalanced chain if and only if in its balanced projective resolution*

$$0 \to B \to C \to G \to 0$$

(where C is completely decomposable) B is a B_2-group. Moreover, if CH holds, then this condition is equivalent to $Bext^2(G,T) = 0$ for all torsion groups T.

The main result of Section 3 will be

Theorem 3. *If (*) holds, then there is a torsion-free group G of cardinality $\aleph_{\omega+1}$ which does not admit an $\aleph_0$-prebalanced chain.*

Using theorem 2 we get

Corollary 4. *If (*) holds, then there is a group G of cardinality $\aleph_{\omega+1}$ such that $Bext^2(G,T) \neq 0$ for some torsion group T.*

By using the balanced projective resolution of G we also get

Corollary 5. *If (*) holds, then there is a balanced subgroup of a completely decomposable group of cardinality $\aleph_{\omega+1}$ which is not a B_2-group.*

2. The Consistency of (*)

In this section we shall prove Theorem 1. We assume familiarity with some basic large cardinals notions like supercompact cardinals and some basic forcing techniques. We start from a ground model V having a supercompact cardinal κ. We can assume without loss of generality that V satisfies G.C.H. We let $\mu = \kappa^{+\omega}$ and $\lambda = \mu^+ = \kappa^{+\omega+1}$. In our final model μ will be $\aleph_\omega$ and λ will be $\aleph_{\omega+1}$. It follows from the results of Menas in [12] that there is a normal ultrafilter U on $P_\kappa(\lambda)$ such that for some set $A \in U$ the map $P \to \sup(P)$ on A is one-to-one. (Recall that $P_\kappa(\lambda)$ is the set of all subsets of λ of cardinality less than κ). Fix such U and A. Also fix an enumeration $\langle g_\alpha \mid \alpha < \lambda \rangle$ of all the ω-sequences in μ. Standard facts about normal ultrafilters on $P_\kappa(\lambda)$ imply that the set of all $P \in P_\kappa(\lambda)$ satisfying the following properties is in U:

(1) The order type of $P \cap \mu$ is a singular cardinal of cofinality ω such that the order type of P is its successor.
(2) For $\alpha \in \lambda$ the range of g_α is a subset of $P \cap \mu$ if and only if $\alpha \in P$.

Hence we can assume without loss of generality that every $P \in A$ satisfies all the above properties. Again standard arguments show that the set $T = \{\sup(P) \mid P \in A\}$ is a stationary subset of λ. For $\alpha \in T$, let P_α be the unique $P \in A$ such that

$\sup(P) = \alpha$. Note that for $P \in A$ and $Q \subseteq P$ we have that if Q is cofinal in $\sup(P)$, then the order type of $Q^* = \cup\{range(g_\alpha) \mid \alpha \in Q\}$ is the same as the order type of $P \cap \mu$. This holds since otherwise Q^* has cardinality smaller than δ =the order type of $P \cap \mu$. Hence, by our G.C.H. assumption, we have less than δ α's such that the range of g_α is in Q^*, hence less than the order type of P, which is a regular cardinal. Therefore Q must be bounded in P.

For $\alpha \in T$ the map $\alpha \to$ the order type of $P_\alpha \cap \mu$ maps T into κ. Hence it is fixed on some subset S which is still stationary in λ. Let δ be the fixed value of this map on S. Note that for $\alpha \in S$ the order type of P_α is δ^+.

Claim 6. *Let α and β be two different members of S. Then $P_\alpha \cap P_\beta \cap \mu$ has order type less than δ.*

Proof. Let $X = P_\alpha \cap P_\beta \cap \mu$. Note that if g is an ω-sequence from X, then $g = g_\rho$ for some $\rho \in P_\alpha \cap P_\beta$. If X has order type δ, then (using the fact that δ is a singular cardinal of cofinality ω) we have δ^+ ω-sequences from X, so that $P_\alpha \cap P_\beta$ must have order type which is at least δ^+. Since the order type of both P_α and P_β is δ^+, P_α and P_β must have the same sup. This is a contradiction. □

The model which will witness (*) will be obtained from V by collapsing δ to be countable, followed by the collpasing all the cardinals between δ^{++} and κ to have cardinality δ^{++}. Denote the resulting model by V_1. Note since V satisfies G.C.H. then the resulting model satisfies G.C.H. Also δ is of course countable, δ^+ is $\aleph_1$, μ is $\aleph_\omega$ and λ is $\aleph_{\omega+1}$. Since the cardinality of the forcing notion is $\kappa < \lambda$, S is still a stationary subset of λ. Note that now we have for every $\alpha \in S$ that the cofinality of α is $\aleph_1$. In order to verify (*) in the resulting model we fix an enumeration $\langle f_\gamma \mid \gamma < \lambda\rangle$ of all the ω-sequences from $\aleph_\omega = \mu$. And as in the previous section let F_γ be the range of f_γ. (Note that in V_1 there are new ω-sequences so that the enumeration $\langle g_\gamma \mid \gamma < \lambda\rangle$ we had in V enumerates only a subset of the set of all ω-sequences). For $\gamma < \lambda$ let $\eta(\gamma)$ be the unique η such that $g_\gamma = f_\eta$. Without loss of generality (by reducing S to a subset which is still stationary in λ) we can assume that for $\alpha \in S$ if $\gamma < \alpha$, then $\eta(\gamma) < \alpha$. We can also assume without loss of generality that for $\alpha \in S$, $Q_\alpha = \{\eta(\gamma) \mid \gamma \in P_\alpha\}$ is cofinal in α. This follows since the set $\{\alpha \in S \mid Q_\alpha$ is bounded in $\alpha\}$ is not stationary. So for each $\alpha \in S$ pick C_α which is cofinal in Q_α and has order type $\aleph_1 = \delta^+$. We claim that S, δ and $\langle C_\alpha \mid \alpha \in S\rangle$ are witnesses to the truth of (*) in V_1. As in the introduction we put

$$E_\alpha = \bigcup_{\gamma \in C_\alpha} F_\gamma.$$

Since we clearly have G.C.H. in V_1, since S is stationary and since δ is an indecomposable ordinal (it is a cardinal in V!), we are left with verifying the following claim:

Claim 7. *In V_1*

A: *For $\alpha \neq \beta \in S$ $E_\alpha \cap E_\beta$ has order type less than δ.*
B: *If $D \subseteq C_\alpha$ is cofinal in α, then $\cup\{F_\gamma \mid \gamma \in D\}$ has order type δ.*

Proof. Clause A follows immediately from the fact that for $\alpha \in S$, $E_\alpha \subseteq P_\alpha \cap \mu$, hence $E_\alpha \cap E_\beta \subseteq P_\alpha \cap P_\beta \cap \mu$ and the last set has order type less than δ if $\alpha \neq \beta$.

For proving B note that if $D \subseteq C_\alpha$ is cofinal in α, then the set $F = \{\gamma \mid \eta(\gamma) \in D\}$ is a subset of P_α of cardinality $\aleph_1 = \delta^+$. Our forcing is an iteration of two forcing notions where the first is of cardinality (in V) δ and the second is δ^{++} closed, hence it introduces no new sets of ordinals of order type δ^+. So F contains a subset $Q \in V$ of cardinality δ^+. Q must be cofinal in P_α since P_α has order type δ^+, so by a previous remark $\cup\{range(g_\gamma) \mid \gamma \in Q\}$ has order type δ. But this last set is clearly a subset of $\cup\{F_\rho \mid \rho \in D\}$, so this set clearly has order type at least δ. It can not have order type greater than δ since it is a subset of $P_\alpha \cap \mu$. $\square$

3. A group which does not admit an $\aleph_0$-prebalanced chain

In this section we prove Theorem 3. So we assume (*). Fix the enumeration $\langle f_\alpha \mid \alpha < \aleph_{\omega+1}\rangle$ of the ω-sequences from $\aleph_\omega$. Let F_α be the range of f_α. Also fix the stationary subset S of $\aleph_{\omega+1}$, the countable ordinal δ and for $\beta \in S$ a set C_β cofinal in β, which witness the truth of (*). As in the statement of (*) (for $\beta \in S$) let

$$E_\beta = \bigcup_{\alpha \in C_\beta} F_\alpha.$$

We know that the order type of E_β is δ. Since $\delta \times \omega$ is countable we can assign to every pair $\mu < \delta, n < \omega$ a unique prime number p^n_μ.

We are ready to define the group G that will not admit a chain of $\aleph_0$-prebalanced subgroups. For each $\alpha < \aleph_{\omega+1}$ and $\beta \in S$ fix distinct symbols x_α and y_β. The group G is a subgroup of

$$\sum_{\alpha<\aleph_{\omega+1}} \oplus \boldsymbol{Q} x_\alpha \oplus \sum_{\beta\in S} \oplus \boldsymbol{Q} y_\beta.$$

G is generated by x_α for $\alpha < \aleph_{\omega+1}$, by y_β for $\beta \in S$ and by $\frac{1}{p^n_\mu}(y_\beta - x_\alpha)$ provided α is in C_β and the $f_\alpha(n)$ is the μ-th member of E_β. For $\delta < \aleph_{\omega+1}$ let G_δ be the subgroup of G generated by x_α,y_γ and $\frac{1}{p^n_\mu}(y_\gamma - x_\alpha)$ where α and γ are less than δ. The sequence $\langle G_\delta \mid \delta < \aleph_{\omega+1}\rangle$ is a filtration of G into a continuous chain of smaller cardinality. If G allows an $\aleph_0$-prebalanced chain, then by standard arguments, the set of $\delta < \aleph_{\omega+1}$ such that G_δ appears in the $\aleph_0$-prebalanced chain contains a closed unbounded subset of $\aleph_{\omega+1}$. This will imply, since S is stationary in $\aleph_{\omega+1}$, that for some $\beta \in S$, G_β is $\aleph_0$ prebalanced in G. The fact that we get a contradiction and that G does not allow an $\aleph_0$-prebalanced chain follows from:

Claim 8. *For $\beta \in S$, G_β is not an $\aleph_0$-prebalanced subgroup of G.*

Proof. Assume that for some fixed $\beta \in S$, G_β is $\aleph_0$-prebalanced in G. We apply the definition of $\aleph_0$-prebalancedness for y_β and get a sequence of elements $z_n \in G_\beta$ such that for every element z of G_β there are e and l such that

$$\boldsymbol{t}(y_\beta - z) \le \boldsymbol{t}(ly_\beta - z_0) \cup \ldots \cup \boldsymbol{t}(ly_\beta - z_e).$$

C_β has order type $\aleph_1$ and hence for some fixed e and l we get that the set

$$D = \{\alpha \in C_\beta \mid \boldsymbol{t}(y_\beta - x_\alpha) \le \boldsymbol{t}(ly_\beta - z_0) \cup \ldots \cup \boldsymbol{t}(ly_\beta - z_e)\} \tag{1}$$

is unbounded in C_β. It means that for $\alpha \in D$ there is a natural number d_α such that if p is a prime number greater than d_α and p divides $y_\beta - x_\alpha$, then p divides $ly_\beta - z_i$ for some $0 \le i \le e$. Without loss of generality we can assume that for $\alpha \in D$, d_α is some fixed natural number d. Let $D^* = \cup_{\gamma\in D} F_\gamma$. We know that $D^* \subseteq E_\beta$ and that the order type of D^* is δ. We need the following lemma.

Lemma 9. *Let z be a member of G_β with*

$$z = \sum_{i=1}^{k} r_i x_{\alpha_i} + \sum_{j=1}^{g} s_j y_{\beta_j},$$

where $r_i, s_j \in \boldsymbol{Q}$ and $\alpha_i, \beta_j < \beta$ for $1 \leq i \leq k, 1 \leq j \leq g$. Assume also that $ly_\beta - z$ is divisible (in G) by p_μ^n where $p_\mu^n > l$. Then either for some $1 \leq j \leq g$, the μ-th member of E_β is the same as the μ-th member of E_{β_j} or for some $1 \leq i \leq k$, the μ-th member of E_β is in F_{α_i}.

Proof. By assumption $ly_\beta - z$ is divisible by $p = p_\mu^n$ in G. Hence

$$ly_\beta - z = p(\sum_{m=1}^{f} r_m x_{\gamma_m} + \sum_{t=1}^{u} s_t y_{\eta_t} + \sum_{q=1}^{v} \frac{w_q}{p_q}(y_{\nu_q} - x_{\xi_q})). \tag{2}$$

where the r_m's , the s_t's and the w_q's are integers.

Let us define a (bipartite) graph P, whose nodes are all the symbols (x's and y's) appearing in equation 2, where y_ρ is connected by an edge to x_ζ iff for some $1 \leq q \leq v$, $\rho = \nu_q$, $\zeta = \xi_q$ and $p_q = p$. Let W be the connected component of y_β in P and let $a \in \boldsymbol{Q}$ be the sum of all the coefficients in the right side of equation 2 of symbols in W. a is easily seen to be a member of $p\boldsymbol{Q}_p$, where $\boldsymbol{Q}_p$ is the ring of rationals whose denominators are prime to p.This is true because the only summands on the right side of 2, that can possibly add to a a rational number which is not in $p\boldsymbol{Q}_p$, is of the form $\frac{w_q}{p_q}(y_{\nu_q} - x_{\xi_q})$ where $p_q = p$. But in this case y_{ν_q} and x_{ξ_q} are connected by an edge of P, so they are both in W or both outside of W. In both cases the contribution of this summand to a is 0.

We use the fact that the sum of the coefficients of symbols in W must be the same for the left side and the right side of 2. Of course $y_\beta \in W$ and its coefficient in equation 2 is l which is not in $p\boldsymbol{Q}_p$, so there must be a symbol in W appearing in the representation of z, so that either $x_{\alpha_i} \in W$ for some $1 \leq i \leq k$, or $y_{\beta_j} \in W$ for some $1 \leq j \leq g$. Our lemma will be verified if we prove

Claim 10. (1) *If $y_\eta \in W$, then the μ-th member of E_η is the same as the μ-th member of E_β.*
(2) *If $x_\gamma \in W$, then $f_\gamma(n)$ is the μ-th member of E_β.*

Proof. The proof is by induction on the length of the path in P leading from y_β to the symbol y_η and x_γ respectively. If this length is 0, we are in the case where the symbol is $y_\eta = y_\beta$, and the claim is obvious. For the induction step, in the first case we are given y_η. Let x_γ be the element preceding y_η in the path leading from y_β to y_η. By the induction assumption $f_\gamma(n)$ is the μ-th member of E_β. x_γ and y_β are connected by an edge of P, so that $\frac{1}{p_\mu^n}(y_\eta - x_\gamma)$ is one of the generators of G. Hence $\gamma \in C_\eta$ and $f_\gamma(n)$ is the μ-th member of E_η, and the claim is verified in this case. The other case (the x_γ case) is argued similary where y_η is now the element in the path preceding x_γ. □

□

For $z \in G_\beta$ let $S(z)$ be the set of all elements γ of E_β such that for some $\mu < \delta$ and $n \in \omega$, γ is the μ-th member of E_β and $ly_\beta - z$ is divisible in G by p_μ^n where $p_\mu^n > l$. It follows from lemma 9 that for $z \in G_\beta$, $S(z)$ is included in a finite union of singletons and of sets of the form $E_\eta \cap E_\beta$ for $\eta < \beta$. So $S(z)$ is a finite union of sets of order type less than δ. δ is an indecomposable ordinal, so for $z \in G_\beta$ the order type of $S(z)$ is less than δ. By definition of D, every element of D^*, except possibly finitely many, is in $\cup_{0 \le i \le e} S(z_i)$. This is because there are only finitely many members of E_β such that if γ is the μ-th member of E_β, then $p_\mu^n \le \max(d,l)$ for some n. So if $\gamma \in D^*$ is not one of these finitely many elements, say γ is the μ-th member of E_β, then $p_\mu^n > \max(d,l)$. Now $\gamma = f_\alpha(n)$ for some $\alpha \in D$ and a natural number n, and hence p_μ^n divides $y_\beta - x_\alpha$, which implies by equation 1 and the definition of d that p_μ^n divides $ly_\beta - z_i$ for some $1 \le i \le e$. We got that D^* is a finite union of sets of order type less that δ, and hence D^* has order type less than δ. We got a contradiction. $\square$

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