A FINITE PARTITION THEOREM WITH DOUBLE EXPONENTIAL BOUND
DEDICATED TO PAUL ERDŐS

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA

Abstract. We prove that double exponentiation is an upper bound to Ramsey theorem for colouring of pairs when we want to predetermine the order of the differences of successive members of the homogeneous set.

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The following problem was raised by Jouko Vaananen for model theoretic reasons (having a natural example of the difference between two kinds of quantifiers, actually his question was a specific case), and propagated by Joel Spencer: Is there for any n, c an m such that

(∗) for every colouring f of the pairs from $\{0, 1, \ldots, m-1\}$ by 2 (or even c) colours, there is a monocromatic subset $\{a_0, \ldots, a_{n-1}\}$, $a_0 < a_1 < \ldots$ such that the sequence $\langle a_{i+1} - a_i : i < n-1 \rangle$ is with no repetition and is with any pregiven order.

Noga Alon [Al] and independently Janos Pach proved that for every n, c there is such an m as in (∗); Alon used van der Waerden numbers (see [Sh 329]) (so obtained weak bounds).
Later Alon improved it to iterated exponential (Alan Stacey and also the author have later and independently obtained a similar improvement). We get a double exponential bound. The proof continues [Sh 37]. Within the "realm" of double exponential in c, n we do not try to save.
We thank Joel Spencer for telling us the problem, and Martin Goldstern for very careful proof reading.

Notation. Let ℓ, k, m, n, c, d belong to the set $\mathbb{N}$ of natural numbers (which include zero). A sequence η is $\langle \eta(0), \ldots, \eta(\ell g \eta - 1) \rangle$, also ρ, ν are sequences. $\eta \triangleleft \nu$ means that η is a proper initial subsequence of ν. We consider sequences as graphs of functions (with domain of the form $n = \{0, \ldots, n-1\}$, so e.g. $\eta \cap \nu$ means the largest initial segment common to η and ν. $\eta {}^\frown \langle s \rangle$ is the sequence $\langle \eta(0), \ldots, \eta(\ell g \eta - 1), s \rangle$ (of length $\ell g(\eta) + 1$).
Let ${}^\ell m := \left\{ \eta : \ell g(\eta) = \ell, \text{ and } \mathrm{Rang}(\eta) \subseteq \{0, \ldots, m-1\} \right\}$,
${}^{\ell >} m = \bigcup_{k < \ell} {}^k m$.
For $\nu \in {}^{\ell >} m$ we write $[\nu]_{{}^\ell m}$ (or $[\nu]$ if ℓ and m are clear from the context) for the set $\{\eta \in {}^\ell m : \nu \triangleleft \eta\}$.
$[A]^n = \{w \subseteq A : |w| = n\}$.
Intervals $[a, b), (a, b), [a, b)$ are the usual intervals of integers. The proof is similar to [Sh 37] but sets are replaced by trees.

0 Definition. $r = r(n, c)$ is the first number m such that:

$(*)^{n,c}_m$ <u>for every</u> $f : [\{0, \ldots, m-1\}]^2 \to \{0, \ldots, c-1\}$ and linear order $<^*$ on $\{0, \ldots, n-2\}$ <u>we can find</u> $a_0 < \cdots < a_n \in [0, m)$ such that:

(a) $f \restriction \{\{a_i, a_j\} : 0 \leq i < j < n\}$ is constant
(b) the numbers $b_\ell := a_{\ell+1} - a_\ell$ (for $\ell < n-1$) are with no repetitions and are ordered by $<^*$, i.e. $i <^* j \Rightarrow b_i < b_j$

We will find a double exponential bound for $r = r(n, c)$, specifically, $r = r(n, c) \leq 2^{(c(n+1)^3)^{nc}}$ (so our bound is double exponential in n and in c).

This is done in conclusion 5. Alon conjectures that the true order of magnitude of $r(n, c)$ is single exponential, and Alon and Spencer have proved this for the case where the sequence $\langle a_{i+1} - a_i : i < n-1 \rangle$ is monotone.

1 Definition. We say S is an (ℓ, m^*, m, u)-tree if:

(a) $u \subseteq \{0, 1, \ldots, \ell-1\}$ and $m^* \geq m$
(b) $S \subseteq {}^{\ell\geq}(m^*)$
(c) S is closed under initial segments
(d) if $\nu \in S$ is $\triangleleft$-maximal then $\ell g(\nu) = \ell$
(e) if $\nu \in S$ and $\ell g(\nu) \in u$ then for m numbers $j < m^*$ we have $\nu^\frown\langle j\rangle \in S$

2 Claim. *Suppose $k, \ell, m, p, m^* \in \mathbb{N}$ satisfies*

$(*)_{k,\ell,m,p,m^*}$ $\qquad m^* \geq p m^{\ell k+1}$,

then for every $i(*) < \ell$, and $u \subseteq [0, \ell-1)$, and $|u| \leq p$ and $f : [{}^\ell(m^*)]^2 \to \{0,1\}$ there is $T \subseteq {}^{\ell\geq}(m^*)$ closed under initial segments, $(T, \triangleleft) \cong ({}^{\ell\geq}m, \triangleleft)$ satisfying

$\bigoplus$ *for every $\eta_1, \ldots, \eta_k \in T \cap {}^\ell(m^*)$ with $\langle \eta_1 \restriction i(*), \ldots, \eta_k \restriction i(*)\rangle$ pairwise distinct we can find a set S such that:*

(a) S is an $(\ell, m^*, m, u \setminus i(*))$-tree
(b) if $j \in [1, k)$ and $\nu \in S \cap {}^\ell(m^*)$ then $f(\{\eta_j, \eta_k\}) = f(\{\eta_j, \nu\})$
(c) $\nu \in S \cap {}^\ell(m^*) \Rightarrow \eta_k \restriction i(*) \triangleleft \nu$

Remark.
(1) With minor change we can demand in $\bigoplus$ "for any $i(*) < \ell$".
(2) We could use here f with range $\{0, \ldots, c-1\}$, and in claim 3 get a longer sequence $\langle \nu_\ell : \ell < n^*\rangle$ such that $f(\{\nu_{\ell_1}, \nu_{\ell_2}\})$ depends just on $\nu_{\ell_1} \cap \nu_{\ell_2}$, then use a partition theorem on such colouring.

Proof. For each $\eta \in {}^{\ell>}(m^*)$ choose randomly a set $A_\eta \subseteq [0, m^*)$, $|A_\eta| = m$, $A_\eta = \{x_1^\eta, \ldots, x_m^\eta\}$ (pairwise distinct, chosen by order) (not all are relevant, some can be fixed).

We define $T = \{\eta : \eta \in {}^{\ell\geq}(m^*)$, and $i < \ell g(\eta) \Rightarrow \eta(i) \in A_{\eta\restriction i}\}$.
We have a natural isomorphism h from ${}^{\ell\geq}m$ onto T:

$$h(\nu) = \eta \Leftrightarrow \bigwedge_{i<\ell g\nu} \eta(i) = x_{\nu(i)}^{\eta\restriction i}.$$

Our problem is to verify $\bigoplus$, we prove that the probability that it fails is < 1, this suffices.
We can represent it as:

$(*)_1$ if $\nu_1, \ldots, \nu_k \in {}^\ell m$ and $\nu_1 \restriction i(*), \ldots, \nu_k \restriction i(*)$ distinct, then for $h(\nu_1), \ldots, h(\nu_k)$ there is S as required there.

So it suffices to prove that for any given such $i(*) < \ell$ and $\nu_1, \ldots, \nu_k$ the probability of failure is $< \dfrac{1}{\binom{|{}^\ell m|}{k}} = \dfrac{1}{\binom{m^\ell}{k}}$ as it suffice the demand in $(*)_1$ to hold for the minimal suitable $i(*)$. Wlog $u = u \setminus i(*)$.
For this we can assume x_j^ρ are fixed whenever $\neg[h(\nu_k) \restriction i(*)) \trianglelefteq \rho]$ or $\ell g(\rho) \notin u$.

Let $Y = \{\eta \in {}^\ell(m^*)$:Prob$[h(\nu_k) = \eta] \neq 0\}$, so $|Y| = m^{|u|}$.
So $h(\nu_1), \ldots, h(\nu_{k-1})$ are determined. Now $h(\nu_1), \ldots, h(\nu_{k-1})$ and f induces an equivalence relation E on Y:

$$\eta' E \eta'' \text{ iff } \bigwedge_{j=1}^{k-1} f(\{h(\nu_j), \eta'\}) = f(\{h(\nu_j), \eta''\}).$$

The number of classes is $\leq 2^{k-1}$, let them be $A_1, \ldots, A_{2^{k-1}}$ (they are pairwise disjoint, some may be empty).

We call A_j large if there is S as required in clauses (a) and (c) of $\bigoplus$ such that $(\forall \rho)[\rho \in S \cap {}^\ell(m^*) \Rightarrow \rho \in A_j]$.

It is enough to show that the probability of $h(\nu_k)$ belonging to a non-large equivalence class is $< \frac{1}{\binom{m^\ell}{k}}$, hence it is enough to prove:

$(*)_2$ A_j not large $\Rightarrow$ Prob$(h(\nu_k) \in A_j) < \frac{1}{\binom{m^\ell}{k} \times 2^k}$.

So assume A_j is not large. Let $Y^* := \{\eta \restriction i : \eta \in Y \text{ and } i \leq \ell\}$.

Let $Z_j := \Big\{ \eta \in Y^*$: there is $S \subseteq Y^*$ satisfying

$(a)'$ S is an $(\ell, m^*, m, u \backslash \ell g(\eta))$-tree

$(b)'$ $\nu \in S \cap {}^\ell(m^*) \Rightarrow \eta \trianglelefteq \nu$

$(c)'$ for every $\nu \in S \cap {}^\ell(m^*)$ we have $\nu \in A_j \Big\}$.

Let $Z_j^* := \{\eta \in Z_j$: there is no $\eta' \triangleleft \eta, \eta' \in Z_j\}$.
Clearly $h(\nu_k \restriction i(*)) \notin Z_j$, (as A_j is not large) hence $h(\nu_k \restriction i(*)) \notin Z_j^*$.
Clearly

$(*)_3$ for $\eta \in Y^* \setminus Z_j$ such that $\ell g(\eta) \in u$ we have

$$|\{i : \eta^\frown\langle i \rangle \in Z_j^* \text{ (or even } \in Z_j)\}| < m.$$

But if $\nu \triangleleft \eta$, $\nu \in Z_j$ then $\eta \notin Z_j^*$. Hence

$(*)_4$ for $\eta \in Y^*$ such that $\ell g(\eta) \in u$ we have

$$|\{i : \eta^\frown\langle i \rangle \in Z_j^*\}| < m.$$

Now

$(*)_5$ if $\eta \in A_j$ (hence $\eta \in Y$; remember that A_j is not large) then $\eta \in Z_j$, hence $\bigvee_{j \in u} \eta \restriction (j+1) \in Z_j^*$.

So

$$\begin{aligned} \text{Prob}(h(\nu_k) \in A_j) \leq & \ \text{Prob}\Big(\bigvee_{j \in u} [h(\nu_k \restriction (j+1)) \in Z_j^*]\Big) \\ & \leq \sum_{j \in u} \text{Prob}(h(\nu_k \restriction (j+1) \in Z_j^*)) \\ & < |u| \times \frac{m}{m^*}. \end{aligned}$$

(first inequality by $(*)_5$, second inequality trivial, last inequality by $(*)_4$ above). So it suffices to show:

$$|u| \times \frac{m}{m^*} \leq \frac{1}{\binom{m^\ell}{k} \times 2^k}$$

equivalently

$$m^* \geq |u| \times m \times \binom{m^\ell}{k} \times 2^k$$

as $\binom{m^\ell}{k} \leq m^{\ell k}/k!$, and $|u| \leq p$ by the hypothesis $(*)_{k,\ell,m,p,m^*}$ we finish. $\square_2$

3 Lemma. *Assume*

(a) $\rho_1, \dots, \rho_n \in {}^{n-1}2$ *are distinct, for* $\ell \in \{2, \dots, n\}$, *we have* $r_\ell \in \{1, \dots, \ell-1\}$ *such that* $\ell g(\rho_\ell \cap \rho_{r_\ell}) = \ell - 1$ *and* $r \in \{1, \dots, \ell-1\} \setminus \{r_\ell\} \Rightarrow \ell g(\rho_\ell \cap \rho_r) < \ell$
(b) $f : [{}^\ell m]^2 \to [0, c) = \{0, \dots, c-1\}$
(c) $m = 2^{(n+1)^{(c+1)n}}$.

Then we can find $\eta_1, \dots, \eta_n \in {}^\ell m$ *such that:*

(α) $f \restriction [\{\eta_1, \dots, \eta_n\}]^2$ *is a constant function*
(β) $\langle \ell g(\eta_{i+1} \cap \eta_{r_{i+1}}) : i = 1, \dots, n-1\} \rangle$ *is a sequence with no repetitions ordered just like* $\langle \ell g(\rho_{i+1} \cap \rho_{r_{i+1}}) : i = 1, n-1\} \rangle$; *also:*

$$\eta_{i+1}(\ell g(\eta_{i+1} \cap \eta_{r_{i+1}})) < \eta_{r_{i+1}}(\ell g(\eta_i \cap \eta_{r_{i+1}})) \Leftrightarrow \rho_{i+1}(\ell g(\rho_{i+1} \cap \rho_{r_{i+1}})) < \rho_{r_{i+1}}(\ell g(\rho_{i+1} \cap \rho_{r_{i+1}})).$$

3A Remark. 1) Note that if $\Gamma \subseteq {}^{n-1}2$, $|\Gamma| = n$ and the set $\{\rho_1 \cap \rho_2 : \rho_1 \neq \rho_2$ are from $\Gamma\}$ has no two distinct members with the same length then we can list Γ as $\langle \rho_1, \dots, \rho_n \rangle$ as required in clause (a) of lemma 3.
2) So if $<^*$ is a linear order on $\{1, \dots, n-1\}$ then we can find distinct $\rho_1, \dots, \rho_n \in {}^{n-1}2$ as in clause (a) of lemma 3 and a permutation σ of $\{1, \dots, n\}$ such that:
for $i \neq j \in \{1, \dots, n-2\}$ we have

$$i <^* j \quad \text{iff} \quad \ell g(\rho_{\sigma(i)} \cap \rho_{\sigma(i+1)}) > \ell g(\rho_{\sigma(j)} \cap \rho_{\sigma(j+1)}).$$

(E.g. use induction on n.)

Proof. Let us define $\langle m_j : 2 \leq j \leq cn \rangle$ by induction on j:

$$m_2 = n^c, \qquad m_{j+1} = n^c m_j^{n^c(n+1)+1}.$$

Check that $m_j \leq 2^{(n+1)^{(c+1)j}}$, so in particular $m_{(cn)} \leq m$. Now we claim that for any number $d \in [1, c]$ the following holds:

$\otimes_d$ Assume $q \in [0, n^c)$ such that q divisible by n^d and $q + n^d \leq cn$, $u = [q, q+n^d)$, T^* is an $(\ell, m, m_{(q+dn)}, u)$-tree and f is a function from $[T^* \cap {}^\ell m]^2$ with range of cardinality d. Then we can find $\eta_1, \dots, \eta_n \in T^* \cap ({}^\ell m)$ such that clauses (α) and (β) of the conclusion of lemma 3 hold.

This suffices: use $q = 0, d = c$. We prove this by induction on d. If $d = 1$, trivial as only one colour occurs. For $d + 1 > 1$, without loss of generality $\mathrm{Rang}(f) = [0, d]$, let $f' : [{}^{\ell}m]^2 \to \{0, 1\}$ be $f'(\{\eta', \eta''\}) = \mathrm{Min}\{f(\{\eta', \eta''\}, 1\}$. Let for $j < n$, $u_j := [q + n^d j, q + n^d j + n^d)$. By downward induction on $j \in [1, n]$ we try to define T_j such that:

(i) T_j is an $(\ell, m, m_{(q+nd+j)}, \bigcup_{i<j} u_i)$-tree

(ii) $T_j \subseteq T_{j+1}$

(iii) for every $j \in [1, n-1]$ and $\eta_1, \dots, \eta_j \in T_j \cap {}^{\ell}m$ with $\langle \eta_1 \restriction (q + n^d j), \dots, \eta_j \restriction (q + n^d j)\rangle$ pairwise distinct, we can find $\eta' \neq \eta'' \in T_{j+1} \cap {}^{\ell}m$ such that:

$$\eta_j \restriction (q + n^d j) \triangleleft \eta', \eta''$$

$$\ell g(\eta' \cap \eta'') \in u_j$$

$$f'(\{\eta', \eta''\}) = 0$$

$$\bigwedge_{t \in [1,j)} \Big[0 = f'(\{\eta_t, \eta_j\}) \Rightarrow 0 = f'(\{\eta_t, \eta'\}) = f'(\{\eta_t, \eta''\}) \Big].$$

<u>This suffices</u> as then we can choose by induction on $j = 1, \dots, n$ a sequence $\nu_j \in T_j \cap ({}^{\ell}m)$ such that (after reordering) the set $\{\nu_1, \dots, \nu_n\}$ will serve as $\{\eta_1, \dots, \eta_n\}$ of $\otimes_d$ (with the constant colour being zero). Let us do it in detail.

By induction on $j = 1, \dots, n$ we choose $\nu_1^j, \dots, \nu_{j-1}^j$ such that:

(a) $\nu_1^j, \dots, \nu_j^j$ are distinct members of $T_j \cap ({}^{\ell}m)$

(b) $f \restriction [\{\nu_1^j, \dots, \nu_j^j\}]^2$ is constantly zero

(c) for $\ell = \{2, \dots, j\}$ we have $\ell g(\nu_\ell^j \cap \nu_{r_\ell}^j) \in u_{\ell-1}$

For $j = 1$ no problem. In the induction step, i.e. for $j + 1$, we apply the condition (iii) above with $\langle \nu_\ell^j : \ell \in [1, j], \ell \neq q_{j+1}\rangle ^\frown \langle \nu_{q_{j+1}}^j \rangle$ here standing for $\eta_1, \dots, \eta_j$ there (we want $\nu_{q_{j+1}}^j$ be the last), the condition $\langle \eta_\ell \restriction (q + n^d j) : \ell = 1, \dots, j\rangle$ with no repetition follows by clause (c), so we get η', $\eta'' \in T_{j+1} \cap ({}^{\ell}m)$ as there. W.l.o.g. $\eta'(\ell g(\eta' \cap \eta'')) < \eta''(\ell g(\eta' \cap \eta''))$.

We now define ν_ℓ^{j+1} for $\ell = 1, \dots, j + 1$:

If $\ell \in \{1, \dots, j + 1\} \setminus \{j + 1, r_{j+1}\}$ then $\nu_\ell^{j+1} = \nu_\ell^j$ (remember $T_j \subseteq T_{j+1}$).

If $\rho_{j+1}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}})) < \rho_{r_{j+1}}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}}))$ then $\nu_{j+1}^{j+1} = \eta'$ and $\nu_{r_{j+1}}^{j+1} = \eta''$.

If $\rho_{r_{j+1}}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}})) < \rho_{j+1}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}}))$ then $\nu_{j+1}^{j+1} = \eta''$ and $\nu_{r_{j+1}}^{j+1} = \eta'$.

Now check. (Note T_0, u_0 could be omitted above.)

Carrying the Inductive Definition. For $j = n$, trivial: let $T_n = T^*$ (given in $\otimes_d$).

For $j \in [2, n)$, where $T_{j+1}, \dots, T_n$ are already defined, we apply Claim 2 with

$$m_{q+nd+j+1},\ m_{q+nd+j},\ n^d(j+1),\ j,\ q+n^d j,\ [q+n^d j, q+n^d j+n^d),\ n^d j$$

here standing for

$$m^*,\ m,\ \ell,\ k,\ i(*),\ u,\ p$$

there. (I.e. the tree in claim 2 is replaced by one isomorphic to it, levels outside $\bigcup_{i<j} u_i$ can be ignored.)
So we need to check $m_{q+nd+j+1} \geq n^d j (m_{q+nd+j})^{n^d(j+1)j+1}$, which holds by the definition of the m_i's (as $d \leq c_1$). But we require above more than in Claim 2 (preferring the colour 0). But if it fails for T_j then for some $\eta_1, \dots, \eta_j$ in T_j we have S as in $\oplus$ of Claim 2, with no $\eta^1 \neq \eta^2 \in S \cap {}^\ell m$ such that $f'(\{\eta', \eta''\}) = 0$. On S we can apply our induction hypothesis on d — allowed as the original f misses a colour (the colour zero) when restricted to S. $\square_3$

4 Fact. *Let $\langle \eta_i : i < (2m-1)^\ell \rangle$ enumerate ${}^\ell(2m-1)$ in lexicographic order. Let*

$$A = \{0, 2, \dots, 2m-2\} \subseteq [0, 2m-1), \text{ so } |A| = m,$$
$$B := \{i < (2m-1)^\ell : \eta_i \in {}^\ell A\}.$$

Let

$$low_{m,\ell}(k) = (2m-1)^{\ell-k-1}, \text{ and}$$
$$high_{m,\ell}(k) = (2m-1)^{\ell-k} - 1.$$

Then:

(0) *if $i \neq j$ are in B, $|\eta_i \cap \eta_j| = k$ then:*
$$i < j \text{ iff } \eta_i(k) < \eta_j(k).$$
(1) *If $i < j$ are in B, $|\eta_i \cap \eta_j| = k$, then:*
$$low_{m,\ell}(k) \leq j - i \leq high_{m,\ell}(k)$$
(2) *If $i_1 < j_1$, $i_2 < j_2$ are all in B, and $|\eta_{i_1} \cap \eta_{j_1}| \neq |\eta_{i_2} \cap \eta_{j_2}|$, then:*
$$j_1 - i_1 < j_2 - i_2 \quad \text{iff} \quad |\eta_{i_1} \cap \eta_{j_1}| > |\eta_{i_2} \cap \eta_{j_2}|.$$

Proof. (0) Check.
(1) Let $\nu := \eta_i \cap \eta_j$ and $k := |\nu|$. We are looking for upper and lower bounds of the cardinality of the set (the order is lexicographic)

$$C := \{\eta \in {}^\ell(2m-1) : \eta_i \leq \eta < \eta_j\}.$$

Clearly each $\eta \in C$ satisfies $\nu \triangleleft \eta$. Moreover, since $\eta_j \in {}^\ell m$, each element $\eta \in C$ must satisfy $\eta(k) \leq \eta_j(k) < 2m-1$. Hence

$$C \subseteq \bigcup_{s<2m-1} [\nu^\frown\langle s\rangle]_{({}^\ell(2m-1))} \setminus \{\eta_j\}$$

so we get $|C| \leq (2m-1) \cdot (2m-1)^{\ell-k-1} - 1 = (2m-1)^{\ell-k} - 1$.
For $k = \ell - 1$ the lower bound claimed in (1) is trivial, so assume $k \leq \ell - 2$. Let $\nu' := (\eta_i \restriction k)^\frown\langle \eta_i(k) + 1\rangle$ (note: as $\eta_i(k) < \eta_j(k)$ are both in A, necessarily $\nu'(k) < \eta_j(k)$). Then we have

$$C \supseteq \bigcup_{s<2m-1} [\nu'^\frown\langle s\rangle]_{({}^\ell(2m-1))} \cup \{\eta_i\},$$

so $|C| \geq (2m-1)^{\ell-k-1} + 1$.
Proof of (2): Check that $high_{m,\ell}(k) < low_{m,\ell}(k+1)$.

$\square_4$

Remark. Also $low_{m,\ell}(k) = (2m-1)^{\ell-k-1} + 1$ is O.K. but with the present bound we can use only η_i with $\eta_i(\ell-1) < m$, $B = \{i : \eta_i \restriction (\ell-1) \in {}^\ell A\}$. So $(2m-1)^\ell$ can be replaced by $m(2m-1)^{\ell-1}$.

5 Conclusion. *If* $f : [0,(2m-1)^\ell)^{[2]} \to [0,c)$, $\ell = nc$, $m := 2^{(n+1)^{(c+1)n}}$, *then we can find* $a_0 < \cdots < a_{n-1} < m$ *such that* $f \restriction [\{a_1,\dots,a_{n-1}\}]^2$ *constant and* $\langle a_{i+1} - a_i : i < n-1\rangle$ *in any pregiven order*

Proof. As in fact 4, let $\langle \eta_i : i < (2m-1)^\ell\rangle$ enumerate ${}^\ell(2m-1)$ in lexicographic order, and let $B := \{i < (2m-1)^\ell : \eta_i \in {}^\ell m\}$ and for $i \in B$ let $\eta'_i = \langle \eta_i(\ell)2 : i < \ell\rangle$. Define a function $f' : [{}^\ell m]^2 \to c$ by requiring $f'(\{\eta'_i,\eta'_j\}) = f(\{i,j\})$ for all $i,j \in B$. Now the conclusion follows from lemma 3, Remark 3A(2) and Fact 4, particularly clause (2).

$\square_5$

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