BORELSETS WITH LARGE SQUARES SH522

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Abstract. For a cardinal μ we give a sufficient condition $\oplus_\mu$ (involving ranks measuring existence of independent sets) for:

$\otimes_\mu$ if a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ contains a μ-square (i.e. a set of the form $A \times A$, $|A| = \mu$) then it contains a $2^{\aleph_0}$-square and even a perfect square.

And also for

$\otimes'_\mu$ if $\psi \in L_{\omega_1,\omega}$ has a model of cardinality μ then it has a model of cardinality continuum generated in a "nice", "absolute" way.

Assuming MA $+ 2^{\aleph_0} > \mu$ for transparency, those three conditions ($\oplus_\mu, \otimes_\mu$ and $\otimes' e_\mu$) are equivalent, and by this we get e.g. $\bigwedge_{\alpha<\omega_1}[2^{\aleph_0} \geq \aleph_\alpha \Rightarrow \neg\otimes_{\aleph_\alpha}]$, and also $\min\{\mu : \otimes_\mu\}$, if $< 2^{\aleph_0}$, has cofinality $\aleph_1$.

We deal also with Borel rectangles and related model theoretic problems.

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Annotated Content

§0 Introduction

[We explain results and history and include a list of notation.]

§1 The rank and the Borel sets

[We define some version of the rank for a model, and then $\lambda_\alpha(\kappa)$ is the first
λ such that there is no model with universe λ, vocabulary of cardinality $\leq \kappa$
and rank $< \alpha$. Now we prove that forcing does not change some ranks of
the model, can only decrease others, and c.c.c. forcing changes little. Now:
{1.9} (1.12) if a Borel or analytic set contains a $\lambda_{\omega_1}(\aleph_0)$-square then it contains a
perfect square; clearly this gives something only if the continuum is large,
{1.10} that is at least $\lambda_{\omega_1}(\aleph_0)$. On the other hand (in 1.13) if $\mu = \mu^{\aleph_0} < \lambda_{\omega_1}(\aleph_0)$
we have in some c.c.c. forcing extension of V: the continuum is arbitrarily
large, and some Borel set contains a μ-square but no μ^+-square. Lastly
{th1.12} (in **??**) assuming MA holds we prove exact results (e.g. equivalence of
conditions).]

§2 Some model theoretic problems

[When we restrict ourselves to models of cardinality up to the continuum,
{2.3} $\lambda_{\omega_1}(\aleph_0)$ is the Hanf number of $L_{\omega_1,\omega}$ (see 2.1). Also (in 2.4) if $\psi \in L_{\omega_1,\omega}$
has a model realizing many types (say in the countable set of formulas,
many means $\geq \lambda_{\omega_1}(\aleph_0)$) even after c.c.c. forcing, then

$$\{\{p : p \text{ a complete } \Delta\text{-type realized in } M\} : M \models \psi\}$$

{2.4} has two to the continuum members. We then (2.5) assume $\psi \in L_{\omega_1,\omega}$
has a two cardinal model, say for (μ, κ) and we want to find a $(\mu', \aleph_0)$-
model, we need $\lambda_{\omega_1}(\kappa) \leq \mu$. Next, more generally, we deal with $\bar{\lambda}$-cardinal
models (i.e. we demand that P^M_ζ have cardinality λ_ζ). We define ranks
{2.6} (2.8), from them we can formulate sufficient conditions for transfer theorem
and compactness. We can prove that the relevant ranks are (essentially)
preserved under c.c.c. forcing as in §1, and the sufficient conditions hold
for $\aleph_{\omega_1}$ under GCH.]

§3 Finer analysis of square existence

{3.2} [We (3.1,3.2) define for a sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of trees (i.e. closed
sets of the plane) a rank, degsq, whose value is a bound for the size of the
{3.3} square it may contain. We then (3.3) deal with analytic, or more generally
κ-Souslin relations, ?? patience incomplete-what has?? and use parallel
degrees. We then prove that statements on the degrees are related to the
existence of squares in κ-Souslin relations in a way parallel to what we have
{3.6} on Borel, using $\lambda_\alpha(\kappa)$. We then (3.7 – 3.11) connect it to the existence of
identities for 2-place colourings. In particular we get results of the form
"there is a Borel set B which contains a μ-square iff $\mu < \lambda_\alpha(\aleph_0)$" when
MA + $\lambda_\alpha(\aleph_0) < 2^{\aleph_0}$.]

§4 Rectangles

[We deal with the problem of the existence of rectangles in Borel and κ-Souslin relations. The equivalence of the rank (for models), the existence of perfect rectangles and the model theoretic statements is more delicate, but is done.]

§ 0. Introduction

We first review the old results (from §1, §2).
The main one is:

$(*)_1$ it is consistent, that for every successor ordinal $\alpha < \omega_1$, there is a Borel subset of ${}^{\omega}2 \times {}^{\omega}2$ containing an $\aleph_\alpha$-square but no perfect square.

In fact:

$(*)_1^+$ the result above follows from $\mathrm{MA} + 2^{\aleph_0} > \aleph_{\omega_1}$.

{1.1} For this we define (Definition 1.1) for any ordinal α a property $\mathrm{Pr}_\alpha(\lambda;\kappa)$ of the cardinals λ, κ. The maximal cardinal with the property of $\aleph_{\omega_1}$ (i.e. for every small cardinal, c.c.c. forcing adds an example as in $(*)_1$) is characterized (as $\lambda_{\omega_1}(\aleph_0)$ where $\lambda_\alpha(\kappa) = \min\{\lambda : \mathrm{Pr}_\alpha(\lambda;\kappa)\}$); essentially it is not changed by c.c.c. forcing; so in $(*)_1$:

$(*)_1'$ if in addition $\mathbf{V} = \mathbf{V}_0^{\mathbf{P}}$, where $\mathbf{P}$ is a c.c.c. forcing then $\lambda_{\omega_1}(\aleph_0) \le (\beth_{\omega_1})^{V_0}$.

{1.5} We will generally investigate $\mathrm{Pr}_\alpha(\lambda;\kappa)$, giving equivalent formulations (1.1 – 1.6),
{1.6} seeing how fast $\lambda_\alpha(\kappa)$ increases, e.g. $\kappa^{+\alpha} < \lambda_\alpha(\kappa) \le \beth_{\omega\times\alpha}(\kappa)$ (in 1.7, 1.8). For two variants we show: $\mathrm{Pr}^2_\alpha(\lambda;\kappa^+)(\alpha \le \kappa^+)$ is preserved by κ^+-c.c. forcing, $\mathrm{Pr}^l_\alpha(\lambda;\kappa^+) \Rightarrow \mathrm{Pr}_\alpha(\lambda;\kappa^+)$ and $\neg\,\mathrm{Pr}_\alpha(\lambda;\kappa^+)$ is preserved by any extension of the universe of set theory. Now $\mathrm{Pr}_{\omega_1}(\lambda;\aleph_0)$ implies that there is no Borel set as above
{1.9} (1.12) but if $\mathrm{Pr}_{\omega_1}(\lambda;\aleph_0)$ fails then some c.c.c. forcing adds a Borel set as above
{th1.12} (1.13). We cannot in $(*)_1$ omit some set theoretic assumption even for $\aleph_2$ - see
{1.13} **??**, 1.16 (add many Cohen reals or many random reals to a universe satisfying e.g. $2^{\aleph_0} = \aleph_1$, then, in the new universe, every Borel set which contains an $\aleph_2$-square, also contains a perfect square). We can replace Borel by analytic or even κ-Souslin (using $\mathrm{Pr}_{\kappa^+}(\kappa)$).

In §2 we deal with related model theoretic questions with less satisfactory results.
{2.1} By 2.1,2.3, giving a kind of answer to a question from [Sh:49],

$(*)_2$ essentially $\lambda = \lambda_{\omega_1}(\aleph_0)$ is the Hanf number for models of sentences in $L_{\omega_1,\omega}$ when we restrict ourselves to models of cardinality $\le 2^{\aleph_0}$. (What is the meaning of "essentially"? If $\lambda_{\omega_1}(\aleph_0) \ge 2^{\aleph_0}$ this fails, but if $\lambda_{\omega_1}(\aleph_0) < 2^{\aleph_0}$ it holds.)

{2.3} In 2.4 we generalize it (the parallel of replacing Borel or analytic sets by κ-Souslin).
{2.3} We conclude (2.4(2)):

$(*)_3$ if $\psi \in L_{\omega_1,\omega}(\tau_1), \tau_0 \subseteq \tau_1$ are countable vocabularies, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1,\omega}(\tau_0)\}$ is countable and ψ has a model which realizes $\ge \lambda_{\omega_1}(\aleph_0)$ complete $(\Delta,1)$-types <u>then</u> $|\{(M{\restriction}\tau_0)/\cong : M \models \psi,\ \|M\| = \lambda\}| \ge \min\{2^\lambda, \beth_2\}$ (for any λ), as we have models as in [Sh:a, ChVII,§4] = [Sh:c, ChVII,§4].

If we allow parameters in the formulas of Δ, and $2^{\aleph_0} < 2^{\aleph_1}$ <u>then</u> $(*)_3$ holds too. However even in the case $2^\lambda = 2^{\aleph_0}$ we prove some results in this direction, see [Sh:262] (better [Sh:e, Ch.VII,§5]. We then turn to three cardinal theorems etc. trying to continue [Sh:49] (where e.g. $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$ was proved).

We knew those results earlier than, or in 1980/1, but failed in efforts to prove the consistency of "ZFC $+\lambda_{\omega_1}(\aleph_0) > \aleph_{\omega_1}$" (or proving ZFC $\vdash$ "$\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}$").

By the mid seventies we knew how to get consistency of results like those in §2 (forcing with $\mathbf{P}$, adding many Cohen reals i.e. in $\mathbf{V}^{\mathbf{P}}$ getting $(*)_3$ for $\lambda = (\beth_{\omega_1})^{\mathbf{V}}$). This (older proof, not the one used) is closely related to Silver's proof of "every Π^1_1-relation with uncountably many equivalence classes has a $2^{\aleph_0}$ ones" (a deeper one is the proof of Harrington of the Lauchli-Halpern theorem; see a generalization of the Lauchli-Halpern theorem, a partition theorem on ${}^{\kappa>}2$, κ large by [Sh:288, §4]).

In fact, about 88 I wrote down for W. Hodges proofs of (a) and (b) stated below.

(a) If, for simplicity, $\mathbf{V}$ satisfies GCH, and we add $> \aleph_{\omega_1}$ Cohen reals then the Hanf number of $L_{\omega_1,\omega}$ below the continuum is $\aleph_{\omega_1}$.

(b) If $\psi \in L_{\omega_1,\omega}(\tau_1)$ and some countable $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1,\omega}(\tau_0)\}$ satisfies: in every forcing extension of $\mathbf{V}$, ψ has a model which realizes $2^{\aleph_0}$ (or at least $\min\{2^{\aleph_0}, \aleph_{\omega_1}\}$) complete Δ-types then the conclusion of $(*)_3$ above holds.

Hodges had intended to write it up. Later Hrushovski and Velickovic independently proved the statement (a).

As indicated above, the results had seemed disappointing as the main question "is $\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}$?" is not answered. But Hjorth asked me about (essentially) $(*)_1$ which was mentioned in [HrSh:152] and urged me to write this down.

In §3 we define degree of Borel sets of the forms $\bigcup\limits_{n<\omega} \lim T_n \subseteq {}^\omega 2 \times {}^\omega 2$ measuring how close are they to having perfect squares, similarly we define degrees for κ-Souslin relations, and get results similar to earlier ones under MA and nail the connection between the set of cardinalities of models of $\psi \in L_{\omega_1,\omega}$ and having squares. In §4 we deal with the existence of rectangles.

We can replace $\mathbb{R}^2$ by $\mathbb{R}^3$ without any difficulty.

In a subsequent paper [Sh:532] which we are writing, we intend to continue the present work and in ?? [Sh:202, §5] and deal with: consistency of the existence of co-κ-Souslin (and even Π^1_2-) equivalence relations with many equivalence classes relationship of $\lambda^1_{\omega_1}, \lambda^1_{\omega_1}$ etc., and also try to deal with independence (concerning {2.9}
2.11 and 4.11(1)) and the existence of many disjoint sections. {4.9}

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§ 0(A). **Notation.**

Set theory:

${}^B A = \{f : f \text{ is a function from } B \text{ to } A\}$, the set of reals is ${}^\omega 2$.

$\mathscr{S}_{<\kappa}(A) = [A]^{<\kappa} = \{B \subseteq A : |B| < \kappa\}$.

By a Borel set B we mean the set it defines in the current universe. A μ-square (or a square of size μ) is a set of the form $A \times A$, where $A \subseteq {}^\omega 2, |A| = \mu$. A (μ_1, μ_2)-rectangle (or rectangle of size (μ_1, μ_2)) is a set of the form $A_1 \times A_2$, for some $A_l \subseteq {}^\omega 2, |A_l| = \mu_l$ (for $l = 0, 1$). A perfect square is $\mathscr{P} \times \mathscr{P}, \mathscr{P} \subseteq {}^\omega 2$ perfect. A perfect rectangle is $\mathscr{P}_1 \times \mathscr{P}_2, \mathscr{P}_l \subseteq {}^\omega 2$ perfect. Note: A perfect rectangle is a $(2^{\aleph_0}, 2^{\aleph_0})$-rectangle.

Note: A perfect square is a $2^{\aleph_0}$-square.

$\mathscr{P}, \mathscr{Q}$ denote perfect sets; $\mathbf{f}, boldP, \mathbf{Q}$ denote forcing notions; P, Q, R denote predicates.
A κ-Souslin set is $\{\eta \in {}^\omega 2 :$ for some ν we have $(\eta, \nu) \in \lim(T)\}$ for some $(2, \kappa)$-tree T (see below). A κ-Souslin relation (say an n-place relation) is defined similarly. For $\bar\lambda = \langle \lambda_\zeta : \zeta < \zeta(*)\rangle$, a $\bar\lambda$-tree is

$$T \subseteq \bigcup_n \prod_{\zeta<\zeta(*)} {}^n(\lambda_\zeta), \text{ ordered by } \bar\eta \triangleleft \bar\nu \Leftrightarrow \bigwedge_{\zeta<\zeta(*)} \eta_\zeta \text{ are right fine now } \triangleleft \nu_\zeta.$$

We usually let $\bar\eta \restriction \ell = \langle \eta_\zeta \restriction \ell : \zeta < \zeta(*)\rangle$.

For a $\bar\lambda$-tree T we define

$$\lim(T) = \{\bar\eta \in \prod_{\zeta<\zeta(*)} {}^\omega(\lambda_\zeta) : n < \omega \Rightarrow \bar\eta \restriction n \in T\}$$

(where $\langle \eta_\zeta : \zeta < \zeta(*)\rangle \restriction n = \langle \eta_\zeta \restriction n : \zeta < \zeta(*)\rangle$) and

$$\begin{aligned}\lim{}^*(T) = \{\bar\eta \in \prod_{\zeta<\zeta(*)} {}^\omega(\lambda_\zeta) : \quad & (\exists \bar\eta' \in \lim T)(\exists k < \omega) \\ & (\bigwedge_{\zeta<\zeta(*)} \eta_\zeta \restriction [k,\omega) = \eta'_\zeta \restriction [k,\omega))\}.\end{aligned}$$

We will use mainly $(2,2)$-trees and $(2,2,\kappa)$-trees; in particular, $\zeta(*)$ is finite.

Let $\eta \sim_n \nu$ mean that η, ν sequences of ordinals, $\ell g(\eta) = \ell g(\nu)$ and

$$(\forall k)[n \le k < \lg(\eta) \Rightarrow \eta(k) = \nu(k)].$$

For a tree T as above, $u \subseteq \zeta(*)$ and $n < \omega$ let

$$\begin{aligned}T^{(\sim_n, u)} = \{\bar\eta : \quad & (\exists k)(\exists \bar\nu \in \lim(T))[\bar\nu \in \prod_{\zeta<\zeta(*)} {}^\omega(\lambda_\zeta) \text{ and} \\ & \bar\eta \in \prod_{\zeta<\zeta(*)} {}^k(\lambda_\zeta) \text{ and } (\forall \zeta \in u)(\eta_\zeta \sim_n \nu_\zeta \restriction k)]\}.\end{aligned}$$

Let $\mathrm{Fr}_n(\lambda, \mu, \kappa)$ mean: if F_α are n-place functions from λ to λ (for $\alpha < \kappa$) then for some $A \in [\lambda]^\mu$ we have

for distinct $a_0, \ldots, a_n \in A$ and $\alpha < \kappa$ we have
$a_n \neq F_\alpha(a_0, \ldots, a_{n-1})$.

§ 0(B). **Model theory.**

Vocabularies are denoted by τ, so languages are denoted by e.g $L_{\kappa,\theta}(\tau)$, models are denoted by M, N. The universe of M is $|M|$, its cardinality $\|M\|$. The vocabulary of M is $\tau(M)$ and the vocabulary of $\mathbf{T}$ (a theory or a sentence) is $\tau(\mathbf{T})$. R^M is the interpretation of R in M (for $R \in \tau(M)$). For a model M, and a set $B \subseteq M$ we have: $a \in c\ell_{<\kappa}(B, M)$ iff for some quantifier free $\varphi = \varphi(y, x_1 \ldots x_n)$, and $b_1, \ldots, b_n \in B$ we have

$$M \models \varphi[a, b_1, \ldots, b_n] \text{ and } (\exists^{<\kappa} x)\varphi(x, b_1, \ldots, b_n).$$

Let $c\ell_\kappa(B, M) = c\ell_{<\kappa^+}(B, M)$ and $c\ell(B, M) = c\ell_{<2}(B, M)$. (Note: if M has Skolem functions then $c\ell_{<\aleph_0}(B, M) = c\ell_{<2}(B, M)$ for every $B \subseteq |M|$.) If κ is an ordinal we mean $|\kappa|$ (is needed just for phrasing absoluteness results that is if we use a cardinal κ in a universe V, and then deal with a generic extension V^P maybe in V^P, κ is no longer a cardinal but we like to still use it as a parameter). Let $\mathbf{T}$ denote a theory, first order if not said otherwise.

§ 1. The rank and the Borel sets

{1.1}

Definition 1.1. 1) For $\ell < 6$, and cardinals $\lambda \geq \kappa$, θ and an ordinal α, let $\mathrm{Pr}^\ell_\alpha(\lambda; < \kappa, \theta)$ mean that for every model M with the universe λ and vocabulary of cardinality $\leq \theta, \mathrm{rk}^\ell(M; < \kappa) \geq \alpha$ (defined below) and let $\mathrm{NPr}^\ell_\alpha(\lambda; < \kappa, \theta)$ be the negation. Instead of "$< \kappa^+$" we may write κ (similarly below); if $\kappa = \theta^+$ we may omit it (so e.g. $\mathrm{Pr}^\ell_\alpha(\lambda; \kappa)$ means $\mathrm{Pr}^\ell_\alpha(\lambda; < \kappa^+, \kappa)$); if $\theta = \aleph_0, \kappa = \aleph_1$ we may omit them.

Lastly, let $\lambda^\ell_\alpha(< \kappa, \theta) = \min\{\lambda : \mathrm{Pr}^\ell_\alpha(\lambda; < \kappa, \theta)\}$.

2) For a model $M, \mathrm{rk}^\ell(M; < \kappa) = \sup\{\mathrm{rk}^\ell(w, M; < \kappa)+1 : w \subseteq |M|$ finite non empty$\}$ where rk^ℓ is defined below in part (3).

3) For a model M, and $w \in [M]^* := \{u : u \subseteq |M|$ is finite nonempty$\}$ we shall define below the truth value of $\mathrm{rk}^\ell(w, M; < \kappa) \geq \alpha$ by induction on the ordinal α (note: if $c\ell_{<\kappa}(w, M) = c\ell_2(w, M)$ for every $w \in [M]^*$ then for $\ell = 0, 1, \kappa$ can be omitted).

Then we can note:

$(*)_0$ $\alpha \leq \beta$ and $\mathrm{rk}^\ell(w, M; < \kappa) \geq \beta \Rightarrow \mathrm{rk}^\ell(w, M; < \kappa) \geq \alpha$

$(*)_1$ $\mathrm{rk}^\ell(w, M; < \kappa) \geq \delta$ (δ limit)$\underline{iff} \bigwedge\limits_{\alpha<\delta} \mathrm{rk}^\ell(w, M; < \kappa) \geq \alpha$

$(*)_2$ $\mathrm{rk}^\ell(w, M; < \kappa) \geq 0 \underline{iff}\, w \in [M]^*$ and no $a \in w$ is in $c\ell_{<\kappa}(w \setminus \{a\}, M)$.

So we can define $\mathrm{rk}^\ell(w, M; < \kappa) = \alpha$ as the maximal α such that $\mathrm{rk}^\ell(w, M; < \kappa) \geq \alpha$, and ∞ if this holds for every α (and -1 whenever $\mathrm{rk}^\ell(w, M; < \kappa) \not\geq 0$).

Now the inductive definition of $\mathrm{rk}^\ell(\omega, M; < \kappa) \geq \alpha$ was already done above for $\alpha = 0$ (by $(*)_2$) and α limit (by $(*)_1$), so for $\alpha = \beta + 1$ we let

$(*)_3$ $\mathrm{rk}^\ell(\omega, M; < \kappa) \geq \beta + 1$ <u>iff</u> (letting $n = |w|$, $w = \{a_0, \ldots, a_{n-1}\}$) for every $k < n$ and a quantifier free formula $\varphi(x_0, \ldots, x_{n-1})$ (in the vocabulary of M) for which $M \models \varphi[a_0, \ldots, a_{n-1}]$ we have:

<u>Case 1</u>: $\ell = 1$. There are $a^i_m \in M$ for $m < n, i < 2$ such that:

- (a) $\mathrm{rk}^\ell(\{a^i_m : i < 2, m < n\}, M; < \kappa) \geq \beta$,
- (b) $M \models \varphi[a^i_0, \ldots, a^i_{n-1}]$ (for $i = 1, 2$), so without loss of generality there is no repetition in $a^i_0, \ldots, a^i_{n-1}$
- (c) $a^0_k \neq a^1_k$ but for $m \neq k$ (such that $m < n$) we have $a^0_m = a^1_m$.

<u>Case 2</u>: $\ell = 0$. As for $\ell = 1$ but in addition

- (d) $\bigwedge\limits_m a_m = a^0_m$

<u>Case 3</u>: $\ell = 3$. We give to κ an additional role and the definition is like case 1 but $i < \kappa$; i.e. there are $a^i_m \in M$ for $m < n, i < \kappa$ such that:

- (a) for $i < j < \kappa$ we have $\mathrm{rk}^\ell(\{a^i_m, a^j_m : m < n\}, M; < \kappa) \geq \beta$
- (b) $M \models \varphi[a^i_0, \ldots, a^i_{n-1}]$ (for $i < \kappa$; so without loss of generality there are no repetitions in $a^i_0, \ldots, a^i_{n-1}$)
- (c) for $i < j < \kappa, a^i_k \neq a^j_k$ but for $m \neq k$ (such that $m < n$) we have $\bigwedge\limits_{i,j<\kappa} a^i_m = a^j_m$

<u>Case 4</u>: $\ell = 2$. Like case 3 but in addition

(d) $a_m = a^0_m$ for $m < n$

Case 5: $\ell = 5$. Like case 3 except that we replace clause (a) by

(a)$^-$ for every function F, Dom $(F) = \kappa, |\mathrm{Rang}(F)| < \kappa$ for some $i < j < \kappa$ we have $F(i) = F(j)$ and $\mathrm{rk}^\ell(\{a^i_m, a^j_m : m < n\}, M; < \kappa) \geq \beta$.

Case 6: $\ell = 4$. Like case 4 (i.e. $\ell = 2$) using clause (a)$^-$ instead of clause (a).

We will actually use the above definition for $\ell = 0$ mainly. As the cardinal $\lambda^\ell_\alpha(< \aleph_1, \aleph_0) = \lambda^\ell_\alpha$ (for $\ell < 2$) may increase when the universe of set theory is extended (new models may be added) we will need some upper bounds which are preserved by suitable forcing. The case $\ell = 2$ provides one (and it is good: it does not increase when the universe is extended by a c.c.c forcing). The case $\ell = 4$ shows how much we can strengthen the definition, to show for which forcing notions lower bounds for the rank for $l = 0$ are preserved. Odd cases show that variants of the definition are immaterial.

Claim 1.2. *1) The truth of each of the statements of* $\mathrm{Pr}^\ell_\alpha(\lambda; < \kappa, \theta)$, $\mathrm{rk}^\ell(M; < \kappa) \geq \alpha$, $\mathrm{rk}^\ell(wM; < \kappa) \geq \alpha$ *is preserved if we replace* $\ell = 0, 2, 3, 2, 2, 2, 3, 5, 4$ *by* $\ell = 1, 3, 1, 0, 1, 4, 5,\ 1, 5$ *respectively (i.e.* $2 \to 4, 3 \to 5 \to 1, 0 \to 1, 2 \to 3, 4 \to 5, 3 \to 1, 2 \to 0, 2 \to 1$*) and also if we decrease* α, κ, θ *or increase* λ *(the last two only when* M *is not a parameter). So the corresponding inequality on* $\lambda^\ell_\alpha(< \kappa, \theta)$ *holds.* {1.2}

2) Also $\mathrm{rk}^\ell(w_1, M; < \kappa) \geq \mathrm{rk}^\ell(w_2, M; < \kappa)$ *for* $w_1 \subseteq w_2$ *from* $[M]^*$*.*

3) Also if we expand M*, the ranks (of* $w \in [M]^*$*, of* M*) can only decrease.*

4) If $A \subseteq M$ *is defined by a quantifier free formula with parameters from a finite subset* w^* *of* M*,* M^+ *is* M *expanded by the relations defined by quantifier free formulas with parameters from* $w^*, M^* = M^+ \restriction A$ *(for simplicity* M^* *has relations only)* then *for* $w \in [A]^*$ *such that* $w \nsubseteq w^*$ *we have* $\mathrm{rk}^\ell(w, M^*; < \kappa) \geq \mathrm{rk}^\ell(w \cup w^*, M; < \kappa)$*. Hence if* $w^* = \emptyset, \mathrm{rk}^\ell(M^*; < \kappa) \geq \mathrm{rk}^\ell(M; < \kappa)$*.*

5) In 1.1(3)$(*)_2$*, if in the definition of* $c\ell_{<\kappa}$ *we allow any first order formula, this means just expanding* M *by relations for any first order formula* $\varphi(\bar{x})$*.* {1.1}

6) For ℓ *odd,* $\mathrm{rk}^\ell(w, M; < \kappa) \geq (|\tau(M)| + \aleph_0)^+$ *implies* $\mathrm{rk}^\ell(u, M; < \kappa) = \infty$*.*

7) $\lambda^\ell_\alpha(< \kappa, \theta)$ *increases (*$\leq$*) with* α*,* θ *and decreases with* κ*.*

8) There is no difference between $\ell = 4$ *and* $\ell = 5$*.*

Proof. Check [e.g. for part (8), we can use function F such that $(\forall \alpha < \kappa)(F(0) \neq F(1 + \alpha)$]. $\square$

Claim 1.3. *1) For* $\ell = 0$*, if* $\alpha = \mathrm{rk}^\ell(M; < \kappa)$ $(< \infty)$ then *for some expansion* M^+ *of* M *by* $\leq \aleph_0 + |\alpha|$ *relations, for every* $w \in [M]^*$ *we have:* {1.3}

$$\mathrm{rk}^{\ell+1}(w, M^+; < \kappa) \leq \mathrm{rk}^\ell(w, M; < \kappa).$$

2) Similarly for $\ell = 2, 4$ *.*

3) If $\mathbf{V}_0$ *is a transitive class of* $\mathbf{V}_1$ *(both models of* ZFC*) and* $M \in \mathbf{V}_0$ *is a model then:*

(a) *for* $\ell < 4$

(α) $[\mathrm{rk}^\ell(w, M; < \kappa)]^{\mathbf{V}_0} \leq [\mathrm{rk}^\ell(w, M; < \kappa)]^{\mathbf{V}_1}$ *for* $w \in [M]^*$

(β) $[\mathrm{rk}^\ell(M; < \kappa)]^{\mathbf{V}_0} \leq [\mathrm{rk}^\ell(M; < \kappa)]^{\mathbf{V}_1}$

(γ) *if* $\ell = 0,1$ *equality holds in* $(\alpha),(\beta)$

(δ) $[\lambda^\ell_\alpha(\kappa)]^{\mathbf{V}_0} \le [\lambda^\ell_\alpha(\kappa)]^{\mathbf{V}_1}$ *if* $\ell = 0,1$.

(b) *Assume:*

(i) *for every* $f : \kappa \to \mathrm{Ord}$ *from* $\mathbf{V}_1$ *there is* $A \in [\kappa]^\kappa$ *such that* $f \restriction A \in \mathbf{V}_0$, *or at least*

(ii) *every graph* H *on* λ *from* $\mathbf{V}_0$ *which in* $\mathbf{V}_1$ *has a complete subgraph of size* κ, *has such a subgraph in* $\mathbf{V}_0$, *which holds if*

(ii)$^+$ $\mathbf{V}_1 = \mathbf{V}_0^{\mathbf{P}}$ *where* $\mathbf{P}$ *is a forcing notion satisfying the* κ*-Knaster Condition.* *Then* *for* $\ell = 2,3$, *in* $(\alpha),(\beta)$ *(of (a)) above equalities hold and the inequality in* (δ) *holds.*

(c) *Assume* $\mathbf{V}_1 = \mathbf{V}_0^{\mathbf{P}}$ *where* $\mathbf{P}$ *is* $\kappa - 2$*-linked.* *Then* *for* $\ell = 4,5$ *in clauses* $(\alpha),(\beta)$ *(of (a)) above we have equality and the inequality in* (δ) *holds.*

Proof. 1) For $\beta < \alpha, n < \omega$, a quantifier free formula $\varphi = \varphi(x_0,\dots,x_{n-1})$ and $k < n$ let

$$R^n_\beta = \{\langle a_0,\dots,a_{n-1}\rangle : \quad a_m \in M \text{ for } m < n \text{ and } \beta = \mathrm{rk}^\ell(\{a_0,\dots,a_{n-1}\}, M; <\kappa)\},$$

$$\begin{aligned} R^{n,k}_{\beta,\varphi} = \{\langle a_0,\dots,a_{n-1}\rangle \in R^n_\beta : \quad & M \models \varphi[a_0,\dots,a_{n-1}] \text{ for no} \\ & a^1_k \in |M| \setminus \{a_0,\dots,a_{n-1}\} \text{ we have} \\ & (\alpha) \quad M \models \varphi[a_0,\dots,a_{k-1},a^1_k,a_{k+1},\dots,a_{n-1}] \\ & (\beta) \quad \mathrm{rk}^\ell(\{a_m : m < n\} \cup \{a^1_k\}, M; <\kappa) \ge \beta\}, \end{aligned}$$

$$M^+ = (M, \dots R^n_\beta, R^{n,k}_{\beta,\varphi} \dots)_{\beta<\alpha, n<\omega, k<n, \varphi}$$

{1.8} Check (or see more details in the proof of 1.10 below).
2) Similarly.
{1.1} 3) The proof should be clear (for (b), looking at Definition 1.1 case 3 the graph is $\{(i,j)$: clause (a) there holds$\}$). $\square$

{1.3A}

{1.3} *Remark* 1.4. 1) In 1.3(1) we can omit "$\alpha = \mathrm{rk}^\ell(M; <\kappa)$" but then weaken the conclusion to $\mathrm{rk}^{\ell+1}(w, M^+; <\kappa) \le \mathrm{rk}^\ell(w, M; <\kappa)$ or both are $> \alpha$.
{1.3} 2) Similarly in 1.3(2).

{1.4} **Conclusion 1.5.** *1)* $\mathrm{Pr}^0_{\omega_1}(\lambda) \Leftrightarrow \mathrm{Pr}^1_{\omega_1}(\lambda) \Leftarrow \mathrm{Pr}^4_{\omega_1}(\lambda) \Leftrightarrow \mathrm{Pr}^5_{\omega_1}(\lambda) \Leftarrow \mathrm{Pr}^2_{\omega_1}(\lambda) \Leftrightarrow \mathrm{Pr}^3_{\omega_1}(\lambda)$.
2) If $\alpha \le \kappa^+$ *then* $\mathrm{Pr}^0_\alpha(\lambda;\kappa) \Leftrightarrow \mathrm{Pr}^1_\alpha(\lambda;\kappa) \Leftarrow \mathrm{Pr}^4_\alpha(\lambda;\kappa) \Leftrightarrow \mathrm{Pr}^5_\alpha(\lambda;\kappa) \Leftarrow \mathrm{Pr}^2_\alpha(\lambda;\kappa) \Leftrightarrow \mathrm{Pr}^3_\alpha(\lambda;\kappa)$.
3) For $\alpha \le \kappa^+$, $\lambda^\ell_\alpha(\kappa) = \lambda^{\ell+1}_\alpha(\kappa)$ *for* $\ell = 0,2,4$, *and* $\lambda^0_\alpha(\kappa) \le \lambda^4_\alpha(\kappa) \le \lambda^2_\alpha(\kappa)$.
4) For $\alpha \ge \kappa^+$ *and* $\ell = 0,2,4$ *we have* $\lambda^{\ell+1}_\alpha(\kappa) = \lambda^{\ell+1}_{\kappa^+}(\kappa)$.

Proof. 1) By 2).
2) For $\alpha = \kappa^+$ it follows from its holding for every $\alpha < \kappa^+$. For $\alpha < \kappa^+$; for
{1.3} $\ell = 0,2,4$ we know that $\mathrm{NPr}^\ell_\alpha(\lambda;\kappa) \Rightarrow \mathrm{NPr}^{\ell+1}_\alpha(\lambda;\kappa)$ by 1.3(1),(2), and $\mathrm{Pr}^\ell_\alpha(\lambda;\kappa) \Rightarrow$
{1.2} $\mathrm{Pr}^{\ell+1}_\alpha(\lambda;\kappa)$ by 1.2(1); together $\mathrm{Pr}^\ell_\alpha(\lambda;\kappa) \Leftrightarrow \mathrm{Pr}^{\ell+1}_\alpha(\lambda;\kappa)$. Now $\mathrm{Pr}^3_\alpha(\lambda;\kappa) \Rightarrow \mathrm{Pr}^5_\alpha(\lambda;\kappa) \Rightarrow$
{2.1} {1.2} $\mathrm{Pr}^1_\alpha(\lambda;\kappa)$ by 1.2(1), together we finish. (By 2.1 we know more.)
3) Follows from part (2) and the definition.
{1.2} 4) By 1.2(6). $\square$

{1.5}

Convention 1.6. Writing $\mathrm{Pr}_\alpha(\lambda;\kappa)$ for $\alpha \le \kappa^+$ (omitting ℓ) we mean $\ell = 0$. Similarly $\lambda_\alpha(<\kappa,\theta)$ and so $\lambda_\alpha(\kappa)$ etc. {1.6}

Claim 1.7. *Let $\ell \in \{0,2,4\}$.*
1) $\mathrm{NPr}^\ell_{\alpha+1}(\kappa^{+\alpha};\kappa)$.
2) If α is a limit ordinal $<\kappa^+$ (in fact, $\aleph_0 \le \mathrm{cf}(\alpha) < \kappa^+$ suffice), and $\mathrm{NPr}_\beta(\lambda_\beta;\kappa)$ for $\beta<\alpha$, then $\mathrm{NPr}^\ell_{\alpha+1}(\sum_{\beta<\alpha}\lambda_\beta;\kappa)$.
3) If $\mathrm{NPr}^\ell_\alpha(\lambda;\kappa)$ *then* $\mathrm{NPr}^\ell_{\alpha+1}(\lambda^+;\kappa)$.
4) If $\mathrm{NPr}^\ell_\alpha(\mu;\kappa)$ *for every $\mu<\lambda$ then* $\mathrm{NPr}^\ell_{\alpha+1}(\lambda;\kappa)$.

Proof. 1) Prove by induction on $\alpha<\kappa^+$, for $\alpha=0$ use a model in which every element is definable (e.g. an individual constant) so $\mathrm{rk}(w;M)=-1$ for $w\in[M]^*$ and hence $\mathrm{rk}^\ell(M)=0$ and consequently $\mathrm{NPr}^\ell_1(\kappa;\kappa)$; for α limit use part (2) and for α successor use part (3).
2) Let M_β witness $\mathrm{NPr}_\beta(\lambda_\beta;\kappa)$ for $\beta<\alpha$, i.e. $\mathrm{rk}^\ell(M_\beta;\kappa)<\beta$ and M_β has universe λ_β and $|\tau(M_\beta)|\le\kappa$. Without loss of generality $\langle\tau(M_\beta):\beta<\alpha\rangle$ are pairwise disjoint and disjoint to $\{P_\beta:\beta<\alpha\}$. Let M have universe $\lambda := \sum\limits_{\beta<\alpha}\lambda_\beta, P^M_\beta=\lambda_\beta$, and $M{\restriction}\lambda_\beta$ expand M_β and $|\tau(M)|\le|\alpha|+\sum\limits_{\beta\le\alpha}|\tau(M_\beta)|\le\kappa$. By 1.2(3),(4), for {1.2}
$w\in[\lambda_\beta]^*, \mathrm{rk}^\ell(w,M;\kappa)\le\mathrm{rk}^\ell(w,M_\beta;\kappa)<\beta\le\alpha$. But $w\in[|M|]^*$ implies $\bigvee\limits_{\beta<\alpha} w\in[\lambda_\beta]^*$. Clearly $\mathrm{rk}(M;\kappa)\le\alpha$ and hence $\mathrm{NPr}^\ell_{\alpha+1}(\lambda;\kappa)$.
3) We define M^+ such that each $\gamma\in[\lambda,\lambda^+)$ codes on $\{\zeta:\zeta<\gamma\}$ an example for $\mathrm{NPr}_\alpha(|\gamma|;\kappa)$. More elaborately, let M be a model with the universe λ such that $\mathrm{rk}^\ell(M;\kappa)<\alpha$. Let $\tau(M)$ be $\{R_i: i<i^*\le\kappa\}, R_i$ an $n(i)$-place predicate (as we replace function symbols and individual constants by predicates), R_0 is a 0-nary predicate representing "the truth". For $\gamma\in[\lambda,\lambda^+)$ let f_γ be a one-to-one function from γ onto λ. Define $\tau^+=\{R_i,Q_i: i<i^*\le\kappa\}, R_i$ is $n(i)$-place, Q_i is $(n(i)+1)$-place. So $|\tau^+|\le\kappa$. We define a τ^+-model M^+: the universe is λ^+, $R_i^{M^+}=R_i^M$, $Q_i^{M^+}=\{\langle\alpha_0,\dots,\alpha_{n(i)}\rangle:\alpha_{n(i)}\in[\lambda,\lambda^+)$ and $\bigwedge\limits_{\ell<n(i)}\alpha_\ell<\alpha_{n(i)}$ and $\langle f_{\alpha_{n(i)}}(\alpha_0),\dots,f_{\alpha_{n(i)}}(\alpha_{n(i)-1})\rangle\in R_i^M\}$ (so $Q_0^{M^+}=[\lambda,\lambda^+)$).
Now note that:

(a) for $w\in[\lambda]^*, \mathrm{rk}^\ell(w,M^+;\kappa)\le\mathrm{rk}^\ell(w,M;\kappa)$
(b) if $w\subseteq\gamma\in[\lambda,\lambda^+), w\ne\emptyset$ then $\mathrm{rk}^\ell(w\cup\{\gamma\},M^+;\kappa)\le\mathrm{rk}^\ell(f''_\gamma[w],M;\kappa)$.

(Easy to check). So if $\gamma<\lambda^+$ then

$(*)_1$ $\gamma<\lambda\Rightarrow\mathrm{rk}^\ell(\{\gamma\},M^+;\kappa)\le\mathrm{rk}^\ell(\{\gamma\},M;\kappa)<\mathrm{rk}^\ell(M;\kappa)$
$(*)_2$ $\gamma\in[\lambda,\lambda^+)\ \&\ \beta\ge\mathrm{rk}^\ell(M;\kappa)\ \Rightarrow\ \mathrm{rk}^\ell(\{\gamma\},M^+;\kappa)\le\beta$.

[Why $(*)_2$? Assume not and let $\kappa^0=2$, $\kappa^2=\kappa^4=\kappa^+$. If $\langle\gamma_i: i<\kappa^l\rangle$ strictly increasing witnesses $\mathrm{rk}^\ell(\{\gamma\},M^+)\ge\beta+1$ for the formula $Q_0(x)$ then for some $i<j<\kappa^l$ we have $\mathrm{rk}^\ell(\{\gamma_i,\gamma_j\},M^+)\ge\beta$ and applying (b) with $\{\gamma_i\}$, γ_j here standing for w, γ there we get $\mathrm{rk}^\ell(\{f_{\gamma_j}(\gamma_i)\},M)\ge\beta$ hence $\beta+1\le\mathrm{rk}^\ell(M)$, contradiction.]
Hence

$(*)_3$ $\mathrm{rk}^\ell(M^+;\kappa) \le \mathrm{rk}^\ell(M;\kappa)+1$.

As $\mathrm{rk}^\ell(M^+;\kappa) < \alpha$ clearly M^+ witnesses $\mathrm{NPr}_{\alpha+1}(\lambda^+;\kappa)$.
4) Like (3). $\square$

{1.7}
Conclusion 1.8. *Remembering that $\lambda_\alpha(\kappa) = \min\{\lambda : \mathrm{Pr}_\alpha(\lambda;\kappa)\}$ we have:*

(1) *for α a limit ordinal $\lambda_\alpha(\kappa) \le \beth_\alpha(\kappa)$ and even $\lambda^2_\alpha(\kappa) \le \beth_\alpha(\kappa)$*
(2) *for ℓ even $\langle \lambda^\ell_\alpha(\kappa) : 0 < \alpha < \infty\rangle$ is strictly increasing, and for a limit ordinal δ, $\lambda_\delta(\kappa) = \sup\limits_{\alpha<\delta} \lambda_\alpha(\kappa)$*
(3) *$\lambda_0(\kappa) = \lambda_1(\kappa) = \kappa, \lambda_2(\kappa) = \kappa^+, \kappa^{+n} \le \lambda_n(\kappa) < \kappa^{+\omega}$ and $\lambda_\omega(\kappa) = \kappa^{+\omega}$.*

{1.7A}
Remark 1.9. [[Saharon $\lambda^2_{\omega\times\alpha}(\kappa) \le \beth_{\omega\times\alpha}(\kappa)$]] $\lambda^2_{\omega\times\alpha}(\kappa) \le \beth_{\omega\times\alpha}(\kappa)$ is proved below essentially like the Morley omitting types theorem (see [Mor65] or see [**?**] or [Sh:a, Ch.VII,§5] = [Sh:c, Ch.VII,§5]).

Proof. 1) We prove by induction on α, that *for every* ordinal $\beta < \alpha$, model M, $|\tau(M)| \le \kappa$, and $A \subseteq |M|, |A| \ge \beth_{\omega\times\alpha}(\kappa)$, and $m, n < \omega$ *there is* $w \subseteq A$, $|w| = n$ such that $\mathrm{rk}^2(w, M;\kappa) \ge \omega\times\beta + m$.

For $\alpha = 0$, α limit this is immediate. For $\alpha = \gamma+1$ (and M, A, β, n, m as above), applying Erdös-Rado theorem we can find distinct $a_i \in A$ for $i < \beth_{\omega\times\gamma}(\kappa)^{++}$ such that:

(a) for all $i_0 < \ldots < i_{m+n}$ the quantifier free type $\langle a_{i_0}, \ldots, a_{i_{n+m}}\rangle$ in M is the same
(b) for each $k \le m+n$ for every $i_0 < \ldots < i_{n+m-k} < \beth_{\omega\times\gamma}(\kappa)$, the ordinal $\min\{\omega\times\alpha, \mathrm{rk}^2(\{a_{i_0}, \ldots, a_{i_{n+m-k}}\}, M;\kappa)\}$ is the same.

By the induction hypothesis, in clause (b) the value is $\ge \omega\times\gamma$. Hence we can prove, by induction on $k \le m+n$, that $\mathrm{rk}^2(\{a_{i_0}, \ldots, a_{i_{n+m-k}}\}, M;\kappa) \ge \omega\times\gamma+k$ whenever $i_0 < \ldots < i_{m+n-k} < \beth_{\omega\times\gamma}(\kappa)$. For $k=0$ this holds by the previous sentence, for $k+1$ use the definition and the induction hypothesis, for rk^2 note that by clause (b) without loss of generality $i_\ell + \kappa^+ < i_{\ell+1}$ and $a_{i_\ell+\zeta}$ for $\zeta < \kappa^+$ are well defined. For $k = m$ we are done.

{1.8} 2) It is increasing by 1.2(1), strict by 1.7(4), continuous because, for limit δ, as on the one hand $\lambda^\ell_\delta(\kappa) \ge \sup\limits_{\alpha<\delta} \lambda^\ell_\alpha(\kappa)$ as $\lambda^l_\delta(\kappa) \ge \lambda^\ell_\alpha(\kappa)$ for $\alpha < \delta$, and on the other hand if M is a model with universe $\lambda := \sup\limits_{\alpha<\delta} \lambda_\alpha(\kappa)$ and $|\tau(M)| \le \kappa$ then $\alpha < \delta \Rightarrow \mathrm{rk}^\ell(M;\kappa) \ge \mathrm{rk}^\ell(M \restriction \lambda_\alpha;\kappa) \ge \alpha$ hence $\mathrm{rk}^\ell(M;\kappa) \ge \delta$. So $\mathrm{Pr}_\alpha(\lambda;\kappa)$ hence $\lambda \ge \lambda^\ell_\delta(\kappa)$ so $\sup\limits_{\alpha<\delta} \lambda^l_\alpha(\kappa) = \lambda \ge \lambda^l_\delta$, together we are done.

3) By [Sh:49] (for the last two clauses, the first two clauses are trivial), will not be really used here. $\square$

{1.8}
Claim 1.10. *1) Assume $\mathbf{P}$ is a forcing notion satisfying the κ^+-c.c.. If $\mathrm{Pr}^3_\alpha(\lambda;\kappa)$ and $\alpha \le \kappa^+$, then this holds in $\mathbf{V}^{\mathbf{P}}$ too.*
2) If $\mathbf{P}$ is a $\kappa^+ - 2$-linked forcing notion (or just: if $p_i \in \mathbf{P}$ for $i < \kappa^+$ then for some $F : \kappa^+ \to \kappa$, $F(i) = F(j) \Rightarrow p_i$, p_j compatible), and $\alpha \le \kappa^+$ and $\mathrm{Pr}^5_\alpha(\lambda;\kappa)$ then this holds in $\mathbf{V}^{\mathbf{P}}$ too.

{1.8A}

Remark 1.11. 1) $\mathrm{NPr}_\alpha(\lambda;\kappa)$ is of course preserved by any extension as the ranks
{claim1.3} $\mathrm{rk}^\ell(M;\kappa)$, $\mathrm{rk}^\ell(w,M;\kappa)$ are absolute for $\ell=0,1$ (see **??**(3)). But the forcing can add new models.

2) So for $\alpha \le \kappa^+$, $\lambda_\alpha(\kappa) \le \lambda^4_\alpha(\kappa) \le \lambda^2_\alpha(\kappa)$ and a κ^+-c.c. forcing notion can only increase the first (by **??**(3)(a)(δ)) and decrease the third by 1.10(1); a κ^+-2-linked {claim1.3} {1.8}
one fixes the second and third (as it can only decrease it by 1.10(1) and can only {1.8}
increase it by **??**(3)(c)(a)(δ) + (b)(γ)). {claim1.3}

3) [[Of course]] We can deal similarly with $\mathrm{Pr}^\ell_\alpha(\lambda;<\kappa,\theta)$, here and in **??** – 1.8. {claim1.3} {1.7}

Proof. We can concentrate on 1), anyhow let $\ell=\{3,5\}$ (for part (1) we use $\ell=3$, for part (2) we shall use $\ell=5$, we shall return to it later). Assume $\mathrm{Pr}^3_\alpha(\lambda;\kappa)$ fails in $\mathbf{V}^{\mathbf{P}}$. So for some $p^*\in\mathbf{P}$ and $\alpha_0<\alpha$ we have:

$$p^* \Vdash_{\mathbf{P}} \text{“}\quad \underset{\sim}{M} \text{ is a model with universe } \lambda,\ \text{vocabulary } \underset{\sim}{\tau} \\ \text{of cardinality } \le\kappa \text{ and } \mathrm{rk}^\ell(\underset{\sim}{M};\kappa)=\alpha_0''.$$

Without loss of generality every quantifier free formula $\varphi(x_0,\dots,x_{n-1})$ is equivalent to one of the form $R(x_0,\dots,x_{n-1})$ and without loss of generality $\underset{\sim}{\tau}=\{R_{n,\zeta}: n<\omega,\zeta<\kappa\}$ with $R_{n,\zeta}$ an n-place predicate. Note that necessarily $\alpha_0<\kappa^+$ hence $|\alpha_0|\le\kappa$.

As we can replace $\mathbf{P}$ by $\mathbf{P}\restriction\{q\in\mathbf{P}: p^*\le q\}$, without loss of generality p^* is the minimal member of $\mathbf{P}$. Now for non zero $n<\omega, k<n, \zeta<\kappa$ and $\beta<\alpha_0$ (or $\beta=-1$) we define an n-place relation $R_{n,\zeta,\beta,k}$ on λ:

$$\begin{aligned} R_{n,\zeta,\beta,k}=\{\quad & \langle a_0,\dots,a_{n-1}\rangle : a_m\in\lambda \text{ with no repetitions and for some}\\ & p\in\mathbf{P},\\ & p\Vdash_{\mathbf{P}} \text{“}[\underset{\sim}{M}\models R_{n,\zeta}[a_0,\dots,a_{n-1}] \text{ and } \mathrm{rk}^\ell(\{a_0,\dots,a_{n-1}\},\underset{\sim}{M};\kappa)=\beta,\\ & \quad\text{where “not } \mathrm{rk}^3(\{a_0,\dots,a_{n-1}\},M;\kappa)\ge\beta+1''\\ & \quad\text{is witnessed by } \varphi=R_{n,\zeta} \text{ and } k]''\}.\end{aligned}$$

Let $M^+=(\lambda,\dots,R_{n,\zeta,\beta,k},\dots)_{n<\omega,\zeta<\kappa,\beta<\alpha_0,k<n}$, so M^+ is a model in $\mathbf{V}$ with the universe λ and the vocabulary of cardinality $\le\kappa$. It suffices to prove that for $\beta<\alpha_0$:

$\otimes_\beta$ if $w=\{a_0,\dots,a_{n-1}\}\in[M^+]^*$, $M^+\models R_{n,\zeta,\beta,k}[a_0,\dots,a_{n-1}]$
then $\mathrm{rk}^\ell(\{a_0,\dots,a_{n-1}\},\ M^+;\kappa)\le\beta$.

(Note that by the choice of $\underset{\sim}{M}$ and $R_{n,\zeta,\beta,k}$, if $w\in[M^+]^*$ then for some n,ζ,β,k we have $M^+\models R_{n,\zeta,\beta,k}[a_0,\dots,a_{n-1}]$). This we prove by induction on β, so assume the conclusion fails; so

$$\mathrm{rk}^\ell(\{a_0,\dots,a_{n-1}\},M^+;\kappa)\ge\beta+1$$

(and eventually we shall get a contradiction). By the definition of rk^3 applied to $\varphi=R_{n,\zeta,\beta,k},\beta$ and k we know that there are a^i_m (for $m<n, i<\kappa^+$) as in Definition 1.1(3) case $\ell=3$. In particular $M^+\models R_{n,\zeta,\beta,k}[a^i_0,\dots,a^i_{n-1}]$. So {1.1}
for each $i<\kappa^+$ by the definition of $R_{n,\zeta,\beta,k}$ necessarily there is $p_i\in\mathbf{P}$ such that $p_i\Vdash_{\mathbf{P}}$ “$\underset{\sim}{M}\models R_{n,\zeta}[a^i_0,\dots,a^i_{n-1}]$ and $\mathrm{rk}^\ell(\{a^i_0,\dots,a^i_{n-1}\},\underset{\sim}{M};\kappa)=\beta$ and [not $\mathrm{rk}^\ell(\{a^i_0,\dots,a^i_{n-1}\},\underset{\sim}{M};\kappa)\ge\beta+1$] is witnessed by $\varphi=R_{n,\zeta}$ and k”.

For part (1), as $\mathbf{P}$ satisfies the $\kappa^+ - cc$, for some $q \in \mathbf{P}, q \Vdash$ "$\underset{\sim}{Y} = \{i : p_i \in \underset{\sim}{G}_{\mathbf{P}}\}$ has cardinality κ^+" (in fact, p_i forces it for every large enough i). Looking at the definition of the rank in $\mathbf{V}^{\mathbf{P}}$ we see that $\langle\langle a^i_0, \dots, a^i_{n-1}\rangle : i \in \underset{\sim}{Y}\rangle$ cannot be a witness for "the demand for $\mathrm{rk}^3(\{a^{i_0}_0, \dots, a^{i_0}_{n-1}\}, \underset{\sim}{M}; \kappa) > \beta$ for $R_{n,\zeta,k}$ hold" for any (or some) $i_0 \in \underset{\sim}{Y}$, so for part (1)

(*) $q \Vdash_{\mathbf{P}}$ "for some $i \neq j$ in $\underset{\sim}{Y}$ we have $\mathrm{rk}^3(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, \underset{\sim}{M}; \kappa) < \beta$" (as the demand on equalities holds trivially).

As we can increase q, without loss of generality q forces a value to those i, j, hence without loss of generality for some $n(*) = n + 1 < \omega, \zeta(*) < \kappa$ and $\beta(*) < \beta$ and for $k(*) < n + 1$ we have

$$q \Vdash_{\mathbf{P}} \text{“}\quad \begin{array}{l} \mathrm{rk}^3(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, \underset{\sim}{M}; \kappa) = \beta(*), \text{ and} \\ \mathrm{rk}^3(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, M; \kappa) \ngeq \beta(*) + 1 \text{ is witnessed by} \\ \varphi = R_{n(*),\zeta(*)}(x_0, \dots, x_n) \text{ and } k(*)\text{”}. \end{array}$$

Hence by the definition of $R_{n(*),\zeta(*),\beta(*),k(*)}$ we have

$$M^+ \models R_{n(*),\zeta(*),\beta(*),k(*)}[a^i_0, \dots, a^i_{n-1}, a^j_k].$$

As $\beta(*) < \beta$ by the induction hypothesis $\otimes_{\beta(*)}$ holds hence

$$\mathrm{rk}^3(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, M^+; \kappa) \leq \beta(*),$$

but this contradicts the choice of $a^i_m (m < n,\ i < \kappa^+)$ above (i.e. clause (a) of
{1.1} Definition 1.1(3) case $\ell = 3$). This contradiction finishes the induction step in the
{1.8} proof of $\otimes_\beta$ hence the proof of 1.10(1).

For part (2), we have $\langle p_i : i < \kappa^+\rangle$ as above. In $\mathbf{V}^{\mathbf{P}}$, if $\underset{\sim}{Y} = \{i : p_i \in G_{\mathbf{P}}\}$ has cardinality κ^+, then $\langle\langle a^i_0, \dots, a^i_{n-1}\rangle : i \in \underset{\sim}{Y}\rangle$ cannot witness $\mathrm{rk}^5(\{a_0, \dots, a_{n-1}\}, M; \kappa) \geq \beta + 1$ so there is a function $\underset{\sim}{F}^0 : \underset{\sim}{Y} \to \kappa$ witnessing it; i.e. $\Vdash_{\mathbf{P}}$ "if $|\underset{\sim}{Y}| = \kappa^+$ then $i \in \underset{\sim}{Y}$ and $j \in \underset{\sim}{Y}$ and $i \neq j$ and $nameF^0(i) = \underset{\sim}{F}^0(j) \Rightarrow \beta > \mathrm{rk}^5(\{a^i_0, \dots, a^i_{n-1}\} \cup \{a^j_0, \dots, a^j_{n-1}\}, M; \kappa)$".

If $|\underset{\sim}{Y}| \leq \kappa$, let $\underset{\sim}{F}^0 : \underset{\sim}{Y} \to \kappa$ be one to one. Let $p_i \leq q_i \in \mathbf{P}, q_i \Vdash \underset{\sim}{F}^0(i) = \gamma_i$. As $\mathbf{P}$ is κ^+-2-linked, for some function $F^1 : \kappa^+ \to \kappa$ we have $(\forall i, j < \kappa^+)(F^1(i) = F^1(j) \Rightarrow q_i, q_j$ are compatible in $\mathbf{P}$). We now define a function F from $\underset{\sim}{Y}$ to κ by $F(i) = \mathrm{pr}(\gamma_i, F^1(i))$ (you can use any pairing function pr on κ). So if $i < j < \kappa^+$ and $F(i) = F(j)$ then there is $q_{i,j}$ such that $\mathbf{P} \models$ "$q_i \leq q_{i,j}$ and $q_j \leq q_{i,j}$", hence $q_{i,j} \Vdash_{\mathbf{P}}$ "$\mathrm{rk}^5(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, \underset{\sim}{M}; \kappa) < \beta$", so possibly increasing $q_{i,j}$, for some $\beta_{i,j} < \beta$ and $\zeta_{i,j} < \kappa$ and $k_{i,j} < n$ we have $q_{i,j} \Vdash$ "$\mathrm{rk}^5(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}, \underset{\sim}{M}; \kappa) = \beta_{i,j}$ and $\mathrm{rk}^5(\{a^i_0, \dots, a^i_{n-1}, a^j_k\}) \ngeq \beta_{i,j} + 1$ is witnessed by $\varphi = R_{n+1,\zeta_{i,1}}(x_0, \dots, x_n)$ and $k_{i,j}$".

Hence by the definition of $R_{n+1,\zeta_{i,j},\beta_{i,j},k_{i,j}}$ we have

$$M^+ \models R_{n+1,\zeta_{i,j},\beta_{i,j},k_{i,j}}[a^i_0, \dots, a^i_{n-1}, a^j_k],$$

but $\beta_{i,j} < \beta$ hence by the induction hypothesis

$$\mathrm{rk}^5(\{a^i_0, \dots, a^j_{n-1}, a^j_k\}, M^+; \kappa) \leq \beta_{i,j}.$$

So F contradicts the choice of $\langle\langle a^i_0,\dots,a^i_{n-1}\rangle : i<\kappa^+\rangle$ i.e. clause (a)$^-$ of Definition
{1.1} 1.1 Case 5. $\square$ {1.9}

Claim 1.12. *Let $B\subseteq {}^\omega 2\times{}^\omega 2$ be a Borel or even analytic set and $\mathrm{Pr}_{\omega_1}(\lambda)$.*
1) If B contains a λ-square <u>then</u> B contains a perfect square.
2) If B contains a (λ,λ)-rectangle <u>then</u> B contains a perfect rectangle.
3) We can replace analytic by κ-Souslin if $\mathrm{Pr}_{\kappa^+}(\lambda;\kappa)$. (This applies to Σ^1_2 sets which are $\aleph_1$-Souslin).

Proof. You can apply the results of section 2 to prove 1.12; specifically 2.1 (1) $\Rightarrow$ (2) {2.9}
proves parts (1),(2) and 2.4(1) proves part 3. of 1.12; those results of §2 say more {2.9}
hence their proof should be clearer.

However, we give a proof of part (1) here for the reader who is going to read this section only. Suppose that $B\subseteq{}^\omega 2\times{}^\omega 2$ is a Borel or even analytic set containing a λ-square. Let T be a $(2,2,\omega)$-tree such that

$$B=\{(\eta_0,\eta_1)\in{}^\omega\times{}^\omega 2:(\exists\rho\in{}^\omega\omega)[(\eta_0,\eta_1,\rho)\in\lim(T)]\},$$

and let $\{\eta_\alpha:\alpha<\lambda\}\subseteq[]^\omega 2$ be such that the square determined by it is contained in B and $\alpha<\beta<\lambda\Rightarrow\eta_\alpha\neq\eta_\beta$. For $\alpha,\beta<\lambda$ let $F(\alpha,\beta)\in{}^\omega\omega$ be such that $(\eta_\alpha,\eta_\beta,F(\alpha,\beta))\in\lim(T)$. Define a model M with the universe λ and the vocabulary $\tau=\{R_{\nu_0,\nu_1,\nu},Q_{\nu_0,\nu}:\nu_0,\nu_1\in{}^{\omega>}2$ and $\nu\in{}^{\omega>}\omega\}$, each $R_{\nu_0,\nu_1,\nu}$ a binary predicate, $Q_{\nu_0,\nu}$ a unary predicate and

$$Q^M_{\nu_0,\nu}=\{\alpha<\lambda:\nu_0\vartriangleleft\eta_\alpha\text{ and }\nu\vartriangleleft F(\alpha,\alpha)\},$$

$$R^M_{\nu_0,\nu_1,\nu}=\{(\alpha,\beta)\in\lambda\times\lambda:\nu_0\vartriangleleft\eta_\alpha\text{ and }\nu_1\vartriangleleft\eta_\beta\text{ and }\nu\vartriangleleft F(\alpha,\beta)\}.$$

By $\mathrm{Pr}_{\omega_1}(\lambda)$ we know that $\mathrm{rk}^0(M)\geq\omega_1$.

A pair (u,h) is called *an n-approximation* if $u\subseteq{}^n 2$, $h:u\times u\to{}^n\omega$ and for every $\gamma<\omega_1$ there is $w\in[\lambda]^*$ such that:

($\oplus_1$) $u=\{\eta_\alpha{\restriction}n:\alpha\in w\}$ and $\eta_\alpha{\restriction}n\neq\eta_\beta{\restriction}n$ for distinct $\alpha,\beta\in w$
($\oplus_2$) $\mathrm{rk}^0(w,M)\geq\gamma$
($\oplus_3$) $F(\alpha,\beta)\restriction n=h(\eta_\alpha\restriction n,\eta_\beta\restriction n)$ for $\alpha,\beta\in w$; hence
$$M\models R_{\eta_\alpha{\restriction}n,\eta_\beta{\restriction}n,h(\eta_\alpha{\restriction}n,\eta_\beta{\restriction}n)}[\alpha,\beta]$$
for $\alpha,\beta\in w$.

Note that $(\{\langle\rangle\},\{((\langle\rangle,\langle\rangle),\langle\rangle)\})$ is a 0-approximation.

Moreover

$(*)_0$ if (u,h) is an n-approximation and $\nu^*\in u$ <u>then</u> there are $m>n$ and an m-approximation (u^+,h^+) such that:
  (i) $\nu\in u\setminus\{\nu^*\}\ \Rightarrow\ (\exists!\nu^+)(\nu\vartriangleleft\nu^+\in u^+)$,
  (ii) $(\exists^{!2}\nu^+)(\nu^*\vartriangleleft\nu^+\in u^+)$ (where $\exists^{!2}x$ means "there are exactly 2 x's)
  (iii) $\nu\in u^+\ \Rightarrow\ \nu{\restriction}n\in u$ and
  (iv) if $\nu_1,\nu_2\in u^+$ then $[h(\nu_1{\restriction}n,\nu_2{\restriction}n)\vartriangleleft h^+(\nu_1,\nu_2)$ or $(\nu_1{\restriction}n=\nu_2{\restriction}n=\nu^*$ and $\nu_1\neq\nu_2)]$.

[Why? For each $\gamma < \omega_1$ choose w_γ satisfying $(\oplus_1)$, $(\oplus_2)$ and $(\oplus_3)$ for $\gamma+1$, now apply the definition of rk^0 (if $w_\gamma = \{\alpha^\gamma_\ell : \ell < |w_\gamma|\}$, $\nu^* \triangleleft \eta_{\alpha^\gamma_k}$, $k < |w_\gamma|$ we apply it to k) to get $w^+_\gamma = w_\gamma \cup \{\alpha_\gamma\}$ satisfying $(\oplus_1)$, $(\oplus_2)$ and $(\oplus_3)$ for γ, then choose $m_\gamma \in (n,\omega)$ such that $\langle \eta_\alpha \restriction m_\gamma : \alpha \in w^+_\gamma\rangle$ is with no repetitions.

Lastly, as there are only countably many possibilities for $\langle m_\gamma, \{\eta_\alpha \restriction m_\gamma : \alpha \in w^+_\gamma\}, \{(\eta_\alpha \restriction m_\gamma, \eta_\beta \restriction m_\gamma, F(\alpha,\beta) \restriction m_\gamma) : \alpha,\beta \in w^+_\gamma\}\rangle$ for $\gamma < \omega_1$, so one value is gotten for uncountably many γ, let γ^* be one of them. Choose $m = m_{\gamma^*}$, $u^+ = \{\eta_\alpha \restriction m : \alpha \in w^+_\gamma\}$ and define h^+ to satisfy $(\oplus_3)$.]

Repeating $|u|$-times the procedure of $(*)_0$ we get:

$(*)_1$ if $u = \{\nu_\ell : \ell < k\} \subseteq {}^n 2$ (no repetition), (u,h) is an n-approximation, then there are $m, u^+ = \{\nu^+_\ell : \ell < 2k\}$ and h^+ such that (u^+,h^+) is an m-approximation for some $m > n$ and
  - (i) $\nu_l \triangleleft \nu^+_{2l}$, $\nu_\ell \triangleleft \nu^+_{2\ell+1}$, $\nu^+_{2\ell} \neq \nu^+_{2\ell+1}$,
  - (ii) if $\ell < k$, $i < 2$ then $h(\nu_\ell,\nu_\ell) \triangleleft h^+(\nu^+_{2\ell+i},\nu^+_{2\ell+i})$ and
  - (iii) if $\ell_1 \neq \ell_2, \ell_1, \ell_2 < k$ and $i,j < 2$ then $h(\nu_{\ell_1},\nu_{\ell_2}) \triangleleft h^+(\nu^+_{2\ell_1+i},\nu^+_{2\ell_2+j})$.

Consequently we have:

$(*)_2$ there are sequences $\langle n_i : i < \omega\rangle \subseteq \omega$ and $\langle (u_i,h_i) : i \in \omega\rangle$ such that $n_i < n_{i+1}$, (u_i,h_i) is an n_i-approximation and $(u_i,h_i),(u_{i+1},h_{i+1})$ are like $(u,h),(u^+,h^+)$ of $(*)_1$.

Now, let $\langle n_i : i < \omega\rangle$ and $\langle (u_i,h_i) : i \in \omega\rangle$ be as in $(*)_2$. Define

$$\mathscr{P} = \{\eta \in {}^\omega 2 : (\forall i \in \omega)(\eta \restriction n_i \in u_i)\}.$$

By $(*)_1$ for (u_{i+1},h_{i+1}) we know that $\mathscr{P}$ is a perfect set. We claim that $\mathscr{P} \times \mathscr{P} \subseteq B$. Suppose that $\eta',\eta'' \in \mathscr{P}$ and $\eta' \neq \eta''$. Then $\eta' \restriction n_{i(*)} \neq \eta'' \restriction n_{i(*)}$ for some $i(*) < \omega$ and the sequence $\langle h_i(\eta' \restriction n_i, \eta'' \restriction n_i) : i(*) \leq i < \omega\rangle$ is $\triangleleft$-increasing and (as (u_i,h_i) are approximations) $(\eta' \restriction n_i, \eta'' \restriction n_i, h_i(\eta' \restriction n_i, \eta'' \restriction n_i)) \in T$ is increasing for $i \in [i(*),\omega)$. The case $\eta' = \eta'' \in \mathscr{P}$ is easier. The claim is proved. $\square_{\{1.10\}}$

Theorem 1.13. *Assume* $\mathrm{NPr}_{\omega_1}(\lambda)$ *and* $\lambda \leq \mu = \mu^{\aleph_0}$. *Then for some c.c.c. forcing notion* $\mathbf{P}, |\mathbf{P}| = \mu$ *and* $\Vdash_{\mathbf{P}}$ "$2^{\aleph_0} = \mu$" *and in* $\mathbf{V}^{\mathbf{P}}$ *we have:*

$(*)$ *there is a Borel set* $B \subseteq {}^\omega 2 \times {}^\omega 2$ *such that:*
  - (a) *it contains a* λ*-square. i.e. there are pairwise distinct* $\eta_\alpha \in {}^\omega 2$ *for* $\alpha < \lambda$ *such that* $(\eta_\alpha,\eta_\beta) \in B$ *for* $\alpha,\beta < \lambda$
  - (b) *let* $\mathbf{V} \models \lambda^{\aleph_0} = \lambda_1$; B *contains no* λ^+_1*-square, i.e. there are no* $\eta_\alpha \in {}^\omega 2$ *(for* $\alpha < \lambda^+_1$*) such that* $[\alpha \neq \beta \Rightarrow \eta_\alpha \neq \eta_\beta]$ *and* $(\eta_\alpha,\eta_\beta) \in B$ *for* $\alpha,\beta < \lambda^+$
  - (c) B *contains no perfect square.*

Actually B *is a countable union of closed sets.*

Proof. Stage A: Clearly for some $\alpha(*) < \omega_1$ we have $\mathrm{NPr}^1_{\alpha(*)}(\lambda)$. Let M be a model with universe λ and a countable vocabulary such that $\mathrm{rk}^1(M) < \alpha(*)$ say with $<^\mu$ the usual order. Let functions φ^M, k^M with domains

$$[\lambda]^* = \{u : u \subseteq \lambda, u \text{ is finite and } u \neq \emptyset\}$$

be such that: if $u = \{\alpha_0, \dots, \alpha_{n-1}\} \in [\lambda]^*$ increasing for definiteness, $\beta = \mathrm{rk}^0(u, M)$ $(< \alpha(*))$ <u>then</u> $\varphi^M(u)$ is a quantifier free formula in the vocabulary of M in the variables $x_0, \dots, x_{n-1}$ for simplicity saying $x_0 < x_1 < \dots < x_{n-1}$, $k^M(u)$ is a natural number $< n = |u|$ such that $\varphi^M(u), k^M(u)$ witness "not $\mathrm{rk}^1(u, M) \geq \beta + 1$" (the same definition makes sense even if $\beta = -1$). In particular

$$M \models \varphi^M(u)[\dots, a, \dots]_{a \in u}.$$

We define the forcing notion $\mathbf{P}$. We can put the diagonal $\{(\eta, \eta) : \eta \in {}^\omega 2\}$ into B so we can ignore it. We want to produce (in $\mathbf{V}^{\mathbf{P}}$) a Borel set $B = \bigcup\limits_{n<\omega} B_n$, each B_n $(\subseteq {}^\omega 2 \times {}^\omega 2)$ closed (in fact perfect), so it is $\lim(T_n)$ for some $(2,2)$-tree T_n, B_0 is the diagonal, and $\bar\eta = \langle \eta_\alpha : \alpha < \mu \rangle$ as witnesses to $2^{\aleph_0} \geq \mu$ and such that $\{\eta_\alpha : \alpha < \lambda\}$ gives the desired square. So for some 2-place function g from λ to $\omega, \alpha \neq \beta \Rightarrow (\eta_\alpha, \eta_\beta) \in \lim(T_{g(\alpha,\beta)})$, all this after we force. But we know that we shall have to use M (by 1.12). In the forcing our problem will be to prove the c.c.c. {1.9}
which will be resolved by using M (and rank) in the definition of the forcing. We shall have a function f which puts the information on the rank into the trees to help in not having a perfect square. Specifically the domain of f is a subset of

$$\{(u, h) : (\exists \ell \in \omega)(u \in [{}^\ell 2]^*) \text{ and } h : u \times u \to \omega\}$$

(the functions h above are thought of as indexing the B_n's). The function f will be such that for any distinct $\alpha_0, \dots, \alpha_{n-1} < \lambda$, if $\langle \eta_{\alpha_t} {\restriction} \ell : t < n \rangle$ are pairwise distinct, $u = \{\eta_{\alpha_t} {\restriction} \ell : t < n\}$, $h(\eta_{\alpha_t} \restriction \ell, \eta_{\alpha_s} \restriction \ell) = g(\alpha_t, \alpha_s)$ and $(u, h) \in \mathrm{Dom}\,(f)$ then $\mathrm{rk}^1(\{\alpha_\ell : \ell < n\}, M) = f_0(u, h)$ and $f_1(u, h)$ is $\eta_{\alpha_k} {\restriction} l$, where $k = k^M(\{\alpha_t : t < n\})$, $f_2(u, h) = \varphi^M(\{\alpha_\ell : \ell < n\})$ writing the variable as x_ν, $\nu \in u$ and $f(u, h) = (f_0(u, h), f_1(u, h), f_2(u, h))$. (Note: f is a way to say $\bigcup\limits_n \lim(T_n)$ contains no perfect square; essentially it is equivalent to fixing appropriate rank.) All this was to motivate the definition of the forcing notion $\mathbf{P}$.

A condition p (of $\mathbf{P}$) is an approximation to all this; it consists of:

(1) $u^p = u[p]$, a finite subset of μ

(2) $n^p = n[p] < \omega$ and $\eta^p_\alpha = \eta_\alpha[p] \in {}^{n[p]}2$ for $\alpha \in u[p]$ such that $\alpha \neq \beta \Rightarrow \eta^p_\alpha \neq \eta^p_\beta$. To clarify let $t^p_* = \{\eta_\alpha {\restriction} \ell : \alpha \in u^p, \ell \leq n_p\}$ is a full subtree of ${}^{n[p] \geq}2$, i.e. maximal models in ${}^{n[p]}2$ only (not really necessary)[1]

(3) $\bar m^p = \langle m^p_\ell : \ell \leq n^p \rangle$ is a strictly increasing sequence of natural numbers with last element $m^p_{n[p]} = m^p = m[p]$ and for $m < m[p]$, we have $t^p_m = t_m[p] \subseteq \bigcup\limits_{\ell \leq n[p]} ({}^\ell 2 \times {}^\ell 2)$ which is downward closed (i.e., $(\nu_0, \nu_1) \in t^p_m \cap ({}^\ell 2 \times {}^\ell 2)$ and $k < \ell \Rightarrow (\nu_0, \nu_1) {\restriction} k = (\nu_0 {\restriction} k, \nu_1 {\restriction} k) \in t^p_m$), also $(\langle \rangle, \langle \rangle) \in t^p_m$ and defining $\triangleleft$ naturally we have: if $(\eta_0, \eta_1) \in t^p_m \cap ({}^\ell 2 \times {}^\ell 2)$ and $\ell < m^p$ then $(\exists \nu_0, \nu_1)[(\eta_0, \eta_1) \triangleleft (\nu_0, \nu_1) \in t^p_m \cap ({}^{\ell+1}2 \times {}^{\ell+1}2)]$

[1]Added for transparency; it is definable from $\langle \eta^p_\alpha : \alpha \in u^p \rangle$; the intention was $p \leq q \Rightarrow t^p_* = t^1_* \cap {}^{n[p]}2$ not stated we may wonder about $(u, h) \in \mathrm{Dom}\,(f^p), u \subseteq 2^\ell \wedge u \not\subseteq t^p_*$. We now exclude them but the relation R excludes them (well when we have two members but recall $|u| \geq 1000$. Alternatively demand ${}^{n[p]}2 = \{\eta_\alpha {\restriction} n^p : \alpha \in u^p\}$ which requires a little more in some places.

(4) a function $f^p = f[p]$, its domain is a subset of: $\{(u,h)$: for some $\ell \le n[p], u \subseteq t^p_* \subseteq {}^\ell 2, |u| \ge 1, h$ is a 2-place function from u to $m[p]$ such that: $[\eta \in u \Rightarrow h(\eta,\eta) = 0]$ $[(\eta,\nu \in u \Rightarrow h(\eta,\nu) < m^p_\ell$ necessary] and $[\eta,\nu \in u \Rightarrow (\eta,\nu) \in t^p_{h(\eta,\nu)}]$ and $\eta \ne \nu \in u \Rightarrow h(\eta,\nu) > 0\}$ and f^p is such that $f^p(u,h) = (f^p_0(u,h), f^p_1(u,h), f^p_2(u,h)) \in [-1,\alpha(*)) \times u \times L_{\omega,\omega}(\tau(M))$.

(5) a function $g = g^p$ with domain $\{(\alpha,\beta) : \alpha,\beta$ from $u^p \cap \lambda\}$ such that $g(\alpha,\alpha) = 0$ and if $\alpha \ne \beta \Rightarrow 0 < g(\alpha,\beta) < m^p$ and $(\eta^p_\alpha, \eta^p_\beta) \in t^p_{g(\alpha,\beta)} \cap ({}^{n(p)}2 \times {}^{n(p)}2)$

(6) $t^p_0 = \{(\eta,\eta) : \eta \in {}^{n^p \ge}2\}$

(7) iff $u \subseteq {}^\ell 2, |u| \ge 1$, $f^p(u,h) = (\beta^*, \rho^*, \varphi^*)$, and $\ell < \ell(*) \le n^p$, e_i are functions with domain u (for $i = 0,1$) such that $(\forall \rho)[\rho \in u \Rightarrow \rho \triangleleft e_i(\rho) \in {}^{\ell(*)}2])$ and $(\forall \rho \in u)[e_0(\rho) = e_1(\rho) \Leftrightarrow \rho \ne \rho^*], u' = \mathrm{Rang}(e_0 \restriction u) \cup \mathrm{Rang}(e_1 \restriction u)$, and $h(\eta,\nu) = h'(e_i(\eta), e_i(\nu))$ for $\eta \ne \nu$ in u and $f^p(u',h') = (\beta', \rho', \varphi')$ (so is well defined) then $\beta' < \beta^*$

(8) iff $\ell \le n^p, w \subseteq u^p \cap \lambda$ is non empty, the sequence $\langle \eta^p_\alpha \restriction \ell : \alpha \in w\rangle$ is with no repetitions and h is defined by $h(\eta^p_\alpha \restriction \ell, \eta^p_\beta \restriction \ell) = g^p(\alpha,\beta)$ for $\alpha \ne \beta$ from w (and $h(\eta^p_\alpha \restriction \ell, \eta^p_\alpha \restriction \ell) = 0$) and $u = \{\eta^p_\alpha \restriction \ell : \alpha \in w\}$, then $f^p(u,h)$ is well-defined hence $[\alpha \ne \beta \in u \Rightarrow g(\alpha,\beta) < m^p_\ell], f^p_2(u,h) = \varphi^M(w), f^p_1(u,h) = \eta^p_\alpha \restriction \ell$ where α is the $k^M(w)$-th member of w and $f^p_0(u,h) = \mathrm{rk}^1(w,M)$; of course in $f^p_2(u,h) = \varphi^M(w)$ the variable x_ν in $f^p_2(u,h)$ corresponds to $x_{|\alpha \cap w|}$ if $\eta_\alpha \restriction l = \nu$ (see last clause of $\oplus_p$ below)

(9) if $(u,h) \in \mathrm{Dom}(f^p)$ then for some w and $\ell, f^p(u,h)$ is gotten as in clause (8)

(10) if $\eta_1 \ne \eta_2$ are in ${}^\ell 2, \ell \le n^p$ and $(\eta_1,\eta_2) \in t^p_m, 0 < m < m^p$ then for some $\alpha_1 \ne \alpha_2$ from $u^p \cap \lambda$ we have $g^p(\alpha_1,\alpha_2) = m$ and $\eta_1 \trianglelefteq \eta^p_{\alpha_1}, \eta_2 \trianglelefteq \eta^p_{\alpha_2}$.

The order is the natural one (including the following requirements: $p \le q$ iff ($p,q \in \mathbf{p}$ and) $n^p \le n^q, m^p \le m^q, \bar m^p = \bar m^q \restriction (n^p+1), u^p \subseteq u^q, \eta^q_\alpha \restriction n^p = \eta^p_\alpha$ for $\alpha \in u^p, t^p_* = t^1_* \cap {}^{n[p] \ge}2, t^p_m = t^q_m \cap \bigcup\limits_{\ell \le n[p]} ({}^\ell 2 \times {}^\ell 2)$ for $m < m^p$, $g^p = g^q \restriction u^p$ and $f^p = f^q \restriction \{(u,h) \in \mathrm{Dom}\,(f^q) : u \subseteq {}^{n^p \ge}2\}$, so if $(u,h) \notin \mathrm{Dom}\,(f^p), u \subseteq {}^{n^p \ge}2$ then $(u,h) \notin \mathrm{Dom}\,(f^q)$). $\square$

Explanation: The function f^p of a condition $p \in \mathbf{P}$ carries no additional information. It is determined by the function g^p and functions φ^M, k^M and the rank. Conditions 8, 9 are to say that:

$\oplus_p$ iff $w_0, w_1 \subseteq \lambda \cap u^p$, $\ell \le n^p$, $u = \{\eta^p_\alpha \restriction \ell : \alpha \in w_0\} = \{\eta^p_\alpha \restriction \ell : \alpha \in w_1\}$ (no repetitions) are non empty and $h : u \times u \longrightarrow m^p$ is such that if either $\alpha,\beta \in w_0$ or $\alpha,\beta \in w_1$ then $h(\eta^p_\alpha \restriction \ell, \eta^p_\beta \restriction \ell) = g^p(\alpha,\beta)$ then $\mathrm{rk}^1(w_0,M) = \mathrm{rk}^1(w_1,M), \varphi^M(w_0) = \varphi^M(w_1), k^M(w_0) = k^M(w_1)$, and if α_i, $\beta_i \in w_i$ for $i = 0,1$ and $\eta_{\alpha_0} \restriction \ell = \eta_{\alpha_1} \restriction \ell$, $\eta_{\beta_0} \restriction \ell = \eta_{\beta_1} \restriction \ell$ then $\alpha_0 < \alpha_1 \Leftrightarrow \beta_0 < \beta_1$.

Moreover, condition 7 gives no additional restriction unless $f^p_0(u,h) = -1$. Indeed, suppose that $u \subseteq {}^\ell 2$, $|u| \ge 1$, $\ell < \ell(*) \le n^p$, $e_i : u \longrightarrow {}^{\ell(*)}2, h, \rho^* \in u, u'$ and h' are as there and $f^p_0(u,h) \ge 0$. As $f^p(u',h')$ is defined we find $w \subseteq \lambda \cap u^p, \alpha_0, \alpha_1 \in w$ $(\alpha_0 \ne \alpha_1)$ such that $u' = \{\eta^p_\alpha \restriction l(*) : \alpha \in w\}$, $h'(\eta^p_\alpha \restriction \ell(*), \eta^p_\beta \restriction \ell(*)) = g^p(\alpha,\beta) < m^p_l$

and $e_i(\rho^*) = \eta^p_{\alpha_i}\restriction\ell(*)$ (for $i = 0, 1$). Looking at $w \setminus \{\alpha_0\}$, $w \setminus \{\alpha_1\}$ and (u, h) we see that

$$\alpha_0 = k^M(w \setminus \{\alpha_1\}), \varphi^M(w \setminus \{\alpha_0\}) = \varphi^M(w \setminus \{\alpha_1\})$$

and

$$\mathrm{rk}^1(w \setminus \{\alpha_1\}, M) = f^p_0(u, h) \geq 0.$$

By the definition of the rank and the choice of φ^M, k^M we get $\mathrm{rk}^1(w, M) = f^p_0(u, h)$ and hence $f^p_0(u', h') < f^p_0(u, h)$. If $f^p_0(u, h) = -1$ then clause 7 says that there are no respective e_0, e_1 introducing a ramification.

Stage B: $\mathbf{P}$ satisfies the c.c.c. Let $p^i \in \mathbf{P}$ for $i < \omega_1$; let $u[p^i] = \{a^i_\ell : \ell < |u[p^i]|\}$ increasing, so with no repetition. Without loss of generality $|u[p^i]|$ does not depend on i, and also $n[p^i]$, $\eta^{p^i}_{a^i_l}$, $\bar{m}^{p^i}$, $\langle t^{p^i}_m : m < m^{p^i}\rangle$, $g^{p^i}(a^i_{l_1}, a^i_{l_2})$, $f[p^i]$, and for a nonempty $v \subseteq |u[p^i]|$ such that $\bigwedge\limits_{\ell\in v} a^i_\ell < \lambda$, $\mathrm{rk}^1(\{a^i_l : l \in v\}, M), \varphi^M(\{a^i_\ell : \ell \in v\}), k^M(\{a^i_\ell : \ell \in v\})$ and the truth value of $a^i_\ell \geq \lambda$ do not depend on i. Note that writing $\varphi[w]$ we always assume that φ carries an information on the order of w.

Also by the Δ-system argument without loss of generality

$$a^{i^1}_{\ell_1} = a^{i^2}_{\ell_2} \text{ and } i^1 \neq i^2 \text{ implies } \ell_1 = \ell_2 \text{ and } \bigwedge_{i,j} a^i_{\ell_1} = a^j_{\ell_2}.$$

We shall show that p^0, p^1 are compatible by defining a common upper bound q:

(i) $n^q = n[p^i] + 1$

(ii) $u^q = \{a^i_\ell : \ell < |u[p^i]|, i < 2\}$

(iii) $\eta^q_{a^i_l}$ is: $\eta^{p^0}_{a^0_\ell}{}^\frown\langle 0\rangle$ if $i = 0$, $\eta^{p^0}_{a^0_\ell}{}^\frown\langle 1\rangle = \eta^{p^1}_{a^1_l}{}^\frown\langle 1\rangle$ if $i = 1$, $a^0_\ell \neq a^1_\ell$

(iv) $m[q] = m[p^0] + 2 \times |\lambda \cap u[p^0] \setminus u[p^1]|^2, \bar{m}^q = \bar{m}^{p^0}{}^\frown\langle m[q]\rangle$

(v) $g^q \supseteq g^{p_0} \cup g^{p_1}$ is such that g^q assigns new (i.e. in $[m^p, m^q)$) distinct values to "new" pairs (α, β) with $\alpha \neq \beta$, i.e. pairs from $(\lambda \times \lambda) \cap (u^q \times u^q) \setminus u^{p^0} \times u^{p^0} \setminus u^{p^1} \times u^{p^1}$

(vi) the trees t^q_m (for $m < m[q]$) are defined as follows:
if $m = 0$ see clause 6,
if $m < m[p^0]$, $m > 0$ then $t^q_m = t^{p^0}_m \cup \{(\eta^q_{a^\varphi_{\ell_1}}, \eta^q_{a^\varphi_{\ell_2}}) : \varphi \in \{0, 1\}$ and distinct $\ell_1, \ell_2 < |u[p^0]|$ satisfying $g^{p^0}(a^0_{\ell_1}, a^0_{\ell_2}) = m\}$ and if $m \in [m[p^0], m[q]), m = g^q(\alpha, \beta)$ and $\alpha \neq \beta$ then

$$t^q_m = \{(\eta^q_\alpha\restriction\ell, \eta^q_\beta\restriction\ell) : \ell \leq n^q\}$$

(vii) if $m \in [m[p^0], m[q])$ then for one and only one pair (α, β) we have $m = g^q(\alpha, \beta)$ and for this pair (α, β) we have $\alpha \neq \beta$, $\{\alpha, \beta\} \nsubseteq u[p^0]$ and $\{\alpha, \beta\} \subseteq u[p^1]$

(viii) The function f^q is determined by the function g^p and clauses 8, 9 of stage A.

Of course, we have to check that no contradiction appears when we define f^q (i.e. we have to check $\oplus_q$ of the Explanation inside stage A for q). So suppose that $w_0, w_1 \subseteq \lambda \cap u[q], \ell \leq n[q], u, h$ are as in $\oplus_q$. If $w_0 \subseteq u[p^i]$ (for some $i < 2$) then $g^q(\alpha,\beta) < m[p^0]$ for $\alpha, \beta \in w_0$ and hence $g^q[w_1 \times w_1] \subseteq m[p^0]$. Consequently either $w_1 \subseteq u[p^0]$ or $w_1 \subseteq u[p^1]$. If $\ell = n^q$ then necessarily $w_0 = w_1$ so we have nothing to prove. If $\ell < n^q$ then $(u,h) \in \mathrm{Dom}(f^{p^0})$ (and $f^{p^0} = f^{p^1}$) and clause 8 of stage A applies.

If w_0 is contained neither in $u[p^0]$ nor in $u[p^1]$ then the function g^q satisfies $g^q(\alpha,\beta) \in [m[p^0], m[q])$ for some $\alpha, \beta \in w_0$ hence $\ell = n^q$ and so as $\{\eta^q_\alpha \restriction \ell : \alpha \in w_0\} = \{\eta^q_\alpha \restriction \ell : \alpha \in w_1\}$ clearly $w_0 = w_1$, so we are done.

Next we have to check condition 7. As we remarked (in the Explanation inside Stage A) we have to consider cases of (u,h) such that $f^q(u,h) = -1$ only. Suppose that $u,\ \ell < \ell(*) \leq n^q, e_i, h, \rho^* \in u, u'$ and h' are as in 7 (and $f^q(u,h) = -1$). Let $w \subseteq u[q] \cap \lambda, \alpha_0, \alpha_1 \in w$ be such that $u' = \{\eta^q_\alpha \restriction \ell(*) : \alpha \in w\}$, $e_i(\rho^*) = \eta^q_{\alpha_i} \restriction l(*)$ (for $i = 0,1$). If $w \subseteq u[p^i]$ for some $i < 2$ then we can apply clause 7 for p^i and get a contradiction (if $\ell(*) = n^q$ then note that $\{\eta^q_\alpha \restriction n^p : \alpha \in w\}$ are already distinct). Since $\alpha \in w \setminus \{\alpha_0, \alpha_1\}$ implies $g^q(\alpha,\alpha_0) = g^q(\alpha,\alpha_1)$ (by the relation between h and h') we are left with the case $w \setminus \{\alpha_0,\alpha_1\} \subseteq u[p^0] \cap u[p^1]$, $\alpha_0 \in u[p^0] \setminus u[p^1]$, $\alpha_1 \in u[p^1] \setminus u[p^0]$ (or conversely). Then necessarily $\alpha_0 = a^0_{k_0}$, $\alpha_1 = a^1_{k_1}$ for some $k_0, k_1 \in [0, |u[p^0]|)$. Now $k_1 = k^M(w \setminus \{\alpha_0\}) = k^M(w \setminus \{\alpha_1\}) = k_0$ by the requirements in condition 7.

Now we see that for each $i < \omega_1$

$$M \models \varphi^M(w \setminus \{\alpha_0\})[w \setminus \{\alpha_0,\alpha_1\} \cup \{a^i_{k_1}\}]$$

and this contradicts the fact that $\varphi^M(w \setminus \{\alpha_0\}), \alpha_1$ witness $\mathrm{rk}^1(w \setminus \{\alpha_0\}, M) = -1$.

<u>Stage C</u>: $|\mathbf{P}| = \mu$ hence $\Vdash_{\mathbf{P}}$ "$2^{\aleph_0} \leq \mu$". We shall get the equality by clause (γ) at stage E below.

<u>Stage D</u>: The following subsets of $\mathbf{P}$?? are dense (for $m, n < \omega, \alpha < \mu$):

$$\mathscr{I}^1_m = \{p \in \mathbf{P} : m[p] \geq m\}$$

$$\mathscr{I}^2_n = \{p \in \mathbf{P} : n^p \geq n\}$$

$$\mathscr{I}^3_\alpha = \{p \in \mathbf{P} : \alpha \in u[p]\}$$

Proof. Let $p \in \mathbf{P}$, $\alpha_0 \in \mu \setminus u[p]$ be given, we shall find q, $p \leq q \in \mathscr{I}^1_{m[p]+1} \cap \mathscr{I}^2_{n[p]+1} \cap \mathscr{I}^3_{\alpha_0}$: this clearly suffices. We may assume that $u[p] \neq \emptyset$ and $\alpha_0 < \lambda$.

Let

(a) $n^q = n^p + 1$, $m^q = m^p + 2 \cdot |(\lambda \cap u[p])|$, $\bar{m}^q = \bar{m}^{p\frown}\langle m^q \rangle$, $u^q = u^p \cup \{\alpha_0\}$,
(b) for $\alpha \in u^p$ we let $\eta^q_\alpha = \eta^{p\frown}_\alpha\langle 0 \rangle$, $\eta^q_{\alpha_0} \in {}^{(n^p+1)}2$ is the sequence constantly equal 1,

(c) g^q is any two-place function from $u^q \cap \lambda$ to m^q extending g^p such that $g^q(\alpha,\alpha)=0, g^q(\alpha,\beta)\neq 0$ for $\alpha\neq\beta$ and $g^q(\alpha,\beta)=g^q(\alpha',\beta')$ and $(\alpha,\beta)\neq(\alpha',\beta')\Rightarrow(\alpha,\beta)\in u^p\times u^p(\alpha',\beta')\in u^p\times u^p$

(d) t^q_m is defined as follows:

(α) if $m<m^p, m\neq 0$ then $t^q_m = t^p_m \cup \{(\eta_0\hat{}\langle 0\rangle, \eta_1\hat{}\langle 0\rangle) : (\eta_0,\eta_1)\in t^p_m \cap ({}^{n[p]}2 \times {}^{n[p]}2)\}$

(β) if $m\in[m^p,m^q), m=g^q(\alpha,\beta), \alpha\neq\beta$ then $t^q_m=\{(\eta^q_\alpha\restriction\ell, \eta^q_\beta\restriction\ell):\ell\leq n^q\}$

(e) f^q extends f^p and satisfies 7,8 and 9 of stage A (note that f^q is determined by g^q).

Now check [similarly as at stage B].

Stage E: We define some $\mathbf{P}$-names

(a) $\utilde{\eta}_\alpha = \bigcup\{\eta^p_\alpha : p\in \utilde{G}_{\mathbf{P}}\}$ for $\alpha<\lambda$

(b) $\utilde{T}_m = \bigcup\{t^p_m : p \in \utilde{G}_{\mathbf{P}}\}$ for $m<\omega$

(c) $\utilde{g}=\bigcup\{g^p : p\in\utilde{G}_{\mathbf{P}}\}$

(d) $\utilde{T}_* = \cup\{t^p_* : p\in\utilde{G}_{\mathbf{P}}\}$.

Clearly it is forced ($\Vdash_{\mathbf{P}}$) such that:

(α) $\utilde{g}$ is a function from $\{(\alpha,\beta):\alpha,\beta<\lambda\}$ to ω.

[Why? Because $\mathscr{I}^3_\alpha$ are dense subsets of $\mathbf{P}$ and by clause 5 of stage A.]

(β) $\utilde{\eta}_\alpha \in {}^\omega 2$.

[Why? Because both $\mathscr{I}^2_n$ and $\mathscr{I}^3_\alpha$ are dense subsets of $\mathbf{P}$.]

(γ) $\utilde{\eta}_\alpha \neq \utilde{\eta}_\beta$ for $\alpha\neq\beta$ $(<\mu)$.

[Why? By clause 2 of the definition of $p\in\mathbf{P}$.]

(δ) $\utilde{T}_m \subseteq \bigcup_{\ell<\omega}({}^\ell 2\times {}^\ell 2)$ is an $(2,2)$-tree.

[Why? By clause 3 of the definition of $p\in\mathbf{P}$ and density of $\mathscr{I}^1_m, \mathscr{I}^2_n$.]

(φ) $(\utilde{\eta}_\alpha,\utilde{\eta}_\beta)\in \lim(\utilde{T}_{\utilde{g}(\alpha,\beta)}) = \{(\nu_0,\nu_1)\in {}^\omega 2\times{}^\omega 2 : (\forall \ell<\omega)((\nu_0\restriction\ell,\nu_1\restriction\ell)\in \utilde{T}_{\utilde{g}(\alpha,\beta)})\}$ (for $\alpha,\beta<\lambda$).

[Why? By clause 5 of the definition of $p\in\mathbf{P}$ and $(\beta)+(\delta)$ above.]

(ζ) if α,β are $<\lambda$ then $(\utilde{\eta}_\alpha,\utilde{\eta}_\beta)\notin\lim(T_m)$ when $m\neq g(\alpha,\beta)$ (and $m<\omega$).

[Why? By clauses 2 + 10 of the definition of $\mathbf{P}$ if $m\neq 0$ and clause 5 if $m=0$.]

(η) $\utilde{T}_*$ is a subtree of ${}^\omega 2$ with no maximal nodes and $\{\utilde{\eta}^p_\alpha : \alpha<\lambda\}\subseteq\lim(\utilde{T}_*)$.

Note that by clause (φ) above the Borel set $\utilde{B}=\bigcup\limits_{m<\omega}\lim(\utilde{T}_m)\subseteq{}^\omega 2\times{}^\omega 2$ satisfies requirement $(*)(a)$ of the Conclusion of 1.13. Moreover, by clause (γ) above we have $\Vdash_{\mathbf{P}}$ "$2^{\aleph_0}\geq\mu$" completing stage C (i.e. $\Vdash_{\mathbf{P}}$ "$2^{\aleph_0}=\mu$"). {1.10}

Stage F: We want to show $(*)(c)$ of the Conclusion of 1.13. Let $\mathbf{P}_\lambda=\{p\in\mathbf{P} : u[p]\subseteq\lambda\}$. Clearly $\mathbf{P}_\lambda \lessdot \mathbf{P}$. Moreover $\utilde{g},\utilde{T}_m,\utilde{B}$ are $\mathbf{P}_\lambda$-names. Since "$\utilde{B}$ contains a {1.10} perfect square" is a Σ^1_2-formula, so absolute, it is enough to prove that in $\mathbf{V}^{\mathbf{P}_\lambda}$ the set $\underset{\sim}{B}$ contains no perfect square.

Suppose that a $\mathbf{P}_\lambda$-name $\underset{\sim}{T}$ for a perfect tree and a condition $p \in \mathbf{P}_\lambda$ are such that:

$(*)^F_1$ $p \Vdash_{\mathbf{P}_\lambda}$ "$(\lim \underset{\sim}{T}) \times (\lim \underset{\sim}{T}) \subseteq \underset{\sim}{B}$".

We have then (a name for) a function $\underset{\sim}{m} : \lim(\underset{\sim}{T}) \times (\lim \underset{\sim}{T}) \longrightarrow \omega$ such that:

$(*)^F_2$ $p \Vdash_{\mathbf{P}_\lambda}$ "if $\eta_0, \eta_1 \in \lim \underset{\sim}{T}$ then $(\eta_0, \eta_1) \in \underset{\sim}{T}_{\underset{\sim}{m}(\eta_0,\eta_1)}$ hence $\eta_0, \eta_1 \in \underset{\sim}{T}_*$".

By shrinking the tree $\underset{\sim}{T}$ we may assume that p forces ($\Vdash_{\mathbf{P}_\lambda}$) the following:

$(*)^F_3$ "if $\eta_0, \eta_1, \eta'_0, \eta'_1 \in \lim \underset{\sim}{T}, \eta_0 \restriction \ell = \eta'_0 \restriction \ell \neq \eta_1 \restriction \ell = \eta'_1 \restriction \ell$ then $\underset{\sim}{m}(\eta_0, \eta_1) = \underset{\sim}{m}(\eta'_0, \eta'_1)$".

Consequently we may think of $\underset{\sim}{m}$ as a function from $\underset{\sim}{T} \times \underset{\sim}{T}$ to ω (with a convention that if $\nu_0, \nu_1 \in \underset{\sim}{T}$ are $\triangleleft$-comparable then $\underset{\sim}{m}(\nu_0, \nu_1) = 0$ and $\eta\hat{\ } langle\ell\rangle \triangleleft \nu_\ell \in \underset{\sim}{T} \Rightarrow \underset{\sim}{m}(\eta\hat{\ }\langle \ell\rangle, \eta\hat{\ }\langle 1-\ell\rangle) = \underset{\sim}{m}(\nu_\ell, \nu_{1-\ell})$ and if $\ell g(\nu_1) = \ell g(\nu_2)$ then $(\nu_1, \nu_2) \in T_{m(\nu_1,\nu_2)}$).

Choose an increasing sequence $\langle n_i : i \in \omega\rangle$ of natural numbers and sequences $\langle p_i : i \in \omega\rangle \subseteq \mathbf{P}_\lambda$,$\langle (t_i, m_i) : i \in \omega\rangle$ such that:

(1) $p \leq p_0 \leq p_1 \leq \ldots lep_i \leq p_{i+1} \leq \ldots$
(2) $t_i \subseteq {}^{n_i \geq}2$ is a full sub-tree, (i.e. $[\eta \triangleleft \nu \in t_i \cap {}^{n_i\geq}2 \Rightarrow \eta \in t_i]$, $\langle\rangle \in t_0$, $[\eta \in {}^{n_i >}2 \cap t_i \Rightarrow \bigvee\limits_{\ell<2} \eta\hat{\ }\langle\ell\rangle \in t_i]$) and $m_i : (t_i \cap {}^{n_i}2)^2 \longrightarrow \omega$ and $|t_i \cap {}^{n_i}2| \geq 1000$
(3) $t_i \subseteq t_{i+1}$ is an end extension (i.e. $t_i = ({}^{n_i\geq}2) \cap t_{i+1}$) such that each node from $t_i \cap {}^{n_i}2$ ramifies in t_{i+1} (i.e. has $\triangleleft$-incomparable extensions)
(4) $p_i \Vdash_{\mathbf{P}_\lambda}$ "$\underset{\sim}{T} \cap {}^{n_i\geq}2 \supseteq t_i$ and $\underset{\sim}{m}\restriction(t_i \cap {}^{n_i}2)^2 = m_i$"
(5) $n[p_i] > n_i, m[p_i] > \max(\mathrm{Rang}(m_i))$.

How do we carry the induction? For $i = 0$, note that $p \Vdash$ "T is perfect then for some $n, |\underset{\sim}{T} \cap {}^n2| > 1000$" let $p_0 \supseteq p$ force n_0 is as above and force a value t_0 to $\underset{\sim}{T} \cap {}^{(n_0)}2$ and force $\underset{\sim}{m}\restriction(t_0 \cap {}^{n_0}2)$ is equal to m_0. If $p_i, t_i, \ldots$ are well defined clearly $p_i \Vdash$ "for some $n > n_i, (\forall\rho \in t_i)(\ell g(\rho) = n_i \Rightarrow (\exists^{\geq 2}\varrho)(\rho \triangleleft \varrho))$. let $p'_i \geq p_i$ force $n'_i > n$ as above; without loss of generality p'_i forces a value to $t'_i, \mathrm{tr}(\underset{\sim}{T}) \cap {}^{(n'_i+1)\geq}2$ and m'_0 to $\underset{\sim}{m}\restriction t'_i \cap {}^{(n'_i+1)}2$. Let $p_{i+1} \geq p'_i, n_{i+1} > n_i$ be such that $m^{p_{i+1}} = \max \mathrm{Rang}(m'_i)$. By $(*)^F_3+$ the paraggrah below this is fine.

Since $p_i \Vdash_{\mathbf{P}_\lambda}$ "$(\nu_0, \nu_1) \in \underset{\sim}{T}_{m_i(\nu_0,\nu_1)}$" for $\nu_0, \nu_1 \in t_i \cap {}^{n_i}2$ we easily get (by clause 8 of the definition of $\mathbf{P}$, stage A) that $u \subseteq t_i \cap {}^{n_i}2, |u| \geq 1000 \Rightarrow (u, m_i\restriction u) \in \mathrm{Dom}\,(f^{p_i})$. Let $\alpha^*_i = \min\{f^{p_i}(u, m_i\restriction u), u \subseteq t_i \cap {}^{n_i}2, 1000 \leq |u| \leq 1000 + i\}$. By clause 7 (of the definition of $\mathbf{P}$) (and 1.2(2)+ clause 8 of the definition of $\mathbf{P}$) we deduce that $\alpha > \alpha_{i+1}$ for each $i < \omega$ and this gives a contradiction (to the ordinals being well ordered).

NOTE: that the $\underset{\sim}{\eta}_\alpha$'s do not appear in this stage. We only use the demand on the f^p's. Note that the domain of f^p does not depend on the η_α's, in fact, only $\underset{\sim}{\eta}_\alpha\restriction n^p$ is well defining knowing p only.

{1.10} Stage G: To prove $(*)(b)$ of Theorem 1.13 we may assume that $\mathbf{V} \models$ "$\lambda^{\aleph_0} = \lambda_1 < \mu$". Let $\mathbf{P}_{\lambda_1} = \{p \in \mathbf{P} : u[p] \subseteq \lambda_1\} \lessdot \mathbf{P}$. Note that the rest of the forcing (i.e. $\mathbf{P}/\mathbf{P}_{\lambda_1}$) is the forcing notion for adding μ Cohen reals so for $v \subseteq \mu \setminus \lambda_1$ the forcing notion $\mathbf{P}_v$ is naturally defined as well as $\mathbf{P}_{\lambda_1 \cup v}$. By stages C, E we know that

$\mathbf{V}^{\mathbf{P}_{\lambda_1}} \models$ "$2^{\aleph_0} = \lambda_1$" and by stage F we have $\mathbf{V}^{\mathbf{P}_{\lambda_1}} \models$ "the Borel set $\underset{\sim}{B}$ does not contain a perfect square". Suppose that after adding μ Cohen reals (over $\mathbf{V}^{\mathbf{P}_{\lambda_1}}$) we have a λ_1^+-square contained in $\underset{\sim}{B}$. We have λ_1^+-branches $\underset{\sim}{\rho}_\alpha$ $(\alpha < \lambda_1^+)$, each is a $\mathbf{P}_{v_\alpha}$-name for some countable $v_\alpha \subseteq \mu \setminus \lambda_1$. By the Δ-system lemma without loss of generality we assume that $\alpha \neq \beta \Rightarrow v_\alpha \cap v_\beta = v^*$. Working in $\mathbf{V}^{\mathbf{P}_{\lambda_1 \cup v^*}}$ we see that $\mathbf{P}_{v_\alpha \setminus v^*}$ is really the Cohen forcing notion and $\underset{\sim}{\rho}_\alpha$ is a $\mathbf{P}_{v_\alpha \setminus v^*}$-name. Without loss of generality $v^* = [\lambda_1, \lambda_1 + \omega), v_\alpha = v^* \cup \{\lambda_1 + \omega + \alpha\}$ and all names $\underset{\sim}{\rho}_\alpha$ are the same (under the natural isomorphism). So we have found a Cohen forcing name $\underset{\sim}{\tau} \in \mathbf{V}^{\mathbf{P}_{\lambda_1+\omega}}$ such that: if c_0, c_1 are (mutually) Cohen reals over $\mathbf{V}^{\mathbf{P}_{\lambda_1+\omega}}$, then $\mathbf{V}^{\mathbf{P}_{\lambda_1}}[c_0, c_1] \models (\underset{\sim}{\tau}^{c_0}, \underset{\sim}{\tau}^{c_1}) \in \underset{\sim}{B}$ and $\underset{\sim}{\tau}^{c_0} \neq \underset{\sim}{\tau}^{c_1}$.

But the Cohen forcing adds a perfect set of (mutually) Cohen reals. By absoluteness this produces a perfect set (in $\mathbf{V}^{\mathbf{P}_{\lambda_1}}$) whose square is contained in $\underset{\sim}{B}$. Once again by absoluteness we conclude that $\underset{\sim}{B}$ contains a perfect square in $\mathbf{V}^{\mathbf{P}_\lambda}$ already, a contradiction. $\square$

Remark 1.14. Note that if B is a subset of the plane $({}^\omega\omega, {}^\omega\omega)$ which is G_δ (i.e. $\bigcap_{n<\omega} U_n$, U_n open, without loss of generality decreasing with n) and it contains an uncountable square $X \times X$ (so $X \subseteq {}^\omega\omega$ is uncountable) *then* it contains a perfect square. Why? {1.11}

Let

$$X' = \{\eta \in X : (\forall n)(\exists^{\aleph_1} \nu)(\nu \in X \text{ and } \nu \restriction n = \eta \restriction n\}.$$

Let

$$\begin{aligned} K = \{(u,n) :\ & \text{for some } \ell \text{ called } \ell(u,n), u \subseteq {}^\ell\omega, \text{ and} \\ & \eta, \nu \in u \text{ and } \eta \triangleleft \eta' \in {}^\omega\omega \text{ and } \nu \triangleleft \nu' \in {}^\omega\omega \Rightarrow (\eta', \nu') \in U_n \\ & \text{and } \eta \in u \text{ and } \eta \triangleleft \eta' \in {}^\omega\omega \Rightarrow (\eta', \eta') \in U_n\} \end{aligned}$$

$$K' = \{(u,n) \in K : \text{ for some } \bar{\nu} = \langle \nu_\rho : \rho \in u \rangle \text{ we have } \nu_\rho \in X', \rho \triangleleft \nu_\rho\}.$$

So

(a) $K' \neq \emptyset$, in fact if $\eta_1, \ldots, \eta_m \in X'$ are pairwise distinct, $n < \omega$, then for any ℓ large enough $(\{\eta_i \restriction \ell : i = 1, \ldots, m\}, n) \in K$,

(b) if $(u,n) \in K'$ as exemplified by $\bar{\nu} = \langle \nu_\rho : \rho \in u \rangle$ and $\rho^* \in u$ $\nu' \in X' \setminus \{\nu_{\rho^*}\}, \nu' \restriction \ell = \nu_{\rho^*} \restriction \ell$ *then* for any $\ell' \in (\ell, \omega)$ and $n' > n$, large enough, we have $(\{\nu_\rho \restriction \ell' : \rho \in u\} \cup \{\nu' \restriction \ell'\}, n') \in K'$.

The following depends on §3:

Theorem 1.15. *Assume MA and $2^{\aleph_0} \geq \lambda_{\omega_1}(\aleph_0)$ or $2^{\aleph_0} > \mu$. Then: there is a Borel subset of the plane with a μ-square but with no perfect square* <u>*iff*</u> $\mu < \lambda_{\omega_1}(\aleph_0)$. {1.12}

Proof. The first clause implies the second clause by 1.12. If the second clause {1.9} holds, let $\mu \leq \lambda_\alpha(\aleph_0)$ and $\alpha < \omega_1$, by 3.2(6) letting $\eta_i \in {}^\omega 2$ for $i < \mu$ be pairwise {3.2} distinct we can find an ω-sequence of $(2,2)$-trees $\bar{T}$ such that $(\eta_i, \eta_j) \in \bigcup_n (\lim T_n)$ for $i, j < \mu$ and $\operatorname{degsq}(\bar{T}) = \alpha$ (just use $A = \{(\eta_i, \eta_j) : i, j < \mu\}$ there). By 3.2(3) {3.2} the set $\bigcup_n (\lim T_n)$ contains no $\lambda_{\alpha+1}(\aleph_0)$-square. $\square$

{1.13}

Fact 1.16. Assume $\mathbf{P}$ is adding $\mu > \kappa$ Cohen reals or random reals and $\kappa > 2^{\aleph_0}$. <u>Then</u> in $\mathbf{V}^{\mathbf{P}}$ we have:

$(*)_\kappa$ there is no Borel set (or analytic) $B \subseteq {}^\omega 2 \times {}^\omega 2$ such that:

(a) there are $\eta_\alpha \in {}^\omega 2$ for $\alpha < \kappa$ such that $[\alpha \neq \beta$ implies $\eta_\alpha \neq \eta_\beta]$, and $(\eta_\alpha, \eta_\beta) \in B$ for $\alpha, \beta < \kappa$

(b) B contains no perfect square.

{1.10} *Proof.* Straight as in the (last) stage G of the proof of theorem 1.13 (except that no relevance of (7) of Stage A there).

Let $\mathbf{P}$ be adding $\langle \underset{\sim}{r}_\alpha : \alpha < \mu \rangle$, assume $p \in \mathbf{P}$ forces that: $\underset{\sim}{B}$ a Borel set, $\langle \underset{\sim}{\eta}_\alpha : \alpha < \kappa \rangle$ are as in clause (a), (b) above. Let $\underset{\sim}{\eta}_\alpha$ be names in $\mathbf{P}_{v_\alpha} = \mathbf{P} \restriction \{\underset{\sim}{r}_\beta : \beta \in v_\alpha\}$, and $\underset{\sim}{B}$ be a name in $\mathbf{P}_v = \mathbf{P} \restriction \{\underset{\sim}{r}_\beta : \beta \in v\}$ where v, v_β are countable subsets of κ. Without loss of generality $\langle v_\alpha : \alpha < (2^{\aleph_0})^+ \rangle$ is a Δ-system with heart v and $\mathrm{otp}(v_\alpha \setminus v) = \mathrm{otp}(v_0 \setminus v)$. In $\mathbf{V}^{\mathbf{P}_v}$ we have $\underset{\sim}{B}$ and $2^{\aleph_0} = (2^{\aleph_0})^{\mathbf{V}}$, so without loss of generality $v = \emptyset$ and $\mathrm{otp}(v_\alpha)$ does not depend on α.

Without loss of generality the order preserving function $f_{\alpha,\beta}$ from v_α onto v_β maps $\underset{\sim}{\eta}_\alpha$ to $\underset{\sim}{\eta}_\beta$. So for Q=Cohen in the Cohen case we have a name $\underset{\sim}{\tau}$ such that $\Vdash_{\mathrm{Cohen}}$ "$\underset{\sim}{\tau}(\underset{\sim}{r}) \in {}^\omega 2$ is new", $\Vdash_{\mathrm{Cohen} \times \mathrm{Cohen}}$ "$(\underset{\sim}{\tau}(\underset{\sim}{r}_1); \underset{\sim}{\tau}(\underset{\sim}{r}_2)) \in B$", and we can finish easily. The random case is similar. $\square$

{1.14}
{1.13} **Conclusion 1.17.** *1) For $\kappa \in (\aleph_1, \aleph_{\omega_1})$ the statement $(*)_\kappa$ of 1.16 is not decided by* $\mathrm{ZFC} + 2^{\aleph_0} > \aleph_{\omega_1}$ *(i.e. it and its negation are consistent with ZFC).*
{1.10} *2) 1.16 applies to the forcing notion of 1.13 (with μ instead of $2^{\aleph_0}$).*

{1.13} *Proof.* 1) Starting with universe $\mathbf{V}$ satisfying CH, Fact 1.16 shows the consistency
{1.6} of "yes". As by 1.7(1) we know that $\lambda_{\omega_1}(\aleph_0) \geq \aleph_{\omega_1}$ <u>and</u> $\aleph_{\omega_1} > \kappa$ (by assump-
{1.12} tion), Theorem 1.15 (with the classical consistency of $\mathrm{MA} + 2^{\aleph_0} > \aleph_{\omega_1}$) gives the consistency of "no" (in fact in both cases it works for all κ simultaneously).
2) Left to the reader. $\square$

§ 2. Some model theoretic related problems

We turn to the model theoretic aspect: getting Hanf numbers below the continuum i.e. if $\psi \in L_{\omega_1,\omega}$ has a model of cardinality $\geq \lambda_{\omega_1}(\aleph_0)$ then it has a model of cardinality continuum. We get that $\mathrm{Pr}_{\omega_1}(\lambda)$ is equivalent to a statement of the form "if $\psi \in L_{\omega_1,\omega}$ has a model of cardinality λ then it has a model generated by an "indiscernible" set indexed by ${}^\omega 2$" (the indiscernibility is with respect to the tree $({}^{\omega\geq}2, \triangleleft, \cap, <_{\mathrm{lx}}, <_\ell g)$, where $\triangleleft$ is being initial segment, $\eta \cap \nu =$ maximal ρ, $\rho \trianglelefteq \eta$ and $\rho \trianglelefteq \nu$, $<_{\mathrm{lx}}$ is lexicographic order, $\eta <_{\mathrm{lx}} \nu$ iff $\ell g(\eta) < \ell g(\nu)$). This gives sufficient conditions for having many non-isomorphic models and also gives an alternative proof of 1.12. {1.9}

We also deal with the generalization to $\bar{\lambda}$-models i.e. fixing the cardinalities of several unary predicates (and point to λ-like models).

Claim 2.1. *The following are equivalent for a cardinal λ.* {2.1}
1) $\mathrm{Pr}_{\omega_1}(\lambda)$.
2) If $\psi \in L_{\omega_1,\omega}$ has a model M with $|R^M| \geq \lambda$ (R is a unary predicate) then ψ has a model of the cardinality continuum, moreover for some countable first order theory $\mathbf{T}_1$ with Skolem functions such that $\tau(\psi) \subseteq \tau(\mathbf{T}_1)$ and a model M_1 of $\mathbf{T}_1$ and $a_\eta \in R^{M_1}$ for $\eta \in {}^\omega 2$ we have:

- $(*)_0$ $M_1 \models \psi$
- $(*)_1$ *M_1, a_η ($\eta \in {}^\omega 2$) are as in* [Sh:a, Ch.II,§4] = [Sh:c, Ch.VII,§4], *i.e.:*
  - (a) *M_1 is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\}$ and $\eta \neq \nu$ implies $a_\eta \neq a_\nu$*
  - (b) *for every $n < \omega$ and a first order formula $\varphi = \varphi(x_0, \ldots, x_{n-1}) \in L(\mathbf{T}_1)$ there is $n^* < \omega$ such that: for every $k \in (n^*, \omega), \eta_0, \ldots, \eta_{n-1} \in {}^\omega 2$ and $\nu_0, \ldots, \nu_{n-1} \in {}^\omega 2$ satisfying $\bigwedge\limits_{m<n} \eta_m \restriction k = \nu_m \restriction k$ and $\bigwedge\limits_{m<\ell<n} \eta_m \restriction k \neq \eta_\ell \restriction k$ we have $M_1 \models$ "$\varphi[a_{\eta_0}, \ldots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \ldots, a_{\nu_{n-1}}]$". Note that necessarily $a_\eta \notin$ Skolem $\mathrm{Hull}_{M_1}\{a_\nu : \nu \in {}^\omega 2 \setminus \{\eta\}\}$*
  - (c) $a_\eta \in R^{M_1}$.

Remark 2.2. We can prove similarly with replacing λ by "for arbitrarily large $\lambda' < \lambda$" here and elsewhere; i.e. in 2) we replace the assumption by "If $\psi \in L_{\omega_1,\omega}$ has, for every $\lambda' < \lambda$, a model M with $|R^M| \geq \lambda'$ <u>then</u> ..." (and still the new version of 2) is equivalent to 1)).

Proof. <u>$1 \Rightarrow 2$</u>
Just as in [Sh:37]+ [Sh:49]: without loss of generality $\|M\| = \lambda$ and moreover $|M| = \lambda$. Let M_1 be an expansion of M by names for subformulas of ψ, a pairing function, and then by Skolem functions. Let $\mathbf{T}_1$ be the first order theory of M_1. There is (see [Kei71]) a set Γ of countably many types $p(x)$ such that: M_1 omits every $p(x) \in \Gamma$ and if M_1' is a model of $\mathbf{T}_1$ omitting every $p(x) \in \Gamma$ then M_1' is a model of ψ (just for each subformula $\bigwedge\limits_{n<\omega} \psi_n(\bar{x})$ of ψ, we have to omit a type; we can use 1-types as we have a pairing function).

Let us define $Y = \{v \subseteq {}^{\omega>}2 : v$ is finite nonempty and its members are pairwise $\triangleleft$-incomparable and for some n, $v \subseteq {}^n 2 \cup {}^{n+1}2\}$, $Z = \{(v, \varphi(\ldots, x_\eta, \ldots)_{\eta \in v}) : v \in Y, \varphi$ a formula in T_1 with the set of free variables included in $\{x_\eta : \eta \in v\}$ and for every $\alpha < \omega_1$ there are $a^\alpha_\eta \in R^M$ for $\eta \in v$ such that: $[\eta \neq \nu$ from $v \Rightarrow a^\alpha_\eta \neq a^\alpha_\nu]$ and $\mathrm{rk}^0(\{a^\alpha_\eta : \eta \in v\}, M) \geq \alpha$ and $M \models \varphi[\ldots, a^\alpha_\eta, \ldots]_{\eta \in v}\}$.

We say for $(v_\ell, \varphi_\ell) \in Z\,(\ell = 1, 2)$ that ¡$(v_2, \varphi_2) \in \mathrm{suc}(v_1, \varphi_1)$ if for some $\eta \in v_1$ (called $\eta(v_1, v_2)$) we have $v_2 = (v_1 \backslash \{\eta\}) \cup \{\eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle\}$ and letting for $i < 2$ the function $h_i : v_1 \to v_2$ be $h_i(\nu)$ is ν if $\nu \neq \eta$ and it is $\eta\hat{\ }\langle i\rangle$ if $\nu = \eta$, we demand for $i = 0, 1$:

$$\varphi_2 \vdash \varphi_1(\ldots, x_{h_i(\nu)}, \ldots)_{\nu \in v_1}.$$

Choose inductively $\langle (v_\ell, \varphi_\ell) : \ell < \omega\rangle$ such that $(v_{\ell+1}, \varphi_{\ell+1}) \in \mathrm{suc}(v_\ell, \varphi_\ell)$ is generic enough, i.e.:

$\otimes_1$ if $\varphi = \varphi(x_0, \ldots, x_{k-1}) \in L(\mathbf{T}_1)$ then for some $\ell < \omega$ for every $m \in [\ell, \omega)$ and $\eta_0, \ldots, \eta_{k-1} \in v_m$ we have: $\varphi_m \vdash \varphi(x_{\eta_0}, \ldots, x_{\eta_{k-1}})$ or $\varphi_m \vdash \neg\varphi(x_{\eta_0}, \ldots, x_{\eta_{k-1}})$

$\otimes_2$ for every $p(x) \in \Gamma$ and for every function symbol $f = f(x_0, \ldots, x_{n-1})$ (note: in $\mathbf{T}_1$ definable function is equivalent to some function symbol), for some $\ell < \omega$ for every $m \in [\ell, \omega)$, for every $\eta_0, \ldots, \eta_{n_1} \in v_m$ for some $\psi(x) \in p(x)$ we have $\varphi_m \vdash \neg\psi(f(x_{\eta_0}), \ldots, f(x_{\eta_{n-1}}))$.

It is straightforward to carry the induction (to simplify you may demand in $(\otimes)_1$, $(\otimes)_2$ just "for arbitrarily large $m \in [\ell, \omega)$", this does not matter and the stronger version of $(\otimes)_1, (\otimes)_2$ can be gotten (replacing the ${}^{\omega>}2$ by a perfect subtree T and then renaming a_η for $\eta \in \lim(T)$ as a_η for $\eta \in {}^{\omega}2$)). Then define the model by the compactness.

<u>$2 \Rightarrow 1$</u>:

If not, then $\mathrm{NPr}_{\omega_1}(\lambda)$ hence for some model M with vocabulary τ, $|\tau| \leq \aleph_0$,
{2.2} cardinality λ we have $\alpha(*) := \mathrm{rk}^0(M) < \omega_1$. Let $\psi_{\alpha(*)} \in L_{\omega_1,\omega}(\tau)$ be as in 2.3 below, so necessarily $M \models \psi_{\alpha(*)}$. Apply to it clause (2) which holds by our present assumption (with $R^M = \lambda$), so $\psi_{\alpha(*)}$ has a model M_1 as there, (so $M_1 \models \psi_{\alpha(*)}$). But $\{a_\eta : \eta \in {}^{\omega}2\}$ easily witnesses $\mathrm{rk}^0(M_1) = \infty$, moreover, for every nonempty finite $w \subseteq \{a_\eta : \eta \in {}^{\omega}2\}$ and an ordinal α we have $\mathrm{rk}^0(w, M) \geq \alpha$. This can be easily proved by induction on α (using $(*)_2$(b) of (2) (and $\eta \neq \nu \in {}^{\omega}2 \Rightarrow a_\eta \neq a_\nu$ of $(*)_2$(a))). $\square$

{2.2}

Fact 2.3. 1) For every $\alpha < \kappa^+$ and vocabulary $\tau, |\tau| \leq \kappa$, there is a sentence $\psi_\alpha \in L_{\kappa^+,\omega}[\tau]$ (of quantifier depth α) such that for any τ-model M:

$$M \models \psi_\alpha \underline{\text{iff}}\ \mathrm{rk}^0(M; < \aleph_0) = \alpha.$$

2) For every $\alpha < \theta^+, \ell \in \{0, 1\}$ and vocabulary τ, $|\tau| \leq \theta$ there is a sentence $\psi \in L_{\theta^+,\omega}(\exists^{\geq\kappa})[\tau]$ ($\exists^{\geq\kappa}$ is the quantifier "there are $\geq \kappa$ many x's) such that for any τ-model M: $M \models \psi^\ell_\alpha$ <u>iff</u> $\mathrm{rk}^\ell(M; < \kappa, \theta) = \alpha$.

Proof. Easy to check. $\square$

Hence (just as in [Sh:a, Ch.VIII,1.8(2)]):

{2.3}

Conclusion 2.4. *Assume τ is a countable vocabulary. If $\psi \in L_{\omega_1,\omega}(\tau)$, R is a unary predicate, $\tau_0 \subseteq \tau$, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1,\omega}(\tau_0)\}$ is countable and for some transitive model $\mathbf{V}_1$ of ZFC (may be a generic extension of $\mathbf{V}$ or an inner model as long as ψ, $\Delta \in \mathbf{V}_1$ and $\mathbf{V}_1 \models$ "$\psi \in L_{\omega_1,\omega}(\tau), \Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1,\omega}(\tau_0)\}$") we* *have* $\mathbf{V}_1 \models$ "$\mathrm{Pr}_{\omega_1}(\lambda)$ *and* ψ *has a model* M *with* $\lambda \le |\{\{\varphi(x) : M \models \varphi[a], \varphi(x) \in \Delta\} : a \in R^M\}|$".

Then we can find a model N *of* ψ *with Skolem functions and* $a_\alpha \in R^N$ *for* $\alpha < 2^{\aleph_0}$ *such that for each* $\alpha < 2^{\aleph_0}$ *the type* $p_\alpha = \{\pm\varphi(x) : N \models \pm\varphi[a_\alpha]$ *and* $\varphi(x) \in \Delta\}$ *is not realized in the Skolem hull of*

$$\{a_\beta : \beta < 2^{\aleph_0} \text{ and } \beta \neq \alpha\}.$$

Hence $\{M/\approx : M \models \psi,\ \|M\| = \lambda\}| \ge \min\{2^\lambda, \beth_2\}$*; really* $\{(M{\restriction}\tau_0)/\approx : M \models \psi$ *and* M *has cardinality* $\lambda\}$ *has cardinality* $\ge \min\{2^\lambda, \beth_2\}$. *Moreover we can find such a family of models no one of them ?? we have embeddable into another by an embedding preserving* $\pm\varphi(x)$ *for* $\varphi \in \Delta$.

A natural generalization of 2.1 is {2.1} {2.4}

Claim 2.5. *1) For cardinals* $\lambda > \kappa \ge \aleph_0$ *the following are equivalent:*

- (a) $\mathrm{Pr}_{\kappa^+}(\lambda; \kappa)$
- (b) *If* M *is a model,* $\tau(M)$ *countable,* $R, R_0 \in \tau(M)$ *unary predicates,* $|R_0^M| \le \kappa, \lambda \le |R^M|$ *then we can find* $M_0, M_1, a_\eta (\eta \in {}^\omega 2)$ *such that:*
  - (i) M_1 *is a model of the (first order) universal theory of* M *(and is a* $\tau(M)$*-model)*
  - (ii) $a_\eta \in R^{M_1}$ *for* $\eta \in {}^\omega 2$ *are pairwise distinct*
  - (iii) M_1 *is the closure of* $\{a_\eta : \eta \in {}^\omega 2\} \cup M_0$ *under the functions of* M_1 *(so*
    - (α) M_1 *also includes the individual constants of* M*; in general* $\|M_1\| = 2^{\aleph_0}$
    - (β) *if* $\tau(M)$ *has predicates only then* $|M_1| = \{a_\eta : \eta \in {}^\omega 2\} \cup |M_0|$*)*
  - (iv) M_0 *is countable,* $M_0 \subseteq M$*,* $M_0 \subseteq M_1$*,* $M_0 = c\ell_M(M_0 \cap R_0^M), R_0^{M_1} = R_0^{M_0} (\subseteq R_0^M)$*; in fact we can have:*
    - ($*$) $(M_1, c)_{c \in M_0}$ *is a model of the universal theory of* $(M, c)_{c \in M_0}$
  - (v) *for every* $n < \omega$ *and a quantifier free first order formula* $\varphi = \varphi(x_0, \dots, x_{n-1}) \in L(\tau(M))$ *there is* $n^* < \omega$ *such that: for every* $k \in (n^*, \omega)$ *and* $\eta_0, \dots, \eta_{n-1} \in {}^\omega 2, \nu_0, \dots, \nu_{n-1} \in {}^\omega 2$ *satisfying* $\bigwedge\limits_{m<n} \eta_m{\restriction}k = \nu_m{\restriction}k$ *and* $\bigwedge\limits_{m<\ell<n} \eta_m{\restriction}k \neq \eta_\ell{\restriction}k$ *we have* $M_1 \models$ "$\varphi[a_{\eta_0}, \dots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \dots, a_{\nu_{n-1}}]$", *we can even allow parameters from* M_0 *in* φ *(but* k *depends on them).*

2) For cardinals $\lambda > \kappa \ge \aleph_0$ *the following are equivalent:*

- (a)$'$ $\mathrm{Pr}_{\omega_1}(\lambda; \kappa)$
- (b)$'$ *like (b) above, but we omit* "$M_0 \subseteq M$".

{2.4A}

Remark 2.6. 1) See 4.6, 4.7 how to use claim 2.5. {2.5}

2) In (b), if M has Skolem functions then we automatically get also:

- (i)$^+$ M_1 a model of the first order theory of M
- (iii)$^+$ M_1 is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\} \cup M_0$
- (iv)$^+$ $M_0 \prec M, M_0 \prec M_1$, M_0 countable (and $R_0^{M_1} = R_0^{M_0} \subseteq R_0^M$)
- (v)$^+$ clause **(v)** above holds even for φ any (first order) formula of $L_{\omega,\omega}(\tau(M))$.

Proof. 1) $\underline{(a) \Rightarrow (b)}$

{2.1} Like the proof of 2.1, (1) $\Rightarrow$ (2), applied to $(M, c)_{c \in R_0^M}$ but the set $M_0 \cap R_0^M$ is chosen by finite approximation i.e. (letting Y be as there and $\tau = \tau(M)$) we let $Z = \{(v, \varphi(\ldots, x_\eta, \ldots)_{\eta \in v}, A) : v \in Y, \varphi$ a quantifier free formula in $L_{\omega,\omega}(\tau)$ with set of free variables included in $\{x_\eta : \eta \in v\}$ and parameters from A, A is a finite subset of R_0^M, and for every ordinal $\alpha < \kappa^+$ there are $a_\eta^\alpha \in R^M$ for $\eta \in v$ such that $[\eta \neq \nu$ from $v \Rightarrow a_\eta^\alpha \neq a_\nu^\alpha]$ and $\mathrm{rk}(\{a_\eta^\alpha : \eta \in v\}, M) \geq \alpha$ and $M \models \varphi[\ldots, a_\eta^\alpha, \ldots]_{\eta \in v}\}$.

We need here the "for every $\alpha < \kappa^+$" because we want to fix elements of R_0^M, and there are possibly κ choices.

$\underline{\neg(a) \Rightarrow \neg(b)}$:

{2.2} Like the proof of 2.3; assume $\mathrm{NPr}_\alpha(\lambda; \kappa), \alpha < \kappa^+$, let M witness it, choose $R_0 = \alpha + 1$, $R = \lambda$, without loss of generality $\tau(M) = \{R_{n,\zeta} : n < \omega, \zeta < \kappa\}$, $R_{n,\zeta}$ is n-place, in M every quantifier free formula is equivalent to some $R_{n,\zeta}$, let $R_{n,k}^* := \{(i_0, \ldots, i_{n-1}, \beta, \zeta) : M \models R_{n,\zeta}(i_0, \ldots, i_{n-1}),\ \{i_0, \ldots, i_{n-1}\}$ is with no repetition, increasing for simplicity, and $\mathrm{rk}(\{i_0, \ldots, i_{n-1}\}, M; \kappa) = \beta$, with $\mathrm{rk}(\{i_0, \ldots, i_{n-1}\}, M; \kappa) \ngeq \beta + 1$ being witnessed by $\varphi(\{i_0, \ldots, i_{n-1}\}) = R_{n,\zeta}$,

{1.10} $k(\{i_0, \ldots, i_{n-1}\}) = k\}$ where the functions φ, k are as in the proof of 1.13. Let M be $(\lambda, <, R, R_0, R_{n,k}^*)_{n \in (0,\omega), k < n}$ expanded by Skolem functions. So assume toward contradiction that (b) holds, hence for this model M there are models M_0, M_1 and

{2.4} $a_\eta \in M_1$ for $\eta \in {}^\omega 2$ as required in clauses (i) – (v) of (b) of claim 2.5. Choose a non-empty finite subset w of ${}^\omega 2$ and β and ζ such that letting $w = \{\eta_0, \ldots, \eta_{m-1}\}$ with $a_{\eta_\ell} <^{M_1} a_{\eta_{\ell+1}}$, we have:

- (α) $M_1 \models R_{m,k}^*(a_{\eta_0}, \ldots, a_{\eta_{m-1}}, \beta, \zeta)$
- (β) $\beta \in R_0^{M_0}$ $(\subseteq \alpha)$
- (γ) β minimal under those constraints.

Note that there are m, $\eta_0, \ldots, \eta_{m-1}$, β, ζ such that (α) holds: for every non-empty $w \subseteq {}^\omega 2$, as M_1 is elementarily equivalent to M there are β, ζ as required in (α). Now (α) implies $\zeta \in R_0^{M_1}$, but $R_0^{M_1} = R_0^{M_0}$, so clause (β) holds too, and so we can satisfy (γ) too as the ordinals are well ordered. Let $\varphi' = \varphi(x_0, \ldots, x_{m-1}, \beta, \zeta)$, note the parameters are from $R_0^{M_1}$ (as M_1 is elementarily equivalent to M) hence

{2.4} from $R_0^{M_0} \subseteq M$, and clause (v) (of (b) of 2.5) applies to φ', $\langle \eta_0, \ldots, \eta_{m-1} \rangle$ giving $n^* < \omega$. We can find $\eta_k' \in {}^\omega 2$, $\eta_k' \neq \eta_k$, $\eta_k' \restriction n^* = \eta_k \restriction n^*$, and easily for $w' = \{\eta, \ldots, \eta_{m-1}, \eta_k'\}$ we can find $\beta' < \beta$, $\zeta' < \kappa$ and k such that $M_1 \models R_{m+1,k'}(a_{\eta_0}, \ldots, a_{\eta_{m-1}}, a_{\eta_k'}, \beta', \zeta')$ and then if $\beta \geq 0$ we get contradiction to clause (γ) above. If $\beta = 0$ we use clause (i) to copy the situation to M and get a contradiction.

2) Similar proof. $\square$

{2:5}

Notation 2.7. Let $\bar\lambda$ denote a finite (or countable) sequence of pairs of infinite cardinals $\langle (\lambda_\zeta; \kappa_\zeta) : \zeta < \zeta(*) \rangle$ such that κ_ζ increases with ζ, so e.g. $\bar\lambda^\oplus = \langle (\lambda_\zeta^\oplus, \kappa_\zeta^\oplus) : \zeta < \zeta^\oplus(*) \rangle$. We shall identify a strictly increasing $\bar\kappa = \langle \kappa_\zeta : \zeta \leq \zeta(*) \rangle$ with $\langle (\kappa_{\zeta+1}; \kappa_\zeta) : \zeta < \zeta(*) \rangle$.

Let $R, R_0, Q_0, \ldots, R_{\zeta(*)-1}, Q_{\zeta(*)-1}$ be fixed unary predicates and $\bar R = \langle (R_\zeta, Q_\zeta) : \zeta < \zeta(*) \rangle$.

A $\bar{\lambda}$-model M is a model M such that: $R, R_\zeta, Q_\zeta \in \tau(M)$ are all unary predicates, $|R^M_\zeta| = \lambda_\zeta$, $|Q^M_\zeta| = \kappa_\zeta$ for $\zeta < \zeta(*)$, $Q^M_\zeta \subseteq R^M_\zeta$ and $\langle R^M_\zeta : \zeta < \zeta(*)\rangle$ are pairwise disjoint, and $R^M = \bigcup\limits_{\zeta<\zeta(*)} R^M_\zeta$.

For $a \in R^M$ let $\zeta(a)$ be the ζ such that $a \in R^M_\zeta$ (e.g. $R^M_\zeta = \lambda_\zeta \backslash \bigcup\limits_{\xi<\zeta} \lambda_\xi$, $\kappa_{\zeta+1} = \lambda_\zeta$). For a $\bar{\lambda}$-model M we say that $a \in c\ell_\kappa(A, M)$ if $A \cup \{a\} \subseteq M$ and for some $n < \omega$, and quantifier free formula $\varphi(x_1, y_1, \ldots, y_n)$ and $b_1, \ldots, b_n \in A$ we have

$$M \models \varphi(a, b_1, \ldots, b_n) \text{ and } (\exists^{\leq\kappa} x)\varphi(x, b_1, \ldots, b_n).$$

{2.6}

Definition 2.8. 1) For $\ell < 6$, $\bar{\lambda}$ as in Notation 2.7, and an ordinal α let $\mathrm{Pr}^\ell_\alpha(\bar{\lambda}; \theta)$ mean that for every $\bar{\lambda}$-model M, with $|\tau(M)| \leq \theta$ we have $\mathrm{rk}^\ell(M, \bar{\lambda}) \geq \alpha$ (and $\mathrm{NPr}^l_\alpha(\bar{\lambda}, \theta)$ is the negation, if θ is omitted it means κ_0, remember $\bar{\lambda} = \langle(\lambda_\zeta, \kappa_\zeta) : \zeta < \zeta(*)\rangle$) where the rank is defined in part (2) below. {2:5}
2) For a $\bar{\lambda}$-model M, $\mathrm{rk}^\ell(M, \bar{\lambda}) = \sup\{\mathrm{rk}^\ell(w, M, \bar{\lambda}) + 1 : \ w \in [\bar{R}^M]^*\}$ where the rank is defined in part (3) below and:

- (α) if $\zeta(*)$ is finite, $[\bar{R}^M]^* = \{w : w$ a finite subset of R^M not disjoint to any $R^M_\zeta\}$
- (β) if $\zeta(*)$ is infinite, $[\bar{R}^M]^* = \{w : w$ a finite non-empty subset of $R^M\}$.

3) For a $\bar{\lambda}$-model M, and $w \in [R^M]^*$ we define the truth value of "$\mathrm{rk}^\ell(w, M; \bar{\lambda}) \geq \alpha$" by induction on α.

Case A: $\alpha = 0$
$\mathrm{rk}^\ell(w, M; \bar{\lambda}) \geq \alpha$ iff no $a \in w \cap R^M$ belongs to $c\ell_{\kappa_{\zeta(a)}}(w\backslash\{a\}, M)$.

Case B: α is a limit ordinal
$\mathrm{rk}^\ell(w, M; \bar{\lambda}) \geq \alpha$ iff $\mathrm{rk}^\ell(w, , M; \bar{\lambda}) \geq \beta$ for every ordinal $\beta < \alpha$.

Case C: $\alpha = \beta + 1$
We demand two conditions:

- (α) exactly as in Definition 1.1(3)$(*)_3$ except that when $\ell = 2, 3, 4, 5$ we use $\kappa = \kappa^+_{\zeta(a_k)}$, {1.1}
- (β) if $\zeta < \zeta(*)$ and $w \cap R^M_\zeta = \emptyset$ then for some $a \in R^M_\zeta$, $\mathrm{rk}^\ell(w \cup \{a\}, M; \theta) \geq \beta$.

{2.7}

Claim 2.9. *The parallel of the following holds: 1.2 (+ statements in 1.1) also 1.3 (use $\alpha < \kappa^+_0$), 1.5(2) (for $\alpha \leq \kappa^+_0$), 1.10 (satisfying κ^+_0-c.c.) and we adapt 1.6.* {1.2} {1.3} {1.8} {1.4} {2.8}

Claim 2.10. *If α is a limit ordinal and $\lambda_\xi \geq \beth_\alpha(\kappa_\xi)$ for every $\xi < \xi(*)$, then* $\mathrm{Pr}_\alpha(\bar{\lambda})$.

Proof. Use indiscernibility and Erdös-Rado as in the proof of 1.8(1). {1.7}

In more details. The induction hypothesis on α is, assuming $\zeta(*) < \omega$: if $A \subseteq R^M$, $\bigwedge\limits_{\zeta<\zeta(*)} |A \cap R^M_\zeta| \geq \beth_{\omega\times\alpha}$ then for every $\beta < \alpha$, $k < \omega$ and every $\bar{m} = \langle m_\zeta : \zeta < \zeta(*)\rangle$, $m_\zeta \in (0, \omega)$ for some $w \subseteq A$ we have $\bigwedge\limits_\zeta |w \cap R^M_\zeta| = m_\zeta$ and $\mathrm{rk}(w; M, \bar{\lambda}) \geq \omega \times \beta + k$. Then for $\alpha = \gamma + 1$, choose distinct $a^\zeta_i \in A \cap R^M_\zeta$ $(i < \beth_{\omega\times\gamma+m+m_\zeta})$ and use polarized partition (see Erdös, Hajnal, Mate, Rado [EHMR84]) on $\langle\langle a^\zeta_i : i < \beth_{\omega\times\alpha}\rangle : \zeta < \zeta(*)\rangle$. For $\zeta(*)$ infinite use $A \subseteq R^M$ such that $w_A = \{\zeta : A \cap R^M_\zeta \neq \emptyset\}$ is finite non-empty, $\zeta \in w_A \Rightarrow |A \cap R^M_\zeta| \geq \beth_{\omega\times\alpha}$ and proceed as above. $\square$

{2.9} **Claim 2.11.** *Let $\zeta(*) < \omega$, $\kappa^\varphi_0 < \cdots < \kappa^\varphi_{\zeta(*)}$, $\bar\lambda^\varphi = \langle(\kappa^\varphi_{\xi+1}, \kappa^\varphi_\xi) : \xi < \zeta(*)\rangle$ (for $\varphi \leq \omega$).*
1) If $\mathrm{Pr}_n(\bar\lambda^n)$ for $n < \omega$, and for some $\theta \leq \kappa^\omega_0$ there is a tree $\mathscr{T} \in {}^{\theta>}(\kappa^\omega_0)$ of cardinality $\leq \kappa^\omega_0$ with $\geq \kappa^\omega_{\zeta(*)}$ θ-branches then:

- $\otimes$ *every first order sentence which has a $\bar\lambda^n$-model for each n, also has a $\bar\lambda^\omega$-model*
- $\otimes'$ *moreover, if $\mathbf{T}$ is a first order theory of cardinality $\leq \kappa^\omega_0$ and every finite $\mathbf{T}' \subseteq \mathbf{T}$ has a $\bar\lambda^n$-model for each n <u>then</u> $\mathbf{T}$ has a $\bar\lambda^\omega$-model.*

2) So if $\bar\lambda^\varphi = \bar\lambda$ for $\varphi \leq \omega$ are as <u>above</u> then we have κ^ω_0-compactness for the class of $\bar\lambda^\omega$-models. Where

- $\oplus$ *a class $\mathfrak{K}$ of models is κ-compact when for every set $\mathbf{T}$ of $\leq \kappa$ first order sentences, if every finite subset of $\mathbf{T}$ has a model in $\mathfrak{K}$ then $\mathbf{T}$ has a model in $\mathfrak{K}$.*

3) In part (1) we can use $\bar\lambda^n$ with domain ω_n, if $\omega_n \subseteq \omega_{n+1}$ and $\zeta(*) = \cup\{w_n : n < \omega\}$.

{2.10} *Proof.* Straight if you have read [Sh:8], [Sh:18], [Sh:37] or read the proof of 2.12 below (only that now the theory is not necessary countable, no types omitted, and by compactness it is enough to deal with the case $\zeta(*)$ is finite). $\square$

{2.10} **Claim 2.12.** *Let $\zeta(*) < \omega_1$, $\bar\lambda^\varepsilon = \langle(\kappa^\varepsilon_{\xi+1}, \kappa^\varepsilon_\xi) : \xi < \zeta(*)\rangle$ for each $\varepsilon \leq \omega_1$ (and κ^ε_ξ strictly increasing with ξ). If $\mathrm{Pr}_\varepsilon(\bar\lambda^\varepsilon)$ for every $\varepsilon < \omega_1$ and $\kappa^{\omega_1}_\xi \leq 2^{\aleph_0}$ and $\psi \in L_{\omega_1,\omega}$ and for each $\varepsilon < \omega_1$ there is a $\bar\lambda^\varepsilon$-model satisfying ψ <u>then</u> there is a $\bar\lambda^{\omega_1}$-model satisfying ψ.*

Proof. For simplicity, again like [Sh:8], [Sh:18], [Sh:37]. Let M_ε be a $\bar\lambda^\varepsilon$ model of ψ for $\varepsilon < \omega_1$. By expanding the M_ε's, by a pairing function and giving names of subformulas of ψ we have a countable first order theory $\mathbf{T}$ with Skolem functions, a countable set Γ of 1-types and M^+_ε such that:

- (a) M^+_ε is a $\bar\lambda^\varepsilon$-model of $\mathbf{T}$ omitting each $p \in \Gamma$
- (b) if M is a model of $\mathbf{T}$ omitting every $p \in \Gamma$ <u>then</u> M is a model of ψ.

{2.1} Now as in the proof of 2.1 we can find a model M^+ and a^ζ_η for $\zeta < \zeta(*)$, $\eta \in {}^\omega 2$ such that

- (α) M^+ a model of $\mathbf{T}$
- (β) $a^\zeta_\eta \in R^M_\zeta$ and $\eta \neq \nu \Rightarrow a^\zeta_\eta \neq a^\zeta_\nu$
- (γ) for every first order formula $\varphi(x_0, \ldots, x_{n-1}) \in L_{\omega,\omega}(\tau(\mathbf{T}))$ and $\zeta(0), \ldots, \zeta(n-1) < \zeta(*)$ ordinals, there is $k < \omega$ such that: if $\eta_0, \ldots, \eta_{n-1} \in {}^\omega 2$, $\nu_0, \ldots, \nu_{n-1} \in {}^\omega 2$ and $\langle \eta_\ell \restriction k : \ell < n\rangle$ is with no repetitions and $\eta_\ell \restriction k = \nu_\ell \restriction k$ *then* $M^+ \models \varphi(a^{\zeta(0)}_{\eta_0}, \ldots, a^{\zeta(n-1)}_{\eta_{n-1}}) \equiv \varphi(a^{\zeta(0)}_{\nu_0}, \ldots, a^{\zeta(n-1)}_{\eta_{n-1}})$

(δ) if $\sigma(x_0,\dots,x_{n-1})$ is a term of $\tau(\mathbf{T})$ and $\zeta(0),\dots,\zeta(n-1)<\zeta(*)$, and $p\in\Gamma$ <u>then</u> for some $k<\omega$ for any $\rho_0,\dots,\rho(n-1)\in {}^k2$ pairwise distinct there is $\varphi(x)\in p(x)$ such that:

  ($*$) $\rho_\ell\triangleleft\eta_\ell\in{}^\omega 2\Rightarrow M^+\models\neg\varphi(a^{\zeta(0)}_{\eta_0},\dots,a^{\zeta(n-1)}_{\eta_{n-1}})$
  (i.e. this is our way to omit the types in Γ)

(φ) if $\zeta(0),\dots,\zeta(n-1)<\zeta(*)$, $\sigma(x_0,\dots,x_{n-1})$ is a term of $\tau(\mathbf{T})$, and $m<n$, <u>then</u> for some $k<\omega$, we have

  ($*$) if $\eta_0,\dots,\eta_{n-1}\in{}^\omega 2$, and $\nu_0,\dots,\nu_{n-1}\in{}^\omega 2$ and $\eta_\ell\restriction k=\nu_\ell\restriction k$ and $\langle\eta_\ell\restriction k:\ell<\omega\rangle$ is without repetitions and $\zeta(\ell)<\zeta(m)\Rightarrow\eta_\ell=\nu_\ell$ then $M^+\models Q_{\zeta(m)}(\sigma(a^{\zeta(0)}_{\eta_0},\dots,a^{\zeta(n-1)}_{\eta_{n-1}}))$ and $Q_{\zeta(m)}(\sigma(a^{\zeta(0)}_{\nu_0},\dots,a^{\zeta(n-1)}_{\nu_{n-1}})\Rightarrow$ $\sigma(a^{\zeta(0)}_{\eta_0},\dots,a^{\zeta(n-1)}_{\eta_{n-1}})=\sigma(a^{\zeta(0)}_{\nu_0},\dots,a^{\zeta(n-1)}_{\nu_{n-1}})$.

Now choose $Y_\zeta\subseteq{}^\omega 2$ of cardinality $\lambda^{\omega_1}_\zeta$ and let M^* be the $\tau(M)$-reduct of the Skolem hull in M^+ of $\{a^\zeta_\eta:\zeta<\zeta(*)$ and $\eta\in Y_\zeta\}$. This is a model as required. $\square$

{2.11}

Conclusion 2.13. *If* $\mathbf{V}_0\models$ GCH, $\mathbf{V}=\mathbf{V}_0^{\mathbf{P}}$ *for some c.c.c. forcing notion* $\mathbf{P}$ *then e.g.*

($*$) *if* $\aleph_{\omega\times 3}<2^{\aleph_0}$, $\langle\aleph_0,\aleph_\omega,\aleph_{\omega+\omega}\rangle\Rightarrow\langle\aleph_\omega,\aleph_{\omega+\omega},\aleph_{\omega+\omega+\omega}\rangle$ *(see* [Sh:8]*) i.e., letting* $\bar\lambda^0=\langle(\aleph_0,\aleph_\omega),(\aleph_\omega,\aleph_{\omega+\omega})\rangle$ *and* $\bar\lambda^1=\langle(\aleph_\omega,\aleph_{\omega+\omega}),(\aleph_{\omega+\omega},\aleph_{\omega+\omega+\omega})\rangle$, *for any countable first order* $\mathbf{T}$, *if every finite* $\mathbf{T}'\subseteq\mathbf{T}$ *has a* $\bar\lambda^0$*-model then* $\mathbf{T}$ *has a* $\bar\lambda^1$*-model*

($**$) $\langle\kappa_\xi:\xi\le\xi(*)\rangle\to\langle\kappa'_\xi:\xi\le\xi(*)\rangle$ *if* $\bigwedge_\xi\kappa_\xi^{+\omega}\le\kappa_{\xi+1}$ *and* $\kappa'_0\le\kappa'_1\le\dots\le\kappa'\xi(*)\le 2^{\aleph_0}$ *(and versions like 2.11(1)).* {2.9}

Proof. Why? By 2.10 if $\bar\lambda=\langle(\lambda_\xi,\kappa_\xi):\xi<\zeta(*)\rangle$, $\lambda_\xi\ge\beth_\omega(\kappa_\xi)$ then $\mathrm{Pr}_n(\bar\lambda)$ (really {2.8}
$\lambda_\xi\ge\beth_k(\kappa_\xi)$ for k depending on n only suffices, see [EHMR84]). Now ccc forcing preserves this and now apply 2.11. Similarly we can use θ^+-cc forcing $\mathbf{P}$ and deal {2.9}
with cardinals in the interval $(\theta,2^\theta)$ in $\mathbf{V}^{\mathbf{P}}$. $\square$

{2.12}

Remark 2.14. We can say parallel things for the compactness of $(\exists^{\ge\lambda})$, for λ singular $\le 2^{\lambda_0}$ (or $\theta+|\mathscr{T}|<\lambda\le$ number of θ-branches of $\mathscr{T}$), e.g. we get the parallel of {2.11}
2.13.

In more details, if $\mathbf{V}_0=\mathbf{V}^{\mathbf{P}}$, $\mathbf{P}$ satisfies the θ^+-c.c. then

($*$) in $\mathbf{V}_0^{\mathbf{P}}$, for any singular $\lambda\in(\theta,2^\theta)$ such that $\mathbf{V}_0\models$ "λ is strong limit" we have

  ⊛ the class $\{(\lambda,<,R_\zeta\dots)_{\zeta<\theta}:R_\zeta$ an n_ζ-relation, $(\lambda,<)$ is λ-like$\}$ of models is θ-compact, and we can axiomatize it.

There are also consistent counterexamples, see [Sh:532].

The point of proving ($*$) is:

⊗ for a vocabulary τ of cardinality $\le\theta$, letting $\mathbf{T}^{\rm sk}_\tau$ be a first order theory with Skolem functions $\tau(T^{\rm sk}_\tau)$ (but the Skolem functions are new), then TFAE for a first order $\mathbf{T}\subseteq L_{\omega,\omega}(\tau)$

  (a) $\mathbf{T}$ has a λ-like model

(b) the following is consistent: $\mathbf{T} \cup \mathbf{T}^{\rm sk}_\tau \cup \{\sigma(\ldots, x^{n(\ell)}_{\eta_\ell}, y_{n(\ell)}, \ldots)_{\ell<k} = \sigma(\ldots, x^{n(\ell)}_{\nu_\ell}, y_{n(\ell)}, \ldots)_{\ell<k} \vee \sigma(\ldots, x^{n(\ell)}_{\eta_\ell}, y_{n(\ell)} \ldots) > y_n : n(\ell) < \omega$, $\eta_\ell \in {}^\omega 2$, and for some $j < \omega$, $\langle \eta_\ell \restriction j : \ell < k\rangle$ is with no repetition, $\eta_\ell \restriction j = \nu_\ell \restriction j$, $n(\ell) < n \Rightarrow \eta_\ell = \nu_\ell\} \cup \{\sigma(\ldots, x^{n(\ell)}_{\eta_\ell}, y_{n(\ell)}, \ldots) < y_n : n(\ell) < n$ and $\eta_\ell \in {}^\omega 2\}$.

§ 3. Finer analysis of square existence

Definition 3.1. 1) For an ω-sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of (2,2)-trees, we define a function degsq (square degree). {3.1}

Its domain is $\mathrm{pfap} = \mathrm{pfap}_{\bar{T}} = \{(u,g) : (\exists n)(u \in [{}^n2]^*, g$ is a 2-place function from u to $\omega)\}$ and its values are ordinals (or ∞ or -1). For this we define the truth value of "$\mathrm{degsq}_{\bar{T}}(u,g) \geq \alpha$" by induction on the ordinal α.

Case 1: $\alpha = -1$

$\mathrm{degsq}_{\bar{T}}(u,g) \geq -1$ iff $(u,g) \in \mathrm{pfap}_{\bar{T}}$ and $\eta,\nu \in u \Rightarrow (\eta,\nu) \in T_{g(\eta,\nu)}$.

Case 2: α is limit

$\mathrm{degsq}_{\bar{T}}(u,g) \geq \alpha$ iff $\mathrm{degsq}_{\bar{T}}(u,g) \geq \beta$ for every $\beta < \alpha$.

Case 3: $\alpha = \beta + 1$

$\mathrm{degsq}_{\bar{T}}(u,g) \geq \alpha$ iff for every $\rho^* \in u$, for some m, $u^* \subseteq {}^m2, g^*$ and functions h_0, h_1, we have:

$h_i : u \to u^*$,
$(\forall \eta \in u)\eta \triangleleft h_i(\eta)$
$(\forall \eta \in u)[h_0(\eta) = h_1(\eta) \Leftrightarrow \eta \neq \rho^*]$
$u^* = \mathrm{Rang}(h_0) \cup \mathrm{Rang}(h_1)$
g^* is a 2-place function from u^* to ω
$g^*(h_i(\eta), h_i(\nu)) = g(\eta,\nu)$ for $i < 2$ and $\eta,\nu \in u$
$\mathrm{degsq}_{\bar{T}}(u^*,g^*) \geq \beta$ (so $(u^*,g^*)a \in \mathrm{pfap}$).

2) We define $\mathrm{degsq}_{\bar{T}}(u,g) = \alpha$ iff for every ordinal $\beta, \mathrm{degsq}(u,g) \geq \beta \Leftrightarrow \beta \leq \alpha$ (so $\alpha = -1, \alpha = \infty$ are legal values).
3) We define $\mathrm{degsq}(\bar{T}) = \bigcup\{\mathrm{degsq}_{\bar{T}}(u,g) + 1 : (u,g) \in \mathrm{pfap}_{\bar{T}}\}$.

Claim 3.2. *Assume $\bar{T}$ is an ω-sequence of (2,2)-trees.* {3.2}
1) For every $(u,g) \in \mathrm{pfap}_{\bar{T}}, \mathrm{degsq}_{\bar{T}}(u,g)$ is an ordinal, ∞ or -1. Any automorphism F of $({}^\omega 2, \triangleleft)$ preserves this (it acts on $\bar{T}$ too, i.e.

$$\mathrm{degsq}_{\bar{T}}(u,g) = \mathrm{degsq}_{\langle F(T_n):n<\omega\rangle}(F(u), g \circ F^{-1}).$$

2) $\mathrm{degsq}(\bar{T}) = \infty$ iff $\mathrm{degsq}(\bar{T}) \geq \omega_1$ iff there is a perfect square contained in $\bigcup\limits_{n<\omega} \lim(T_n)$ iff for some ccc forcing notion $\mathbf{P}, \Vdash_{\mathbf{P}}$ "$\bigcup\limits_{n<\omega} \lim(T_n)$ contains a $\lambda_{\omega_1}(\aleph_0)$-square" (so those properties are absolute).
3) If $\mathrm{degsq}(\bar{T}) = \alpha(*) < \omega_1$ then $\bigcup\limits_{n<\omega} \lim(T_n)$ contains no $\lambda_{\alpha(*)+1}(\aleph_0)$-square.
4) For each $\alpha(*) < \omega_1$ there is an ω-sequence of $(2,2)$-trees $\bar{T} = \langle T_n : n < \omega\rangle$ with $\mathrm{degsq}(\bar{T}) = \alpha(*)$.
5) If $\bar{T} = \langle T_n : n < \omega\rangle$ is a sequence of (2,2)-trees then the existence of an $\aleph_1$-square in $\bigcup\limits_{n<\omega} \lim(T_n)$ is absolute.
6) Moreover for $\alpha(*) < \omega_1$ we have: if $\mu < \lambda_{\alpha(*)}(\aleph_0), A, B$ disjoint subsets of ${}^\omega 2 \times {}^\omega 2$ of cardinality $\leq \mu$, then some c.c.c. forcing notion $\mathbf{P}$ adds $\bar{T}$ as in (4) (i.e. an ω-sequence of $(2,2)$-trees $\bar{T} = \langle T_n : n < \omega\rangle$ with $\mathrm{degsq}_{\bar{T}}(\bar{T}) = \alpha(*)$) such that: $A \subseteq \bigcup\limits_{n<\omega} \lim[f(T_n)]$, $B \cap \bigcup\limits_{n<\omega} \lim[f(T_n)] = \emptyset$.

Proof. Easy to prove. E.g.

3) Let $\lambda = \lambda_{\alpha(*)+1}(\aleph_0)$ and assume $\{\eta_i : i < \lambda\} \subseteq {}^\omega 2$, $[i < j \Rightarrow \eta_i \neq \eta_j]$ and $(\eta_i, \eta_j) \in \bigcup\limits_n \lim(T_n)$ and let $(\eta_i, \eta_j) \in \lim(T_{g(\eta_i,\eta_j)})$. For $(u, f) \in \mathrm{pfap}_{\bar T}$, $u = \{\nu_0, \ldots, \nu_{k-1}\}$ (with no repetition, $<_{lx}$-increasing) let $R_{(u,f)} = \{(\alpha_0, \ldots, \alpha_{k-1}) : \alpha_\ell < \lambda$ and $\nu_\ell \triangleleft \eta_{\alpha_\ell}$ for $\ell < k$ and $f(\nu_\ell, \nu_m) = g(\eta_{\alpha_\ell}, \eta_{\alpha_m})$ for $\ell, m < k\}$. Let $M = (\lambda, R_{(u,f)})_{(u,f)\in \mathrm{pfap}_{\bar T}}$. Clearly if we have $\alpha_0, \ldots, \alpha_{k-1} < \lambda$ and n such that $\langle \eta_{\alpha_\ell} \restriction n : \ell < k\rangle$ is with no repetition, $g(\eta_{\alpha_{\ell(1)}}, \eta_{\alpha_{\ell(2)}}) = f(\eta_{\alpha_{\ell(1)}} \restriction n, \eta_{\alpha_{\ell(2)}} \restriction n)$ then $R_{(u,f)}(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ and we can then prove

$$\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{k-1}\}, M) \leq \mathrm{degsq}_{\bar T}(u, f)$$

(by induction on the left ordinal). But M is a model with countable vocabulary and cardinality $\lambda = \lambda_{\alpha(*)+1}(\aleph_0)$. Hence by the definition of $\lambda_{\alpha(*)+1}$ we have $\mathrm{rk}(M) \geq \alpha(*) + 1$, so $\alpha(*) + 1 \leq \mathrm{rk}(M) \leq \mathrm{degsq}(\bar T) \leq \alpha(*)$ (by previous sentence, earlier sentence and a hypothesis respectively). Contradiction.
4) Let $W = \{\eta : \eta$ is a (strictly) decreasing sequence of ordinals, possibly empty$\}$.
  We choose by induction on $i < \omega$, n_i and an indexed set $\langle (u^i_x, f^i_x, \alpha^i_x) : x \in X_i\rangle$ such that:

(a) $n_i < \omega$, $n_0 = 0$, $n_i < n_{i+1}$
(b) X_i finite including $\bigcup\limits_{j<i} X_j$
(c) for $x \in X_i$, $u^i_x \subseteq {}^{n_i}2$, f^i_x a two place function from u^i_x to ω and $\alpha^i_x \in w_i$
(d) for some $x \in X_i$, $u^i_x = \{0_{n_i}\}$
(e) h_i is a function from X_i into W and $h_i \subseteq h_{i+1}$
(f) $|X_0| = 1$, and h_0 is constantly $\langle\rangle$
(g) if $x \in X_i$ <u>then</u>: $\alpha^{i+1}_x = \alpha^i_x$ and the function $\nu \mapsto \nu \restriction n_i$ is one to one from u^{i+1}_x onto u^i_x and $\nu \in u^{i+1}_x \Rightarrow \nu \restriction [n_i, n_{i+1}) = 0_{[n_i,n_{i+1})}$ and $\eta, \nu \in u^{i+1}_x \Rightarrow f^{i+1}_x(\eta, \nu) = f^i_x(\eta \restriction n_i, \nu \restriction n_i)$
(h) for some $x = x_i \in X_i$, $\beta = \beta_i \in w_{i+1} \cap \alpha^i_x$ and $\rho^* = \rho^*_i \in u^i_x$ and $\Upsilon_i \in W$ such that $h_i(x_i) \triangleleft \Upsilon_i$ we can find $y = y_i$ such that:
  (α) $X_{i+1} = X_i \cup \{y_i\}$, $y_i \notin X_i$
  (β) $\alpha^{i+1}_y = \beta$
  (γ) the function $\nu \mapsto \nu \restriction n_i$ is a function from u^{i+1}_y onto u^i_x, almost one to one: ρ^* has exactly two predecessors say ρ^y_1, ρ^y_2, and any other $\rho \in u^i_x \setminus \{\rho^*\}$ has exactly one predecessor in u^{i+1}_y
  (δ) for $\nu, \eta \in u^{i+1}_y$ if $(\nu, \eta) \neq (\rho^y_1, \rho^y_2)$ and $(\nu, \eta) \neq (\rho^y_2, \rho^y_1)$ then $f^{i+1}_{y_i}(\eta, \nu) = f^i_x(\eta \restriction n_i, \nu \restriction n_i)$
  (φ) $h_{i+1} = h_i \cup \{(y_i, \Upsilon_i)\}$
(i) if $x_1, x_2 \in X_i$ and $u^i_{x_1} \cap u^i_{x_2} \neq \emptyset$ *then* $u^i_{x_1} \cap u^i_{x_2} = \{0_{n_i}\}$
(j) if $x \in X_i$, $\beta \in w_i \cap \alpha^i_x$, $\rho^* \in u^i_x$ and $h(x) \triangleleft \Upsilon \in W$, then for some $j \in (i, \omega)$ we have $x_j = x$, $\beta_j = \beta$, $\rho^* \trianglelefteq \rho^*_j$ and $\Upsilon_j = \Upsilon$
(k) the numbers $f^{i+1}_{y_i}(\rho^{y^1}_1, \rho^{y^1}_2)$, $f^{i+1}_{y_i}(\rho^{y^2}_2, \rho^{y^2}_1)$ are distinct and do not belong to $\bigcup\{\mathrm{Rang}(f^i_x) : x \in X_i\}$.

There is no problem to carry the definition.
We then let

$$T_n = \{(\eta,\nu) : \quad \begin{array}{l} \text{for some } i < \omega \text{ and } x \in X_i \text{ and } \eta',\nu' \in u^i_x \\ \text{we have } f^i_x(\eta',\nu') = n \text{ and for some } \ell \le n_i \text{ we have} \\ (\eta,\nu) = (\eta' \restriction \ell, \nu' \restriction \ell)\} \end{array}$$

and $\bar{T} = \langle T_n : n < \omega\rangle$. Now it is straight to compute the rank.
5) By the completeness theorem for $L_{\omega_1,\omega}(Q)$ (see Keisler [Kei71])
6) By the proof of 1.13. $\square$ {1.10}

* * *

Now we turn to κ-Souslin sets. {3.3}

Definition 3.3. Let T be a $(2,2,\kappa)$-tree. Let $\mathrm{set}(T)$ be the set of all pairs (u,f) such that $(\exists n = n(u,f))[u \subseteq {}^n2$ and $f : u \times u \to {}^n\kappa$ and $\eta,\nu \in u \Rightarrow (\eta,\nu,f(\eta,\nu)) \in T]$.

We want to define $\mathrm{degsq}_T(x)$ for $x \in \mathrm{set}(T)$. For this we define by induction on the ordinal α when $\mathrm{degsq}_T(x) \ge \alpha$.

<u>Case 1</u>: $\alpha = -1$
$\mathrm{degsq}_T(u,f) \ge \alpha$ <u>iff</u> $(u,f) \in \mathrm{set}(T)$.

<u>Case 2</u>: α limit
$\mathrm{degsq}_T(u,f) \ge \alpha$ <u>iff</u> $\mathrm{degsq}_T(u,f) \ge \beta$ for every $\beta < \alpha$.

<u>Case 3</u>: $\alpha = \beta + 1$
$\mathrm{degsq}_T(u,f) \ge \alpha$ <u>iff</u> for every $\eta^* \in u$, for some $m > n(u,f)$ there are $(u^*,f^*) \in \mathrm{set}(T)$ and functions h_0,h_1 such that $\mathrm{degsq}_T(u^*,f^*) \ge \beta$ and:

(i) $n(u^*,f^*) = m$
(ii) h_i is a function from u to ${}^m2$
(iii) $\eta \triangleleft h_i(\eta)$ for $i < 2$
(iv) for $\eta \in u$ we have $h_0(\eta) \ne h_1(\eta) \Leftrightarrow \eta = \eta^*$
(v) for $\eta_1 \ne \eta_2 \in u, i < 2$ we have $f(\eta_1,\eta_2) \triangleleft f^*(h_i(\eta_1),h_i(\eta_2))$
(vi) for $\eta \in u^*$ we have $f(\eta{\restriction}n,\ \eta{\restriction}n) \triangleleft f^*(\eta,\eta)$

Lastly, $\mathrm{degsq}_T(u,f) = \alpha$ <u>iff</u> for every ordinal β we have $[\beta \le \alpha \Leftrightarrow \mathrm{degsq}_T(u,f) \ge \beta]$.

Also let $\mathrm{degsq}(T) = \mathrm{degsq}_T(\{\langle\rangle\},\{<\langle\rangle,\langle\rangle>\})$. {3.4}

Claim 3.4. *1) For a $(2,2,\kappa)$-tree T, for $(u,f) \in \mathrm{set}(T)$, $\mathrm{degsq}_T(u,f)$ is an ordinal or infinity or $= -1$. And similarly $\mathrm{degsq}(T)$. All are absolute. Also $\mathrm{degsq}(T) \ge \kappa^+$ implies $\mathrm{degsq}(T) = \infty$ and similarly for $\mathrm{degsq}_T(u,f)$.*
2) $\mathrm{degsq}(T) = \infty$ <u>iff</u> $\Vdash_{\mathbf{P}}$ "$\mathrm{prj}\lim T$ $(= \{(\eta,\nu) \in {}^\omega2 \times {}^\omega2$: for some $\rho \in {}^\omega\kappa$, $\bigwedge_{n<\omega} (\eta{\restriction}n, \nu{\restriction}n, \rho{\restriction}n) \in T\})$ contains a perfect square" for every forcing notion $\mathbf{P}$ including a trivial one i.e. $\mathbf{V}^{\mathbf{P}} = \mathbf{V}$ <u>iff</u> $\models_{\mathbf{P}}$ "$\mathrm{prj}\lim(T)$ contains a $2^{\aleph_0}$-square" for the forcing notion $\mathbf{P}$ which is adding λ Cohen reals for $\lambda = \lambda_{\omega_1}(\kappa)$ some λ <u>iff</u> for some $\mathbf{P}, \Vdash_{\mathbf{P}}$ "$\mathrm{prj}\lim(T)$ contains ?? $\lambda_{\kappa^+}(\aleph_0)$-square".

3) If $\alpha(*) = \text{degsq}(T) < \kappa^+$*, then* $\text{prj}\lim(T)$ *does not contain a* $\lambda_{\alpha(*)+1}(\kappa)$*-square.*

Proof. 1) Easy.
2) Assume $\text{degsq}(T) = \infty$, and note that $\alpha^* = \{\text{degsq}_T(u, f) : (u, f) \in \text{set}(T)$ and $\text{degsq}_T(u, f) < \infty\} \setminus \{\infty\}$ is an ordinal so $(u, f) \in \text{set}(T)$ and $\text{degsq}_T(u, f) \geq \alpha^* \Rightarrow \text{degsq}_T(u, f) = \infty$ (in fact any ordinal $\alpha \geq \sup\{\text{degsq}_T(u, f) + 1 : (u, f) \in \text{set}(T)\}$ will do). Let $\text{set}^\infty(T) = \{(u, f) \in \text{set}(T) : \text{degsq}_T(u, f) = \infty\}$.

Now

- $(*)_1$ there is $(u, f) \in \text{set}^\infty(T)$
- $(*)_2$ for every $(u, f) \in \text{set}^\infty(T)$ and $\rho \in u$ we can find $(u^+, f^+) \in \text{set}^\infty(T)$ and for $e = 1, 2, h_e : u \to u^+$ such that $(\forall \eta \in u)(\eta \triangleleft h_e(\eta))$, $(\forall \eta, \nu \in u)(f(\eta, \nu) \triangleleft f^+(h_e(\eta), h_e(\nu))$, $(\forall \eta \in u)[h_1(\eta) = h_2(\eta) \Leftrightarrow \eta = \rho]$.

[Why? As $\text{degsq}_T(u, f) = \infty$ it is $\geq \alpha^* + 1$ so by the definition we can find (u^+, f^+), h_1, h_2 as above but only with $\text{degsq}_T(u^+, f^+) \geq \alpha^*$, but this implies $\text{degsq}_T(u^+, f^+) = \infty$.]

- $(*)_3$ for every $(u, f) \in \text{set}^\infty(T)$ with $u = \{\eta_\rho : \rho \in {}^n2\} \subseteq {}^{(n_1)}2$ (no repetition) we can find $n_2 > n_1$ and $(u^+, f^+) \in \text{set}^\infty(T)$ with $u^+ = \{\eta_\rho : \rho \in {}^{n+1}2\} \subseteq {}^{(n_2)}2$ (no repetitions) such that
  - (i) $\rho \in {}^{n+1}2 \Rightarrow \eta_{\rho \restriction n} \triangleleft \eta_\rho$
  - (ii) for $\rho_1, \rho_2 \in {}^{n+1}2$, $\rho_1 \restriction n \neq \rho_2 \restriction n \Rightarrow f(\eta_{\rho_1 \restriction n}, \eta_{\rho_2 \restriction n}) \triangleleft f^+(\eta_{\rho_1}, \eta_{\rho_2})$
  - (iii) for $\rho \in {}^{n+1}2$, $f(\eta_{\rho \restriction n}, \eta_{\rho \restriction n}) \triangleleft f(\eta_\rho, \eta_\rho)$.

[Why? Repeat $(*)_2$ 2^n times.]

So we can find $\langle \eta_\rho : \eta \in {}^n2 \rangle$, f_n by induction on n such that $\langle \eta_\rho : \rho \in {}^n2 \rangle$ is with no repetition, $\text{degsq}_T(\{\eta_\rho : \rho \in {}^n2\}, f_n) = \infty$, and for each n clauses (i), (ii), (iii) of $(*)_3$ hold, i.e. $\rho_1, \rho_2 \in {}^{n+1}2$, $\rho_1 \restriction n \neq \rho_2 \restriction n \Rightarrow f_n(\eta_{\rho_1 \restriction n}, \eta_{\rho_2 \restriction n}) \triangleleft f_{n+1}(\eta_{\rho_1}, \eta_{\rho_2})$ and for $\rho \in {}^{n+1}2$ we have $f_n(\eta_{\rho \restriction n}, \eta_{\rho \restriction n}) \triangleleft f_{n+1}(\eta_\rho, \eta_\rho)$ and of course $\{\eta_\rho : \rho \in {}^m2\} \subseteq {}^{(k_n)}2$, $k_n < k_{n+1} < \omega$.

So we have proved that the first clause implies the second (about the forcing: the $\text{degsq}(T) = \infty$ is absolute so holds also in $\mathbf{V}^{\mathbf{P}}$ for any forcing notion $\mathbf{P}$). Trivially the second clause implies the third and fourth.

{1.7} So assume the third clause and we shall prove the first. By 1.8 $\lambda^2_{\kappa^+}(\kappa)$ is well
{1.4} defined (e.g. $\leq \beth_{\kappa^+}$), but $\lambda^3_{\kappa^+}(\kappa) = \lambda^2_{\kappa^+}(\kappa)$ by 1.5(3), let $\mathbf{P}$ be the forcing notion
{1.1} adding $\lambda^2_{\kappa^+}(\kappa)$ Cohen reals. By 1.1(2) in $\mathbf{V}^{\mathbf{P}}$, $\lambda_{\kappa^+}(\kappa) \leq \lambda^2_{\kappa^+}(\kappa) \leq 2^{\aleph_0}$, and so there are pairwise disjoint $\eta_i \in {}^\omega2$ for $i < \lambda_{\kappa^+}(\kappa)$ such that $(\eta_i, \eta_j) \in \text{prj}\lim(T)$ for i, $j < \lambda_{\kappa^+}(\kappa)$.

Lastly, we prove forcing implies first in $\mathbf{V}^{\mathbf{P}}$. By part (3) of the claim proved below we get for every $\alpha < \kappa^+$, as $\lambda_\alpha(\kappa) \leq \lambda_{\kappa^+}(\kappa)$ that $\neg[\alpha = \text{degsq}(T)]$. Hence $\text{degsq}(T) \geq \kappa^+$, but by part (1) this implies $\text{degsq}(T) = \infty$.

{3.2} 3) Just like the proof of 3.2(3). □

We shall prove in [Sh:532]

{3.5}

Claim 3.5. *Assume* $\alpha(*) < \kappa^+$ *and* $\lambda < \lambda_{\alpha(*)}(\kappa)$.
1) For some c.c.c. forcing notion $\mathbf{P}$*, in* $V^{\mathbf{P}}$ *there is a* κ*-Souslin subset* $A = \text{prj}\lim(T)$ *(where* T *is a* $(2, 2, \kappa)$*-tree) such that:*

- $(*)$ *A contains a* λ*-square but* $\text{degsq}(T) \leq \alpha(*)$.

2) For given $B \subseteq ({}^\omega 2 \times {}^\omega 2)^{\mathbf{V}}$ of cardinality $\le \lambda$ we can replace $(*)$ by

$(*)'$ $A \cap ({}^\omega 2 \times {}^\omega 2)^{\mathbf{V}} = B$ *but* $\mathrm{degsq}(T) \le \alpha(*)$.

{3.5A}

Remark 3.6. The following says in fact that "colouring of pairs is enough": say for the Hanf number of $L_{\omega_1,\omega}$ below the continuum, for clarification see 3.8. {3.6A} {3.6}

Claim 3.7. *[MA] Assume $\lambda < 2^{\aleph_0}$ and $\alpha(*) < \omega_1$ is a limit ordinal, $\lambda < \mu := \lambda_{\alpha(*)}(\aleph_0)$. Then for some symmetric 2-place function F from λ to ω we have:*

$(*)_{\lambda,\mu,F}$ *for no two place (symmetric) function F' from μ to ω do we have:*

$(**)$ *for every $n < \omega$, and pairwise distinct $\beta_0, \dots, \beta_{n-1} < \mu$ there are pairwise distinct $\alpha_0, \dots, \alpha_{n-1} < \lambda$ such that*

$$\bigwedge_{k<\ell<n} F'(\beta_k, \beta_\ell) = F(\alpha_k, \alpha_\ell).$$

{3.6A}

Remark 3.8. 1) This is close to Gilchrist Shelah [GcSh:491].
2) The proof of 3.7 says that letting $R_n = \{(\alpha,\beta) : F(\alpha,\beta) = n\}$, and $N = (\lambda, R_n)_{n<\omega}$, we have $\mathrm{rk}(N) < \alpha(*)$. {3.6}
3) So 3.7 improves §2 by saying that the examples for the Hanf number of $L_{\omega_1,\omega}$ below the continuum being large can be very simple, speaking only on "finite patterns" of colouring pairs by countably many colours. {3.6}

Proof. Let M be a model of cardinality λ with a countable vocabulary and $\mathrm{rk}^1(M) < \alpha(*)$. Without loss of generality it has the universe λ, has the relation $<$ and individual constants c_α for $\alpha \le \omega$. Let k^M, φ^M be as in the proof of 1.13. {1.10}

Let $\mathbf{P}$ be the set of triples $(u, f, \mathbf{w})$ such that:

(a) u is a finite subset of λ

(b) f is a symmetric two place function from u to ω

(c) $\mathbf{w}$ is a family of nonempty subsets of u

such that

(d) if $\alpha \in u$ then $\{\alpha\} \in \mathbf{w}$

(e) if $w = \{\alpha_0, \dots, \alpha_{n-1}\} \in \mathbf{w}$ (the increasing enumeration), $k = k^M(w)$, $\alpha \in u \setminus w$ and $(\forall \ell)[\ell < n$ and $\ell \ne k \Rightarrow f(\alpha_\ell, \alpha) = f(\alpha_\ell, \alpha_k)]$, then $w \cup \{\alpha\}$ belongs to $\mathbf{w}$, $(\forall m \ne k)(\alpha < \alpha_m \Leftrightarrow \alpha_k < \alpha_m)$ and $k = k^M(w \cup \{\alpha\} \setminus \{\alpha_k\})$ and $M \models \varphi^M(w)[\alpha_0, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots, \alpha_{n-1}]$

(f) if $w^i = \{\alpha^i_0, \dots, \alpha^i_{n-1}\} \subseteq u$ (increasing enumeration, so with no repetition), $i = 0, 1$, and $(\forall \ell < k < n)[f(\alpha^0_l, \alpha^0_k) = f(\alpha^1_\ell, \alpha^1_k)]$ then $w^0 \in \mathbf{w} \Leftrightarrow w^1 \in \mathbf{w}$ and if $w^i \in \mathbf{w}$ then $\varphi^M(w^0) = \varphi^M(w^1)$, $k^M(w^0) = k^M(w^1)$ and $\mathrm{rk}^1(w^0, M) = \mathrm{rk}^1(w^1, M)$.

The order is the natural one.

It is easy to check that:

$\oplus_1$ $\mathbf{P}$ satisfies the c.c.c.

$\oplus_2$ for every $\alpha < \lambda, \mathscr{L}_\alpha = \{(u, f, \mathbf{w}) \in \mathbf{P} : \alpha \in u\}$ is dense.

Hence there is a directed $G \subseteq \mathbf{P}$ not disjoint from $\mathscr{L}_\alpha$ for every $\alpha < \lambda$. Let $F = \bigcup\{f : \text{for some } u, \mathbf{w} \text{ we have } (u, f, \mathbf{w}) \in G\}$. We shall show that it is as required. Clearly F is a symmetric two place function from λ to ω; so the only thing that can fail is if there is a symmetric two place function F' from μ to ω such that $(**)$ of 3.7 holds. By the compactness theorem for propositional logic, there {3.6} is a linear order $<^*$ of μ such that

$(*)'$ for every $n < \omega$ and $\beta_0 <^* \cdots <^* \beta_{n-1}$ from μ there are $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$ such that $\bigwedge_{k<\ell<n} F'(\beta_k, \beta_\ell) = F(\alpha_k, \alpha_\ell)$.

Let

$$\mathbf{W} = \bigcup\{\mathbf{w} : \text{ for some } u, f \text{ we have } (u, f, \mathbf{w}) \in G\}.$$

Let $N = (\lambda, R_n)_{n<\omega}$ where $R_n = \{(\alpha, \beta) : F(\alpha, \beta) = n\}$, so $\mathrm{rk}^1(w, N)$ for $w \in [\lambda]^*$ is well defined; in fact we can restrict ourselves to formulas of the form $\varphi(x_0, \ldots, x_{n-1}) = \bigwedge\limits_{\ell<k<n} F(x_\ell, x_k) = c_{m(\ell,k)}$. Let $N' = (\mu, R'_n)_{n<\omega}$, where $R'_n = \{(\alpha, \beta) : F'(\alpha, \beta) = n\}$.

Now first note that

$\otimes_1$ if $w \in \mathbf{W}$ then $\mathrm{rk}^1(w, N) \leq \mathrm{rk}^1(w, M)$

(this is the role of clause (e) in the definition of $\mathbf{P}$; we prove it by induction on $\mathrm{rk}^1(w, M)$ using the same "witness" k^M).

Secondly,

$\otimes_2$ for $\alpha < \lambda$ we have $\{\alpha\} \in \mathbf{W}$

{1.1} (this is the role of clause (d) in the definition of $\mathbf{P}$). Hence we conclude (by 1.1(2), $\otimes_1$, $\otimes_2$):

$\otimes_3$ $\mathrm{rk}^1(N) < \alpha(*)$.

Lastly

$\otimes_4$ <u>iff</u> $\alpha_0 <^* \ldots <^* \alpha_{m-1}$ are $< \mu$, $\beta_0 < \ldots < \beta_{m-1} < \lambda$, $\bigwedge\limits_{k<\ell<m} F'(\alpha_k, \alpha_l) = F(\beta_k, \beta_l)$ and $\{\beta_0, \ldots, \beta_{m-1}\} \in \mathbf{W}$, <u>then</u> $\mathrm{rk}^1(\{\alpha_0, \ldots, \alpha_{m-1}\}, N') \leq \mathrm{rk}^1(\{\beta_0, \ldots, \beta_{m-1}\}, N)$

(again it can be proved by induction on $\mathrm{rk}^1(\{\beta_\ell, \ldots, \beta_{m-1}\}, N)$, the choice of N' and our assumption toward contradiction that $(**)$ from the claim holds). Now by {1.1} $\otimes_3$, $\otimes_4$ and 1.1(2) (and $\otimes_2$) we have $\mathrm{rk}^1(N') \leq \mathrm{rk}^1(N) < \alpha(*)$ but this contradicts $\|N'\| = \mu = \lambda_{\alpha(*)}(\aleph_0)$. $\square$

{3.7}

{3.6} **Claim 3.9.** *[MA] If λ, μ, F are as in 3.7 (i.e. $(*)_{\lambda,\mu,F}$ holds) <u>then</u> some Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ (actually of the form $\bigcup\limits_{n<\omega} \lim(T_n)$) has a λ-square but no μ-square.*

{3.7A}

Remark 3.10. 1) The converse holds too, of course.
2) Instead of λ we can use "all $\lambda < \mu$".

Proof. Without loss of generality $\mathrm{cf}(\lambda) > \aleph_0$ (otherwise combine ω examples). Let F be a symmetric two place function from λ to ω such that $(*)_{\lambda,\mu,F}$. For simplicity let $f^* : \omega \to \omega$ be such that $\forall n \exists^\infty m f^*(m) = n$. We define a forcing {1.10} notion $\mathbf{P}$ as in 1.13 except that we require in addition for $p \in \mathbf{P}$:

$\otimes_1$ $f^*(g^p(\alpha,\beta)) = F(\alpha,\beta)$

$\otimes_2$ if $\alpha' \neq \beta', \alpha'' \neq \beta''$ are from $u^p, k < \omega, \eta^p_{\alpha'} \restriction k = \eta^p_{\alpha''} \restriction k \neq \eta^p_{\beta'} \restriction k = \eta^p_{\beta''} \restriction k$ both not constantly 1 then $F(\alpha',\beta') = F(\alpha'',\beta'')$

$\otimes_3$ if $\eta,\nu \in {}^{n[p]}2$ then for at most one $m < m[p]$ we have $(\eta,\nu) \in t^p_m$

$\otimes_4$ iff $n < \omega$, $n > 1$, $\eta_0 <_{lx} \cdots <_{lx} \eta_{n-1}$ are pairwise distinct members of ${}^{n(p)}2$ and $k < \ell \Rightarrow (\eta_k,\eta_\ell) \in t^p_{g(k,\ell)}$,then for some pairwise distinct $\alpha_0,\ldots,\alpha_{n-1}$ from $u[p^n]$, we have $\bigwedge\limits_{k<\ell<n} f^*(h(k,\ell)) = F(\alpha_k,\alpha_\ell)$.

We have

$\oplus_1$ if $\alpha < \beta < \lambda$, there is a unique $n < \omega$ such that $(\underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}_\beta) \in \lim \underset{\sim}{T}_n$.

Thus $\bigcup\limits_{n\in\omega} T_n$ contains a λ-square. In proving that it does not contain a μ-square we apply $(*)_{\lambda,\mu,F}$. For this the crucial fact is:

$\oplus_2$ if $n < \omega$, $\eta_0,\ldots,\eta_{n-1} \in {}^\omega 2$ are distinct, $(\eta_k,\eta_\ell) \in \lim(T_{h(k,\ell)})$, then for some pairwise distinct $\alpha_0,\ldots,\alpha_{n-1} < \lambda$

$$\bigwedge_{k<\ell<n} f^*(h(k,\ell)) = F(\alpha_k,\alpha_\ell).$$

Instead of "$\mathscr{I}^3_\alpha$ is dense" it is enough to show

$\oplus_3$ for some $p \in \mathbf{P}$, $p \Vdash$ "the number of $\alpha < \lambda$ such that for some $q \in \underset{\sim}{G}_{\mathbf{P}}$ $\alpha \in u^q$ is λ".

$\square$

Similarly we can show

Claim 3.11 (MA)**.** *Assume $\aleph_0 < \lambda < 2^{\aleph_0}$. Then the following are equivalent:* {3.8}

(a) *For some symmetric 2-place functions F_μ from μ to ω (for $\mu < \lambda$) we have*

$(*)_{\langle F_\mu:\mu<\lambda\rangle}$ *for no two place function F' from λ to ω do we have:*

$(**)$ *for every $n < \omega$ and pairwise distinct $\beta_0,\ldots,\beta_{n-1} < \lambda$ there are $\mu < \lambda$ and pairwise distinct $\alpha_0,\ldots,\alpha_{n-1} < \mu$ such that*

$$\bigwedge_{k<\ell<n} F'(\beta_k,\beta_\ell) = F_\mu(\alpha_k,\alpha_\ell).$$

(b) *Some Borel subset of ${}^\omega 2 \times {}^\omega 2$ contains a μ-square iff $\mu < \lambda$ (in fact B is an F_σ set).*

§ 4. Rectangles

Simpler than squares are rectangles: subsets of ${}^{\omega}2 \times {}^{\omega}2$ of the form $X_0 \times X_1$, so a pair of cardinals characterize them: $\langle \lambda_0, \lambda_1 \rangle$ where $\lambda_\ell = |X_\ell|$. So we would like to define ranks and cardinals which characterize their existence just as $\mathrm{rk}^\ell(w, M; \kappa)$, $\lambda^\ell_\alpha(\kappa)$,$\mathrm{degsq}_{\bar{T}}(-)$, $\mathrm{degsq}_T(-)$ have done for squares. Though the demands are weaker, the formulation is more cumbersome: we have to have two "kinds" of variables one corresponding to λ_0 one to λ_1. So the models have two distinguished predicates, R_0, R_1 (corresponding to X_0, X_1 respectively) and in the definition of rank we connect only elements from distinct sides (in fact in §33944 we already concentrate on two place relations explaining not much is lost). This is very natural as except for inequality nothing connects two members of X_0 or two members of X_1.

{4.1}

Definition 4.1. We shall define $\mathrm{Prk}^\ell_\alpha(\lambda_1, \lambda_2; < \kappa, \theta_0, \theta_1), \mathrm{rkk}^\ell((w_1, w_2), M, \lambda_1, \lambda_2; \kappa, \theta_0, \theta_1)$
{1.1} as in 1.1 (but $w_\ell \in [R^M_\ell]^*$ and $|R^M_\ell| = \lambda_\ell$), replacing rk by rkk etc. Let $\bar{\lambda} = (\lambda_1, \lambda_2)$, $\bar{w} = (w_1, w_2)$.
1) For $\ell < 6$, and cardinals $\bar{\lambda} = (\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \geq \kappa$ and $\bar{\theta} = (\theta_0, \theta_1)$, $\theta_0 \leq \theta_1 < \lambda_1, \lambda_2$ and an ordinal α let $\mathrm{Prk}^\ell_\alpha(\bar{\lambda}; < \kappa, \bar{\theta})$ mean that: for every model M with vocabulary of cardinality $\leq \theta_0$, such that $\bigwedge\limits_{i=1}^{2} |R^M_i| = \lambda_i$, $R^M_1 \cap R^M_2 = \emptyset$, F^M is a two place function with range included in $\theta_1 = Q^M$, we have $\mathrm{rkk}^\ell(M; < \kappa) \geq \alpha$ (defined below).

Let $\mathrm{NPrk}^\ell_\alpha(\bar{\lambda}; < \kappa, \bar{\theta})$ be the negation. Instead of $< \kappa^+$ we may write κ; if $\kappa = \theta_0^+$ we may omit θ_0; if $\theta_0 = \aleph_0$, $\kappa = \aleph_1$, we may omit them. We may write θ_0, θ_1 instead $\bar{\theta} = (\theta_0, \theta_1)$ and similarly for $\bar{\lambda}$.

Lastly, we let $\lambda\mathrm{rc}^\ell_\alpha(\kappa, \bar{\theta}) = \min\{\lambda : \mathrm{Prk}^\ell_\alpha(\lambda, \lambda; < \kappa, \bar{\theta})\}$.
2) For a model $M, \mathrm{rkrc}^\ell(M; < \kappa) = \sup\{\mathrm{rkrc}^\ell(\bar{w}, M; < \kappa) + 1 : \bar{w} = \langle w_1, w_2 \rangle$ where $w_i \subseteq R^M_i$ are finite non empty for $i = 1, 2$ and $(\exists c \in Q^M)(\forall a, b)[a \in w_1$ and $b \in w_2 \Rightarrow F(a, b) = c]\}$ where rkk is as defined in part (3) below.
3) For a model M, and $\bar{w} \in [M]^{\otimes} := \{\bar{u} : \bar{u} = (u_1, u_2), u_i \subseteq R^M_i$ are finite nonempty and $(\exists c)(\forall a, b)[a \in u_1$ and $b \in u_2 \Rightarrow F(a, b) = c]\}$ we shall define the truth value of $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$ by induction on the ordinal α (for $\ell = 0, 1, \kappa$ can be omitted). If we write w instead of w_1, w_2 we mean $w_1 = w \cap R^M_1$, $w_2 = w \cap R^M_2$ (here $R^M_1 \cap R^M_2 = \emptyset$ helps).

Then we can note:

$(*)_0$ $\alpha \leq \beta$ and $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \beta \Rightarrow \mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$

$(*)_1$ $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \delta$ (δ limit) <u>iff</u> $\bigwedge\limits_{\alpha < \delta} \mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$

$(*)_2$ $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq 0 \underline{iff}\, \bar{w} \in [M]^{\otimes}$.

So we can define $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) = \alpha$ for the maximal α such that $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$, and ∞ if this holds for every α (and -1 if $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \not\geq 0$). Now the inductive definition of $\mathrm{rkk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$ was already done above for $\alpha = 0$ and α limit, so for $\alpha = \beta + 1$ we let

$(*)_3$ $\mathrm{rkk}^\ell(\bar{w}, M; <\kappa) \geq \beta+1$ iff (letting $w = w_1 \cup w_2$, $n = |w|$, $w = \{a_0, \ldots, a_{n-1}\}$), we have: for every $k < n$ and quantifier free formula $\varphi(x_0, \ldots, x_{n-1}) = \bigwedge_{i<j} x_i \neq x_j$ and $\bigwedge\{R_1(x_i) \wedge R_2(x_j) \wedge \varphi_{i,j}(x_i, x_j) : R_1(a_i)$ and $R_2(a_j)\}$.

(In the vocabulary of M) for which $M \models \varphi[a_0, \ldots, a_{n-1}]$ we have:

Case 1 : $\ell = 1$.
There are $a^i_m \in M$ for $m < n$, $i < 2$ such that:

(a) $\mathrm{rkk}^\ell(\{a^i_m : i < 2, m < n\}, M; <\kappa) \geq \beta$,
(b) $M \models \varphi[a^i_0, \ldots, a^i_{n-1}]$ (for $i = 1, 2$), so there is no repetition in $a^i_0, \ldots, a^i_{n-1}$ and $[a^i_m \in R^M_j \Leftrightarrow a_m \in R^M_j]$ for $j = 1, 2$
(c) $a^0_k \neq a^1_k$ but if $m < n$ and $(a_m \in R^M_1 \Leftrightarrow a_k \notin R^M_1)$ then $a^0_m = a^1_m$
(d) if $a_{m_1} \in R^M_1$, $a_{m_2} \in R^M_2$ then for any i, j $(\in \{1, 2\})$ we have $F^M(a^i_{m_1}, a^j_{m_2}) = F^M(a_{m_1}, a_{m_2})$.

Case 2: $\ell = 0$
As for $l = 1$ but in addition

(e) $\bigwedge_m a_m = a^0_m$.

Case 3: $\ell = 3$
The definition is like case 1 but $i < \kappa$; i.e. there are $a^i_m \in M$ for $m < n$, $i < \kappa$ such that:

(a) for $i < j < \kappa$ we have $\mathrm{rkk}^\ell(\{a^i_m, a^j_m : m < n\}, M; <\kappa) \geq \beta$
(b) $M \models \varphi[a^i_0, \ldots, a^i_{n-1}]$ (for $i < \kappa$; so there are no repetitions in $a^i_0, \ldots, a^i_{n-1}$)
(c) for $i < j < \kappa, a^i_k \neq a^j_k$ but if $m < n$ and $(a_m \in R^M_1 \Leftrightarrow a_k \notin R^M_1)$ then $a^i_m = a^j_m$
(d) if $a_{m_1} \in R^M_1$, $a_{m_2} \in R^M_2$ then for any i, j, $F^M(a^i_{m_1}, a^j_{m_2}) = F^M(a_{m_1}, a_{m_2})$.

Case 4: $\ell = 2$
Like case 3 but in addition:

(e) $a_m = a^0_m$ for $m < n$.

Case 5: $\ell = 5$
Like case 3 except that we replace clause (a) by:

(a)$^-$ for every function H, $\mathrm{Dom}(H) = \kappa$, $|\mathrm{Rang}(H)| < \kappa$ for some $i < j < \kappa$ we have $H(i) = H(j)$ and $\mathrm{rkk}^\ell(\{a^i_m, a^j_m : m < n\}, M; <\kappa) \geq \beta$.

Case 6: $\ell = 4$
Like case 4 using clause (a)$^-$ instead (a).

4) For M as above and $c \in Q^M$ we define $\mathrm{rkk}^\ell(M, c; <\kappa)$ as $\sup\{\mathrm{rkk}^\ell(\bar{w}, M; <\kappa) + 1 : \bar{w} \in [M]^{\otimes}$ and $(\forall a \in w_1)(\forall b \in w_2)[F(a, b) = c]\}$.
5) Let $\mathrm{Prrd}^\ell_\alpha(\bar{\lambda}, \kappa, \bar{\theta})$ mean $\mathrm{rkk}^\ell(M, c; <\kappa, \bar{\theta}) \geq \alpha$ for every M for some $c \in M$ when M is such that $|R^M_1| = \lambda_1$, $|R^M_2| = \lambda_2$, $|\tau(M)| \leq \theta_0$, $F^M : R^M_1 \times R^M_2 \to Q^M$,

$|Q^M| = \theta_1$. Let $\mathrm{NPrrd}^\ell_\alpha(\bar\lambda, \kappa, \bar\theta)$ mean its negation and $\lambda\mathrm{rd}^\ell_\alpha(\kappa, \bar\theta)$ be the minimal λ such that $\mathrm{Prrd}^\ell_\alpha(\lambda, \lambda, \kappa, \bar\theta)$.

{4.1A} *Remark* 4.2. The reader may wonder why in addition to Prk we use the variant Prrd. The point is that for the existence of the rectangle $X_1 \times X_2$ with $F \restriction (X_1 \times X_2)$ constantly c^*, this constant plays a special role. So in our main claim 4.6, to get a model as there, we need to choose it, one out of θ_1, but the other choices are out of κ. So though the difference between the two variants is small (see 4.5 below) we actually prefer the Prrd version.

{4.4} {4.3A} {4.2} {claim1.8}

Claim 4.3. *The parallels of 1.2 (+statements in 1.1), also* **??**, *1.5(2), 1.6, 1.10 hold.*

{4.3}

Claim 4.4. *1) If $w_i \in [R^M_i]^*$ for $i = 1, 2$ then $\omega \times \mathrm{rkk}^l(\langle w_1, w_2\rangle, M; \kappa) \geq \mathrm{rk}^l(w_1 \cup w_2, M; \kappa)$.*
2) If $R^M_1 = R^M_2$ (abuse of the notation) then $\mathrm{rkk}^\ell(M; < \kappa) \geq \mathrm{rk}^\ell(M; \kappa)$.
3) If $\lambda_1 = \lambda_2 = \lambda$ then $\mathrm{Pr}_\alpha(\lambda, \kappa) \Rightarrow \mathrm{Prk}_\alpha(\lambda_1, \lambda_2; \kappa, \kappa)$.

{4.3A}

Claim 4.5. $\lambda\mathrm{rd}^\ell_\alpha(\kappa, \theta) = \lambda\mathrm{rc}^\ell_\alpha(\kappa, \theta)$ *iff α is a successor ordinal or* $\mathrm{cf}(\alpha) > \theta$.

{4.4}

Claim 4.6. *Assume $\kappa \leq \theta < \lambda_1, \lambda_2$. Then the following are equivalent:*
1) $\mathrm{Prrd}^1_{\kappa^+}(\lambda_1, \lambda_2; \kappa, \theta)$.
2) Assume M is a model with a countable vocabulary, $|R^M_\ell| = \lambda_\ell$ for $\ell = 1, 2$, $P^M = \kappa, Q^M = \theta$, and F^M a two-place function (really just $F{\restriction}(R^M_1 \times R^M_2)$ interests us) and the range of $F{\restriction}(R^M_1 \times R^M_2)$ is included in Q^M and G is a function from $[R^M_1]^* \times [R^M_2]^*$ to P^M. Then we can find $\tau(M)$-models M_0, N and elements, c^*, a_η, b_η (for $\eta \in {}^\omega 2$) such that:

(i) *N is a model with the vocabulary of M (but functions may be interpreted as partial ones, i.e. as relations)*
(ii) *$a_\eta \in R^N_1$, $b_\eta \in R^N_2$ are pairwise distinct and $F^N(a_\eta, b_\eta) = c^*(\in N)$*
(iii) *M_0 countable, $M_0 \subseteq M$, $c^* \in Q^M$, M_0 is the closure of $(M_0 \cap P^M) \cup \{c^*\}$ in M, in fact for some $M'_0 \prec M$ we have $M_0 =$ closure of $P^{M'_0} \cup \{c^*\}$, $c^* \in M'_0$,*
(iv) *$M_0 \subseteq N$, $P^{M_0} = P^N$,*
(v) *$|N| = \{\sigma(a_\eta, b_\nu, \bar d) : \sigma$ is a $\tau(M)$-term, $\eta \in {}^\omega 2, \nu \in {}^\omega 2$ and $\bar d \subseteq M_0\}$*
(vi) *for $\{\eta_\ell : \ell < \ell(*)\}, \{\nu_m : m < m(*)\} \subseteq {}^\omega 2$ (both without repetitions non-empty) there is $d^1 \in P^{M_0}$ such that if $\bar d \subseteq P^{M_0}$ and quantifier free formulas $\varphi_{l,m}$ are such that $N \models \bigwedge\limits_{\ell<\ell(*), m<m(*)} \varphi_{\ell,m}[a_{\eta_\ell}, b_{\nu_m}, \bar d]$, then for some $\{a_\ell : \ell < \ell(*)\} \subseteq R^M_1$, $\{b_m : m < m(*)\} \subseteq R^M_2$ (both with no repetition) we have $M \models \bigwedge\limits_{\ell<\ell(*), m<m(*)} \varphi_{\ell,m}[a_\ell, b_m, \bar d]$ and $G(\{a_l : \ell < \ell(*)\}, \{b_m : m < m(*)\}) = d^1$*
(vii) *for every quantifier free first order $\varphi = \varphi(x, y, z_0, \ldots) \in L(\tau(M))$ and $d_0, \ldots \in M_0$ there is $k < \omega$ such that: for every $\eta_1, \eta_2, \nu_1, \nu_2 \in {}^\omega 2$ such that $\eta_1 \restriction k = \eta_2 \restriction k$, $\nu_1 \restriction k = \nu_2 \restriction k$ we have*
$$N \models \varphi[a_{\eta_1}, b_{\nu_1}, d_1, \ldots] = \varphi[a_{\eta_2}, b_{\nu_2}, d_1, \ldots].$$

Moreover

(vii)$^+$ *for every* $n < \omega$, *first order* $\varphi = \varphi(x_0, \ldots, x_{n-1}) \in L(\tau(M))$ *quantifier free and* $d_2, d_3, \ldots \in M_0$ *there is* $n^* < \omega$ *such that: for every* $k \in (n^*, \omega)$, $\eta_0, \eta_1 \in {}^{\omega}2, \nu_0, \nu_1 \in {}^{\omega}2$ *satisfying* $\eta_0 \restriction k = \eta_1 \restriction k$ *and* $\nu_0 \restriction k = \nu_1 \restriction k$ *we have* $N \models \varphi[a_{\eta_0}, b_{\nu_0}, d_2, \ldots] \equiv \varphi[a_{\eta_1}, b_{\nu_1}, d_2, \ldots,]$

(viii) *if* φ *is an existential sentence in* $\tau(M)$ *satisfied by* N *then* φ *is satisfied by* M.

3) Like 2 omitting (vii)$^+$, *(viii).*

Proof. $(B)^- \Rightarrow (A)$:

Toward contradiction assume $\mathrm{NPrrd}_{\kappa^+}(\lambda_1, \lambda_2; \kappa)$ hence there is a model M' witnessing it, so $|\tau(M')| \le \kappa$. So $c \in Q^M \Rightarrow \mathrm{rkk}^1(M', c; \kappa) < \kappa^+$ (note that Prrd was defined by cases of $\mathrm{rkk}(M, c, \kappa)$).

Let $\{\varphi_i(x, y) : i < \kappa\}$ list the quantifier free formulas in $L_{\omega,\omega}(\tau(M'))$ with free variables x, y. Let $\{u_i : i < \kappa\}$ list the finite subsets of κ. For $c \in Q^{M'}$ and $a_0, \ldots, a_{\ell(*)-1} \in R_1^{M'}, b_0, \ldots, b_{m(*)-1} \in R_2^{M'}$ ($\bar a = \langle a_0, \ldots, a_{\ell(*)-1}\rangle, \bar b = \langle b_0, \ldots, b_{m(*)-1}\rangle$ and for notation let $a_{n+1+\ell} = b_\ell$) let

$$\alpha_{c,\bar a,\bar b} = \mathrm{rkk}^1((\{a_0, \ldots, a_{\ell(*)-1}\}, \{b_0, \ldots, b_{m(*)-1}\}), M', c; \kappa),$$

and $k_{c,\bar a,\bar b}, \varphi_{c,\bar a,\bar b}$ be witnesses for $\mathrm{rkk}^1((\bar a, \bar b), M', c; \kappa, \theta) \not\ge \alpha_{c,\bar a,\bar b} + 1$. Let $i(c, \bar a, \bar b) < \kappa$ be such that $\varphi_{c,\bar a,\bar b}$ is a conjunction of formulas of the form $\varphi_j(x_\ell, y_m)$ for $j \in u_{i(c,\bar a,\bar b)}$.

We define M:

the universe is $|M'|$,

the function $F^{M'}$, relations $R_1^{M'}, R_2^{M'}, Q^{M'}, P^{M'}$, the pairing function on ordinals,

$$R_n = \{(i, a, b) : \quad a \in R_1^M, b \in R_2^M \text{ and if } |u_i| > n \text{ then } M \models \varphi_j[a, b] \text{ where } j \text{ is the n-th member of } u_i\}$$

and let H_c be one to one from $\omega \times \mathrm{rkk}(M', c; \kappa, \theta) \times \kappa$ into κ; we define the function $G : G_c(\bar a, \bar b) = H(G_{c,0}(\bar a, \bar b), G_{c,1}(\bar a, \bar b), G_{c,2}(\bar a, \bar b)) = H(k_{c,\bar a,\bar b}, \alpha_{c,\bar a,\bar b}, i_{c,\bar a,\bar b})$.

Now we can apply statement (B)-of 4.6 which we are assuming and get $M_0, N, c^*, a_\eta, b_\eta$ {4.4} (for $\eta \in {}^{\omega}2$) satisfying clauses (i)—(vii) there. So $c^* \in M_0 \subseteq M' \cap N$, so $\beta^* = \mathrm{rkk}(M', c; \kappa)$ satisfies $\beta^* < \infty$, even $< \kappa^+$. Clearly $\mathrm{rkk}^1(N, c^*; \kappa) \le \beta^*$.

Consider all sequences $\langle\langle \eta_\ell : \ell < \ell(*)\rangle, \langle \nu_m : m < m(*)\rangle, d_1, \bar d, \langle \varphi_{\ell,m} : \ell < \ell(*), m < m(*)\rangle, \langle a_\ell : \ell < \ell(*)\rangle, \langle b_m : m < m(*)\rangle\rangle$ which are as in clause (vi) of (B).

Among those tuples choose one with $\alpha^* = \mathrm{rkk}^1(\{a_\ell : \ell < \ell(*)\}, \{b_m : m < m(*)\}, N, c^*; \kappa)$ minimal. Let this rank not being $\ge \alpha^* + 1$ be exemplified by φ and $k < \ell(*) + m(*)$, so by symmetry without loss of generality $k < \ell(*)$.

Choose $k^* < \omega$ large enough for clause (vi) of (B) for all formulas $\varphi(x, y)$ appearing in $\{\varphi_j(x, y) : j \in u_{i_{c^*,\bar a,\bar b}}\}$ where $i_{c^*,\bar a,\bar b} = G_{c^*,2}(\bar a, \bar b)$ and $\langle \eta_\ell \restriction k^* : \ell < \ell(*)\rangle, \langle \nu_m \restriction k^* : m < m(*)\rangle$ and with no repetition. Choose $\eta_{\ell(*)} \in 2^\omega \setminus \{\eta_k\}$ such that $\eta_{\ell(*)} \restriction k^* = \eta_k \restriction k^*$. Now apply clause (vi) of (B)-to $\bar\eta' = \langle \eta_\ell : \ell \le \ell(*)\rangle$, $\bar\nu' = \langle \nu_m : m < m(*)\rangle$, $\bigwedge\limits_m \varphi_{\ell(*),m}(x_{\ell(*)}, x_m), d_1', \bar d'$.

By the choice of φ, these clearly satisfy $\mathrm{rkk}^1(\{a'_\ell : \ell < \ell(*)\}, \{b'_m : m < m(*)\}, N, c^*; \kappa, \bar\theta) < \mathrm{rkk}^1(\{a_\ell : \ell < \ell(*)\}, \{b_m : m < m(*)\}, N, c^*; \kappa, \bar\theta) = \alpha^*$, but by this we easily contradict the choice of α^* as minimal.

(A) $\Rightarrow$ (B)

As in the proof of 2.5, 2.1 (choosing a fixed c). {2.4}

(B) $\Rightarrow$ (B)$^-$

Trivial. $\square$

{4.5}

{2.4} **Discussion 4.7.** 1) When applying 4.6 (1) $\Rightarrow$ (2), or 2.5 we can use M which is an expansion of $(\mathscr{H}(\chi), \in, <^*)$ by Skolem functions, $P^M = \kappa$, χ large enough, so for $\eta, \nu \in {}^\omega 2, N_{\eta,\nu} := c\ell_N(M_0 \cup \{a_\eta, a_\nu\})$ is a model of ZFC, not well founded but with standard ω and more: its $\{i : i < \kappa\}$ is a part of the true κ. In [Sh:532] we
{2.1} will have as in 2.1 $M_0 \subseteq N$, $M_0 \prec M$, $M_0 \prec N_{\eta,\nu}$, and if $N_{\eta,\nu} \models$ "$\varphi(\eta,\nu), \nu, \eta$ are (essentially) in ${}^\omega 2, \varphi \in M_0$ a κ-Souslin relation", then $\mathbf{V} \models \varphi(\eta,\nu)$.
2) We can give a rank to subsets of $\lambda_1 \times \lambda_2$ and have parallel theorems.

{4.6}

Claim 4.8. *If $\varphi \subseteq {}^\omega 2 \times {}^\omega 2$ is $\bigvee\limits_{i<\theta} \varphi_i$, each φ_i is κ-Souslin, φ contains a (λ_1, λ_2)-rectangle, and $\mathrm{Prrd}^1_{\kappa^+ + 1}(\lambda_1, \lambda_2; \theta)$, then φ contains a perfect rectangle.*

Proof. Let $\varphi_i(\eta,\nu) = (\exists \rho)((\eta,\nu,\rho) \in \lim(T_i))$ where T_i is a $(2,2,\kappa)$-tree. Let M be $(\mathscr{H}(\chi), \in, <^*_\chi, \langle T_i : i < \theta\rangle, h, \Upsilon, \lambda_1, \lambda_2, R_1, R_2, Q, n)_{n<\omega}$ expanded by Skolem functions, where $Q^M = \theta$ and choosing $\eta_i \in {}^\omega 2$ for $i < \lambda_1$ pairwise distinct, $\nu_j \in {}^\omega 2$ for $j < \lambda_2$ are pairwise distinct, $R^M_1 = \{\eta_i : i < \lambda_1\}$, $R^M_2 = \{\nu_j : j < \lambda_2\}$ and let Υ, h be functions such that $(\eta_i, \nu_j, \Upsilon(\eta_i,\nu_j)) \in \lim(T_{h(\eta_i,\nu_j)})$. So let N,
{4.4} M_0, c^*, a_η (for $\eta \in {}^\omega 2$), b_η (for $\eta \in {}^\omega 2$) be as in clause (B) of 4.6. Now M has elimination of quantifiers, so there are quantifier free formulas $\varphi^\ell_n(x)$ saying (in M) that $x \in R_\ell$ & $x(n) = 1$, and $H_n(x,y)$ be such that $x \in R^M_1$ & $y \in R^M_2 \Rightarrow (\Upsilon(x,y))(n) = H_n(x,y) \in \kappa = P^M$.

So for $\eta \in {}^\omega 2$ we can define $\sigma^1_\eta \in {}^\omega 2$ by $\sigma^1_\eta(n) = 1 \Leftrightarrow N \models \varphi^1_n(a_\eta)$ and $\sigma^1_\eta \in {}^\omega 2$ by $\sigma^2_\eta(n) = 1 \Leftrightarrow N \models \varphi^2_n(b_\eta)$ and we define, for η, $\nu \in {}^\omega 2$, a sequence $\sigma_{\eta,\nu} \in {}^{\omega>}(P^{M_0}) \subseteq {}^{\omega>}\kappa$ by $\sigma_{\eta,\nu(n)} = H_n(a_\eta, b_\nu)$.

Now $A =: \{\sigma^1_\eta : \eta \in {}^\omega 2\}$, $B =: \{\sigma^2_\eta : \eta \in {}^\omega 2\}$ are perfect and for η, $\nu \in {}^\omega 2$, $(\sigma^1_\eta, \sigma^2_\eta, \sigma_{\eta,\nu}) \in \lim(T)$ hence $A \times B$ is a perfect rectangle inside $\mathrm{prj}\lim(T)$. $\square$

{4.7}

Fact 4.9. 1) Assume that $\varphi \subseteq {}^\omega 2 \times {}^\omega 2$ is θ_1-Souslin, $\kappa < \theta_1$, $\theta =$ cf $(S_{\leq\kappa}(\theta_1), \subseteq)$. Then φ can be represented as $\bigvee\limits_{i<\theta} \varphi_i$, each φ_i is κ-Souslin.
2) If φ is co-κ-Souslin, then it can be represented as $\bigvee\limits_{i<\kappa^+} \varphi_i$, each φ_i is κ-Borel (i.e. can be obtained from clopen sets by unions and intersections of size $\leq \kappa$).

Proof. 1) Easy.
2) Let $\neg\varphi$ be represented as $\mathrm{prj}\lim(T)$, T a $(2,2,\kappa)$-tree.

Now $\varphi(\eta,\nu)$ iff $T_{(\eta,\nu)} = \{\rho$: for some $n, (\eta \restriction n, \nu \restriction n, \rho) \in T\}$ is well founded which is equivalent to the existence of $\alpha < \kappa^+$, $f : T_{(\eta,\nu)} \to \alpha$ such that $\rho_1 < \rho_2 \in T \Rightarrow f(\rho_1) > f(\rho_2)$. For each $\alpha < \kappa^+$, this property is κ-Borel. $\square$

{4.8}

Conclusion 4.10. *If φ is an $\aleph_n$-Souslin subset of ${}^\omega 2 \times {}^\omega 2$ containing a $(\lambda \mathrm{rd}_{\omega_1}(\aleph_n), \lambda \mathrm{rd}_{\omega_1}(\aleph_n))$-rectangle, then it contains a perfect rectangle. (Note: $\aleph_n$ can replaced by κ if* cf $(S_{\leq\aleph_0}(\kappa), \subseteq) = \kappa$, *e.g. $\aleph_{\omega_4}$, by* [Sh:g, Ch.IX,§4].*)*

{4.9}

Conclusion 4.11. *1) For $\ell < 6, \mathrm{Prk}^\ell_\infty(\kappa^+, (2^{\kappa^+})^+; \kappa)$.*
2) If $\mathbf{V} = \mathbf{V}^{\mathbf{P}}_0$, $\mathbf{P} \models$ c.c.c. and $\mathbf{V}_0 \models$ GCH, then $\mathrm{Prk}_{\omega_1}(\aleph_1, \aleph_3)$.

Proof. 1) For a model M, letting $(\lambda_1, \lambda_2) = (\kappa^+, (2^{\kappa^+})^+)$ choose (for $m = 1, 2$) $\bar{a}^m_i$ a nonempty sequence from R^M_m for $i < \lambda_m$, $\{\bar{a}^m_i : i < \lambda_m\}$ pairwise disjoint. For

$(i,j) \in \lambda_1 \times \lambda_2$ let $\beta_{i,j} = \mathrm{rkk}(\bar{a}^1_i, \bar{a}^2_j, M)$ with witnesses $k^M(\bar{a}^1_i, \bar{a}^2_j)$, $\varphi^M(\bar{a}^1_i, \bar{a}^2_j)$ for $\neg\mathrm{rkk}^1(\bar{a}^1_i, \bar{a}^2_j, M) > \beta_{i,j}$. As $\lambda_2 = (2^{\lambda_1})^+$, $|\tau(M)| \le \kappa$, for some $B_2 \subseteq \lambda_2$, $|B_2| = \lambda_2$ and for every $i < \lambda_1$ the following does not depend on $j \in B_2$:

$$k^M(\bar{a}^1_i, \bar{a}^2_j), \varphi^M(\bar{a}^1_i, \bar{a}^2_j).$$

Similarly, there is $B_1 \subseteq \lambda_1$, $|B_1| = \lambda_1 (= \kappa^+)$ such that for $j = \min(B_2)$ the values

$$k^M(\bar{a}^1_i, \bar{a}^2_j), \varphi^M(\bar{a}^1_i, \bar{a}^2_j)$$

are the same for all $i \in B_1$; but they do not depend on $j \in B_2$ either. So for $(i,j) \in B_1 \times B_2$ we have $k^M(\bar{a}^1_i, \bar{a}^2_j) = k^*$, $\varphi^M(\bar{a}^1_i, \bar{a}^2_j) = \varphi^*$. Let k^* "speak" on $\bar{a}^1_i$, for definiteness only. Choose distinct i_ζ in B_1 (for $\zeta < \kappa^+$). Without loss of generality $\mathrm{rkk}^1(\bar{a}^1_{i_0}, \bar{a}^1_j, M) \le \mathrm{rkk}^1(\bar{a}^1_{i_\zeta}, \bar{a}^1_j, M)$.

Now $\bar{a}^1_\zeta$ give contradiction to $\mathrm{rkk}^l(\bar{a}^1_{i_0}, \bar{a}^2_j, M) \not\geq> \beta_{i_0,j}$.

2) This can be proved directly (or see [Sh:532] through preservation by c.c.c. forcing notion of rank which are relations of rkrc similarly to 1.10. $\square$ {1.8} {4.9A}

Remark 4.12. 1) If T is an (ω,ω)-tree and $A \times B \subseteq \lim(T)$, with $A, B \subseteq {}^\omega\omega$ uncountable (or just not scattered) then $\lim(T)$ contains a perfect rectangle. Instead $\lim(T)$ (i.e. a closed set) we can use countable intersection of open sets. The proof is just like 1.17. {1.14}

2) We can define a rank for $(2,2,\kappa)$-trees measuring whether $\mathrm{prj}\lim(T) \subseteq {}^\omega 2 \times {}^\omega 2$ contains a perfect rectangle, and similarly for (ω,ω)-tree T measuring whether $\lim(T)$ contains a perfect rectangle. We then have theorems parallel to those of §1. See below and in [Sh:532].

* * *

The use of ${}^\omega\omega$ below is just notational change. {4.10}

Definition 4.13. For T a (ω,ω)-tree we define a function degrc_T (rectangle degree). Its domain is $\mathrm{rcpr}(T) := \{(u_1,u_2)$: for some $\ell < \omega, u_1, u_2$ are finite nonempty subsets of ${}^\ell\omega$ and g a function from $u_1 \times u_2$ to ω such that $(\eta_0, \eta_1) \in T$ for $\eta_i \in u_i\}$. Its value is an ordinal $\mathrm{degrc}_T(u_1,u_2)$ (or -1 or ∞). For this we define the truth value of $\mathrm{degrc}_T(u_1,u_2) \ge \alpha$ by induction on the ordinal α.

<u>Case 1</u>: $\alpha = -1$

$\mathrm{degrc}_T(u_1,u_2) \ge -1$ <u>iff</u> (u_1,u_2) is in $\mathrm{rcpr}(T)$.

<u>Case 2</u>: α limit

$\mathrm{degrc}_T(u_1,u_2) \ge \alpha$ <u>iff</u> $\mathrm{degrc}_T(u_1,u_2) \ge \beta$ for every $\beta < \alpha$.

<u>Case 3</u>: $\alpha = \beta + 1$

$\mathrm{degrc}_T(u_1,u_2) \ge \alpha$ <u>iff</u> for $k \in \{1,2\}$, $\eta^* \in u_k$ we can find $\ell(*) < \omega$, and functions h_0, h_1, such that: Dom $(h_i) = u_1 \cup u_2$, $[\eta \in u_1 \cup u_2 \Rightarrow \eta \triangleleft h_i(\eta) \in {}^{\ell(*)}\omega]$ such that $h_0(\eta^*) \ne h_1(\eta^*)$, $\eta \in u_{1-k} \Rightarrow h_0(\eta) = h_1(\eta)$ and letting $u^1_i = \mathrm{Rang}(h_0 \restriction u_i) \cup \mathrm{Rang}(h_1 \restriction u_i)$ we have $\mathrm{degrc}_T(u^1_0, u^1_1) \ge \beta$.

Lastly define: $\mathrm{degrc}_T(u_0,u_1) = \alpha$ <u>iff</u> $\bigwedge\limits_\beta [\mathrm{degrc}_T(u_0,u_1) \ge \beta \Leftrightarrow \alpha \ge \beta]$ (α an ordinal or ∞).

Also $\mathrm{degrc}(T) = \mathrm{degrc}_T(\{\langle\rangle\}, \{\langle\rangle\})$.

{4.11}

Claim 4.14. *Assume T is in an (ω,ω)-tree.*
1) For every $(u_0, u_1) \in \mathrm{rcpr}(T)$, $\mathrm{degrc}_T(u_0, u_1)$ is an ordinal or ∞ or -1; iff f is an automorphism of $({}^{\omega>}\omega, \triangleleft)$, then $\mathrm{degrc}_T(u_0, u_1) = \mathrm{degrc}_{f(T)}(f(u_0), f(u_1))$.
2) $\mathrm{drc}(T) = \infty$ iff there is a perfect rectangle in $\lim(T)$ iff $\mathrm{degrc}_T(u_0, u_1) \geq \omega_1$ for some (u_0, u_1) (so those statements are absolute).
3) If $\mathrm{drc}(T) = \alpha(*) < \omega_1$, then $\lim(T)$ contains no $(\lambda\mathrm{rc}_{\alpha(*)+1}(\aleph_0), \lambda\mathrm{rc}_{\alpha(*)+1}(\aleph_0))$-rectangle.
4) If $\bar{T} = \langle T_n : n < \omega\rangle$ is a sequence of (ω,ω)-trees and $\mathrm{degrc}(T_n) \leq \alpha(*)$, and $A = \bigcup\limits_{n<\omega} \lim(T_n)$ then A contains no $(\lambda\mathrm{rc}_{\alpha(*)+1}(\aleph_0), \lambda\mathrm{rc}_{\alpha(*)+1}(\aleph_0)$-rectangle.
5) In part 4. we can replace ω by any infinite cardinal θ.

Proof. 1), 2), 3) Left to the reader.
4) Follows from part 5).
5) Let $\lambda = \lambda\mathrm{rc}_{\alpha(*)+1}(\theta)$, and let $\bar{T} = \langle T_i : i < \theta\rangle$, $\mathrm{degrc}(T_i) \leq \alpha(*)$ and $A = \bigcup\limits_{i<\theta} \lim(T_i)$. Let $\{\eta_\alpha : \alpha < \lambda\} \times \{\nu_\beta : \beta < \lambda\} \subseteq A$ where $\alpha < \beta \Rightarrow \eta_\alpha \neq \eta_\beta$ and $\nu_\alpha \neq \nu_\beta$, and for simplicity $\{\eta_\alpha : \alpha < \lambda\} \cap \{\nu_\beta : \beta < \lambda\} = \emptyset$.

We define a model M, with universe $\mathscr{H}((2^{\aleph_0})^+)$ and relation: all those definable in $(\mathscr{H}((2^{\aleph_1})^+), \in, <^*, R_1, R_2, g, \bar{T}, i)_{i\leq\theta}$ where $R_1^M = \{\eta_\alpha : \alpha < \lambda\}$, $R_2^M = \{\nu_\beta : \beta < \lambda\}$, $g(\eta_\alpha, \nu_\beta) = \min\{i \leq \theta : (\eta_\alpha, \nu_\beta) \in \lim(T_i)\}$.

Next we prove

$(*)$ if $w_\ell \in [R_\ell^M]$ for $\ell = 1, 2$, then

$$\begin{aligned}\mathrm{rkk}(\langle w_1, w_2\rangle, M) \leq \min\{ \quad & \mathrm{degrc}_{T_i}(\{\eta_\alpha \restriction k : \alpha \in u_1\}, \{\nu_\beta \restriction k : \beta \in u_2\}) : \\ & u_1 \subseteq w_1, u_2 \subseteq w_2, u_1 \neq \emptyset, u_2 \neq \emptyset, k < \omega \text{ and} \\ & \langle \eta_\alpha \restriction k : \alpha \in u_1\rangle \text{ is with no repetitions, and} \\ & \langle \nu_\beta \restriction k : \beta \in u_2\rangle \text{ is with no repetitions}\}.\end{aligned}$$

We prove $(*)$ by induction on the left side of the inequality. Now by the definitions we are done. $\square$

{4.15}

Claim 4.15. *1) For each $\alpha(*) < \omega_1$, there is an ω-sequence $\bar{T} = \langle T_n : n < \omega\rangle$ of $(\omega,\ \omega)$-trees such that:*

- (α) *for every $\mu < \lambda\mathrm{rc}_{\alpha(*)}(\aleph_0)$, some c.c.c. forcing notion adds a (μ,μ)-rectangle to $\bigcup\limits_{n<\omega} \lim(T_n)$,*
- (β) $\mathrm{degrc}(T_n) = \alpha(*)$.

2) If $\mathrm{NPrk}_{\alpha(*)}(\lambda_1, \lambda_2; \aleph_0)$ then for some (ω,ω)-tree T: some c.c.c. forcing notion adds a (λ_1, λ_2)-rectangle to $\lim(T)$ such that $\alpha(*) = \mathrm{degrc}(\bar{T})$ (consequently, if $\mathrm{Prk}_{\alpha(*)}(\lambda_1', \lambda_1'; \aleph_0)$ then there is no (λ_1', λ_1')–rectangle in $\lim(T)$).
3) Moreover, we can have for the tree T of (4): if $\mu < \lambda\mathrm{rc}_{\alpha(*)}(\aleph_0)$, A, B disjoint subsets of ${}^\omega 2 \times {}^\omega 2$ of cardinality $\leq \mu$, then some c.c.c. forcing notion $\mathbf{P}$, adds an automorphism f of $({}^{\omega>}\omega, \triangleleft)$ such that: $A \subseteq \lim^*[f(T)]$, $B \cap \lim^*[f(T)] = \emptyset$ (the $\lim^*$ means closure under finite changes).

Proof. 1) We define the forcing for part 2. and delay the others to [Sh:532].
{1.10} 2) It is enough to do it for successor $\alpha(*)$, say $\beta(*)+1$. It is like 1.13; we will
{4.1} give the basic definition and the new points. Let M be a model as in Definition 4.1, $|R^M_\ell| = \lambda_\ell, \mathrm{rkk}^1(M) < \alpha(*)$ so $\mathrm{rkk}^1(M) \le \beta(*)$. We assume that R^M_1, R^M_2 are disjoint sets of ordinals. For non empty $\bar a_l \subseteq R^M_l$ ($l < 2$; no repetition inside $\bar a_l$), let $\varphi^M(\bar a_1, \bar a_2)$, $k^M(\bar a_1, \bar a_2) \in \mathrm{Rang}(\bar a_1) \cup \mathrm{Rang}(\bar a_2)$ be witnesses to the value of $\mathrm{rkk}^1(\bar a_1, \bar a_2, M)$ which is $< \alpha(*)$.

We define the forcing notion $\mathbf{P}$: a condition p consists of:

(1) $\bar u^p = \langle u^p_0, u^p_1\rangle$ and $u^p_\varphi = u_\varphi[p]$ is a finite subset of R^M_φ for $\varphi < 2$,

(2) $n^p = n[p] < \omega$ and $\eta^p_\alpha = \eta_\alpha[p] \in {}^{n[p]}\omega$ for $\alpha \in u^p_\varphi$ such that $\alpha \ne \beta \in u^p_\varphi \Rightarrow \eta^p_\alpha \ne \eta^p_\beta$,

(3) $0 < m^p < \omega$ and $t^p_m \subseteq \bigcup\{{}^\ell\omega \times {}^\ell\omega : \ell \le n^p\}$ closed under initial segments and such that the $\triangleleft$-maximal elements have the length n^p and $\langle\rangle \in t^p$,

(4) the domain of f^p is $\{\bar u = (u_1, u_2) :$ for some $\ell = \ell(u_1, u_2) \le n^p$ and $m = m(u_1, u_2) < m^p$ we have $u_\varphi \subseteq t^p_\varphi \cap {}^\ell\omega$ and if $\alpha_1 \in u_1, \alpha_2 \in u_2$, $\eta^p_{\alpha_1} \restriction l \in u_1$, $\eta^p_{\alpha_2} \restriction l \in u_2$ then $g^p(\alpha_1, \alpha_2) = m\}$ and $f^p(\bar u) = (f^p_0(\bar u), f^p_1(\bar u), f^p_2(\bar u)) \in \alpha(*) \times (u_1 \cup u_2) \times L_{\omega_1,\omega}(\tau(M))$,

(5) a function $g^p : u^p_1 \times u^p_2 \longrightarrow \{0, \dots, m^p - 1\}$, $m^p < \omega$,

(6) $t^p_m \cap {}^{(n^p)}\omega = \{(\eta^p_\alpha, \eta^p_\beta) : \alpha \in u^p_0, \beta \in u^p_1$ and $m = g^p(\alpha, \beta)\}$

(7) <u>iff</u> $\emptyset \ne u_\varphi \subseteq t_\varphi \cap {}^\ell\omega$, $f^p(u_1, u_2) = (\beta^*, \rho^*, \varphi^*)$, $\ell < \ell(*) \le n^p$, for $i = 0, 1$ a function $e_{i,\varphi}$ has the domain u_φ, $[(\forall l)(\rho \in u_\varphi \Rightarrow \rho \triangleleft e_{i,\varphi}(\rho) \in t^p_\varphi \cap {}^{\ell(*)}\omega)]$, $[\rho^* \notin u_\varphi$ and $\rho \in u_\varphi \Rightarrow e_{0,\varphi}(\rho) = e_{1,\varphi}(\rho)]$,$[\rho^* \in u_\varphi \Rightarrow e_{0,\varphi}(\rho^*) \ne e_{1,\varphi}(\rho^*)]$ and $f^p(e_{0,1}(u_1) \cup e_{1,1}(u_1), e_{0,2}(u_2) \cup e_{1,2}(u_2)) = (\beta', \rho', \varphi')$ (so well defined), <u>then</u> $\beta' < \beta^*$,

(8) <u>iff</u> $\ell \le n^p$, for $\varphi = 1, 2$ we have $u_\varphi \subseteq u^p_\varphi$ are non empty, the sequence $\langle \eta^p_\alpha \restriction \ell : \alpha \in u_\varphi\rangle$ is with no repetition, and $u'_\varphi = \{\eta^p_\alpha \restriction \ell : \alpha \in u_\varphi\}$ and $f^p(u'_1, u'_2)$ is well defined, <u>then</u>
$f^p_2(u'_1, u'_2) = \varphi^M(u_1, u_2)$,
$f^p_1(u'_1, u'_2) = \eta^p_\alpha \restriction l$
where α is $k^M(u_1, u_2)$ and
$f^p_0(u'_1, u'_2, h) = \mathrm{rkk}(u_1, u_2, M)$,

(9) if $(u'_1, u'_2) \in$ Dom (f^p) <u>then</u> there are l, u_1, u_2 as above,

(10) if $(\eta_1, \eta_2) \in t^p_m \cap ({}^n2 \times {}^n2)$ <u>then</u> for some $\alpha_1 \in u^p_1$, $\alpha_2 \in u^p_2$ we have $g^p(\alpha_1, \alpha_2) = m$ and $\eta_1 \trianglelefteq \eta^p_{\alpha_1}$, $\eta_2 \trianglelefteq \eta^p_{\alpha_2}$.

□ {4.11B}

Remark 4.16. We can generalize 4.13, 4.14, **??** to Souslin relations. {4.16new}

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