

# ON THE EXISTENCE OF RIGID $\aleph_1$ -FREE ABELIAN GROUPS OF CARDINALITY $\aleph_1$

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## 1. INTRODUCTION

An abelian group is said to be  $\aleph_1$ -free if all its countable subgroups are free. A crucial special case of our main result can be stated immediately.

*Indecomposable  $\aleph_1$ -free abelian groups of cardinality  $\aleph_1$  do exist.*

The first example of any  $\aleph_1$ -free group which is not free is the Baer-Specker group  $\mathbb{Z}^\omega$ , which is the cartesian product of countably many copies of the group  $\mathbb{Z}$  of integers, known for almost sixty years; cf. Baer [1] or [14, p.94]. Assuming CH, this group of cardinality  $2^{\aleph_0} = \aleph_1$  is an example of a non-free abelian group of cardinality  $\aleph_1$ . Under the same set-theoretic assumption of the continuum hypothesis it can be shown that any countable ring  $R$  with free additive group can be realized as the endomorphism ring of an  $\aleph_1$ -free abelian group  $G$  of cardinality  $\aleph_1$ . The chronologically earlier realization theorem of this kind uses the weak diamond prediction principle which follows from  $2^{\aleph_0} < 2^{\aleph_1}$ , cf. Devlin and Shelah [6] for the weak diamond, Shelah [28] for the case  $\text{End } G = \mathbb{Z}$  and Dugas, Göbel [7] for the case  $R = \text{End } G$  and extensions to larger cardinals. Using, what is called Shelah's Black Box, the existence of  $\aleph_1$ -free groups  $G$  with  $|G| = 2^{\aleph_0}$  also follows from Corner, Göbel [5] using Dugas, Göbel [7] and combinatorial fine tuning from Shelah [29].

Without the assumption of CH, the existence of non-free,  $\aleph_1$ -free groups of cardinality  $\aleph_1$  follows from a more general result by Griffith [18], Hill [21], Eklof [11], Mekler [24] and Shelah in Eklof [12, p.82, Theorem 8.8]. By an induction it can be shown, that there are  $\aleph_n$ -free groups, non-free of cardinality  $\aleph_n$ . The non-abelian case is due to Higman [19, 20].

By Shelah's singular compactness theorem it is known that  $\lambda$ -free abelian groups of cardinality  $\lambda$  do not exist if  $\lambda$  is singular, e.g. if  $\lambda = \aleph_\omega$ , cf. Eklof, Mekler [13]. Hence induction breaks down and it is more complicated to show the existence of  $\lambda$ -free, non-free abelian groups of cardinality  $\lambda > \aleph_\omega$ . This question is investigated in Magidor, Shelah [23] and we just refer to this paper and restrict ourselves to cardinals  $\lambda \leq 2^{\aleph_0}$  again, and we will focus on  $\lambda = \aleph_1$ . Only very little is known about algebraic properties of  $\aleph_1$ -free groups of cardinality  $\aleph_1$ , see Eklof [11] and Eklof, Mekler [13]. Shelah's construction [27] (see also [30]) of groups also mentioned in [12, 13] which are not separable was refined in Eda [10] prove the existence of

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an  $\aleph_1$ -free group  $G$  of cardinality  $\aleph_1$  such that  $\text{Hom}(G, \mathbb{Z}) = 0$ , a result derived independently but later by Corner, Göbel [5]. Counterexamples for Kaplansky's test problems among  $\aleph_1$ -free groups of cardinality  $\aleph_1$  are given recently in Göbel, Goldsmith [17], realizing rings modulo some large ideal, see also [16]. Moreover,  $\aleph_1$ -separable groups of cardinality  $\aleph_1$  serving as counterexamples of Kaplansky's test problems were constructed in [31]. These results about  $\aleph_1$ -free groups become special cases of our quite satisfying main theorem.

**Main Theorem 4.1** *If  $R$  is a ring with  $R^+$  free and  $|R| < \lambda \leq 2^{\aleph_0}$ , then there exists an  $\aleph_1$ -free abelian group  $G$  of cardinality  $\lambda$  with  $\text{End } G = R$ .*

We have identified  $R$  with endomorphisms acting on the  $R$ -module  $G$  by scalar multiplication. This result has many applications. If  $R = \mathbb{Z}$ , we derive the existence of  $\aleph_1$ -free abelian groups of cardinality  $\aleph_1$ , a result which was unknown.

If  $\Gamma$  is any abelian semigroup, then we use Corner's ring  $R_\Gamma$ , implicitly discussed in Corner, Göbel [4], and constructed for particular  $\Gamma$ 's in [3] with special idempotents (expressed below), with free additive group and  $|R_\Gamma| = \max\{|\Gamma|, \aleph_0\}$ . If  $|\Gamma| < 2^{\aleph_0}$ , we may apply the main theorem and find a family of  $\aleph_1$ -free abelian groups  $G_\alpha$  ( $\alpha \in \Gamma$ ) of cardinality  $\aleph_1$  such that for  $\alpha, \beta \in \Gamma$ ,

$$G_\alpha \oplus G_\beta \cong G_{\alpha+\beta} \text{ and } G_\alpha \cong G_\beta \text{ if and only if } \alpha = \beta.$$

Observe that this induces all kinds of counterexamples to Kaplansky's test problems for suitable  $\Gamma$ 's. If we consider Corner's ring in [2], see Fuchs [15, p.145], then it is easy to see that  $R^+$  is free and  $|R| = \aleph_0$ . The particular idempotents in  $R$  and our main theorem provide the existence of an  $\aleph_1$ -free superdecomposable group of cardinality  $\aleph_1$ , which was unknown as well. Recall that a group is superdecomposable if any non-trivial summand decomposes into a proper direct sum.

Finally, we remark that as the reader might suspect, it is easy to replace  $G$  in Theorem 4.1 by a rigid family of  $2^\lambda$  such groups with only the trivial homomorphism between distinct members. The main theorem cannot be generalized, replacing  $\aleph_1$  by another cardinal. In Section 5 we will show that there are many models of ZFC (e.g. assuming MA and  $\aleph_2 < 2^{\aleph_0}$ ) in which no  $\aleph_2$ -free group of cardinality  $< 2^{\aleph_0}$  has endomorphism ring  $\mathbb{Z}$ ; it is even possible that all such groups are separable and the best one can do now is a realization theorem of the form  $\text{End } G = R \oplus \text{Ines } G$  with  $\text{Ines } G \neq 0$  an ideal containing all endomorphisms of finite rank.

This is in contrast to the result [7], that under  $\diamond_\lambda$  any countable ring  $R$  with  $R^+$  free is of the form  $R \cong \text{End } G$  for all uncountable regular, not weakly compact cardinal  $\lambda = |G| > |R|$  such that  $G$  is  $\lambda$ -free. In particular, *the existence of indecomposable  $\aleph_2$ -free groups of cardinality  $\aleph_2$  or the existence of such groups with endomorphism ring  $\mathbb{Z}$  is undecidable.*

## 2. THE BUILDING BLOCKS, $\aleph_1$ -FREE MODULES WITH A DISTINGUISHED CYCLIC SUBMODULE

Let  $R$  be a ring of cardinality  $|R| < 2^{\aleph_0}$  such that  $R^+$  is a free abelian group. In view of Pontrjagin's theorem we say that an  $R$ -module is  $\aleph_1$ -free if any subgroup of finite rank is contained in a free  $R$ -submodule.

We have the immediate application of Pontrjagin's theorem [14, p.93, Theorem 19.1.].

**Observation 2.1.** *If  $M$  is  $\aleph_1$ -free as  $R$ -module and  $R^+$  is free, then  $M$  is  $\aleph_1$ -free as abelian group, this means all countable subgroups are free.*

**Remark 2.2.** *If  $U$  is a finitely generated submodule of an  $\aleph_1$ -free  $R$ -module  $M$  of infinite rank and  $M/U$  is flat, then  $M/U$  is an  $\aleph_1$ -free  $R$ -module as well.*

**Proof** If  $S/U$  is a subgroup of finite rank in  $M/U$ , then  $S_*/U$  denotes its purification and  $S_*$  is a pure subgroup of finite rank in  $M$ , hence it is contained in a free  $R$ -submodule  $F$  of  $M$ . Moreover, we find a finitely generated summand  $F'$  of the  $R$ -module  $F$  with  $S_* \subseteq F'$  and  $F/F'$  is  $R$ -free. Also  $F'/U$  is flat because  $M/U$  is flat and  $F'/U$  can be finitely presented by

$$F'' \rightarrow F' \rightarrow F'/U \rightarrow 0$$

for some finitely generated free module  $F''$  mapping onto  $U \subseteq F'$ . Hence  $F'/U$  is projective by Rotman [25, p. 90, 91], and  $F/U \cong F/F' \oplus F'/U$  is projective.

Finally we may assume that  $F/U$  has infinite rank and  $F/U$  is free by a well-known argument of Kaplansky's, cf. [17], for instance. Hence  $M/U$  must be an  $\aleph_1$ -free  $R$ -module.

Recall that Remark 2.2 does not hold if  $U$  is not finitely generated. Consider a free resolution of any torsion-free abelian group  $A$  which is not  $\aleph_1$ -free:  $0 \rightarrow U \rightarrow M \rightarrow A \rightarrow 0$ . By Remark 2.2 in particular quotients of  $\aleph_1$ -free groups modulo pure, cyclic subgroups are  $\aleph_1$ -free again.

Next we will construct particular  $\aleph_1$ -free  $R$ -modules  $A$  with distinguished cyclic submodules  $cR$ .

First we will fix some more notation. Let  $\mathcal{P}$  be a family of  $2^{\aleph_0}$  almost disjoint infinite subsets of an infinite set of primes. At present, we choose a fixed  $X \in \mathcal{P}$  with an enumeration  $X = \{p_n : n \in \omega\}$  without repetitions. Let  $T = {}^\omega 2$  denote the tree of all finite branches  $\eta : n \rightarrow 2, n < \omega$ , where  $\ell(\eta) = n$  denotes the length of the branch  $\eta$ . The branch of length 0 is denoted by  $\perp = \emptyset \in T$  and we also write  $\eta = (\eta \upharpoonright n - 1)^\wedge \eta(n - 1)$ . Finally  ${}^\omega 2 = Br(T)$  denotes all infinite branches  $\eta : \omega \rightarrow 2$  and clearly  $\eta \upharpoonright n \in T$  for all  $\eta \in Br(T), n \in \omega$ .

Let  $\lambda$  be an infinite cardinal  $\leq 2^{\aleph_0}$  and  $Y \subseteq Br(T)$  with  $|Y| = \lambda$  and  $|R| < \lambda$ . Then  $V'$  will denote the vector space over the rationals  $\mathbb{Q}$  with basis  $T \cup Y$ . Finally  $R$  becomes a vector space by  $R \otimes_{\mathbb{Z}} \mathbb{Q} = \hat{R}$  and  $V = V' \otimes_{\mathbb{Q}} \hat{R}$  is a vector space of dimension  $\lambda$ . We now select an  $R$ -submodule  $A \subseteq V$  which is generated by  $T$  together with elements

$$\eta_0 = \eta, \eta_{n+1} = \frac{1}{p_n}(\eta_n + \eta \upharpoonright n + \eta(n) \perp) \in V \quad (X)$$

defined inductively for all  $\eta \in Y, n \in \omega$ . Hence

$$A = A_X = A_{XY} = \langle \sigma R, \eta_n R : \sigma \in T, \eta \in Y, n \in \omega \rangle \subseteq V$$

depends on  $X \in \mathcal{P}$  and  $Y \subseteq Br(T)$ . The required cyclic  $R$ -submodule is  $\perp R$ . We will show that  $(A, \perp R)$  belongs to the category of modules we are interested in, i.e. the following Lemma holds.

**Lemma 2.3.** *Let  $(A, \perp R)$  be the pair of  $R$ -modules defined above, let  $B = \langle T \rangle$  and  $\bar{\cdot} : A \rightarrow A/B$  be the canonical homomorphism. Then we have*  
 (a)  *$B$  is a free  $R$ -module and  $A/B = \bigoplus_{\eta \in Y} \bar{\eta}(\bar{X} \otimes_{\mathbb{Z}} R)$  with  $\bar{X} \subseteq \mathbb{Q}$  of characteristic  $\chi : \omega \rightarrow 2$  with support  $X$ .*

- (b)  $A$  is an  $\aleph_1$ -free  $R$ -module.  
(c)  $A/\perp R$  is an  $\aleph_1$ -free  $R$ -module.

**Proof** (a) Clearly  $B = \bigoplus_{\sigma \in T} \sigma R$  and if  $g \in A$ , then we use (X) to find  $k \in \omega$  and finite sets  $T_1 \subseteq T$ ,  $Y_1 \subseteq Y$  with

$$g = \sum_{\eta \in Y_1} \eta_k g_\eta + \sum_{\sigma \in T_1} \sigma g_\sigma$$

for some  $g_\eta, g_\sigma \in R$ . Using (X) again, we have

$$g \equiv \sum_{Y_1} \eta \frac{g_\eta}{q_k} \pmod{B}$$

where  $q_k = \prod_{i=1}^{k-1} p_i$  by the enumeration in X and  $\frac{g_\eta}{q_k} \in \bar{X} \otimes R$ . Clearly  $\{\bar{\eta} : \eta \in Y\}$  is  $\mathbb{Q} \otimes R$ -independent and hence  $\bar{X} \otimes R$ -independent and (a) follows.

(b) Obviously  $|A| = |Y| = \lambda$ . Next we show that

(\*) any finite subset of  $A$  lies in a submodule  $U$  which is free and pure in  $A$ .

For any finite subset  $E$  of  $A$  we can find some  $n \in \omega$  and a finite subset  $Y_0 \subseteq Y$  such that

$$E \subseteq U = \langle \sigma R, \eta_n R : \sigma \in T, \ell(\sigma) < n, \eta \in Y_0 \rangle.$$

Obviously  $U$  is freely generated by the elements  $\sigma, \eta_n$ . In order to show that  $U$  is pure in  $A$ , consider  $g \in A$  and  $m \in \mathbb{N}$  minimal with  $gm = u \in U$ .

We may write

$$g = \sum_{\eta \in Y_1} \eta_k g_\eta + \sum_{\sigma \in T_1} \sigma g_\sigma \quad \text{and} \quad u = \sum_{\eta \in Y_2} \eta_n u_\eta + \sum_{\sigma \in T_2} \sigma u_\sigma$$

with  $g_\eta, g_\sigma, u_\eta, u_\sigma \in R$  and  $k = k(\eta)$  minimal for each  $\eta \in Y_1$ . Since  $gm = u$  we have  $Y_1 = Y_2$  and  $\eta_n u_\eta = \eta_k g_\eta m$  for all  $\eta \in Y_1$  from (a). If  $k < n$  for some  $\eta \in Y_1 = Y_2$ , then we can reduce  $Y_1$  to a smaller set  $Y_1 \setminus \{\eta\}$  by the observation  $\eta_k g_\eta \in U$  and  $\eta_k g_\eta m = \eta_n u_\eta$  and  $g \in U$  follows by induction. We derive  $k \geq n$  for all  $\eta \in Y_1$ , and suppose  $k > n$  for some  $\eta$ .

We have  $p_{k-1} | q = \prod_{i=n}^{k-1} p_i$  and minimality of  $m$  requires  $p_{k-1}$  does not divide  $m$ . On the other hand  $g_\eta m = q u_\eta$  and  $p_{k-1} | q$  hence  $p_{k-1} | g_\eta$  which contradicts minimality of  $k = k(\eta)$ . We derive  $k = n$  for all  $\eta$  and  $g$  decomposes into a  $Y$ -part  $g_Y \in U$  with  $g_Y m = \sum_{Y_2} \eta_n u_\eta$  and a  $T$ -part  $g_T \in B$  with  $g_T m = \sum_{T_1} \sigma g_\sigma$ . However  $g_T \in U$ , hence  $g = g_Y + g_T \in U$  as well and  $U$  is pure in  $A$ , i.e. (\*) holds. Finally  $A$  is an  $\aleph_1$ -free  $R$ -module by the argument in Remark 2.2 and Pontrjagin's collection of a direct sum of projective modules, see Fuchs [14, p.93, Theorem 19.1.]. Now (b) and also (c) follow from (\*).

**Observation 2.4.** *If  $(A, \perp R)$  is as above, then  $A$  and  $A/\perp R$  are  $\aleph_1$ -free abelian groups with  $R \subseteq \text{End } A$ ,  $R \subseteq \text{End } (A/\perp R)$  identifying  $r = r \cdot \text{id}$  for all  $r \in R$ .*

Observation 2.4 is immediate from Observation 2.1 and Lemma 2.3, which is all we need in Section 3.

Moreover we will require enough splitting in  $A$  which is established by the following

**Proposition 2.5.** *Let  $(A, \perp R)$  be as above, where  $A = A_X$ ,  $X \neq P \in \mathcal{P}$  and  $\bar{P} = \mathbb{Z}_P$  the obvious localization at  $P$ . Then  $A_X \otimes R_P$  is a free  $R_P$ -module with  $\perp$  a basis element, where  $R_P = \mathbb{Z}_P \otimes_{\mathbb{Z}} R$  is the localization of  $R$  at  $P$ .*

**Proof** Recall that

$$A_X = \langle \sigma R, \eta_n R : \sigma \in T, \eta \in Y, n \in \omega \rangle.$$

Moreover  $X \cap P$  is finite by our choice of  $\mathcal{P}$ . We find  $k \in \omega$  such that  $\{p_n \in X : n \geq k\} \cap P = \emptyset$ . Now we claim that

$$T \cup \{\eta_k : \eta \in Y\}$$

is a basis of the  $R_P$ -module  $A_X \otimes R_P$ . Note that  $\perp \in T$  and Proposition 2.5 will follow.

The set  $M = T \cup \{\eta_k : \eta \in Y\}$  is clearly independent over  $\mathbb{Q} \otimes_{\mathbb{Z}} R$  in  $V$  and hence freely generates the  $R_P$ -submodule

$$U = \bigoplus_{m \in M} mR_P = F \otimes R_P \subseteq A_X \otimes R_P$$

with  $F = \bigoplus_{m \in M} mR$ . It remains to show  $U = A_X \otimes R_P$ .

The submodule  $F \subset A_X$  induces a natural sequence

$$0 \rightarrow F \rightarrow A_X \rightarrow A_X/F \rightarrow 0$$

of  $R$ -modules, where  $A_X/F$  is generated by  $\{\eta_n + F : \eta \in Y, n > k\}$ , see Lemma 2.3(a). Using (X) we derive  $p_{n-1} \cdot \dots \cdot p_{k+1} \eta_n \equiv \eta_k \equiv 0 \pmod{F}$  where the enumeration of primes is taken in  $X$ . These primes belong to  $\{p_n \in X : n \geq k\}$  and cannot belong to  $P$  by our choice of  $k$ . We observe that  $A_X/F$  is a  $P'$ -group in the well-known sense, that  $A_X/F$  is torsion and the order of elements is a product of primes in  $P'$ , the complement of  $P$ . On the other hand  $R_P$  is  $P'$ -divisible, hence  $(A_X/F) \otimes R_P = 0$ . Using flatness of  $R_P$  the above sequence becomes

$$0 \rightarrow F \otimes R_P \rightarrow A_X \otimes R_P \rightarrow (A_X/F) \otimes R_P \rightarrow 0$$

and  $(A_X/F) \otimes R_P = 0$  forces  $A_X \otimes R_P = F \otimes R_P$  as desired.

### 3. REPEATING THE BUILDING BLOCKS

Let  $R, \mathcal{P}$  and  $|R| < \lambda \leq 2^{\aleph_0}$  be as in Section 2. Then we enumerate  $\mathcal{P} = \{X_\alpha : \alpha < \lambda\}$  without repetition and it is easy to find a family  $\mathcal{F} = \{L_\alpha \subset \omega : \alpha < \lambda\}$  of infinite, almost disjoint subsets  $L_\alpha$  of  $\omega$  without repetitions. Since  $Br(T) = {}^\omega 2$  and  $|{}^\omega 2| = 2^{\aleph_0}$ , we can also find a family  $\{Y_\alpha \subset Br(T) : \alpha < \lambda\}$  of sets  $Y_\alpha$  of branches with the following additional properties

(b1)  $|Y_\alpha| = \lambda$  for all  $\alpha < \lambda$ .

(b2)  $Y_\alpha$  has  $\lambda$  branch points above every level:

If  $\eta \in Y_\alpha$  and  $n \in \omega$ , there are  $\lambda$  distinct branches  $\nu \in Y_\alpha$  with  $\eta \upharpoonright n = \nu \upharpoonright n$ .

(b3) The length of a branch point of branches in  $Y_\alpha$  is in  $L_\alpha$  :

If  $\nu \neq \eta \in Y_\alpha$ , then  $\ell(\nu \cap \eta) \in L_\alpha$ .

We use these three families to enumerate a family of  $R$ -modules  $A_{XY}$  constructed in Section 2 defining  $A_\alpha = A_{X_\alpha Y_\alpha}$  for all  $\alpha < \lambda$ . Moreover we denote  $R_{X_\alpha} = R_\alpha$  the localization of  $R$  at the primes  $X_\alpha$  from Section 2.

Inductively we define an ascending, continuous chain of  $R$ -modules  $G_\alpha$  ( $\alpha < \lambda$ ) with distinguished cyclic submodules  $c_\alpha R \subset G_\alpha$  for non-limit ordinals  $\alpha < \lambda$ . The module we are interested in will then be the  $R$ -module  $G = G_\lambda = \bigcup_{\alpha < \lambda} G_\alpha$ . If  $\alpha = 0$ , let  $G_0 = \bigoplus_{\nu < \lambda} e_\nu R$  be free  $R$ -module of rank  $\lambda$ , which is also a free abelian group of rank  $\lambda$  because  $R^+$  is free of rank  $< \lambda$ . We will choose elements  $c_\alpha \in G_\alpha$  for non-limit ordinals  $\alpha$  subject to the following conditions

(c1)  $G_\alpha/c_\alpha R$  is an  $\aleph_1$ -free  $R$ -module

(c2) If  $c \in G$  and  $G/cR$  is an  $\aleph_1$ -free  $R$ -module, then  $|\{\alpha < \lambda : c = c_\alpha\}| = \lambda$ .

The extension  $G_{\alpha+1}$  will be constructed such that condition (c1) ensures that  $G$  is  $\aleph_1$ -free and (c2) can easily be arranged by an enumeration of elements  $c \in G_\alpha$  with  $G_\alpha/cR$   $\aleph_1$ -free with  $|\alpha|$  repetitions for all  $\alpha < \lambda$ . If  $\alpha = 0$ , then for (c1) we may choose a basic element  $c_0$  and we do not care for (c2).

If  $c_\nu \in G_\nu$  are defined for all  $\nu < \alpha$  and  $\alpha$  is a limit, then  $G_\alpha = \bigcup_{\nu < \alpha} G_\nu$  by continuity and it remains to construct  $G_\alpha$  from  $c_\beta \in G_\beta$  for  $\alpha = \beta + 1$ . From our choice (c1) of  $c_\beta$  we know that  $G_\beta/c_\beta R$  is an  $\aleph_1$ -free  $R$ -module. We consider a pushout diagram. There exists a (unique) pushout  $R$ -module  $G_\alpha$  with the well-known pushout mapping properties [14, p.52] or [25] in case of  $R$ -modules.

$$\begin{array}{ccc} c_\beta R & \longrightarrow & G_\beta \\ \downarrow & & \downarrow \\ A_\beta & \longrightarrow & G_\alpha \end{array}$$

The first row is the canonical embedding and the first column is an embedding by the identification  $c_\beta = \perp$ . By the pushout property we now may assume that

$$(p_\alpha) \quad G_\alpha = A_\beta + G_\beta \text{ and } A_\beta \cap G_\beta = c_\beta R$$

hence  $G_\alpha/c_\beta R \cong G_\beta/c_\beta R \oplus A_\beta/\perp R$ . The construction of  $G$  is complete.

First we will discuss freeness properties of  $G$ .

**Lemma 3.1.**  *$G$  is an  $\aleph_1$ -free  $R$ -module of cardinality  $\lambda$ .*

**Proof** If  $G = \bigcup_{\alpha < \aleph_1} G_\alpha$  as above, then we only have to show that  $G_\alpha$  is  $\aleph_1$ -free for any  $\alpha$  which we prove by induction. Since  $G_0$  is free we consider  $\alpha > 0$  and assume that all  $G_\beta$  ( $\beta < \alpha$ ) are  $\aleph_1$ -free. If  $\alpha = \beta + 1$ , then  $G_\alpha = A_\beta + G_\beta$  and  $(p_\alpha)$  holds, hence

$$G_\alpha/c_\beta R \cong G_\beta/c_\beta R \oplus A_\beta/\perp R.$$

The right hand side is  $\aleph_1$ -free by Lemma 2.3 and assumption on the choice of  $c_\beta$ . However, if  $G_\alpha/c_\beta R$  is  $\aleph_1$ -free, then  $G_\alpha$  must be  $\aleph_1$ -free as well.

If  $\alpha$  is a limit ordinal, then any subgroup of finite rank in  $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$  is a subgroup of  $G_\beta$  for some  $\beta < \alpha$  and  $\aleph_1$ -freeness follows.

The following observation plays a role in our next proposition, which provides splittings of  $G$  coming from Proposition 2.5 and is based on

$$R_\alpha \cap R_\beta \text{ is divisible by all primes not in } X_\alpha \cap X_\beta \text{ which is finite for } \alpha \neq \beta.$$

**Proposition 3.2.** *If  $G = \bigcup_{\alpha < \lambda} G_\alpha$  is the  $R$ -module above, then  $G_\alpha \otimes R_\beta$  is a free  $R_\beta$ -module for all  $\alpha \leq \beta < \lambda$ .*

**Proof** If  $\alpha < \beta$ , then  $(G_{\alpha+1} \otimes R_\beta)/(G_\alpha \otimes R_\beta) = (G_{\alpha+1}/G_\alpha) \otimes R_\beta$  because  $R_\beta$  is a flat  $R$ -module. We also have  $G_{\alpha+1}/G_\alpha = A_\alpha/c_\alpha R$  by the pushout property  $(p_{\alpha+1})$  and  $(A_\alpha/c_\alpha R) \otimes R_\beta$  is a free  $R_\beta$ -module by  $\alpha \neq \beta$  and Proposition 2.5. We derive that  $(G_{\alpha+1} \otimes R_\beta)/(G_\alpha \otimes R_\beta)$  is a free  $R_\beta$ -module, hence projective and the rest follows inductively by an obvious basis collection. Taking into account that  $G_0 \otimes R_\beta$  is a free  $R_\beta$ -module, the same holds for  $G_\alpha \otimes R_\beta$ .

**Proposition 3.3.** *With the notation as above we have*

- (a)  $A_\beta \otimes R_\beta$  is a direct summand of  $G_{\beta+1} \otimes R_\beta$
- (b)  $G_{\beta+1} \otimes R_\beta$  is a direct summand of  $G \otimes R_\beta$
- (c)  $A_\beta \otimes R_\beta$  is a direct summand of  $G \otimes R_\beta$ .

**Proof** Obviously (c) follows from (a) and (b) and it remains to show the first two assertions.

(a) Observe that  $G_\beta \otimes R_\beta$  is free by Proposition 3.2 and we may write  $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus H_\beta$  as  $R_\beta$ -modules by our choice of  $c_\beta$ . The pushout property  $(p_{\beta+1})$  gives  $G_{\beta+1} \otimes R_\beta = H_\beta \oplus (A_\beta \otimes R_\beta)$  and (a) follows.

(b) Inductively we will find an ascending, continuous chain of complements  $C_\alpha$  of  $G_{\beta+1} \otimes R_\beta$  in  $G_\alpha \otimes R_\beta$  for  $\beta+1 \leq \alpha \leq \lambda$  and  $C_\lambda$  will verify (b). If  $\alpha = \beta+1$ , then  $C_\alpha = 0$  and if  $\alpha$  is a limit ordinal between  $\beta+1$  and  $\lambda$  and all  $C_\gamma (\gamma < \alpha)$  are defined, then  $C_\alpha = \bigcup_{\gamma < \alpha} C_\gamma$  is already defined by continuity and  $C_\alpha$  is a complement of  $G_{\beta+1} \otimes R_\beta$  in  $G_\alpha \otimes R_\beta$  indeed, because  $G_\gamma \otimes R_\beta (\gamma \leq \alpha)$  is continuous at  $\alpha$  as well. It remains to define  $C_\alpha$  for  $\alpha = \gamma+1$  where  $C_\gamma$  is given. We are in the case  $\alpha > \beta+1$ , hence  $\gamma > \beta$  and  $\gamma \neq \beta$  follows. From Proposition 2.5 we see that  $c_\beta R_\beta = \perp R_\beta$  is a summand of the free  $R_\beta$ -module  $A_\gamma \otimes R_\beta$  and we may write  $A_\gamma \otimes R_\beta = c_\gamma R_\beta \oplus D_\gamma$ . Obviously  $C_\alpha = C_\gamma \oplus D_\gamma$  is a complement of  $G_{\beta+1} \otimes R_\beta$  in  $G_\alpha \otimes R_\beta$  by the pushout property  $(p_{\beta+1})$ .

#### 4. PROOF OF THE MAIN THEOREM

The main result of this paper is the following

**Theorem 4.1.** *If  $R$  is a ring with  $R^+$  free and  $|R| < \lambda \leq 2^{\aleph_0}$ , then there exists an  $\aleph_1$ -free abelian group  $G$  of cardinality  $\lambda$  with  $\text{End } G = R$ .*

Remark:  $G$  will be the  $R$ -module constructed in Section 3 and we have identified  $r \in R$  with  $r \cdot id_G$ .

**Proof** From Lemma 3.1 we have an  $R$ -module  $G$  of cardinality  $\lambda$  which is  $\aleph_1$ -free as  $R$ -module, hence  $\aleph_1$ -free as abelian group. Moreover  $R \subseteq \text{End } G$  by our identification and we must show that

$\varphi \in \text{End } G \setminus R$  does not exist.

Such a homomorphism  $\varphi$  has a unique extension  $\hat{\varphi} : G \otimes R_\beta \rightarrow G \otimes R_\beta$  because  $\hat{\varphi} = \varphi \otimes id$  extends and  $G \otimes R_\beta / G = (G \otimes R_\beta) / (G \otimes R) \cong R_\beta / R$  being torsion forces uniqueness.

If  $c_\alpha \varphi \in c_\alpha R$  for all  $\alpha < \lambda$ , then  $c_\alpha \varphi = c_\alpha r_\alpha$  for some  $r_\alpha \in R$ . If  $\alpha < \lambda$  is fixed, we can choose an element  $c \in G$  (even in  $G_0$ ) such that  $G/cR$  is an  $\aleph_1$ -free  $R$ -module  $cR \oplus c_\alpha R$  is a direct sum and  $G/(c + c_\alpha)R$  is an  $\aleph_1$ -free  $R$ -module as well. There exist some  $\gamma, \delta < \lambda$  with  $c = c_\gamma$  and  $c + c_\alpha = c_\delta$ . We have

$$c_\gamma r_\gamma + c_\alpha r_\alpha = c_\gamma \varphi + c_\alpha \varphi = (c_\gamma + c_\alpha) \varphi = c_\delta \varphi = c_\delta r_\delta = c_\gamma r_\delta + c_\alpha r_\delta$$

and  $r_\gamma = r_\delta = r_\alpha$  follows. We find a uniform  $r \in R$  such that  $c_\alpha \varphi = c_\alpha r$  for all  $\alpha < \lambda$ . However,  $G$  is generated by the set  $\{c_\alpha : \alpha < \lambda\}$ , hence  $\varphi = r$  which was excluded.

There exists  $\alpha < \lambda$  such that  $c_\alpha \varphi \notin c_\alpha R$ . We also find  $\gamma > \alpha$  such that  $c_\alpha \varphi \in G_\gamma$  and the repetition (c2) (Section 2) of the enumeration of  $c_\alpha$ 's provides  $\gamma < \beta < \lambda$  such that  $c_\beta = c_\alpha$ , hence

(i)  $c_\beta \varphi \notin c_\beta R$  and  $c_\beta \varphi \in G_\beta$ .

However,  $G_\beta \otimes R_\beta$  is a free  $R_\beta$ -module by Proposition 3.2 and  $c_\beta$  is a basic element of the  $R_\beta$ -module  $G_\beta \otimes R_\beta$ ; we find a free decomposition  $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus C$ . The pushout  $G_{\beta+1} = G_\beta + A_\beta$  gives  $G_{\beta+1} \otimes R_\beta = (A_\beta \otimes R_\beta) \oplus C$  and Proposition 3.3(b) provides an  $R_\beta$ -module  $D$  such that  $L = C \oplus D$  satisfies

(ii)  $(A_\beta \otimes R_\beta) \oplus L = G \otimes R_\beta$ ,  $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus C$

where  $C = L \cap (G_\beta \otimes R_\beta)$  by the modular law.

The element  $c_\beta \varphi \in G_\beta \subseteq G_\beta \otimes R_\beta$  has a unique decomposition  $c_\beta \varphi = c_\beta r + c$  with  $r \in R_\beta$  and  $c \in C$ . If  $c = 0$ , then  $c_\beta \varphi \in c_\beta R_\beta \cap G_\beta = c_\beta R$  by purity of  $c_\beta$  is a contradiction. Hence  $0 \neq c \in C$  which is a free  $R_\beta$ -module with a basis  $B$ . The element  $c = \sum_{b \in [c]} bc_b$  has a unique decomposition and a  $B$ -support  $[c] = \{b \in B : c_b \in R_\beta \setminus \{0\}\} \neq \emptyset$ .

On the other hand  $c \in C \subseteq G_\beta \otimes R_\beta$  and  $cm = \sum_{[c]} bc_b m \in G_\beta \cap C$  for some  $m \neq 0$ . However  $G_\beta \cap C \subset G_\alpha$  for some  $\alpha < \beta$ , which is contained in the free  $R_\alpha$ -module  $G_\alpha \otimes R_\alpha$ . Since  $\alpha \neq \beta$ , our choice of  $R_\alpha, R_\beta$  provides an  $h < \omega$  such that

(iii)  $p_j$  does not divide  $c \in C$  for all  $j > h$ ,

where the enumeration of primes is taken in  $X_\beta = \{p_n : n < \omega\}$ .

If  $\pi : G_{\beta+1} \otimes R_\beta \rightarrow C$  denotes the canonical projection induced by (ii), then

(iv)  $0 \neq c = c_\beta \varphi \pi$ .

Moreover, the image  $\eta \varphi \pi$  of any  $\eta \in Y_\beta$  viewed as  $\eta \in A_\beta \otimes R_\beta \subseteq G_{\beta+1} \otimes R_\beta$  can be expressed by

$$\eta \varphi \pi = \sum_{b \in [\eta]} br_b^\eta \text{ with } r_b^\eta \in R_\beta \setminus \{0\}$$

with a finite subset  $[\eta]$  of  $B$ . Abusing notation we shall call  $[\eta]$  the  $B$ -support of  $\eta$  as well. Recall that  $|Y_\beta| = \lambda > |R_\beta| \geq \aleph_0$ , and it is easy to find  $Y' \subseteq Y_\beta$ ,  $n \in \mathbb{N}$  and  $r_b \in R_\beta$  for all  $b \in B$  such that  $|Y'| = \lambda$  and  $|\eta| = n$ ,  $r_b^\eta = r_b$  for all  $\eta \in Y'$  and  $b \in B$ . Next we apply the  $\Delta$ -Lemma to  $\{[\eta] : \eta \in Y'\}$  (cf. Jech [22, p.225]) and find  $Y'' \subseteq Y'$ ,  $E \subset B$  such that  $|Y''| = \lambda$  and  $[\eta] \cap [\eta'] = E$  for all  $\eta \neq \eta' \in Y''$ . Since  $[c] \subset B$  is finite, we also find  $Y \subset Y''$  such that  $|Y| = \lambda$  and  $[\eta] \cap [c] \subseteq E$  for all  $\eta \in Y$ .

From  $|Y| = \lambda > \aleph_0$  we find two distinct branches  $\eta, \eta' \in Y$  with  $\eta \upharpoonright h = \eta' \upharpoonright h$ . The branch point  $j > h$  of  $\eta, \eta'$  belongs to  $L_\beta$  by (b3), hence  $p_j \in X_\beta$ , where  $j$  is from the enumeration along branches. The definition branch point gives  $\eta \upharpoonright j = \eta' \upharpoonright j$  and  $\eta(j) = 1$ ,  $\eta'(j) = 0$  without loss of generality. From the relations  $(X_\beta)$  in  $A_\beta$  (Section 2) we have  $p_j | (\eta'_j + \eta' \upharpoonright j + \eta'(j) \perp)$  in  $A_\beta$  and  $p_j | (\eta'_j + \eta' \upharpoonright j + \eta'(j) \perp)$  in  $A_\beta$ , hence  $p_j | \eta_j - \eta'_j + \eta(j) \perp = \eta_j - \eta'_j + c_\beta$  in  $G_\beta$  and therefore  $p_j | (\eta_j \varphi \pi - \eta'_j \varphi \pi) + c_\beta \varphi \pi$ . However  $[c] = [c_\beta \varphi \pi]$  and if  $d = \eta_j \varphi \pi - \eta'_j \varphi \pi$ , then  $d \upharpoonright E = 0$  by our choice of  $\eta, \eta' \in Y$  with  $\eta \neq \eta'$ , hence  $d$  and  $c$  are linearly independent. We conclude  $p_j | c$  in  $C$  which contradicts (iii) and Theorem 4.1 follows.

## 5. A COUNTEREXAMPLE

The reader might suspect that  $\aleph_1$  in Theorem 4.1 can be replaced by  $\aleph_2$  for instance. This is the case if we assume prediction principles as  $\diamond$  (which imply CH), see Dugas, Göbel [7]. However, in general it is no longer true as follows from

**Theorem 5.1.** *Assuming Martin's axiom, any  $\aleph_2$ -free group of cardinality  $< 2^{\aleph_0}$  is separable.*

Recall that an  $\aleph_1$ -free group is separable if any pure cyclic subgroup is a summand. Preliminaries on (MA) can be seen in Jech [22] or Eklof, Mekler [13].

**Proof** If  $G$  is an  $\aleph_2$ -free group of cardinality  $|G| < 2^{\aleph_0}$ ,  $0 \neq e \in G$  pure in  $G$  and  $\sigma : e\mathbb{Z} \rightarrow \mathbb{Z}$  taking  $e\sigma = 1$ , then we must extend  $\sigma$  to an homomorphism  $\Phi : G \rightarrow \mathbb{Z}$ .



Let  $P = \{\varphi; \varphi : D_\varphi \rightarrow \mathbb{Z}, e \in D_\varphi, e\varphi = 1\}$  where  $D_\varphi$  is a pure and finitely generated subgroup of  $G$ . Obviously  $|P| < 2^{\aleph_0}$  from  $|G| < 2^{\aleph_0}$  and  $(P, \subseteq)$  is partially ordered by extensions of maps. Suppose for a moment that  $P$  satisfies the hypothesis for MA and  $P_g = \{\varphi \in P : g \in D_\varphi\}$  is dense for all  $g \in G$ . Then by MA there is a compatible set  $F \subseteq P$  such that  $F \cap P_g \neq \emptyset$  for all  $g \in G$ . So  $\bigcup F = \Phi$  is a partial homomorphism from  $G$  to  $\mathbb{Z}$ . Since  $F \cap P_g \neq \emptyset$ , also  $g \in \text{dom } \Phi$  for all  $g \in G$ , hence  $\Phi \in \text{Hom}(G, \mathbb{Z})$  and  $\Phi$  extends  $\sigma$  by definition of P. Thus it remains to show that  $(P, \subseteq)$  satisfies the hypothesis of MA:

In order to show that  $P_g$  is dense in  $P$ , we consider any  $\varphi \in P$  and find  $\varphi \subset \varphi' \in P$  such that  $g \in \text{dom } \varphi'$ . Since  $G$  is  $\aleph_2$ -free, there is  $D' \supseteq \text{dom } \varphi$  such that  $g \in D'$  and  $D'$  is pure and finitely generated in  $G$  by Pontrjagin's theorem. Recall that  $\text{dom } \varphi$  is pure in  $G$ , hence pure in  $D'$  and  $D'/\text{dom } \varphi$  must be finitely generated and torsion-free. We apply Gauß' theorem to see that  $D'/\text{dom } \varphi$  is free, hence  $D' = \text{dom } \varphi \oplus C$  for some  $C \subseteq D'$  with  $C \cong D'/\text{dom } \varphi$ . Now it is easy to extend  $\varphi$  to a homomorphism  $\varphi' : D' \rightarrow \mathbb{Z}$ . Finally, we must show that  $(P, \subseteq)$  satisfies ccc, the countable antichain condition. Let  $F \subseteq P$  be an uncountable subset of P. We must find two distinct elements  $\varphi_i \in F$  and  $\Phi \in P$  such that  $\varphi_i \subseteq \Phi$  for  $i = 1, 2$ . We may assume  $|F| = \aleph_1$ , hence  $(\sum_{\varphi \in F} \text{dom } \varphi)_* = U$ , the pure subgroup of  $G$  generated (purely) by all  $\text{dom } \varphi$  has cardinality  $\aleph_1$  and must be free by hypothesis on  $G$ . We select a basis  $B$  of  $U$  and replace any  $\varphi \in F$  by  $\varphi'$  with  $\text{dom } \varphi' = \langle B_\varphi \rangle \supseteq \text{dom } \varphi$  with a finite subset of  $B_\varphi$  of  $B$ . The argument given above allows to extend  $\varphi$  to a homomorphism  $\varphi'$ .

Clearly, it is enough to find two compatible elements  $\varphi_i$  in the new  $F$ . By the  $\Delta$ -Lemma (Jech [22, p.225]) we also find  $E \subset B$  and  $F' \subseteq F$  such that  $|F'| = \aleph_1$  and  $\text{dom } \varphi \cap \text{dom } \varphi' = E$  for all  $\varphi \neq \varphi' \in F'$ . By a pigeon-hole argument we can also find  $F'' \subseteq F'$  such that  $|F''| = \aleph_1$  and  $\varphi \upharpoonright E = \varphi' \upharpoonright E$  for all  $\varphi, \varphi' \in F''$ . Now it is clear that we can extend two of these maps  $\varphi, \varphi'$  to  $\text{dom } \varphi + \text{dom } \varphi'$  as required.

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