BOREL SETS WITH LARGE SQUARES
SH522

SAHARON SHELAH

ABSTRACT. For a cardinal $\mu$ we give a sufficient condition $\oplus_\mu$ (involving ranks measuring existence of independent sets) for:

$\otimes_\mu$ if a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ contains a $\mu$-square (i.e. a set of the form $A \times A$, $|A| = \mu$) then it contains a $2^{\aleph_0}$-square and even a perfect square.

And also for $\otimes'_\mu$ if $\psi \in L_{\omega_1, \omega}$ has a model of cardinality $\mu$ then it has a model of cardinality continuum generated in a “nice”, “absolute” way.

Assuming MA + $2^{\aleph_0} > \mu$ for transparency, those three conditions ($\oplus_\mu$, $\otimes_\mu$ and $\otimes'_\mu$) are equivalent, and by this we get e.g. $\bigwedge_{\alpha < \omega_1} [2^{\aleph_0} \geq \aleph_\alpha \Rightarrow \neg \otimes_\aleph_\alpha]$, and also $\min \{\mu : \otimes_\mu\}$, if $2^{\aleph_0}$ has cofinality $\aleph_1$.

We deal also with Borel rectangles and related model theoretic problems.
§0 Introduction

[We explain results and history and include a list of notation.]

§1 The rank and the Borel sets

[We define some version of the rank for a model, and then \( \lambda_\alpha(\kappa) \) is the first \( \lambda \) such that there is no model with universe \( \lambda \), vocabulary of cardinality \( \leq \kappa \) and rank \( < \alpha \). Now we prove that forcing does not change some ranks of the model, can only decrease others, and c.c.c. forcing changes little. Now:

(1.12) if a Borel or analytic set contains a \( \lambda_\omega(\kappa_0) \)-square then it contains a perfect square; clearly this gives something only if the continuum is large, that is at least \( \lambda_\omega(\kappa_0) \). On the other hand (in 1.13) if \( \mu = \mu^{\omega_1} < \lambda_\omega(\kappa_0) \) we have in some c.c.c. forcing extension of \( V \): the continuum is arbitrarily large, and some Borel set contains a \( \mu \)-square but no \( \mu^+ \)-square. Lastly (in ??) assuming MA holds we prove exact results (e.g. equivalence of conditions).]

§2 Some model theoretic problems

[When we restrict ourselves to models of cardinality up to the continuum, \( \lambda_\omega(\kappa_0) \) is the Hanf number of \( L_{\omega_1, \omega} \) (see 2.1). Also (in 2.4) if \( \psi \in L_{\omega_1, \omega} \) has a model realizing many types (say in the countable set of formulas, many means \( \geq \lambda_\omega(\kappa_0) \)) even after c.c.c. forcing, then

\{ \{ p : p a complete \( \Delta \)-type realized in \( M \) \} : \M \models \psi \} \]

has two to the continuum members. We then (2.5) assume \( \psi \in L_{\omega_1, \omega} \) has a two cardinal model, say for \( (\mu, \kappa) \) and we want to find a \( (\mu', \kappa_0) \)-model, we need \( \lambda_{\omega_1}(\kappa) \leq \mu \). Next, more generally, we deal with \( \bar{\lambda}_\alpha \)-cardinal models (i.e. we demand that \( P^M \) have cardinality \( \lambda_\alpha \)). We define ranks (2.8), from them we can formulate sufficient conditions for transfer theorem and compactness. We can prove that the relevant ranks are (essentially) preserved under c.c.c. forcing as in §1, and the sufficient conditions hold for \( \aleph_\omega \) under GCH.]

§3 Finer analysis of square existence

[We (3.1,3.2) define for a sequence \( \bar{T} = \langle T_n : n < \omega \rangle \) of trees (i.e. closed sets of the plane) a rank, \text{deg sq}, whose value is a bound for the size of the square it may contain. We then (3.3) deal with analytic, or more generally \( \kappa \)-Souslin relations, ?? patience incomplete-what has?? and use parallel degrees. We then prove that statements on the degrees are related to the existence of squares in \( \kappa \)-Souslin relations in a way parallel to what we have on Borel, using \( \lambda_\alpha(\kappa) \). We then (3.7 – 3.11) connect it to the existence of identities for 2-place colourings. In particular we get results of the form “there is a Borel set \( B \) which contains a \( \mu \)-square iff \( \mu < \lambda_\alpha(\kappa_0) \)” when \( \text{MA + } \lambda_\alpha(\kappa_0) < 2^{\aleph_0} \).]

§4 Rectangles
[We deal with the problem of the existence of rectangles in Borel and $\kappa$-Souslin relations. The equivalence of the rank (for models), the existence of perfect rectangles and the model theoretic statements is more delicate, but is done.]
§ 0. Introduction

We first review the old results (from §1, §2).

The main one is:

\((\ast)_1\) it is consistent, that for every successor ordinal \(\alpha < \omega_1\), there is a Borel subset of \(\omega_2 \times \omega_2\) containing an \(\aleph_\alpha\)-square but no perfect square.

In fact:

\((\ast)_1^+\) the result above follows from \(\text{MA} + 2^{\aleph_0} > \aleph_{\omega_1}\).

\(\{1.1\}\)

For this we define (Definition 1.1) for any ordinal \(\alpha\) a property \(\text{Pr}_\alpha(\lambda; \kappa)\) of the cardinals \(\lambda, \kappa\). The maximal cardinal with the property of \(\aleph_\alpha\) (i.e. for every small cardinal, c.c.c. forcing adds an example as in \((\ast)_1\)) is characterized (as \(\lambda_{\omega_1}(\aleph_0)\)) where \(\lambda_\alpha(\kappa) = \min\{\lambda : \text{Pr}_\alpha(\lambda; \kappa)\}\); essentially it is not changed by c.c.c. forcing; so in \((\ast)_1\):

\((\ast)_1'\) if in addition \(V = V^P\), where \(P\) is a c.c.c. forcing then \(\lambda_{\omega_1}(\aleph_0) \leq (\aleph_{\omega_1})^{V_0}\).

\(\{1.5\}\)

\(\{1.6\}\)

We will generally investigate \(\text{Pr}_\alpha(\lambda; \kappa)\), giving equivalent formulations \((1.1 - 1.6)\), seeing how fast \(\lambda_\alpha(\kappa)\) increases, e.g. \(\kappa^\alpha < \lambda_\alpha(\kappa) \leq (\aleph_\omega)^{\omega_\alpha}(\kappa)\) (in 1.7, 1.8).

For two variants we show: \(\text{Pr}_\alpha(\lambda; \kappa^+)\) is preserved by \(\kappa^+\)-c.c.c. forcing, \(\text{Pr}_\alpha(\lambda; \kappa^+) \Rightarrow \text{Pr}_\alpha(\lambda; \kappa^{++})\) and \(-\text{Pr}_\alpha(\lambda; \kappa^+)\) is preserved by any extension of the universe of set theory. Now \(\text{Pr}_\omega(\lambda; \aleph_0)\) implies that there is no Borel set as above \((1.12)\) but if \(\text{Pr}_\omega(\lambda; \aleph_0)\) fails then some c.c.c. forcing adds a Borel set as above \((1.13)\). We cannot in \((\ast)_2\) omit some set theoretic assumption even for \(\aleph_2\) - see ??, 1.16 (add many Cohen reals or many random reals to a universe satisfying e.g. \(2^{\aleph_0} = \aleph_1\), then, in the new universe, every Borel set which contains an \(\aleph_2\)-square, also contains a perfect square). We can replace Borel by analytic or even \(\kappa\)-Souslin (using \(\text{Pr}_\omega(\kappa)\)).

In §2 we deal with related model theoretic questions with less satisfactory results.

\(\{2.2\}\)

By 2.1.2.3, giving a kind of answer to a question from [Sh:49],

\((\ast)_3\) essentially \(\lambda = \lambda_{\omega_1}(\aleph_0)\) is the Hanf number for models of sentences in \(L_{\omega_1, \omega}\) when we restrict ourselves to models of cardinality \(\leq 2^{\aleph_0}\). (What is the meaning of “essentially”? If \(\lambda_{\omega_1}(\aleph_0) \geq 2^{\aleph_0}\) this fails, but if \(\lambda_{\omega_1}(\aleph_0) < 2^{\aleph_0}\) it holds.)

\(\{2.3\}\)

In 2.4 we generalize it (the parallel of replacing Borel or analytic sets by \(\kappa\)-Souslin).

We conclude (2.4(2)):

\((\ast)_3'\) if \(\psi \in L_{\omega_1, \omega}(\tau_1), \tau_0 \subseteq \tau_1\) are countable vocabularies, \(\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}\) is countable and \(\psi\) has a model which realizes \(\geq \lambda_{\omega_1}(\aleph_0)\) complete \((\Delta, 1)\)-types then \(\langle\{\langle M, \tau_0\rangle : M \models \psi, \|M\| = \lambda\\rangle\rangle \geq \min\{2^\lambda, \aleph_2\}\) (for any \(\lambda\)), as we have models as in [Sh:a, ChVII,§4] = [Sh:c, ChVII,§4].

If we allow parameters in the formulas of \(\Delta\), and \(2^{\aleph_0} < 2^{\aleph_1}\) then \((\ast)_3\) holds too. However even in the case \(2^\lambda = 2^{\aleph_0}\) we prove some results in this direction, see [Sh:262] (better [Sh:c, ChVII,§5]). We then turn to three cardinal theorems etc. trying to continue [Sh:49] (where e.g. \((\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)\) was proved).

We knew those results earlier than, or in 1980/1, but failed in efforts to prove the consistency of “\(\text{ZFC} + \lambda_{\omega_1}(\aleph_0) > \aleph_{\omega_1}\)” (or proving \(\text{ZFC} \vdash \text{“}\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}\text{“}\)).
By the mid seventies we knew how to get consistency of results like those in §2 (forcing with P, adding many Cohen reals i.e. in V getting (∗)3 for λ = (ℵω)\(^V\)).

This (older proof, not the one used) is closely related to Silver’s proof of “every \(\Pi^1_1\)-relation with uncountably many equivalence classes has a 2\(^\aleph_0\) one” (a deeper one is the proof of Harrington of the Lauchli-Halpern theorem; see a generalization of the Lauchli-Halpern theorem, a partition theorem on \(\kappa>2\), \(\kappa\) large by [Sh:288, §4]).

In fact, about 88 I wrote down for W. Hodges proofs of (a) and (b) stated below.

(a) If, for simplicity, \(V\) satisfies GCH, and we add \(\aleph_\omega\) Cohen reals then the Hanf number of \(L_{\omega_1,\omega}\) below the continuum is \(\aleph_\omega\).

(b) If \(\psi \in L_{\omega_1,\omega}(\tau_1)\) and some countable \(\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1,\omega}(\tau_0)\}\) satisfies: in every forcing extension of \(V\), \(\psi\) has a model which realizes 2\(^\aleph_0\) (or at least \(\min\{2^{\aleph_\omega}, \aleph_\omega\}\)) complete \(\Delta\)-types then the conclusion of (∗)3 above holds.

Hodges had intended to write it up. Later Hrushovski and Velickovic independently proved the statement (a).

As indicated above, the results had seemed disappointing as the main question “is \(\lambda_{\omega_1}(\aleph_0) = \aleph_\omega\)?” is not answered. But Hjorth asked me about (essentially) (∗)1 which was mentioned in [HrSh:152] and urged me to write this down.

In §3 we define degree of Borel sets of the forms \(\bigcup_{n<\omega} \lim T_n \subseteq \omega^2 \times \omega^2\) measuring how close are they to having perfect squares, similarly we define degrees for \(\kappa\)-Souslin relations, and get results similar to earlier ones under MA and nail the connection between the set of cardinalities of models of \(\psi \in L_{\omega_1,\omega}\) and having squares. In §4 we deal with the existence of rectangles.

We can replace \(\mathbb{R}^2\) by \(\mathbb{R}^3\) without any difficulty.

In a subsequent paper [Sh:532] which we are writing, we intend to continue the present work and in ?? [Sh:202, §5] and deal with: consistency of the existence of co-\(\kappa\)-Souslin (and even \(\Pi^1_1\))-equivalence relations with many equivalence classes relationship of \(\lambda_{\omega_1}^\omega, \lambda_{\omega_1}^1\) etc., and also try to deal with independence (concerning 2.11 and 4.11(1)) and the existence of many disjoint sections.

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§ 0(A). Notation.

Set theory:

- \(BA = \{f : f\) is a function from \(B\) to \(A\}\), the set of reals is \(\omega^2\).
- \(\mathcal{F}_{<\kappa}(A) = [A]^{<\kappa} = \{B \subseteq A : |B| < \kappa\}\).

By a Borel set \(B\) we mean the set it defines in the current universe. A \(\mu\)-square (or a square of size \(\mu\)) is a set of the form \(A \times A\), where \(A \subseteq \omega^2, |A| = \mu\). A \((\mu_1, \mu_2)\)-rectangle (or rectangle of size \((\mu_1, \mu_2)\)) is a set of the form \(A_1 \times A_2\), for some \(A_1 \subseteq \omega^2, |A_1| = \mu_1\) (for \(l = 0, 1\)). A perfect square is \(\mathcal{P} \times \mathcal{P}, \mathcal{P} \subseteq \omega^2\) perfect. A perfect rectangle is \(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}_l \subseteq \omega^2\) perfect. Note: A perfect rectangle is a \(2^{\aleph_0}, 2^{\aleph_0}\)-rectangle.

Note: A perfect square is a \(2^{\aleph_0}\)-square.
\( \mathcal{P}, \mathcal{Q} \) denote perfect sets; \( \text{forcing notions} \); \( P, Q, R \) denote predicates.

A \( \kappa \)-Souslin set is \( \{ \eta \in \omega^2 : \text{for some } \nu \text{ we have } (\eta, \nu) \in \lim(T) \} \) for some \( (2, \kappa) \)-tree \( T \) (see below). A \( \kappa \)-Souslin relation (say an \( n \)-place relation) is defined similarly.

For \( \lambda = \langle \lambda_\zeta : \zeta < \zeta(*) \rangle \), a \( \lambda \)-tree is

\[
T \subseteq \bigcup \prod n(\lambda_\zeta), \text{ ordered by } \eta \prec \nu \iff \bigwedge_{\zeta < \zeta(*)} \eta_\zeta \text{ are right fine now } \prec \nu_\zeta.
\]

We usually let \( \langle \eta_\zeta : \zeta < \zeta(*) \rangle \).

For a \( \lambda \)-tree \( T \) we define

\[
\text{lim}(T) = \{ \bar{\eta} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) : n < \omega \Rightarrow \bar{\eta}[n \in T] \}
\]

(where \( \langle \eta_\zeta : \zeta < \zeta(*) \rangle \upharpoonright n = \langle \eta_\zeta \upharpoonright n : \zeta < \zeta(*) \rangle \) and

\[
\text{lim}^*(T) = \{ \bar{\eta} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) : (\exists \bar{\eta'}) \in \lim(T)(\exists k < \omega)
\]

\[
\bigwedge_{\zeta < \zeta(*)} \eta_\zeta\upharpoonright [k, \omega) = \eta_\zeta\upharpoonright [k, \omega) \}.\]

We will use mainly \( (2, 2) \)-trees and \( (2, 2, \kappa) \)-trees; in particular, \( \zeta(*) \) is finite.

Let \( \eta \sim \nu \) mean that \( \eta, \nu \) sequences of ordinals, \( \ell g(\eta) = \ell g(\nu) \) and

\[
(\forall k)[n < k < \lg(\eta) \Rightarrow \eta(k) = \nu(k)].
\]

For a tree \( T \) as above, \( u \subseteq \zeta(*) \) and \( n < \omega \) let

\[
\text{T}^{(\sim n, u)} = \{ \bar{\eta} : (\exists k)(\exists \bar{\eta} \in \lim(T))[\bar{\eta} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) \text{ and } \bar{\eta} \in \prod_{\zeta < \zeta(*)} k(\lambda_\zeta) \text{ and } (\forall \xi \in u)(\eta_\xi \sim \nu_\xi [k])\}.
\]

Let \( \text{Fr}_n(\lambda, \mu, \kappa) \) mean: if \( F_\alpha \) are \( n \)-place functions from \( \lambda \) to \( \lambda \) (for \( \alpha < \kappa \)) then for some \( A \in [\lambda]^\mu \) we have

for distinct \( a_0, \ldots, a_n \in A \) and \( \alpha < \kappa \) we have

\[ a_n \neq F_\alpha(a_0, \ldots, a_{n-1}) \].

\[ \S 0(B) \). Model theory.

Vocabularies are denoted by \( \tau \), so languages are denoted by e.g. \( L_{\kappa, \theta}(\tau) \), models are denoted by \( M, N \). The universe of \( M \) is \( |M| \), its cardinality \( ||M|| \). The vocabulary of \( M \) is \( \tau(M) \) and the vocabulary of \( T \) (a theory or a sentence) is \( \tau(T) \). \( R^M \) is the interpretation of \( R \) in \( M \) (for \( R \in \tau(M) \)). For a model \( M \), and a set \( B \subseteq M \) we have: \( a \in c\ell_{\kappa}(B, M) \) iff for some quantifier free \( \varphi = \varphi(y, x_1 \ldots x_n) \), and \( b_1, \ldots, b_n \in B \) we have

\[ M \models \varphi[a, b_1, \ldots, b_n] \text{ and } (\exists x) \varphi(x, b_1, \ldots, b_n) \].

Let \( c\ell_{\kappa}(B, M) = c\ell_{\kappa^+}(B, M) \) and \( c\ell(B, M) = c\ell_{<2}(B, M) \). (Note: if \( M \) has Skolem functions then \( c\ell_{\kappa^0}(B, M) = c\ell_{<2}(B, M) \) for every \( B \subseteq |M| \).) If \( \kappa \) is an
ordinal we mean $|\kappa|$ (is needed just for phrasing absoluteness results that is if we use a cardinal $\kappa$ in a universe $V$, and then deal with a generic extension $V^P$ maybe in $V^P$, $\kappa$ is no longer a cardinal but we like to still use it as a parameter). Let $T$ denote a theory, first order if not said otherwise.
§ 1. The rank and the Borel sets

{1.1}

Definition 1.1. 1) For $\ell < 6$, and cardinals $\lambda \geq \kappa$, $\theta$ and an ordinal $\alpha$, let $\Pr_{\lambda}^\ell(\lambda; \kappa, \theta)$ mean that for every model $M$ with the universe $\lambda$ and vocabulary of cardinality $\leq \theta$, $\text{rk}^\ell(M; \kappa) \geq \alpha$ (defined below) and let $\text{NPr}^\ell_{\lambda}(\lambda; \kappa, \theta)$ be the negation. Instead of $^\omega < \kappa^+$ we may write $\kappa$ (similarly below); if $\kappa = \theta^+$ we may omit it (so e.g. $\Pr^\ell_{\lambda}(\lambda; \kappa)$ means $\Pr^\ell_{\lambda}(\lambda; < \kappa^+, \kappa)$); if $\theta = \aleph_0, \kappa = \aleph_1$ we may omit them.

Lastly, let $\lambda^*(\kappa, \theta) = \min \{\lambda : \Pr^\ell_{\lambda}(\lambda; \kappa, \theta)\}$.

2) For a model $M$, $\text{rk}^\ell(M; \kappa) = \sup \{\text{rk}^\ell(w, M; \kappa) + 1 : w \subseteq |M| \text{ finite non empty}\}$ where $\text{rk}^\ell$ is defined below in part (3).

3) For a model $M$, and $w \in [M]^* := \{u : u \subseteq |M| \text{ finite nonempty}\}$ we shall define below the truth value of $\text{rk}^\ell(w, M; \kappa)$ by induction on the ordinal $\alpha$ (note: if $\text{cl}_{<\kappa}(w, M) = \text{cl}_2(w, M)$ for every $w \in [M]^*$ then for $\ell = 0, 1, \kappa$ can be omitted).

Then we can note:

$(*)_0 \alpha \leq \beta$ and $\text{rk}^\ell(w, M; \kappa) \geq \beta \Rightarrow \text{rk}^\ell(w, M; \kappa) \geq \alpha$

$(*)_1 \text{rk}^\ell(w, M; \kappa) \geq \delta \text{ (}\delta \text{ limit)} \iff \bigwedge_{\alpha < \delta} \text{rk}^\ell(w, M; \kappa) \geq \alpha$

$(*)_2 \text{rk}^\ell(w, M; \kappa) \geq 0 \iff w \in [M]^*$ and no $a \in w$ is in $\text{cl}_{<\kappa}(w \setminus \{a\}, M)$.

So we can define $\text{rk}^\ell(\omega, M; \kappa) = \alpha$ as the maximal $\alpha$ such that $\text{rk}^\ell(w, M; \kappa) \geq \alpha$, and $\infty$ if this holds for every $\alpha$ (and $-1$ whenever $\text{rk}^\ell(w, M; \kappa) \geq 0$).

Now the inductive definition of $\text{rk}^\ell(\omega, M; \kappa)$ was already done above for $\alpha = 0$ (by $(*)_2$) and $\alpha$ limit (by $(*)_1$), so for $\alpha = \beta + 1$ we let

$(*)_3 \text{rk}^\ell(\omega, M; \kappa) \geq \beta + 1 \iff \text{(letting } n = |w|, \ w = \{a_0, \ldots, a_{n-1}\}\text{)}$ (in the vocabulary of $M$) for which $M \models \varphi[a_0, \ldots, a_{n-1}]$ we have:

Case 1: $\ell = 1$. There are $a^0_m \notin M$ for $m < n, i < 2$ such that:

(a) $\text{rk}^\ell(\{a^0_m : i < 2, m < n\}, M; \kappa) \geq \beta$,
(b) $M \models \varphi[a_0, \ldots, a_{n-1}]$ (for $i = 1, 2$), so without loss of generality there is no repetition in $a^0_0, \ldots, a^0_{n-1}$
(c) $a^0_k \neq a^1_k$ but for $m \neq k$ (such that $m < n$) we have $a^0_m = a^1_m$.

Case 2: $\ell = 0$. As for $\ell = 1$ but in addition

(d) $\bigwedge_m a^0_m = a^1_m$

Case 3: $\ell = 3$. We give to $\kappa$ an additional role and the definition is like case 1 but $i < \kappa$; i.e. there are $a^0_m \in M$ for $m < n, i < \kappa$ such that:

(a) for $i < j < \kappa$ we have $\text{rk}^\ell(\{a^0_m, a^j_m : m < n\}, M; \kappa) \geq \beta$,
(b) $M \models \varphi[a_0, \ldots, a_{n-1}]$ (for $i < \kappa$; so without loss of generality there are no repetitions in $a^0_0, \ldots, a^0_{n-1}$)
(c) for $i < j < \kappa, a^0_k \neq a^j_k$ but for $m \neq k$ (such that $m < n$) we have $\bigwedge_{i, j < \kappa} a^0_m = a^i_m$

Case 4: $\ell = 2$. Like case 3 but in addition
(4) \( a_m = a_m^0 \) for \( m < n \)

**Case 5:** \( \ell = 5. \) Like case 3 except that we replace clause (a) by (a)\(^{-} \) for every function \( F \) : Dom \( (F) = \kappa, |\text{Rang}(F)| < \kappa \) for some \( i < j < \kappa \) we have \( F(i) = F(j) \) and \( \text{rk}^\ell(\{a_m^0 : m < n\}, M; < \kappa) \geq \beta. \)

**Case 6:** \( \ell = 4. \) Like case 4 (i.e. \( \ell = 2 \)) using clause (a)\(^{-} \) instead of clause (a).

We will actually use the above definition for \( \ell = 0 \) mainly. As the cardinal \( \lambda_\alpha^\ell(\kappa; < \kappa, \theta) = \lambda_\alpha^\ell \) for \( \ell < 2 \) may increase when the universe of set theory is extended (new models may be added) we will need some upper bounds which are preserved by suitable forcing. The case \( \ell = 2 \) provides one (and it is good: it does not increase when the universe is extended by a c.c.c forcing). The case \( \ell = 4 \) shows how much we can strengthen the definition, to show for which forcing notions lower bounds for the rank for \( l = 0 \) are preserved. Odd cases show that variants of the definition are immaterial.

**Claim 1.2.** 1) The truth of each of the statements of \( \text{Pr}_\alpha^\ell(\lambda; < \kappa, \theta) \), \( \text{rk}^\ell(M; < \kappa) \geq \alpha, \text{rk}^\ell(wM; \kappa) \geq \alpha \) is preserved if we replace \( \ell = 0, 2, 3, 2, 2, 3, 5, 4 \) by \( \ell = 1, 3, 1, 0, 1, 4, 5, 1, 5 \) respectively (i.e. \( 2 \to 4, 3 \to 5 \to 1 \to 2 \to 3, 4 \to 5, 3 \to 1, 2 \to 0 \to 2 \to 1 \) and also if we decrease \( \alpha, \kappa, \theta \) or increase \( \lambda \) (the last two only when \( M \) is not a parameter). So the corresponding inequality on \( \lambda_\alpha^\ell(\kappa; < \kappa, \theta) \) holds.

2) Also \( \text{rk}^\ell(w_1, M; < \kappa) \geq \text{rk}^\ell(w_2, M; < \kappa) \) for \( w_1 \subseteq w_2 \) from \([M]^*\).

3) Also if we expand \( M \), the ranks (of \( w \in [M]^* \), of \( M \)) can only decrease.

4) If \( A \subseteq M \) is defined by a quantifier free formula with parameters from a finite subset \( w^* \) of \( M \), \( M^+ \) is \( M \) expanded by the relations defined by quantifier free formulas with parameters from \( w^*, M^+, M^+ \mid A \) (for simplicity \( M^* \) has relations only) then for \( w \in [A]^* \) such that \( w \not\subseteq w^* \) we have \( \text{rk}^\ell(w, M^+, < \kappa) \geq \text{rk}^\ell(w \cup w^*, M; < \kappa) \).

5) In 1.1(3)(*) we allow any first order formula, this means just expanding \( M \) by relations for any first order formula \( \varphi(x) \).

6) For \( \ell \text{ odd}, \text{rk}^\ell(w, M; < \kappa) \geq |(\mathcal{P}(M)| + \aleph_0| + \aleph_0| ) \) implies \( \text{rk}^\ell(w, M; < \kappa) = \infty \).

7) \( \lambda_\alpha^\ell(\kappa; < \kappa, \theta) \) increases (\( \leq \)) with \( \alpha, \theta \) and decreases with \( \kappa \).

8) There is no difference between \( \ell = 4 \) and \( \ell = 5 \).

**Proof.** Check [e.g. for part (8), we can use function \( F \) such that \( (\forall \alpha < \kappa)(F(0) \neq F(1 + \alpha)) \)].

**Claim 1.3.** 1) For \( \ell = 0 \), if \( \alpha = \text{rk}^\ell(M; < \kappa) \) \((< \infty) \) then for some expansion \( M^+ \) of \( M \) by \( \leq \aleph_0 + |\alpha| \) relations, for every \( w \in [M]^* \) we have:

\[
\text{rk}^{\ell+1}(w, M^+, < \kappa) \leq \text{rk}^\ell(w, M; < \kappa).
\]

2) Similarly for \( \ell = 2, 4 \).

3) If \( V_0 \) is a transitive class of \( V_1 \) (both models of ZFC) and \( M \in V_0 \) is a model then:

(a) for \( \ell < 4 \)

\[
\text{[rk}^\ell(w, M; < \kappa)]^{V_0} \leq [\text{rk}^\ell(w, M; < \kappa)]^{V_1} \text{ for } w \in [M]^*.
\]

(b) \( \text{[rk}^\ell(M; < \kappa)]^{V_0} \leq [\text{rk}^\ell(M; < \kappa)]^{V_1} \).
(γ) if ℓ = 0, 1 equality holds in (α), (β).

(b) Assume:

(i) for every f : κ → Ord from $V_1$ there is $A \in [\kappa]^\kappa$ such that $f \upharpoonright A \in V_0$, or at least

(ii) every graph $H$ on $\lambda$ from $V_0$ which in $V_1$ has a complete subgraph of size $\kappa$, has such a subgraph in $V_0$, which holds if

(ii) $\lambda_1 V_1 = V_0^P$ where $P$ is a forcing notion satisfying the $\kappa$-Knaster Condition. Then for $\ell = 2, 3$, in (α), (β) (of (a)) above equalities hold and the inequality in (δ) holds.

(c) Assume $V_1 = V_0^P$ where $P$ is $\kappa$-2-linked. Then for $\ell = 4, 5$ in clauses (α), (β) (of (a)) above we have equality and the inequality in (δ) holds.

Proof. 1) For $\beta < \alpha, n < \omega$, a quantifier free formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$ and $k < n$ let

$$R^\beta_n = \{ (a_0, \ldots, a_{n-1}) : a_m \in M for m < n and$$

$$\beta = \text{rk}^\beta(\{ a_0, \ldots, a_{n-1} \}, M; < \kappa) \},$$

$$R_{\alpha, \varphi}^{\beta, k} = \{ (a_0, \ldots, a_{n-1}) \in R^\beta_n : M \models \varphi[a_0, \ldots, a_{n-1}] for no$$

$$a_k \in [M \setminus \{ a_0, \ldots, a_{n-1} \}] we have$$

(a) $M \models \varphi[a_0, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_{n-1}]$

(β) $\text{rk}^\beta(\{ a_m : m < n \} \cup \{ a_k \}, M; < \kappa) \geq \beta,$

$$M^+ = (M, R^\beta_n, R_{\alpha, \varphi}^{\beta, k}, \ldots)_{\beta < \alpha, n < \omega, k < n, \varphi}$$

{1.8} Check (or see more details in the proof of 1.10 below).

2) Similarly.

{1.1} 3) The proof should be clear (for (b), looking at Definition 1.1 case 3 the graph is $\{(i, j) : \text{clause (a) there holds}\}$.

{1.3} Remark 1.4. 1) In 1.3(1) we can omit “$\alpha = \text{rk}^\beta(M; < \kappa)$” but then weaken the conclusion to $\text{rk}^\beta(w, M^+; < \kappa) \leq \text{rk}^\beta(w, M; < \kappa)$ or both are $> \alpha$.

{1.3} 2) Similarly in 1.3(2).

{1.4} Conclusion 1.5. 1) $Pr_{\omega_1}^0(\lambda) \leftrightarrow Pr_{\omega_1}^1(\lambda) \leftrightarrow Pr_{\omega_1}^4(\lambda) \leftrightarrow Pr_{\omega_1}^5(\lambda) \leftrightarrow Pr_{\omega_1}^2(\lambda) \leftrightarrow Pr_{\omega_1}^3(\lambda)$.

2) If $\alpha < \kappa^+$ then $Pr_{\alpha}^0(\lambda; \kappa) \leftrightarrow Pr_{\alpha}^1(\lambda; \kappa) \leftrightarrow Pr_{\alpha}^4(\lambda; \kappa) \leftrightarrow Pr_{\alpha}^5(\lambda; \kappa) \leftrightarrow Pr_{\alpha}^2(\lambda; \kappa) \leftrightarrow Pr_{\alpha}^3(\lambda; \kappa)$.

3) For $\alpha < \kappa^+$, $\lambda_0^0(\kappa) = \lambda_0^{\ell+1}(\kappa)$ for $\ell = 0, 2, 4$, and $\lambda_0^0(\kappa) \leq \lambda_0^2(\kappa) \leq \lambda_0^3(\kappa)$.

4) For $\alpha \geq \kappa^+$ and $\ell = 0, 2, 4$ we have $\lambda_\alpha^{\ell+1}(\kappa) = \lambda_\alpha^{\ell+1}(\kappa)$.

Proof. 1) By 2).

2) For $\alpha = \kappa^+$ it follows from its holding for every $\alpha < \kappa^+$. For $\alpha < \kappa^+$; for

{1.3} $\ell = 0, 2, 4$ we know that $\text{NPPr}_{\alpha}^0(\lambda; \kappa) \Rightarrow \text{NPPr}_{\alpha}^0(\lambda; \kappa)$ by 1.3(1), (2), and $Pr_{\alpha}^4(\lambda; \kappa) \Rightarrow

{1.2} Pr_{\alpha}^4(\lambda; \kappa)$ by 1.2(1); together $Pr_{\alpha}^0(\lambda; \kappa) \Rightarrow Pr_{\alpha}^4(\lambda; \kappa).$ Now $Pr_{\alpha}^0(\lambda; \kappa) \Rightarrow Pr_{\alpha}^4(\lambda; \kappa) \Rightarrow

{2.2} Pr_{\alpha}^4(\lambda; \kappa)$ by 1.2(1), together we finish. (By 2.1 we know more.)

3) Follows from part (2) and the definition.

{1.2} 4) By 1.2(6).
Convention 1.6. Writing $Pr_\alpha(\lambda; \kappa)$ for $\alpha \leq \kappa^+$ (omitting $\ell$) we mean $\ell = 0$. Similarly $\lambda_\gamma(< \kappa, \theta)$ and so $\lambda_\gamma(\kappa)$ etc.

Claim 1.7. Let $\ell \in \{0, 2, 4\}$.

1) $NP_{\alpha+1}(\kappa^+; \kappa)$.

2) If $\alpha$ is a limit ordinal $< \kappa^+$ (in fact, $\aleph_0 \leq cf(\alpha) < \kappa^+$ suffice), and $NP_{\beta}(\lambda; \kappa)$ for $\beta < \alpha$, then $NP_{\alpha+1}(\sum_{\beta < \alpha} \lambda_\beta; \kappa)$.

3) If $NP_{\alpha}^\ell(\lambda; \kappa)$ then $NP_{\alpha+1}^\ell(\lambda^+; \kappa)$.

4) If $NP_{\alpha}^\ell(\mu; \kappa)$ for every $\mu < \lambda$ then $NP_{\alpha+1}^\ell(\lambda; \kappa)$.

Proof. 1) Prove by induction on $\alpha < \kappa^+$, for $\alpha = 0$ use a model in which every element is definable (e.g. an individual constant) so $rk(w, M) = -1$ for $w \in [M]^*$ and hence $rk^\ell(M) = 0$ and consequently $NP_{\alpha}(\kappa; \kappa)$; for $\alpha$ limit use part (2) and for $\alpha$ successor use part (3).

2) Let $\beta$ witness $NP_{\beta}(\lambda; \kappa)$ for $\beta < \alpha$, i.e. $rk^\ell(M_\beta; \kappa) < \beta$ and $M_\beta$ has universe $\lambda_\beta$ and $|\tau(M_\beta)| \leq \kappa$. Without loss of generality $|\tau(M_\beta) : \beta < \alpha$ are pairwise disjoint and disjoint to $\{P_\beta : \beta < \alpha\}$. Let $M$ have universe $\lambda := \sum_{\beta < \alpha} \lambda_\beta$, $P_\beta^M = \lambda_\beta$, and $M|\lambda_\beta$ expand $M_\beta$ and $|\tau(M)| \leq |\alpha| + \sum_{\beta < \alpha} |\tau(M)| \leq \kappa$. By 1.3(2),(4), for $w \in [\lambda_\beta]^*$, $rk^\ell(w, M; \kappa) \leq rk^\ell(w, M_\beta; \kappa) < \beta \leq \alpha$. But $w \in [M]^*$ implies $\bigvee_{\beta < \alpha} w \in [\lambda_\beta]^*$.

3) We define $M^+$ such that each $\gamma \in [\lambda, \lambda^+]$ codes on $\{\xi : \xi < \gamma\}$ an example for $NP_{\alpha}(\gamma; \kappa)$. More elaborately, let $M$ be a model with the universe $\lambda$ such that $rk^\ell(M; \kappa) < \alpha$. Let $\tau(M)$ be $\{R_i : i < i^* \leq \kappa\}, R_i$ an $n(i)$-place predicate (as we replace function symbols and individual constants by predicates), $R_0$ is a 0-nary predicate representing “the truth”. For $\gamma \in [\lambda, \lambda^+]$ let $f_\gamma$ be a one-to-one function from $\gamma$ onto $\lambda$. Define $\tau^+ = \{R_i, Q_i : i < i^* \leq \kappa\}, R_i$ is $n(i)$-place, $Q_i$ is $(n(i) + 1)$-place. So $|\tau^+| \leq \kappa$. We define a $\tau^+$-model $M^+$: the universe is $\lambda^+$, $R^M_i = R_i^M, Q_i^{M^+} = \{\alpha_0, \ldots, \alpha_{n(i)} : \alpha_n(i) \in [\lambda, \lambda^+]$ and $\bigwedge_{\ell < n(i)} \alpha_\ell < \alpha_{n(i)}$ and $\langle f_{\alpha(n(i))}(\alpha_0), \ldots, f_{\alpha(n)}(\alpha_{n(i)-1}) \rangle \in R^M_i \}$ (so $Q^{M^+}_0 = [\lambda, \lambda^+]$).

Now note that:

(a) for $w \in [\lambda]^*, rk^\ell(w, M^+; \kappa) \leq rk^\ell(w, \kappa).

(b) if $w \subseteq \gamma \in [\lambda, \lambda^+]$, $w \neq \emptyset$ then $rk^\ell(w \cup \{\gamma\}, M^+; \kappa) \leq rk^\ell(f_{\gamma}(w), M; \kappa)$.

(Easy to check). So if $\gamma < \lambda^+$ then

(*) $\gamma < \lambda \Rightarrow rk^\ell(\{\gamma\}, M^+; \kappa) \leq rk^\ell(\{\gamma\}, M; \kappa) < rk^\ell(M; \kappa).

(**) $\gamma \in [\lambda, \lambda^+]$ & $\beta \geq rk^\ell(M; \kappa) \Rightarrow rk^\ell(\{\gamma\}, M^+; \kappa) \leq \beta$.

[Why (**)? Assume not and let $\kappa_0 = 2, \kappa_2 = \kappa^4 = \kappa^+$. If $\langle \gamma_i : i < \kappa^4 \rangle$ strictly increasing witnesses $rk^\ell(\{\gamma_i\}, M^+) \geq \beta + 1$ for the formula $Q_0(x)$ then for some $i < j < \kappa^4$ we have $rk^\ell(\{\gamma_i, \gamma_j\}, M^+) \geq \beta$ and applying (b) with $\{\gamma_i\}, \gamma_j$ here standing for $w, \gamma$ there we get $rk^\ell(\{f_{\gamma_i}(\gamma_j)\}, M) \geq \beta$ hence $\beta + 1 \leq rk^\ell(M)$, contradiction.]

Hence
(\ast)_3 \, \text{rk}^\ell (M^+; \kappa) \leq \text{rk}^\ell (M; \kappa) + 1.

As \text{rk}^\ell (M^+; \kappa) < \alpha$ clearly $M^+$ witnesses NPr_{\alpha+1}(\lambda^+; \kappa).

4) Like (3).

\begin{equation}
(1.7)
\end{equation}

**Conclusion 1.8.** Remembering that $\lambda_\alpha(\kappa) = \min \{ \lambda : \text{Pr}_\alpha(\lambda; \kappa) \}$ we have:

\begin{enumerate}
  \item for $\alpha$ a limit ordinal $\lambda_\alpha(\kappa) \leq \mathbb{D}_\alpha(\kappa)$ and even $\lambda^2_\alpha(\kappa) \leq \mathbb{D}_\alpha(\kappa)$
  \item for $\ell$ even $\langle \lambda^\ell_\alpha(\kappa) : 0 < \alpha < \infty \rangle$ is strictly increasing, and for a limit ordinal $\delta$, $\lambda_\delta(\kappa) = \sup_{\alpha < \delta} \lambda_\alpha(\kappa)$
  \item $\lambda_0(\kappa) = \lambda_1(\kappa) = \lambda_2(\kappa) = \kappa$, $\lambda^+ \leq \lambda_n(\kappa) < \kappa^{+\omega}$ and $\lambda_\omega(\kappa) = \kappa^{+\omega}$.
\end{enumerate}

\begin{equation}
(1.7a)
\end{equation}

**Remark 1.9.** [[Saharon $\lambda^2_{\omega \times \alpha}(\kappa) \leq \mathbb{D}_{\omega \times \alpha}(\kappa)] \lambda^2_{\omega \times \alpha}(\kappa) \leq \mathbb{D}_{\omega \times \alpha}(\kappa)$ is proved below essentially like the Morley omitting types theorem (see [Mor65] or see [?]) or [Sh, Ch VII, §5] = [Shv, Ch VII, §5]].

**Proof.** 1) We prove by induction on $\alpha$, that for every ordinal $\beta < \alpha$, model $M$, $|\tau(M)| \leq \kappa$, and $A \subseteq |M|, |A| \geq \mathbb{D}_{\omega \times \alpha}(\kappa)$, and $m, n < \omega$ there is $w \subseteq A, |w| = n$ such that $\text{rk}^2(w, M; \kappa) = \omega \times \beta + m$.

For $\alpha = 0$, $\alpha$ limit this is immediate. For $\alpha = \gamma + 1$ and $(M, A, \beta, n, m$ as above), applying Erdős-Rado theorem we can find distinct $a_i \in A$ for $i < \mathbb{D}_{\omega \times \gamma}(\kappa)^{++}$ such that:

\begin{enumerate}
  \item for all $i_0 < \ldots < i_{m+n}$ the quantifier free type $\langle a_{i_0}, \ldots, a_{i_{m+n}} \rangle$ in $M$ is the same
  \item for each $k \leq m + n$ for every $i_0 < \ldots < i_{m+n-k} \leq \mathbb{D}_{\omega \times \gamma}(\kappa)$, the ordinal $\min \{ \omega \times \alpha, \text{rk}^2(\langle a_{i_0}, \ldots, a_{i_{m+n-k}} \rangle, M; \kappa) \}$ is the same.
\end{enumerate}

By the induction hypothesis, in clause (b) the value is $\geq \omega \times \gamma$. Hence we can prove, by induction on $k \leq m + n$, that $\text{rk}^2(\langle a_{i_0}, \ldots, a_{i_{m+n-k}} \rangle, M; \kappa) = \omega \times \gamma + k$ whenever $i_0 < \ldots < i_{m+n-k} \leq \mathbb{D}_{\omega \times \gamma}(\kappa)$. For $k = 0$ this holds by the previous sentence, for $k + 1$ use the definition and the induction hypothesis, for $\text{rk}^2$ note that by clause (b) without loss of generality $i_{\ell} + \kappa^+ < i_{\ell+1}$ and $a_{i_{\ell+\zeta}}$ for $\zeta < \kappa^+$ are well defined. For $k = m$ we are done.

\begin{equation}
(1.8)
\end{equation}

2) It is increasing by 1.2(1), strict by 1.7(4), continuous because, for limit $\delta$, as on the one hand $\lambda^\delta_\alpha(\kappa) \geq \sup_{\alpha < \delta} \lambda^\delta_\alpha(\kappa)$ as $\lambda^\delta_\alpha(\kappa) \geq \lambda^\epsilon_\alpha(\kappa)$ for $\alpha < \delta$, and on the other hand if $M$ is a model with universe $\lambda := \sup_{\alpha < \delta} \lambda_\alpha(\kappa)$ and $|\tau(M)| \leq \kappa$ then $\alpha < \delta \Rightarrow \text{rk}^\ell(M; \kappa) \geq \text{rk}^\ell(M \upharpoonright \lambda; \kappa) \geq \alpha$ hence $\text{rk}^\ell(M; \kappa) \geq \delta$. So $\text{Pr}_\alpha(\lambda; \kappa)$ hence $\lambda \geq \lambda^\delta_\alpha(\kappa)$ so $\sup_{\alpha < \delta} \lambda^\delta_\alpha(\kappa) = \lambda \geq \lambda^\delta_\alpha(\kappa), \text{together we are done.}$

3) By [Sh:49] (for the last two clauses, the first two clauses are trivial), will not be really used here.

\begin{equation}
(1.8a)
\end{equation}

**Claim 1.10.** 1) Assume $P$ is a forcing notion satisfying the $\kappa^+ - c.c.$ If $\text{Pr}_\alpha^Y(\lambda; \kappa)$ and $\alpha \leq \kappa^+$, then this holds in $V^P$ too.

2) If $P$ is a $\kappa^+ - 2$-linked forcing notion (or just: if $p_i \in P$ for $i < \kappa^+$ then for some $F : \kappa^+ \rightarrow \kappa$, $F(i) = F(j) \Rightarrow p_i, p_j$ compatible), and $\alpha \leq \kappa^+$ and $\text{Pr}_\alpha^Y(\lambda; \kappa)$ then this holds in $V^P$ too.
Remark 1.11. 1) NPPr$_n$(λ; κ) is of course preserved by any extension as the ranks
rk$^\ell$(M; κ), rk$^\ell$(w; M; κ) are absolute for ℓ = 0, 1 (see ??(3)). But the forcing can add new models.

2) So for α ≤ κ$^+$, λ$^\alpha$(κ) ≤ λ$^\alpha_{n+1}$(κ) ≤ λ$^\alpha_2$(κ) and a κ$^+$-c.c. forcing notion can only
increase the first (by ??(3)(a)(δ)) and decrease the third by 1.10(1): a κ$^+$-2-linked
one fixes the second and third (as it can only decrease it by 1.10(1) and can only increase it by ??(3)(c)(a)(δ) + (b)(γ)).

3) [[Of course]] We can deal similarly with Pr$^\ell_n$(λ; κ, θ), here and in ?? - 1.8.

Proof. We can concentrate on 1), anyhow let ℓ = {3, 5} (for part (1) we use ℓ = 3,
for part (2) we shall use ℓ = 5, we shall return to it later). Assume Pr$^\ell_n$(λ; κ) fails
in V$^P$. So for some p$^* \in$ P and α$^0 < α$ we have:

\[
p^* \Vdash_\mathbb{P} " M \text{ is a model with universe } \lambda, \text{ vocabulary } \tau \text{ of cardinality } \leq \kappa \text{ and } \text{rk}^\ell(M; \kappa) = \alpha^0_0\.
\]

Without loss of generality every quantifier free formula φ(x$_0$, ..., x$_{n-1}$) is equivalent to one of the form $R(x_0, ..., x_{n-1})$ and without loss of generality $\tau = \{ R_n, \zeta : n < \omega, \zeta < \kappa \}$ with $R_n, \zeta$ an n-place predicate. Note that necessarily α$^0_0 < \kappa$ hence $|\alpha_0| \leq \kappa$.

As we can replace P by $P[\{ q \in P : p^* \leq q \}]$, without loss of generality $p^*$ is the minimal member of P. Now for non zero $n < \omega, k < n, \zeta < \kappa$ and $\beta < \alpha_0$ (or $\beta = -1$) we define an n-place relation $R_n, \zeta, \beta, k$ on λ:

\[
R_n, \zeta, \beta, k = \{ (a_0, ..., a_{n-1}) : a_m \in \lambda \text{ with no repetitions and for some } p \in P, \]
\[
p^* \Vdash_\mathbb{P} " [M] = R_n, \zeta[a_0, ..., a_{n-1}] \text{ and } \text{rk}^\ell(\{a_0, ..., a_{n-1}\}, M; \kappa) = \beta,
\]

where "not rk$^\ell(\{a_0, ..., a_{n-1}\}, M; \kappa) ≥ \beta + 1^\alpha_0" \text{ is witnessed by } \varphi = R_n, \zeta \text{ and } k'^\alpha_0\}.

Let $M^+ = (\lambda, ..., R_n, \zeta, \beta, k, ...)_{n < \omega, \zeta, \beta, k < \alpha_0, k < \kappa}$, so $M^+$ is a model in V with the universe $\lambda$ and the vocabulary of cardinality $\leq \kappa$. It suffices to prove that for $\beta < \alpha_0$:

\[\otimes_\beta \text{ if } w = \{a_0, ..., a_{n-1}\} \in [M^+]^\alpha, M^+ \Vdash R_n, \zeta, \beta, k[a_0, ..., a_{n-1}]
\]
\[\text{then } \text{rk}$^\ell(\{a_0, ..., a_{n-1}\}, M^+; \kappa) ≤ \beta.
\]

(Not e that by the choice of M and $R_n, \zeta, \beta, k$, if w $\in [M^+]^\alpha$ then for some $n, \zeta, \beta, k$ we have $M^+ \Vdash R_n, \zeta, \beta, k[a_0, ..., a_{n-1}]$. This we prove by induction on $\beta$, so assume the conclusion fails; so

\[\text{rk}$^\ell(\{a_0, ..., a_{n-1}\}, M^+; \kappa) ≥ \beta + 1
\]

and eventually we shall get a contradiction). By the definition of rk$^\ell$ applied to $\varphi = R_n, \zeta, \beta, k, \beta$ and with k we know that there are $a^\ell$ (for $m < n, i < \kappa^+$) as in Definition 1.1(3) case ℓ = 3. In particular

\[M^+ \Vdash R_n, \zeta, \beta, k[a_0, ..., a_{n-1}], \text{ So for each } i < \kappa^+	ext{ by the definition of } R_n, \zeta, \beta, k \text{ necessarily there is } p_i \in P \text{ such that } p_i \Vdash_\mathbb{P} " M \Vdash R_n, \zeta[a_0^i, ..., a_{n-1}^i] \text{ and } \text{rk}$^\ell(\{a_0, ..., a_{n-1}\}, M^+; \kappa) = \beta \text{ and } \not\text{rk}$^\ell(\{a_0, ..., a_{n-1}^i\}, M^+; \kappa) ≥ \beta + 1 \text{ is witnessed by } \varphi = R_n, \zeta \text{ and } k^\alpha_0\].
For part (1), as $P$ satisfies the $\kappa^+ - cc$, for some $q \in P$, $q \Vdash "Y = \{i : p_i \in G_P\}$ has cardinality $\kappa^+$ (in fact, $p_i$ forces it for every large enough $i$). Looking at the definition of the rank in $V^P$ we see that $\langle a_0^i, \ldots, a_{n-1}^i \rangle : i \in Y$ cannot be a witness for the demand for $rk^3(\{a_0^i, \ldots, a_{n-1}^i\}, \kappa) > \beta$ for $R_{n,j,k}$ hold” for any (or some) $i_0 \in Y$, so for part (1)

(*) $q \Vdash_P \text{"for some } i \neq j \text{ in } Y \text{ we have } \text{rk}^3(\{a_0^i, \ldots, a_{n-1}^i, a_k^i\}, M; \kappa) < \beta\"$

(As the demand on equalities holds trivially).

As we can increase $q$, without loss of generality $q$ forces a value to those $i, j$, hence without loss of generality for some $n(\kappa) = n + 1 < \omega, \zeta(\kappa) < \kappa$ and $\beta(\kappa) < \beta$ and for $k(\kappa) < n + 1$ we have

$q \Vdash \text{rk}^3(\{a_0^i, \ldots, a_{n-1}^i, a_k^i\}, M; \kappa) = \beta(\kappa)$, and
\n$\text{rk}^3(\{a_0^j, \ldots, a_{n-1}^j, a_k^j\}, M; \kappa) \geq \beta(\kappa) + 1$ is witnessed by 
\n$\varphi = R_{n(\kappa), \zeta(\kappa)}(x_0, \ldots, x_n)$ and $k(\kappa)^\kappa$.

Hence by the definition of $R_{n(\kappa), \zeta(\kappa), \beta(\kappa), k(\kappa)}$ we have

$M^+ \models R_{n(\kappa), \zeta(\kappa), \beta(\kappa), k(\kappa)}[a_0^i, \ldots, a_{n-1}^i, a_k^i]$.

As $\beta(\kappa) < \beta$ by the induction hypothesis $\otimes_{\beta(\kappa)}$ holds hence

$\text{rk}^3(\{a_0^i, \ldots, a_{n-1}^i, a_k^i\}, M^+; \kappa) \leq \beta(\kappa)$,

but this contradicts the choice of $a_m^i (m < n, i < \kappa^+)$ above (i.e. clause (a) of Definition 1.1(3) case $\ell = 3$). This contradiction finishes the induction step in the proof of $\otimes_{\beta}$ hence the proof of 1.10(1).

For part (2), we have $(p_i : i < \kappa^+)$ as above. In $V^P$, if $Y = \{i : p_i \in G_P\}$ has cardinality $\kappa^+$, then $(\langle a_0^i, \ldots, a_{n-1}^i \rangle : i \in Y)$ cannot witness $\text{rk}^3(\{a_0, \ldots, a_{n-1}\}, M; \kappa) \geq \beta + 1$ so there is a function $F^0 : Y \rightarrow \kappa$ witnessing it; i.e. $\not\Vdash_P \text{"if } |Y| = \kappa^+ \text{ then } i \in Y \text{ and } j \in Y \text{ and } i \neq j \text{ and } \text{nameF}(i) = \text{F}(0)(j) \Rightarrow \beta > \text{rk}^5(\{a_0^i, \ldots, a_{n-1}^i\} \cup \{a_0^j, \ldots, a_{n-1}^j\}, M; \kappa)^\kappa$.

If $|Y| \leq \kappa$, let $F^0 : Y \rightarrow \kappa$ be one to one. Let $p_i \leq q_i \in P, q_i \Vdash F^0(i) = \gamma_i$. As $P$ is $\kappa^+$-2-linked, for some function $F^1 : \kappa^+ \rightarrow \kappa$ we have $(\forall i, j < \kappa^+)(F^1(i) = F^1(j) \Rightarrow q_i, q_j$ are compatible in $P)$. We now define a function $F$ from $Y$ to $\kappa$ by $F(i) = \text{pr}(\gamma_i, F^1(i))$ (you can use any pairing function $\text{pr}$ on $\kappa$). So if $i < j < \kappa^+$ and $F(i) = F(j)$ then there is $q_{i,j}$ such that $P \Vdash "q_i \leq q_{i,j}$ and $q_j \leq q_{i,j}$", hence $q_{i,j} \Vdash_P \text{rk}^5(\{a_0^i, \ldots, a_{n-1}^i, a_k^j\}, M; \kappa) < \beta^\kappa$, so possibly increasing $q_{i,j}$, for some $\beta_{i,j} < \beta$ and $\zeta_{i,j} < \kappa$ and $k_{i,j} < n$ we have $q_{i,j} \Vdash \text{rk}^5(\{a_0^i, \ldots, a_{n-1}^i, a_k^j\}, M; \kappa) = \beta_{i,j}$ and $\text{rk}^5(\{a_0^i, \ldots, a_{n-1}^i, a_k^j\}) \not\leq \beta_{i,j} + 1$ is witnessed by $\varphi = R_{n+1, \zeta_{i,j}}(x_0, \ldots, x_n)$ and $k_{i,j}$.

Hence by the definition of $R_{n+1, \zeta_{i,j}, \beta_{i,j}, k_{i,j}}$ we have

$M^+ \models R_{n+1, \zeta_{i,j}, \beta_{i,j}, k_{i,j}}[a_0^i, \ldots, a_{n-1}^i, a_k^j]$,

but $\beta_{i,j} < \beta$ hence by the induction hypothesis

$\text{rk}^5(\{a_0^i, \ldots, a_{n-1}^i, a_k^j\}, M^+; \kappa) \leq \beta_{i,j}$. 

}
1.1 Case 5.

Claim 1.12. Let $B \subseteq \omega^2 \times \omega^2$ be a Borel or even analytic set and $\Pr_{\omega_1}(\lambda)$. 

1) If $B$ contains a $\lambda$-square then $B$ contains a perfect square.

2) If $B$ contains a $(\lambda, \lambda)$-rectangle then $B$ contains a perfect rectangle.

3) We can replace analytic by $\kappa$-Souslin if $\Pr_{\kappa^+}(\lambda; \kappa)$. (This applies to $\Sigma^1_1$ sets which are $\aleph_1$-Souslin).

Proof. You can apply the results of section 2 to prove 1.12; specifically 2.1 (1) $\Rightarrow$ (2) proves parts (1),(2) and 2.4(1) proves part 3. of 1.12; those results of §2 say more hence their proof should be clearer.

However, we give a proof of part (1) here for the reader who is going to read this section only. Suppose that $B \subseteq \omega^2 \times \omega^2$ is a Borel or even analytic set containing a $\lambda$-square. Let $T$ be a $(2,2,\omega)$-tree such that

$$B = \{(\eta_0, \eta_1) \in \omega \times \omega : (\exists \rho \in \omega)(\eta_0, \eta_1, \rho) \in \lim(T)\},$$

and let $\{\eta_0 : \alpha < \lambda\} \subseteq [\omega^2]$ be such that the square determined by it is contained in $B$ and $\alpha < \beta < \lambda \Rightarrow \eta_0 \neq \eta_1$. For $\alpha, \beta < \lambda$ let $F(\alpha, \beta) \in \omega^2$ be such that $(\eta_0, \eta_1, \rho) \in \lim(T)$. Define a model $M$ with the universe $\lambda$ and the vocabulary $\tau = \{R_{\nu_0, \nu_1, \nu}, Q_{\nu_0, \nu} : \nu_0, \nu_1 \in \omega^2 \text{ and } \nu \in \omega^2\}$, each $R_{\nu_0, \nu_1, \nu}$ a binary predicate, $Q_{\nu_0, \nu}$ a unary predicate and

$$Q^M_{\nu_0, \nu} = \{\alpha < \lambda : \nu_0 < \eta_0 \text{ and } \nu < F(\alpha, \alpha)\},$$

$$R^M_{\nu_0, \nu_1, \nu} = \{\alpha, \beta) \in \lambda \times \lambda : \nu_0 < \eta_0 \text{ and } \nu_1 < \eta_3 \text{ and } \nu < F(\alpha, \beta)\}.$$ 

By $\Pr_{\omega_1}(\lambda)$ we know that $\text{rk}^0(M) \geq \omega_1$.

A pair $(u, h)$ is called an $n$-approximation if $u \subseteq \omega$, $h : u \times u \to \nu_0 \omega$ and for every $\gamma < \omega_1$ there is $w \in [\lambda]^\gamma$ such that:

$(\Sigma_1)$ $u = \{\eta_0|n : \alpha \in w\}$ and $\eta_0|n \neq \eta_3|n$ for distinct $\alpha, \beta \in w$

$(\Sigma_2)$ $\text{rk}^0(M, w) \geq \gamma$

$(\Sigma_3) F(\alpha, \beta) \mid n = h(\eta_0|n, \eta_3|n) \text{ for } \alpha, \beta \in w$; hence

$$M \models R_{\eta_0|n, \eta_3|n, h(\eta_0|n, \eta_3|n)[\alpha, \beta]}$$

for $\alpha, \beta \in w$.

Note that $\{\{\}, \{((\emptyset, \emptyset), (\emptyset, \emptyset))\}\}$ is a 0-approximation.

Moreover

$(\ast)_0$ if $(u, h)$ is an $n$-approximation and $\nu^* \in u$ then there are $m > n$ and an $m$-approximation $(u^+, h^+)$ such that:

(i) $\nu \in u \setminus \{\nu^*\} \Rightarrow (\exists \nu^+)(\nu < \nu^+ \in u^+)$,

(ii) $(\exists^2 \nu^+)(\nu^* < \nu^+ \in u^+)$ (where $\exists^2 x$ means “there are exactly 2 $x$’s”)

(iii) $\nu \in u^+ \Rightarrow \nu|n \in u$ and

(iv) if $\nu_1, \nu_2 \in u^+$ then $[h|\nu_1|n, \nu_2|n, h+(\nu_1, \nu_2)$ or $(\nu_1|n = \nu_2|n = \nu^*$ and $\nu_1 \neq \nu_2)$].
(522)
be such that: if \( u = \{\alpha_0, \ldots, \alpha_{n-1}\} \in [\lambda]^n \) increasing for definiteness, \( \beta = \text{rk}^n(u, M) \) (< \( \alpha(\ast) \)) then \( \varphi^M(u) \) is a quantifier free formula in the vocabulary of \( M \) in the variables \( x_0, \ldots, x_{n-1} \) for simplicity saying \( x_0 < x_1 \ldots < x_{n-1} \), \( k^M(u) \) is a natural number < \( n = |u| \) such that \( \varphi^M(u), k^M(u) \) witness “not rk\(^n(u, M) \geq \beta + 1 \)” (the same definition makes sense even if \( \beta = -1 \)). In particular

\[
M \models \varphi^M(u)[\ldots, a, \ldots]_{a \in u}.
\]

We define the forcing notion \( P \). We can put the diagonal \( \{(\eta, \eta) : \eta \in \omega^2\} \) into \( B \) so we can ignore it. We want to produce (in \( V^P \)) a Borel set \( B = \bigcup_{n<\omega} B_n \), each \( B_n \) \((\subseteq \omega^2 \times \omega^2) \) closed (in fact perfect), so it is \( \lim(T_n) \) for some \((2, 2)\)-tree \( T_n \), \( B_0 \) is the diagonal, and \( \tilde{\eta} = (\eta_\alpha : \alpha < \mu) \) as witnesses to \( 2^{\aleph_\alpha} \geq \mu \) and such that \( \{\eta_\alpha : \alpha < \lambda\} \) gives the desired square. So for some 2-place function \( g \) from \( \lambda \) to \( \omega, \alpha \neq \beta \Rightarrow (\eta_\alpha, \eta_\beta) \in \lim(T_{g(\alpha, \beta)}) \), all this after we force. But we know that we shall have to use \( M \) (by 1.12). In the forcing our problem will be to prove the c.c.c. which will be resolved by using \( M \) (and rank) in the definition of the forcing. We shall have a function \( f \) which puts the information on the rank into the trees to help in not having a perfect square. Specifically the domain of \( f \) is a subset of

\[
\{(u, h) : (\exists \ell \in \omega)(u \in \ell^2) \mbox{ and } h : u \times u \rightarrow \omega\}
\]

(the functions \( h \) above are thought of as indexing the \( B_n \)'s). The function \( f \) will be such that for any distinct \( \alpha_0, \ldots, \alpha_{n-1} < \lambda \), if \( \eta_{\alpha_0}[\ell] = (t < n) \) are pairwise distinct, \( u = \{\eta_\alpha : t < n\}, h(\eta_\alpha, \tilde{\eta_\alpha} | \ell) = g(\alpha_\ell, \alpha_\omega) \) and \( (u, h) \in \text{Dom}(f) \) then \( \text{rk}^1(\{\alpha_\ell : t < n\}, M) = f_0(u, h) \) and \( f_1(u, h) = \eta_\alpha | \ell \), where \( k^M(\{\alpha_\ell : t < n\}) \), \( f_2(u, h) = \varphi^M(\{\alpha_\ell : t < n\}) \) writing the variable as \( x_\nu, \nu \in u \) and \( f(u, h) = (f_0(u, h), f_1(u, h), f_2(u, h)) \). (Note: \( f \) is a way to say \( \bigcup_n \lim(T_n) \) contains no perfect square; essentially it is equivalent to fixing appropriate rank.) All this was to motivate the definition of the forcing notion \( P \).

A condition \( p \) of \( P \) is an approximation to all this; it consists of:

\begin{enumerate}
\item \( u^p = u[p] \), a finite subset of \( \mu \)
\item \( n^p = n[p] < \omega \) and \( \eta^p_\alpha = \eta_\alpha[p] \in n[p]^2 \) for \( \alpha \in u[p] \) such that \( \alpha \neq \beta \Rightarrow \eta^p_\alpha \neq \eta^p_\beta \). To clarify let \( \ell^p_\alpha = \{\eta_\alpha[\ell] : \alpha \in u^p, \ell \leq n[p]\} \) is a full subtree of \( n[p]\)\(^2 \) only (not really necessary)\(^1\)
\item \( \tilde{m}^p = (\ell^p \geq \ell \leq n^p) \) is a strictly increasing sequence of natural numbers with last element \( m^p_{\text{max}} = m^p = m[p] \) and for \( m < m[p] \), we have \( \ell^p_m = \langle \ell^p \rangle_{\leq m[p]} \subseteq \bigcup_{\ell \leq m[p]} (\ell^2 \times \ell^2) \) which is downward closed (i.e., \( (\nu_0, \nu_1) \in \ell_{m[p]}^p \cap (\ell^2 \times \ell^2) \) and \( k \leq \ell \Rightarrow \nu_0, \nu_1 \) \( k = (\nu_0|k, \nu_1|k) \in \ell_{m[p]}^p \)), also \( \langle \langle \rangle \rangle \) \( \in \ell_{m[p]}^p \) and defining \( < \) naturally we have: if \( (\eta_0, \eta_1) \in \ell_{m[p]}^p \cap (\ell^2 \times \ell^2) \) and \( \ell < m^p \) then \( (\exists \nu_{0, 1})(\langle \eta_0, \eta_1 \rangle \prec (\nu_0, \nu_1) \in \ell_{m[p]}^p \cap (\ell+1^2 \times \ell+1^2)) \)
\end{enumerate}

\(^1\)Added for transparency; it is definable from \( \eta^p_\alpha : \alpha \in u^p \); the intention was \( p < q \Rightarrow \ell^p = \ell^q \cap n[p]^2 \) not stated we may wonder about \( (u, h) \in \text{Dom}(f^p), u \subseteq 2^\omega \) and \( \nu \notin \ell^p \). We now exclude them but the relation \( R \) excludes them (well when we have two members but recall \( |u| \geq 1000 \). Alternatively demand \( n[p]^2 = \{\eta_\alpha[n] : \alpha \in u^p \} \) which requires a little more in some places.
(4) a function \( f^p = f[p] \), its domain is a subset of \( \{u, h\} \): for some \( \ell \leq n[p] \), \( u \subseteq t_\ell^p \subseteq \ell^2, |u| \geq 1 \), \( h \) is a 2-place function from \( u \) to \( m[p] \) such that: \( \[ \forall \eta \in u \Rightarrow h(\eta, \eta) = 0 \] \) \( \{ (\eta, \nu) \in u^2 \mid \nu \neq \eta \Rightarrow h(\eta, \nu) > 0 \} \) and \( f^p \) is such that \( f^p(u, h) = (f_{\eta^p}^p(u, h), f_{\nu^p}^p(u, h), f_{\eta, \nu^p}^p(u, h, h)) \in [-1, \alpha(\ast)] \times u \times L_{\omega, \omega}(\tau(M)) \).

(5) a function \( g = g^p \) with domain \( \{\alpha, \beta\} : \alpha, \beta \) from \( u^p \cap \lambda \) such that \( g(\alpha, \beta) = 0 \) and if \( \alpha \neq \beta \Rightarrow 0 < g(\alpha, \beta) < m^p \) and \( (n_\alpha^p, n_\beta^p) \in I_{p(\alpha, \beta)} \cap (n[p] \times n[p]) \)

(6) \( t_0^p = \{ (\eta, \eta) : \eta \in n^p \geq 2 \} \)

(7) if \( u \subseteq \ell^2, |u| \geq 1 \), \( f^p(u, h) = (\beta^p, \rho^p, \varphi^p) \), and \( \ell < \ell(\ast) \leq n^p \), \( e_i \) are functions with domain \( u \) (for \( i = 0, 1 \)) such that \( (\forall \rho)(\rho \in u \Rightarrow \rho \neq e_i(\rho) \in (\ell^2) \)) and \( (\forall \rho \in u)[e_0(\rho) = e_1(\rho) \Rightarrow \rho \neq \rho^\ast] \), \( u' = \text{Rang}(e_0[u] \cup \text{Rang}(e_1[u]) \), and \( h(\eta, \nu) = h'_i(\eta, \nu) \) for \( \eta \neq \nu \) in \( u \) and \( f^p(u', h') = (\beta^p, \rho^p, \varphi^p) \) (is well defined) \( \Rightarrow \beta^p < \beta^\ast \).

(8) if \( \ell \leq n^p \), \( w \subseteq u^p \cap \lambda \) is non empty, the sequence \( (n_\alpha^p \mid \alpha \in u) \) is with no repetitions and \( h \) is defined by \( h(n_\alpha^p \mid \alpha \in u) = g^p(\alpha, \beta) \) for \( \alpha \neq \beta \) from \( w \) and \( h(n_\alpha^p \mid \alpha \in u) ) = 0 \) and \( u = (n_\alpha^p \mid \alpha \in u) \), then \( f^p(u, h) \) is well-defined hence \( (\forall \beta \neq \beta \in u \Rightarrow g(\alpha, \beta) < m^p, f_{\alpha, \beta^p}^p(u, h) = \varphi^M(w), f_{\alpha, \beta^p}^p(u, h) = n_\alpha^p \ell \) where \( \alpha \) is the \( k^M(w) \)-th member of \( w \) and \( f_{\alpha, \beta^p}^p(u, h) = \rho^\ast \).

(9) if \( (u, h) \in \text{Dom}(f^p) \) then for some \( w \) and \( \ell, f^p(u, h) \) is gotten as in clause (8) otherwise.

(10) if \( \eta_1 \neq \eta_2 \) in \( \ell^2, \ell \leq n^p \) and \( (\eta_1, \eta_2) \in t_\ell^0 \), \( 0 < m < m^p \) then for some \( \alpha_1 \neq \alpha_2 \) from \( u^\ast \cap \lambda \) we have \( g^p(\alpha_1, \alpha_2) = m \) and \( \eta_1 \leq \eta_\alpha^1, \eta_2 \leq \eta_\alpha^2 \) exist.

The order is the natural one (including the following requirements: \( p \leq q \) \( \Rightarrow \) \( (p, q) \in P \) and \( n^p \leq n^q, m^p \leq m^q, \rho^p = \rho^q \mid (n^p + 1), u^p \subseteq u^q, n_\alpha^p | n^p = n_\alpha^q \) for \( \alpha \in n^p, t_\ell^p = t_\ell^q \cap n^p, t_m^p = t_m^q \cap \bigcup_{\ell \leq n^p} (\ell^2 \times \ell^2) \) for \( m < m^p, g^p = g^q \mid u^p \) and \( f^p = f^q \mid \{ (u, h) \in \text{Dom}(f^q) : u \subseteq n^p \geq 2 \}, \) so if \( (u, h) \notin \text{Dom}(f^q), u \subseteq n^p \geq 2 \) then \( (u, h) \notin \text{Dom}(f^q) \). □

**Explanation:** The function \( f^p \) of a condition \( p \in P \) carries no additional information. It is determined by the function \( g^p \) and functions \( \varphi^M, k^M \) and the rank.

Conditions 8, 9 are to say that:

\( \oplus_p \) \( \Rightarrow \) \( w_0, w_1 \subseteq \lambda \cap u^p, \ell \leq n^p, u = (n_\alpha^p \mid \alpha \in w_0) = \{ n_\alpha^p \mid \alpha \in w_1 \} \) (no repetitions) are non empty and \( h : u \times u \rightarrow m^p \) is such that if either \( \alpha, \beta \in w_0 \) or \( \alpha, \beta \in w_1 \) then \( h(n_\alpha^p \mid \alpha \in w_0, 0 \leq \rho^\ast \rightarrow g^p(\alpha, \beta) \) (is well defined) \( \Rightarrow \beta^p < \beta^\ast \).

Moreover, condition 7 gives no additional restriction unless \( f_{\alpha, \beta^p}^p(u, h) = -1 \). Indeed, suppose that \( u \subseteq \ell^2, |u| \geq 1, \ell < \ell(\ast) \leq n^p, e_i : u \rightarrow (\ell^2) \rightarrow U, \rho^* \in u, u' \) and \( h' \) are as there and \( f_{\alpha, \beta^p}^p(u, h) \geq 0 \). As \( f^p(u', h') \) is defined we find \( w \subseteq \lambda \cap u^p, \alpha_0, \alpha_1 \in w \) (\( \alpha_0 \neq \alpha_1 \)) such that \( u' = (n_\alpha^p \mid \alpha \in w_1) \), \( h'(n_\alpha^p \mid \alpha \in u_1, \rho^p \mid \alpha \in u_1) = g^p(\alpha, \beta) \leq m^p \).
and \(e_i(p^*) = \eta^{\alpha}_i, \ell_i \) (for \( i = 0, 1 \)). Looking at \( w \setminus \{ \alpha_0 \}, w \setminus \{ \alpha_1 \} \) and \((u, h)\) we see that

\[
\alpha_0 = k^M(w \setminus \{ \alpha_1 \}), \nu^M(w \setminus \{ \alpha_0 \}) = \nu^M(w \setminus \{ \alpha_1 \})
\]

and

\[
\text{rk}^1(w \setminus \{ \alpha_1 \}, M) = f^0_\ell(u, h) > 0.
\]

By the definition of the rank and the choice of \( \nu^M, k^M \) we get \( \text{rk}^1(w, M) = f^0_\ell(u, h) \) and hence \( f^0_\ell(u, h') < f^0_\ell(u, h) \). If \( f^0_\ell(u, h) = -1 \) then clause 7 says that there are no respective \( e_0, e_1 \) introducing a ramification.

**Stage B:** Let \( p^i \in \mathbb{P} \) for \( i < \omega_1 \); let \( u[p^i] = \{ a^i_\ell : \ell < |u[p^i]| \} \) increasing, so with no repetition. Without loss of generality \( |u[p^i]| \) does not depend on \( i \), and also \( n[p^i], q^{\nu^i}, m^{\nu^i}, (\ell^n_{\alpha,i} : m < m^{p^i}), g^p(a^{i}_1, a^{i}_2), f[p^i], \) and for a nonempty \( v \subseteq |u[p^i]| \) such that \( \lambda \subseteq |u[p^i]| \) and clauses 8, 9 of stage A say that there are no respective \( e_0, e_1 \) introducing a ramification.

Also by the \( \Delta \)-system argument without loss of generality

\[
a^{i_1}_\ell = a^{i_2}_\ell \text{ and } i_1 \neq i_2 \text{ implies } \ell_1 = \ell_2 \text{ and } \bigwedge_{i,j} a^{i}_\ell = a^{j}_\ell.
\]

We shall show that \( p^0, p^1 \) are compatible by defining a common upper bound \( q \):

\[
\begin{align*}
(i) \quad & n^q = n[p^1] + 1 \\
(ii) \quad & u^q = \{ a^i_\ell : \ell < |u[p^i]|, i < 2 \} \\
(iii) \quad & \eta^q\alpha^q_i =: \eta^0\alpha^0_i(0) \text{ if } i = 0, \eta^0\alpha^0_i(1) = \eta^0\alpha^0_i(1) \text{ if } i = 1, a^q_i \neq a^i_1 \\
(iv) \quad & m[q] = m[p^0] + 2 \times |\lambda \cap u[p^0] \setminus u[p^1]|^2, m^q = m^0 \times m^1 \\
(v) \quad & g^q \geq g^\nu \cup g^p \text{ is such that } g^q \text{ assigns new (i.e. in } [m^p, m^q]) \text{ distinct values} \\
\text{ to } \text{ “new” } \text{ pairs } (\alpha, \beta) \text{ with } \alpha \neq \beta, \text{ i.e. pairs from } (\lambda \times \lambda) \cap (u^q \times u^q) \text{ satisfying } g^q(a^q_1, a^q_2) = m \text{ and if } m \in [m[p^0], m[q]), m = g^q(\alpha, \beta) \text{ and } \alpha \neq \beta \text{ then} \\
\text{ the trees } t^n_m \text{ (for } m < m[q]) \text{ are defined as follows:} \\
\text{ if } m = 0 \text{ see clause 8,} \\
\text{ if } m < m[p^0], m > 0 \text{ then } t^n_m = t^n_m \cup \{ \eta^0\alpha^0_i, \eta^0\alpha^0_i : \nu^q \in \{0, 1\} \} \text{ and distinct } \\
\ell_1, \ell_2 \subseteq |u[p^0]| \text{ satisfying } g^q(a^q_1, a^q_2) = m \text{ and if } m \in [m[p^0], m[q]), m = g^q(\alpha, \beta) \text{ and } \alpha \neq \beta \text{ then} \\
\text{ the trees } t^n_m = \{ \eta^0\alpha^0_i, \eta^0\alpha^0_i : \nu^q, \ell \subseteq 2 \} \\
(vii) \quad & \text{ if } m \in [m[p^0], m[q]), \text{ then for one and only one pair } (\alpha, \beta) \text{ we have } m = g^q(\alpha, \beta) \text{ and for this pair } (\alpha, \beta) \text{ we have } \alpha \neq \beta, \{0, 1\} \subseteq u[p^0] \text{ and } \{0, 1\} \subseteq u[p^0] \text{ and for this pair } (\alpha, \beta) \text{ we have } \alpha \neq \beta, \{0, 1\} \subseteq u[p^0] \text{ and for this pair } (\alpha, \beta) \text{ we have } \alpha \neq \beta, \{0, 1\} \subseteq u[p^0] \text{ and for this pair } (\alpha, \beta) \text{ we have } \alpha \neq \beta, \{0, 1\} \subseteq u[p^0] \\
(viii) \quad & \text{ The function } f^q \text{ is determined by the function } g^p \text{ and clauses 8, 9 of stage A.}
\end{align*}
\]
Of course, we have to check that no contradiction appears when we define \( f^q \) (i.e. we have to check \( \oplus_g \) of the Explanation inside stage A for \( q \)). So suppose that \( w_0, w_1 \subseteq \lambda \cap u[q], \ell \leq n[q], u, h \) are as in \( \oplus_g \). If \( w_0 \subseteq u[p^i] \) (for some \( i < 2 \)) then \( g^q(\alpha, \beta) \subseteq m[p^i] \) for \( \alpha, \beta \in w_0 \) and hence \( g^q[w_1 \times w_1] \subseteq m[p^0] \). Consequently either \( w_1 \subseteq u[p^0] \) or \( w_1 \subseteq u[p^1] \). If \( \ell = n^q \) then necessarily \( w_0 = w_1 \) so we have nothing to prove. If \( \ell < n^q \) then \( (u, h) \in \text{Dom}(f^p) \) (and \( f^p = f^q \)) and clause 8 of stage A applies.

If \( w_0 \) is contained neither in \( u[p^0] \) nor in \( u[p^1] \) then the function \( g^q \) satisfies \( g^q(\alpha, \beta) \subseteq [m[p^0], m[q]] \) for some \( \alpha, \beta \in w_0 \) hence \( \ell = n^q \) and so as \( \{ \eta^q_n \mid \ell : \alpha \in w_0 \} = \{ \eta^q_n \mid \ell : \alpha \in w_1 \} \) clearly \( w_0 = w_1 \), so we are done.

Next we have to check condition 7. As we remarked (in the Explanation inside Stage A) we have to consider cases of \( (u, h) \) such that \( f^q(u, h) = -1 \) only. Suppose that \( u, \ell < \ell(*) \leq n^q, e_i, h, \rho^* \in u, u' \) and \( h' \) are as in 7 (and \( f^q(u, h) = -1 \)). Let \( w \subseteq u[q] \cap \lambda, \alpha_0, \alpha_1 \in w \) be such that \( u' = \{ \eta^q_n \mid \ell(*) : \alpha \in w \}, e_i(\rho^*) = \eta^q_n \mid \ell(*) \) (for \( i = 0, 1 \)). If \( w \subseteq u[p^i] \) for some \( i < 2 \) then we can apply clause 7 for \( p^i \) and get a contradiction (if \( \ell(*) = n^q \) then note that \( \{ \eta^q_n \mid p^i : \alpha \in w \} \) are already distinct). Since \( \alpha \in w \setminus \{ \alpha_0, \alpha_1 \} \) implies \( g^q(\alpha, \alpha_0) = g^q(\alpha, \alpha_1) \) (by the relation between \( h \) and \( h' \)) we are left with the case \( w \setminus \{ \alpha_0, \alpha_1 \} \subseteq u[p^0] \cap u[p^1], \alpha_0 \in u[p^0] \setminus u[p^1], \alpha_1 \in u[p^1] \setminus u[p^0] \) (or conversely). Then necessarily \( \alpha_0 = a_{\alpha_0}^k, \alpha_1 = a_{\alpha_1}^k \) for some \( k_0, k_1 \in [0, |u[p^0]|) \). Now \( k_1 = k^M(w \setminus \{ \alpha_0 \}) = k^M(w \setminus \{ \alpha_1 \}) = k_0 \) by the requirements in condition 7.

Now we see that for each \( i < \omega_1 \)

\[ M \models \varphi^M(w \setminus \{ \alpha_0 \})[w \setminus \{ \alpha_0, \alpha_1 \} \cup \{ a_{\alpha_1}^k \}] \]

and this contradicts the fact that \( \varphi^M(w \setminus \{ \alpha_0 \}), \alpha_1 \) witness \( \text{rk}^1(w \setminus \{ \alpha_0 \}, M) = -1 \).

Stage C: \( |\mathcal{P}| = \mu \) hence \( \models \eta \leq \mu^\omega \). We shall get the equality by clause (\( \gamma \)) at stage E below.

Stage D: The following subsets of \( \mathcal{P}^\omega \) are dense (for \( m, n < \omega, \alpha < \mu \)):

\[ \mathcal{J}_{\alpha}^1 = \{ p \in \mathcal{P} : m[p] \geq m \} \]

\[ \mathcal{J}_{\alpha}^2 = \{ p \in \mathcal{P} : n^p \geq n \} \]

\[ \mathcal{J}_{\alpha}^3 = \{ p \in \mathcal{P} : \alpha \in u[p] \} \]

Proof. Let \( p \in \mathcal{P}, \alpha_0 \in \mu \setminus u[p] \) be given, we shall find \( q, p \leq q \in \mathcal{J}_{\alpha_0}^1 \cap \mathcal{J}_{\alpha_0}^2 \cap \mathcal{J}_{\alpha_0}^3 \); this clearly suffices. We may assume that \( u[p] \neq \emptyset \) and \( \alpha_0 < \lambda \).

Let

(a) \( n^q = n^p + 1, m^q = m^p + 2 \cdot (|\lambda \cap u[p]|), m^q = m^p \cdot m^q, u^q = u^p \cup \{ \alpha_0 \}, \)

(b) for \( \alpha \in u^p \) we let \( \eta^q_\alpha = \eta^q_\alpha(0), \eta^q_\alpha \in (n^{\alpha+1})^1 \) 2 is the sequence constantly equal 1,
(c) \(g^q\) is any two-place function from \(u^q \cap \lambda\) to \(m^q\) extending \(g^p\) such that \(g^q(\alpha, \alpha) = 0, g^q(\alpha, \beta) \neq 0\) for \(\alpha \neq \beta\) and \(g^q(\alpha, \beta) = g^q(\alpha', \beta')\) and \((\alpha, \beta) \neq (\alpha', \beta') \Rightarrow (\alpha, \beta) \in \omega^p \cup u^p(\alpha', \beta') \in u^p \cup u^p.

(d) \(t^p_m\) is defined as follows:

\[
(\alpha) \text{ if } m < m^p, m \neq 0 \text{ then } t^p_m = t^p_m \cup \{(\eta_0, \eta_1) : (\eta_0, \eta_1) \in t^p_m \cap \left(\bigcup_{n^p} \{n^p \times n^p\}\right)\}
\]

\[
(\beta) \text{ if } m \in [m^p, m^q], m = g^q(\alpha, \beta), \alpha \neq \beta \text{ then } t^p_m = \{(\eta^a_0, \eta^a_1, \ell) : \ell \leq n^q\}
\]

(e) \(f^q\) extends \(f^p\) and satisfies 7, 8 and 9 of stage A (note that \(f^q\) is determined by \(g^q\)).

Now check [similarly as at stage B].

Stage E: We define some \(\mathbf{P}\)-names

\[
\begin{align*}
(a) \quad & \eta^a_\alpha = \bigcup \{\eta^p_\alpha : p \in \mathcal{G}_p\} \text{ for } \alpha < \lambda \\
(b) \quad & T^m = \bigcup \{t^p_m : p \in \mathcal{G}_p\} \text{ for } m < \omega \\
(c) \quad & g = \bigcup \{g^p : p \in \mathcal{G}_p\} \\
(d) \quad & T_* = \bigcup \{t^p : p \in \mathcal{G}_p\}.
\end{align*}
\]

Clearly it is forced (\(\mathbb{P}^\mathbf{P}\)) such that:

\[
(\alpha) \quad g \text{ is a function from } \{(\alpha, \beta) : \alpha, \beta < \lambda\} \text{ to } \omega.
\]

[Why? Because \(\mathcal{F}_\beta^3\) are dense subsets of \(\mathbf{P}\) and by clause 5 of stage A.]

\[
(\beta) \quad \eta^a_\alpha \in \omega^2.
\]

[Why? Because both \(\mathcal{F}_\beta^3\) and \(\mathcal{F}_\beta^3\) are dense subsets of \(\mathbf{P}\).]

\[
(\gamma) \quad \eta^a_\alpha \neq \eta^a_\beta \text{ for } \alpha \neq \beta (< \mu).
\]

[Why? By clause 2 of the definition of \(p \in \mathbf{P}\).]

\[
(\delta) \quad T^m \subseteq \bigcup_{n<\omega} (\omega^2 \times \omega^2) \text{ is an } (2, 2)\text{-tree.}
\]

[Why? By clause 3 of the definition of \(p \in \mathbf{P}\) and density of \(\mathcal{F}_m^1, \mathcal{F}_n^2\).]

\[
(\varphi) \quad (\eta^a_\alpha, \eta^a_\beta) \in \lim(T^q_{(\alpha, \beta)}) = \{(\nu_0, \nu_1) : (\forall \ell < \omega)(\nu_0, \nu_1, \ell) \in T^q_{(\alpha, \beta)}) \text{ for } (\alpha, \beta < \lambda).}
\]

[Why? By clause 5 of the definition of \(p \in \mathbf{P}\) and \((\beta) + (\delta)\) above.]}

\[
(\zeta) \quad \text{if } \alpha, \beta \text{ are } < \lambda \text{ then } (\eta^a_\alpha, \eta^a_\beta) \notin \lim(T^m) \text{ when } m \neq g(\alpha, \beta) \text{ (and } m < \omega).}
\]

[Why? By clauses 2 + 10 of the definition of \(\mathbf{P}\) if \(m \neq 0\) and clause 5 if \(m = 0\).]

\[
(\eta) \quad T_* \text{ is a subtree of } \omega^2 \text{ with no maximal nodes and } \{\eta^p_\alpha : \alpha < \lambda\} \subseteq \lim(T_*).
\]

Note that by clause \((\varphi)\) above the Borel set \(B = \bigcup_{m<\omega} \lim(T^m) \subseteq \omega^2 \times \omega^2\) satisfies requirement \((*)\) of the Conclusion of 1.13. Moreover, by clause \((\gamma)\) above we have \(\mathbb{P}^\mathbf{P} \triangleright \omega^\mathfrak{b} = \mu\) completing stage C (i.e. \(\mathbb{P}^\mathbf{P} \triangleright \omega^\mathfrak{b} = \mu\)).

\[
\begin{align*}
\text{Stage F: We want to show } (*) \text{ of the Conclusion of 1.13. Let } \mathbf{P}_\lambda = \{p \in \mathbf{P} : u[p] \leq \lambda\}. \text{ Clearly } \mathbf{P}_\lambda \ll \mathbf{P}. \text{ Moreover } g, T^m, B \text{ are } \mathbf{P}_\lambda\text{-names. Since } \triangleleft B \text{ contains a }
\end{align*}
\]

\[
\{1.10\}
\]

\[
\{1.10\}
\]
perfect square” is a $\Sigma^1_2$-formula, so absolute, it is enough to prove that in $V^{P\lambda}$ the set $B$ contains no perfect square.

Suppose that a $P\lambda$-name $T$ for a perfect tree and a condition $p \in P\lambda$ are such that:

$$(\ast)_1^P \quad p \Vdash_{P\lambda} \text{“(} \text{lim} T \times (\text{lim} T) \subseteq B \text{”}.}$$

We have then (a name for) a function $m : \text{lim}(T) \times (\text{lim} T) \rightarrow \omega$ such that:

$$(\ast)_2^P \quad p \Vdash_{P\lambda} \text{“if } \eta_0, \eta_1 \in \text{lim} T \text{ then } (\eta_0, \eta_1) \in T_{m(\eta_0, \eta_1)} \text{ hence } \eta_0, \eta_1 \in T_\ast”.}$$

By shrinking the tree $T$ we may assume that $p$ forces $(\ast)_2^P$ the following:

$$(\ast)_3^P \quad \text{“if } \eta_0, \eta_1, \eta_0', \eta_1' \in \text{lim} T, \eta_0|\ell = \eta_0'|\ell \neq \eta_1|\ell = \eta_1'|\ell \text{ then } m(\eta_0, \eta_1) = m(\eta_0', \eta_1')\”.}$$

Consequently we may think of $m$ as a function from $T \times T$ to $\omega$ (with a convention that if $\eta_0, \eta_1 \in T$ are $\prec$-comparable then $m(\eta_0, \eta_1) = 0$ and $\eta' \text{ lang}(\ell) \prec \eta \in T \Rightarrow m(\eta', \ell, (1 - \ell)) = m(\eta, \eta, \ell)$ and if $\ell g(\eta_1) = \ell g(\eta_2)$ then $(\eta_1, \eta_2) \in T_{m(\eta_1, \eta_2)}$).

Choose an increasing sequence $\langle t_i : i \in \omega \rangle \subseteq P\lambda, \langle (t_i, m_i) : i \in \omega \rangle$ such that:

1. $p \leq p_0 \leq p_1 \leq \ldots \leq p_i \leq \ldots$
2. $t_i \subseteq n_i \geq 2$ is a full sub-tree, (i.e. $\eta \prec \nu \in t_i \cap n_i \geq 2 \Rightarrow \eta \in t_i$, $\ell \in t_0, [\eta \in n_i \geq 2 \cap t_i \Rightarrow n_\eta^\ast(\ell) \in t_i]$) and $m_i : (t_i \cap n_i \geq 2) \rightarrow \omega$ and $|t_i \cap n_i \geq 2| \geq 1000$
3. $t_i \subseteq t_{i+1}$ is an end extension (i.e. $t_i = (n_i \geq 2) \cap t_{i+1}$) such that each node from $t_i \cap n_i \geq 2$ ramifications in $t_{i+1}$ (i.e. has $\prec$-incomparable extensions)
4. $p_i \Vdash_{P\lambda} \text{“} T \cap n_i \geq 2 \supseteq t_i \text{ and } m_i(t_i \cap n_i \geq 2) = m_i^n \text{”}
5. $n[p_i] > n_i, m[p_i] > \text{max(Rang}(m_i))$.

How do we carry the induction? For $i = 0$, note that $p \Vdash \text{“} T \text{ is perfect then for some } n, [T \cap n] > 1000\text{’}$ let $p_0 \geq p \text{ force } n_0 = \text{as above and force a value } t_0 \text{ to } T \cap (n_0 \geq 2) \text{ and force } m_i_0(t_0 \cap n_0 \geq 2) = n_0$. If $p_i, t_i, \ldots \text{ are well defined clearly } p_i \Vdash \text{“for some } n > n_i, (\nu \prec \rho \in t_i) (\ell g(\rho) = n_i \Rightarrow (\exists \eta_\ell \ast g)(\rho \prec g)) \text{. let } p_i' > p_i \text{ force } n_i' > n \text{ as above; without loss of generality } p_i' \text{ forces a value to } t_i', \text{ tr}(T) \cap (n_i' \geq 1) \geq 2 \text{ and } m_i' \text{ to } m_i(t_i' \cap n_i' \geq 1)$. Let $p_{i+1} \geq p_i', n_{i+1} > n_i \text{ be such that } m_{i+1} = \text{max(Rang}(m_i')$. By $(\ast)_3^P + \text{the paragraph below this is fine.}$

Since $p_i \Vdash_{P\lambda} \text{“}(\nu_0, \nu_1) \in T_{m_i(\nu_0, \nu_1)} \text{” for } \nu_0, \nu_1 \in t_i \cap n_i \geq 2 \text{ easily get (by clause 8 of the definition of } P, \text{ stage A) that } u \subseteq t_i \cap n_i \geq 2, |u| \geq 1000 \Rightarrow \text{ (u, m_i|u) } \in \text{Dom}(f^P)\text{. Let } \alpha_i^\ast = \text{min}\{f^P(u, m_i|u), u \subseteq t_i \cap n_i \geq 2, 1000 \leq |u| \leq 1000 + i\}. \text{ By clause 7 (of the definition of } P) \text{ (and 1.2(2)+ clause 8 of the definition of } P) \text{ we deduce that } \alpha > \alpha_{i+1} \text{ for each } i < \omega \text{ and this gives a contradiction (to the ordinals being well ordered).}$

NOTE: that the $\eta_\alpha$’s do not appear in this stage. We only use the demand on the $f^P$’s. Note that the domain of $f^P$ does not depend on the $\eta_\alpha$’s, in fact, only $\eta_\alpha|n^P$ is well defining knowing $p$ only.

{1.10} Stage G: To prove $(\ast)(b)$ of Theorem 1.13 we may assume that $V \models \text{“} \forall \lambda \in \lambda_1 < \mu^+. \text{ Let } P_{\lambda_1} = \{p \in P : u[p] \subseteq \lambda_1\} \subseteq P$. Note that the rest of the forcing (i.e. $P_{\lambda_1}$) is the forcing notion for adding $\mu$ Cohen reals so for $v \subseteq \mu \setminus \lambda_1$ the forcing notion $P_v$ is naturally defined as well as $P_{\lambda_1 \cup v}$. By stages C, E we know that
\[ V^{P_{\lambda_1}} \models "2^{\aleph_0} = \lambda_1" \] and by stage $F$ we have $V^{P_{\lambda_1}} \models "\text{the Borel set } B \text{ does not contain a perfect square}"$. Suppose that after adding $\mu$ Cohen reals (over $V^{P_{\lambda_1}}$) we have a $\lambda^+_1$-square contained in $B$. We have $\lambda^+_1$-branches $\rho_\alpha$ ($\alpha < \lambda^+_1$), each is a $P_{\nu_\alpha}$-name for some countable $\nu_\alpha \subseteq \mu \setminus \lambda_1$. By the $\Delta$-system lemma without loss of generality we assume that $\alpha \neq \beta \Rightarrow \nu_\alpha \cap \nu_\beta = \emptyset$. Working in $V^{P_{\lambda_1}+\mathbb{R}^*}$ we see that $P_{\nu_\alpha \cap \nu_\beta}$ is really the Cohen forcing notion and $\rho_\alpha$ is a $P_{\nu_\alpha \setminus \nu_\beta}$-name. Without loss of generality $\nu^* = [\lambda_1, \lambda_1 + \omega], \nu_\alpha = \nu^* \cup \{\lambda_1 + \omega + \alpha\}$ and all names $\rho_\alpha$ are the same (under the natural isomorphism). So we have found a Cohen forcing name $\tau \in V^{P_{\lambda_1}+\mathbb{R}^*}$ such that: if $c_0, c_1$ are (mutually) Cohen reals over $V^{P_{\lambda_1}+\mathbb{R}^*}$, then $V^{P_{\lambda_1}}[c_0, c_1] = (\tau^{c_0}, \tau^{c_1}) \in B$ and $\tau^{c_0} \neq \tau^{c_1}$.

But the Cohen forcing adds a perfect set of (mutually) Cohen reals. By absoluteness this produces a perfect set (in $V^{P_{\lambda_1}}$) whose square is contained in $B$. Once again by absoluteness we conclude that $B$ contains a perfect square in $V^{P_{\lambda_1}}$, already, a contradiction. \hfill \{1.11\}

Remark 1.14. Note that if $B$ is a subset of the plane $([\omega, \omega] \times \omega)$ which is $G_\delta$ (i.e. $\bigcap_{n \in \omega} U_n$, $U_n$ open, without loss of generality decreasing with $n$) and it contains an uncountable square $X \times X$ (so $X \subseteq [\omega, \omega]$ is uncountable) then it contains a perfect square. Why?

Let

\[ X' = \{ \eta \in X : (\forall n)(\exists^{\aleph_1} \nu)(\nu \in X \text{ and } \nu \upharpoonright n = \eta \upharpoonright n) \}. \]

Let

\[ K = \{(u, n) : \text{ for some } \ell \text{ called } \ell(u, n), u \subseteq \ell \omega, \text{ and } \eta, \nu \in u \text{ and } \eta \triangleleft \eta' \in \omega^\omega \text{ and } \nu \triangleleft \nu' \in \omega^\omega \Rightarrow (\eta', \nu') \in U_n \text{ and } \eta \in u \text{ and } \eta \triangleleft \eta' \in \omega^\omega \Rightarrow (\eta', \nu') \in U_n \} \]

\[ K' = \{(u, n) \in K : \text{ for some } \nu = \langle \nu_\rho : \rho \in u \rangle \text{ we have } \nu_\rho \in X', \rho \triangleleft \nu_\rho \}. \]

So

(a) $K' \neq \emptyset$, in fact if $\eta_1, \ldots, \eta_m \in X'$ are pairwise distinct, $n < \omega$, then for any $\ell$ large enough ($\{\eta_i : i = 1, \ldots, m\}, n) \in K$,

(b) if $(u, n) \in K'$ as exemplified by $\nu = \langle \nu_\rho : \rho \in u \rangle$ and $\rho^* \in u \nu' \in X' \setminus \{\nu_\rho\}, \nu' \upharpoonright \ell = \nu_\rho \upharpoonright \ell$ then for any $\nu' \in (\ell, \omega)$ and $n' > n$, large enough, we have $(\nu_\rho \upharpoonright \ell' : \rho \in u) \cup \{\nu' \upharpoonright \ell'\}, n') \in K'$.

The following depends on §3:

Theorem 1.15. Assume $MA$ and $2^{\aleph_0} \geq \lambda_{\omega_i}(\aleph_0)$ or $2^{\aleph_0} > \mu$. Then: there is a Borel subset of the plane with a $\mu$-square but with no perfect square \iff $\mu < \lambda_{\omega_1}(\aleph_0)$.

Proof. The first clause implies the second clause by 1.12. If the second clause holds, let $\mu \leq \lambda_0(\aleph_0)$ and $\alpha < \omega_1$, by 3.2(6) letting $\eta_i \in ^\omega 2$ for $i < \mu$ be pairwise distinct we can find an $\omega$-sequence of $(2, 2)$-trees $T$ such that $(\eta_i, \eta_j) \in \bigcup_n (\lim T_n)$ for $i, j < \mu$ and $\deg_{\omega}(T) = \alpha$ (just use $A = \{(\eta_i, \eta_j) : i, j < \mu\}$ there). By 3.2(3) the set $\bigcup_n (\lim T_n)$ contains no $\lambda_{\omega_1}(\aleph_0)$-square. \hfill \{1.9\} \hfill \{3.2\} \hfill \{3.2\}
Fact 1.16. Assume $P$ is adding $\mu > \kappa$ Cohen reals or random reals and $\kappa > 2^{\aleph_0}$.

Then in $V^P$ we have:

$\star$ there is no Borel set (or analytic) $B \subseteq \omega^2 \times \omega^2$ such that:

(a) there are $\eta_\alpha \in \omega^2$ for $\alpha < \kappa$ such that $[\alpha \neq \beta$ implies $\eta_\alpha \neq \eta_\beta]$, and
(b) $B$ contains no perfect square.

Proof. Straight as in the (last) stage G of the proof of theorem 1.13 (except that no relevance of (7) of Stage A there).

Let $P$ be adding $\langle \tilde{r}_\alpha : \alpha < \mu \rangle$, assume $p \in P$ forces that: $B$ a Borel set, $\langle \eta_\alpha : \alpha < \kappa \rangle$ are as in clause (a), (b) above. Let $\eta_\alpha$ be names in $P_{v_\alpha} = P \restriction \{ \tilde{r}_\beta : \beta \in v_\alpha \}$, and $B$ be a name in $P_v = P \restriction \{ \tilde{r}_\beta : \beta \in v \}$ where $v, v_\beta$ are countable subsets of $\kappa$. Without loss of generality $\langle v_\alpha : \alpha < (2^{\aleph_0})^+ \rangle$ is a $\Delta$-system with heart $v$ and $\text{otp}(v_\alpha \setminus v) = \text{otp}(v_0 \setminus v)$. In $V^P$, we have $\tilde{B}$ and $2^{\aleph_0} = (2^{\aleph_0})^V$, so without loss of generality $\tilde{v} = \emptyset$ and $\text{otp}(v_\alpha)$ does not depend on $\alpha$.

Without loss of generality the order preserving function $f_{\alpha,\beta}$ from $v_\alpha$ onto $v_\beta$ maps $\eta_\alpha$ to $\eta_\beta$. So for $Q=Cohen$ in the Cohen case we have a name $\tilde{\tau}$ such that $\models_{\text{Cohen}} "\tau(\tilde{r}) \in \omega^2$ is new", $\models_{\text{Cohen} \times \text{Cohen}} "(\tau(\tilde{r_1});\tau(\tilde{r_2})) \in \tilde{B}"$, and we can finish easily. The random case is similar. \qed

Conclusion 1.17. 1) For $\kappa \in (\aleph_1, \aleph_\omega)$ the statement $\star$ of 1.16 is not decided by ZFC $+ 2^{\aleph_0} > \aleph_\omega$ (i.e. it and its negation are consistent with ZFC).

2) 1.16 applies to the forcing notion of 1.13 (with $\mu$ instead of $2^{\aleph_0}$).

Proof. 1) Starting with universe $V$ satisfying CH, Fact 1.16 shows the consistency of “yes”. As by 1.7(1) we know that $\lambda_{\omega_1}(\aleph_0) \geq \aleph_{\omega_1}$ and $\aleph_{\omega_1} > \kappa$ (by assumption), Theorem 1.15 (with the classical consistency of MA $+ 2^{\aleph_0} > \aleph_{\omega_1}$) gives the consistency of “no” (in fact in both cases it works for all $\kappa$ simultaneously).

2) Left to the reader. \qed
§ 2. Some model theoretic related problems

We turn to the model theoretic aspect: getting Hanf numbers below the continuum i.e. if \( \psi \in L_{\omega_1,\omega} \) has a model of cardinality \( \geq \lambda_\omega(\aleph_0) \) then it has a model of cardinality continuum. We get that \( \text{Pr}_{\omega_1}(\lambda) \) is equivalent to a statement of the form “if \( \psi \in L_{\omega_1,\omega} \) has a model of cardinality \( \lambda \) then it has a model generated by an “indiscernible” set indexed by “2” (the indiscernibility is with respect to the tree \( \langle \omega^2, <, \subset, <, <_1, <_2, <_3, <_4 \rangle \), where \( \omega \) is being initial segment, \( \eta \cap \nu = \text{maximal } \rho, \rho \leq \eta \) and \( \rho \leq \nu, <_1 \nu \) is lexicographic order, \( \eta <_1 \nu \) if \( \ell(g(\eta)) < \ell(g(\nu)) \). This gives sufficient conditions for having many non-isomorphic models and also gives an alternative proof of 1.12.

We also deal with the generalization to \( \lambda \)-models i.e. fixing the cardinalities of several unary predicates (and point to \( \lambda \)-like models).

Claim 2.1. The following are equivalent for a cardinal \( \lambda \).

1) \( \text{Pr}_{\omega_1}(\lambda) \).

2) If \( \psi \in L_{\omega_1,\omega} \) has a model \( M \) with \( |M| \geq \lambda \) (\( R \) is a unary predicate) then \( \psi \) has a model of the cardinality continuum, moreover for some countable first order theory \( T_1 \) with Skolem functions such that \( \tau(\psi) \subseteq \tau(T_1) \) and a model \( M_1 \) of \( T_1 \) and \( a_\eta \in R^{M_1} \) for \( \eta \in \omega^2 \) we have:

\[
(*)_0 \ M_1 \models \psi \\
(*)_1 \ M_1, a_\eta (\eta \in \omega^2) \text{ are as in [Sh:a, Ch.II,§4] = [Sh:c, Ch.VII,§4], i.e.:
(a) } M_1 \text{ is the Skolem hull of } \{ a_\eta : \eta \in \omega^2 \} \text{ and } \eta \neq \nu \text{ implies } a_\eta \neq a_\nu \\
(b) \text{ for every } n < \omega \text{ and a first order formula } \varphi = \varphi(x_0, \ldots, x_{n-1}) \in L(T_1) \text{ there is } n^* < \omega \text{ such that: for every } k \in (n^*, \omega), \eta_0, \ldots, \eta_{n-1} \in \omega^2 \text{ and } \nu_0, \ldots, \nu_{n-1} \in \omega^2 \text{ satisfying } \bigwedge_{m < n} \eta_m[k] = \nu_m[k] \text{ and } \bigwedge_{m < \ell < n} \eta_m[k] \neq \nu_m[k] \forall k \text{ we have } M_1 \models \varphi[a_{\eta_0}, \ldots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \ldots, a_{\nu_{n-1}}] \text{. Note that necessarily } a_\eta \notin \text{Skolem Hull}_{M_1}(a_\nu : \nu \in \omega^2 \setminus \{ \eta \}) \text{.}
(c) a_\eta \in R^{M_1}.

Remark 2.2. We can prove similarly with replacing \( \lambda \) by “for arbitrarily large \( \lambda' < \lambda' \) here and elsewhere; i.e. in 2) we replace the assumption by “If \( \psi \in L_{\omega_1,\omega} \) has, for every \( \lambda' < \lambda' \), a model \( M \) with \( |M| \geq \lambda' \) then ...” (and still the new version of 2) is equivalent to 1)).

Proof. 1 \( \Rightarrow \) 2

Just as in [Sh:37] + [Sh:49]: without loss of generality \( |M'| = \lambda \) and moreover \( |M'| = \lambda \). Let \( M_1 \) be an expansion of \( M \) by names for subformulas of \( \psi \), a pairing function, and then by Skolem functions. Let \( T_1 \) be the first order theory of \( M_1 \). There is (see [Kei71]) a set \( \Gamma \) of countably many types \( p(x) \) such that: \( M_1 \) omits every \( p(x) \in \Gamma \) and if \( M_1' \) is a model of \( T_1 \), omitting every \( p(x) \in \Gamma \) then \( M_1' \) is a model of \( \psi \) (just for each subformula \( \bigwedge_{n<\omega} \psi_n(\bar{x}) \) of \( \psi \), we have to omit a type; we can use 1-types as we have a pairing function).

Let us define \( Y = \{ v \in \omega^2 : v \text{ is finite nonempty and its members are pairwise } \preceq \text{-incomparable and for some } n, v \in \omega^2 \cup \{ n+1 \}, Z = \{ (v, \varphi, \ldots, x_n, \ldots, \eta_{n+1}) : v \in Y, \varphi \text{ a formula in } T_1 \text{ with the set of free variables included in } \{ x_n : \eta \in v \} \text{ and for every } \alpha < \omega_1 \text{ there are } a^\alpha_\eta \in R^M \text{ for } \eta \in v \text{ such that: } [\eta \neq \nu \text{ from } v \Rightarrow a^\alpha_\eta \neq a^\alpha_\nu] \text{ and } \text{rk}^\alpha(\{ a^\alpha_\eta : \eta \in v \}, M) \geq \alpha \text{ and } M \models \varphi[\ldots, a^\alpha_n, \ldots]_{\eta \in v} \}. \)
We say for $v_1, \varphi \in \mathbb{Z}$ ($\ell = 1, 2$) that $i(v_2, \varphi_2) \in \text{suc}(v_1, \varphi_1)$ if for some $\eta \in v_1$ (called $\eta(v_1, v_2)$) we have $v_2 = (v_1 \setminus \{\eta\}) \cup \{\eta' (0), \eta' (1)\}$ and letting for $i < 2$ the function $h_i : v_1 \rightarrow v_2$ be $h_i(\nu)$ is $\nu$ if $\nu \neq \eta$ and it is $\eta' (i)$ if $\nu = \eta$, we demand for $i = 0, 1$:

$$\varphi_2 \vdash \varphi_1(\ldots, x_{h_i(\nu)}, \ldots)_{\nu \in v_1}.$$  

Choose inductively $((v_1, \varphi) : \ell < \omega)$ such that $(v_{\ell+1}, \varphi_{\ell+1}) \in \text{suc}(v_1, \varphi_1)$ is generic enough, i.e.:

1) if $\varphi = \varphi(x_0, \ldots, x_{k-1}) \in L(T_1)$ then for some $\ell < \omega$ for every $m \in [\ell, \omega)$ and $\eta_0, \ldots, \eta_{k-1} \in v_m$ we have: $\varphi_m \vdash \varphi(x_{\eta_0}, \ldots, x_{\eta_{k-1}})$ or $\varphi_m \vdash \neg \varphi(x_{\eta_0}, \ldots, x_{\eta_{k-1}})$

2) for every $p(x) \in \Gamma$ and for every function symbol $f = f(x_0, \ldots, x_{n-1})$ (note: in $T_1$ definable function is equivalent to some function symbol), for some $\ell < \omega$ for every $m \in [\ell, \omega)$, for every $\eta_0, \ldots, \eta_n \in v_m$ for some $\psi(x) \in p(x)$ we have $\varphi_m \vdash \neg \psi(x_{\eta_0}, \ldots, f(x_{\eta_{n-1}}))$.

It is straightforward to carry the induction (to simplify you may demand in $(\circ)_1$, $(\circ)_2$ just “for arbitrarily large $m \in [\ell, \omega)$”, this does not matter and the stronger version of $(\circ)_1$, $(\circ)_2$ can be gotten (replacing the $\omega \uparrow 2$ by a perfect subtree $T$ and then renaming $a_\eta$ for $\eta \in \text{lim}(T)$ as $a_\eta$ for $\eta \in \omega \uparrow 2$). Then define the model by the compactness.

2) 1:

If not, then $\text{NPr}_{\omega_i}(\lambda)$ hence for some model $M$ with vocabulary $\tau$, $|\tau| \leq \aleph_0$, cardinality $\lambda$ we have $\alpha(\tau) := \text{rk}_0(M) < \omega_1$. Let $\psi_{\alpha(\tau)} \in L_{\omega_1, \omega}(\tau)$ be as in 2.3 below, so necessarily $M \models \psi_{\alpha(\tau)}$. Apply to it clause (2) which holds by our present assumption (with $R^M = \lambda$), so $\psi_{\alpha(\tau)}$ has a model $M_1$ as there, (so $M_1 \models \psi_{\alpha(\tau)}$).

But $\{a_\eta : \eta \in \omega \uparrow 2\}$ easily witnesses $\text{rk}_0(M_1) = \omega_1$ moreover, for every nonempty finite $w \subseteq \{a_\eta : \eta \in \omega \uparrow 2\}$ and an ordinal $\alpha$ we have $\text{rk}_0(w, M_1) \geq \alpha$. This can be easily proved by induction on $\alpha$ (using $(\ast)_2(b)$ of (2) (and $\eta \neq \nu \in \omega \uparrow 2 \Rightarrow a_\eta \neq a_\nu$ of $(\ast)_2(a)$)).

2.2

Fact 2.3. 1) For every $\alpha < \kappa^+$ and vocabulary $\tau$, $|\tau| \leq \kappa$, there is a sentence $\psi_\alpha \in L_{\kappa^+, \omega}[\tau]$ (of quantifier depth $\alpha$) such that for any $\tau$-model $M$:

$$M \models \psi_\alpha \text{ iff } \text{rk}_0(M; < \aleph_0) = \alpha.$$  

2) For every $\alpha < \theta^+, \ell \in \{0, 1\}$ and vocabulary $\tau$, $|\tau| \leq \theta$ there is a sentence $\psi \in L_{\theta^+, \omega}[\tau]$ (of quantifier depth $\alpha$) such that for any $\tau$-model $M$:

$$M \models \psi_\alpha \text{ iff } \text{rk}_\ell(M; < \kappa, \theta) = \alpha.$$  

Proof. Easy to check.

Hence (just as in [Sh:Ch.VIII.1.8(2)]):

2.3

Conclusion 2.4. Assume $\tau$ is a countable vocabulary. If $\psi \in L_{\omega_1, \omega}(\tau)$, $R$ is a unary predicate, $\tau_0 \subseteq \tau$, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$ is countable and for some transitive model $V_1$ of ZFC (may be a generic extension of $V$ or an inner model as long as $\psi$, $\Delta \in V_1$ and $V_1 \models 0 \in L_{\omega_1, \omega}(\tau)$, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$) we
have $V_1 \models \text{“Pr}_{\omega_1}(\lambda)$ and $\psi$ has a model $M$ with $\lambda \leq |\{\varphi(x) : M = \varphi[a], \varphi(x) \in \Delta]\} : a \in R^M|$”.

Then we can find a model $N$ of $\psi$ with Skolem functions and $a_\alpha \in R^N$ for $\alpha < 2^{R_0}$ such that for each $\alpha < 2^{R_0}$ the type $p_\alpha = \{\pm \varphi(x) : N = \pm \varphi[a_\alpha] \text{ and } \varphi(x) \in \Delta\}$ is not realized in the Skolem hull of

$$\{a_\beta : \beta < 2^{R_0} \text{ and } \beta \neq \alpha\}.$$ 

Hence $\{M/ \equiv N \mid |M| = \lambda\} \geq \min\{2^\lambda, 2^2\}$; really $\{(M/|\tau_0)/ \equiv N \mid |M| = \psi \text{ and } M \text{ has cardinality } \lambda\} \text{ has cardinality } \geq \min\{2^\lambda, 2^2\}$. Moreover we can find such a family of models no one of them if we have embeddable into another by an embedding preserving $\pm \varphi(x)$ for $\varphi \in \Delta$.

A natural generalization of 2.1 is

Claim 2.5. 1) For cardinals $\lambda > \kappa \geq R_0$ the following are equivalent:

(a) $\text{Pr}_{\omega_1}(\lambda; \kappa)$

(b) If $M$ is a model, $\tau(M)$ countable, $R, R_0 \in \tau(M)$ unary predicates, $|R_0^M| \leq \kappa, \lambda \leq |R_0^M|$ then we can find $M_0, M_1, a_\eta(\eta \in \omega^2)$ such that:

(i) $M_1$ is a model of the (first order) universal theory of $M$ (and is a $\tau(M)$-model)

(ii) $a_\eta \in R_0^M$ for $\eta \in \omega^2$ are pairwise distinct

(iii) $M_1$ is the closure of $\{a_\eta : \eta \in \omega^2\} \cup M_0$ under the functions of $M_1$ (so

$$(\alpha) M_1 \text{ also includes the individual constants of } M \text{; in general } |M_1| = 2^{R_0^M}$$

$$(\beta) \text{ if } \tau(M) \text{ has predicates only then } |M_1| = \{a_\eta : \eta \in \omega^2 \cup |M_0|\}$$

(iv) $M_0$ is countable, $M_0 \subseteq M, M_0 \subseteq M_1, M_0 = \text{ct}_M(M_0 \cap R_0^M), R_0^{M_1} = R_0^{|M_0|}(\subseteq R_0^M)$; in fact we can have:

(*) $(M_1, c), c \in M_0$ is a model of the universal theory of $(M, c), c \in M_0$

(v) for every $n < \omega$ and a quantifier free first order formula $\varphi = \varphi(x_0, \ldots, x_{n-1}) \in L(\tau(M))$ there is $n^* < \omega$ such that: for every $k \in (n^*, \omega)$ and $\eta_0, \ldots, \eta_{n-1} \in \omega^2, \nu_0, \ldots, \nu_{n-1} \in \omega^2$ satisfying $\bigwedge_{m < n} \eta_m[k = \nu_m[k$ and $\bigwedge_{m < \ell < n} \eta_m[k \neq \eta_\ell[k$ we have $M_1 \models \varphi[a_{\eta_0}, \ldots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \ldots, a_{\nu_{n-1}}]$;

we can even allow parameters from $M_0$ in $\varphi$ (but $k$ depends on them).

2) For cardinals $\lambda > \kappa \geq R_0$ the following are equivalent:

(a) $\text{Pr}_{\omega_1}(\lambda; \kappa)$

(b) like (b) above, but we omit “$M_0 \subseteq M$”.

Remark 2.6. 1) See 4.6, 4.7 how to use claim 2.5.

2) In (b), if $M$ has Skolem functions then we automatically get also:

(i) $M_1$ a model of the first order theory of $M$

(ii) $M_1$ is the Skolem hull of $\{a_\eta : \eta \in \omega^2\} \cup M_0$

(iv) $M_0 \prec M, M_0 \prec M_1, M_0$ countable (and $R_0^{M_1} = R_0^{|M_0|} \subseteq R_0^M$)

(v) clause (v) above holds even for $\varphi$ any (first order) formula of $L_{\omega, \omega}(\tau(M))$. 


\[ \begin{align*}
\text{Proof.} \ 1) \ (a) \Rightarrow (b) \\
\{2.1\} \text{ Like the proof of 2.1, (1) } (a) \Rightarrow (2), \text{ applied to } (M, c) \in R_0^M \text{ but the set } M_0 \cap R_0^M \text{ is chosen by finite approximation i.e. (letting } Y \text{ be as there and } \tau = \tau(M)) \text{ we let } \\
Z = \{(v, \varphi(\ldots, x_n, \ldots), \eta) \in V, A) : v \in Y, \varphi \text{ a quantifier free formula in } L_{\omega, \omega}(\tau) \text{ with set of free variables included in } \{x_0 : \eta \in v\} \text{ and parameters from } A, A \text{ is a finite subset of } R_0^M, \text{ and for every ordinal } \alpha < \kappa^+ \text{ there are } a_0^\alpha \in R_0^M \text{ for } \eta \in v \text{ such that } [\eta \neq \nu \\text{ from } v \Rightarrow a_0^\alpha \neq a_0^\nu] \text{ and } rk(a_0^\alpha : \eta \in v), M) \geq \alpha \text{ and } M \models \varphi[\ldots, a_0^\alpha, \ldots]_{\eta \in v}. \\
\text{We need here the } \text{"for every } \alpha < \kappa^+" \text{ because we want to fix elements of } R_0^M, \text{ and there are possibly } \kappa \text{ choices.}
\text{(a)} \Rightarrow (b): \\
\{2.2\} \text{ Like the proof of 2.3; assume } NP_{\pi^2}(\lambda; \kappa), \alpha < \kappa^+, \text{ let } M \text{ witness it, choose } \\
R_0 = \alpha + 1, R = \lambda, \text{ without loss of generality } \tau(M) = \{R_0, \zeta : n < \omega, \zeta < \kappa\}, \text{ } R_{n, \zeta} \text{ is } n\text{-place, } M \text{ in } \tau(M) \text{ quantifier free formula is equivalent to some } R_{n, \zeta}, \text{ let } R_{n, k} := \{\langle i_0, \ldots, i_{n-1}, \beta, \zeta \rangle : M \models R_{n, \zeta}(i_0, \ldots, i_{n-1}), \{i_0, \ldots, i_{n-1}\} \text{ is with no repetition, increasing for simplicity, and } rk\{i_0, \ldots, i_{n-1}\}, M; \kappa = \beta, \text{ with } \text{rk} \{i_0, \ldots, i_{n-1}\}, M; \kappa \not\geq \beta + 1 \text{ being witnessed by } \varphi\{i_0, \ldots, i_{n-1}\} = R_{n, \zeta}, \\
k\{i_0, \ldots, i_{n-1}\} = k \} \text{ where the functions } \varphi, \zeta \text{ are as in the proof of 1.13. Let } M \text{ be } (\lambda, <, R, R_0, R_{n, k})_{n \in (0, \omega), k \in n} \text{ expanded by Skolem functions. So assume toward contradiction that } (b) \text{ holds, hence for this model } M \text{ there are models } M_0, M_1 \text{ and } \\
a_\eta \in M_1 \text{ for } \eta \in \omega^2 \text{ as required in clauses (i) - (v) of (b) of claim 2.5. Choose a non-empty finite subset } w \in \omega^2 \text{ and } \beta, \zeta \text{ such that letting } w = \{\eta_0, \ldots, \eta_{m-1}\} \text{ with } a_{\eta_0} < M_1, a_{\eta_{n+1}}, \text{ we have:} \\
(\alpha) \ M_1 \models R_{m, k}^*(a_{\eta_0}, \ldots, a_{\eta_{m-1}}, \beta, \zeta) \\
(\beta) \ \beta \in R_0^M (\subseteq \alpha) \\
(\gamma) \ \beta \text{ minimal under those constraints.}
\end{align*} \\
\text{Note that there are } m, \eta_0, \ldots, \eta_{m-1}, \beta, \zeta \text{ such that } (\alpha) \text{ holds: for every non-empty } w \subseteq \omega^2, \text{ as } M_1 \text{ is elementarily equivalent to } M \text{ there are } \beta, \zeta \text{ as required in } (\alpha). \text{ Now } (\alpha) \text{ implies } \zeta \in R_0^M, \text{ but } R_0^M = R_0^M, \text{ so clause } (\beta) \text{ holds too, and so we can satisfy } (\gamma) \text{ too as the ordinals are well ordered. Let } \varphi' = \varphi(x_0, \ldots, x_m, \beta, \zeta), \text{ note the parameters are from } R_0^M \text{ (as } M_1 \text{ is elementarily equivalent to } M) \text{ hence from } R_0^M \subseteq M, \text{ and clause (v) of (b) of 2.5 applies to } \varphi', \{\eta_0, \ldots, \eta_{m-1}\} \text{ giving } n^* < \omega. \text{ We can find } \eta^*_k \in \omega^2, \eta^*_k \neq \eta_0, \eta^*_k \models n^* = \eta_k \models n^*, \text{ and easily for } w' = \{\eta_0, \ldots, \eta_{m-1}, \eta^*_k\} \text{ we can find } \beta' < \beta, \zeta' < \kappa \text{ and } k \text{ such that } M_1 \models R_{m+1,k}(\alpha_0, \alpha_{m-1}, a_{\eta^*_k}, \beta', \zeta') \text{ and then if } \beta \geq 0 \text{ we get contradiction to clause } (\gamma) \text{ above. If } \beta = 0 \text{ we use clause } (i) \text{ to copy the situation to } M \text{ and get a contradiction.} \\\n2) \text{ Similar proof.} \]
\{2.5\} \text{ Notation 2.7. Let } \lambda \text{ denote a finite (or countable) sequence of pairs of infinite cardinals } \langle(\lambda_\zeta; \kappa_\zeta) : \zeta < \zeta^*(\cdot)\rangle \text{ such that } \kappa_\zeta \text{ increases with } \zeta, \text{ so e.g. } \lambda_\zeta = \langle(\lambda_\zeta^0; \kappa_\zeta^0) : \zeta < \zeta^0(\cdot)\rangle \text{. We shall identify a strictly increasing } \bar{k} = \{k_\zeta : \zeta \leq \zeta^*(\cdot)\} \text{ with } \langle(k_{\zeta^0}; \kappa_\zeta^0) : \zeta < \zeta^0(\cdot)\rangle. \\
\text{Let } R, R_0, Q_0, \ldots, R_{\zeta^*(\cdot)-1}, Q_{\zeta^*(\cdot)-1} \text{ be fixed unary predicates and } \bar{R} = \langle(R_\zeta, Q_\zeta) : \zeta < \zeta^*(\cdot)\rangle.\]
A \( \lambda \)-model \( M \) is a model \( M \) such that: \( R, R_\zeta, Q_\zeta \in \tau(M) \) are all unary predicates, \( |R_M^M| = \lambda_\zeta \), \( |Q_\zeta^M| = \kappa_\zeta \) for \( \zeta < \zeta(*) \), \( Q_\zeta \subseteq R_\zeta \) and \( (R_\zeta^M : \zeta < \zeta(*)) \) are pairwise disjoint, and \( R^M = \bigcup \{ R_\zeta^M \} \).

For \( a \in R^M \) let \( \zeta(a) \) be the \( \zeta \) such that \( a \in R^M_\zeta \) (e.g. \( R^M_\zeta = \lambda_\zeta \setminus \bigcup \lambda_\xi \), \( \kappa_{\xi+1} = \lambda_\xi \)). For a \( \lambda \)-model \( M \) we say that \( a \in c\ell_\alpha(A, M) \) if \( A \cup \{ a \} \subseteq M \) and for some \( n < \omega \), and quantifier free formula \( \varphi(x_1, y_1, \ldots, y_n) \) and \( b_1, \ldots, b_n \in A \) we have

\[
M \models \varphi(a, b_1, \ldots, b_n) \text{ and } (\exists x_{\leq n} \varphi(x, b_1, \ldots, b_n)).
\]

**Definition 2.8.** 1) For \( \ell < 6 \), \( \lambda \) as in Notation 2.7, and an ordinal \( \alpha \) let \( \text{Pr}_\alpha^\ell(\lambda; \theta) \) mean that for every \( \lambda \)-model \( M \), with \( |\tau(M)| \leq \theta \) we have \( \text{rk}'(M, \lambda) \geq \alpha \) (and \( \text{NP}^\ell_\alpha(\lambda, \theta) \) is the negation, if \( \theta \) is omitted it means \( \kappa_0 \), remember \( \lambda = (\lambda_\zeta, \kappa_\zeta) : \zeta < \zeta(*) \) where the rank is defined in part (1) below.

2) For a \( \lambda \)-model \( M \), \( \text{rk}'(M, \lambda) = \sup\{ \text{rk}'(w, M, \lambda) + 1 : w \in [R^M]^* \} \) where the rank is defined in part (1) below and:

- (a) if \( \zeta(*) \) is finite, \( [R^M]^* = \{ w : w \text{ a finite subset of } R^M \} \) not disjoint to any \( R^M_\zeta \}
- (b) if \( \zeta(*) \) is infinite, \( [R^M]^* = \{ w : w \text{ a finite non-empty subset of } R^M \} \).

3) For a \( \lambda \)-model \( M \), and \( w \in [R^M]^* \) we define the truth value of “\( \text{rk}'(w, M; \lambda) \geq \alpha \)” by induction on \( \alpha \).

**Case A:** \( \alpha = 0 \)

\( \text{rk}'(w, M; \lambda) \geq \alpha \) iff no \( a \in w \cap R^M \) belongs to \( c\ell_\kappa(a)(w \setminus \{ a \}, M) \).

**Case B:** \( \alpha \) is a limit ordinal

\( \text{rk}'(w, M; \lambda) \geq \alpha \) iff \( \text{rk}'(w, M; \lambda) \geq \beta \) for every ordinal \( \beta < \alpha \).

**Case C:** \( \alpha = \beta + 1 \)

We demand two conditions:

- (a) exactly as in Definition 1.1(3)(*) except that when \( \ell = 2, 3, 4, 5 \) we use \( \kappa = \kappa_\zeta(a_\alpha) \), \( \{1.1\} \)
- (b) if \( \zeta < \zeta(*) \) and \( w \cap R^M = \emptyset \) then for some \( a \in R^M_\zeta \), \( \text{rk}'(w \cup \{ a \}, M; \theta) \geq \beta \).

**Claim 2.9.** The parallel of the following holds: 1.2 (+ statements in 1.1) also 1.3 (use \( \alpha < \kappa_0^+ \), 1.5(2) for \( \alpha \leq \kappa_0^+ \), 1.10 (satisfying \( \kappa_0^+ \)-c.c.) and we adapt 1.6. \( \{2.7\} \)

**Claim 2.10.** If \( \alpha \) is a limit ordinal and \( \lambda_\xi \geq \bigcup_\alpha(\kappa_\xi) \) for every \( \xi < \zeta(*) \), then \( \text{Pr}_\alpha(\lambda) \).

**Proof.** Use indiscernibility and Erdös-Rado as in the proof of 1.8(1).

In more details, the induction hypothesis on \( \alpha \) is, assuming \( \zeta(*) < \omega \): if \( A \subseteq R^M \), \( \bigwedge_{\zeta < \zeta(*)} \lfloor A \cap R^M_{\zeta} \rfloor \geq \bigcup_{\alpha} \lfloor a \rfloor_{x \alpha} \) then for every \( \beta < \alpha \), \( k < \omega \) and every \( m = \langle m_\zeta : \zeta < \zeta(*) \rangle \) for some \( m < A \) we have \( \Delta_{\zeta} \lfloor w \cap R^M_{\zeta} \rfloor = m_\zeta \) and \( \text{rk}(w; M; \lambda) \geq \omega \times \beta + k \). Then for \( \alpha = \gamma + 1 \), choose distinct \( a^\zeta_\xi \in A \cap R^M_{\zeta} \) (i.e. \( \bigcup_{\omega \times \gamma + m + m_\zeta} \)) and
use polarized partition (see Erdös, Hajnal, Mate, Rado [EHMR84]) on \((\langle a^*_i : i < \bigcup_{\alpha \in \lambda} \rangle : \zeta < \zeta(\ast))\). For \(\zeta(\ast)\) infinite use \(A \subseteq R^M\) such that \(w_A = \{ \zeta : A \cap R^M_\zeta \neq \emptyset \}\) is finite non-empty, \(\zeta \in w_A \Rightarrow |A \cap R^M_\zeta| \geq \bigcup_{\alpha \in \lambda} \) and proceed as above.

\[\tag{2.9}\]

\[\text{Claim 2.11. Let } \zeta(\ast) < \omega, \kappa^0_\zeta < \cdots < \kappa^\omega_\zeta, \lambda^\varphi = \langle \langle \kappa^\varphi_{\xi+1}, \kappa^\varphi_\xi : \xi < \zeta(\ast) \rangle \rangle \text{ for } \varphi \leq \omega.\]

1) If \(\text{Pr}_n(\lambda^\varphi)\) for \(n < \omega\), and for some \(\theta \leq \kappa^0_\zeta\) there is a tree \(\mathcal{T} \in \theta^>(\kappa^0_\zeta)\) of cardinality \(\leq \kappa^0_\zeta\) with \(\geq \kappa^\omega_\zeta\) \(\theta\)-branches then:

\[\forall \text{ every first order sentence which has a } \lambda^\varphi\text{-model for each } n, \text{ also has a } \lambda^\omega\text{-model}\]

\[\forall' \text{ moreover, if } T \text{ is a first order theory of cardinality } \leq \kappa^0_\zeta \text{ and every finite } T' \subseteq T \text{ has a } \lambda^\varphi\text{-model for each } n \text{ then } T \text{ has a } \lambda^\omega\text{-model.}\]

2) So if \(\lambda^\varphi = \bar{\lambda}\) for \(\varphi \leq \omega\) are as above then we have \(\kappa^0_\zeta\)-compactness for the class of \(\lambda^\omega\)-models. Where

\[\forall \text{ a class } \mathcal{R} \text{ of models is } \kappa\text{-compact when for every set } T \text{ of } \leq \kappa \text{ first order sentences, if every finite subset of } T \text{ has a model in } \mathcal{R} \text{ then } T \text{ has a model in } \mathcal{R}.\]

3) In part (1) we can use \(\bar{\lambda}^\varphi\) with domain \(\omega_n\), if \(\omega_n \subseteq \omega_{n+1}\) and \(\zeta(\ast) = \cup \{ w_n : n < \omega \}\).

\[\tag{2.10}\]

\[\text{Proof. Straight if you have read [Sh:8], [Sh:18], [Sh:37] or read the proof of 2.12 below (only that now the theory is not necessary countable, no types omitted, and by compactness it is enough to deal with the case } \zeta(\ast) \text{ is finite).}\]

\[\tag{2.10}\]

\[\text{Claim 2.12. Let } \zeta(\ast) < \omega_1, \lambda^\varphi = \langle \langle \kappa^\varphi_{\xi+1}, \kappa^\varphi_\xi : \xi < \zeta(\ast) \rangle \rangle \text{ for each } \xi < \omega_1 \text{ (and } \kappa^\varphi_\xi \text{ strictly increasing with } \xi). \text{ Pr}_n(\lambda^\varphi) \text{ for each } \varphi \leq \omega_1 \text{ and } \kappa^\omega_1 \leq 2^{\omega_0} \text{ and } \psi \in L_{\omega_1,\omega} \text{ and for each } \varphi \leq \omega_1 \text{ there is a } \lambda^\varphi\text{-model satisfying } \psi \text{ then there is a } \lambda^\omega\text{-model satisfying } \psi.\]

\[\text{Proof. For simplicity, again like [Sh:8], [Sh:18], [Sh:37]. Let } M^\varphi \text{ be a } \lambda^\varphi\text{-model of } \psi \text{ for } \varphi < \omega_1. \text{ By expanding the } M^\varphi\text{'s, by a pairing function and giving names of subformulas of } \psi \text{ we have a countable first order theory } T \text{ with Skolem functions, a countable set } \Gamma \text{ of } 1\text{-types and } M^\varphi_\Gamma \text{ such that:}\]

\[(a) \ M^\varphi_\Gamma \text{ is a } \lambda^\varphi\text{-model of } T \text{ omitting each } p \in \Gamma\]

\[(b) \text{ if } M \text{ is a model of } T \text{ omitting every } p \in \Gamma \text{ then } M \text{ is a model of } \psi.\]

\[\tag{2.1}\]

Now as in the proof of 2.1 we can find a model \(M^\varphi\) and \(a^\varphi_\zeta\) for \(\zeta < \zeta(\ast), \eta \in \omega_2\) such that

\[(a) \ M^\varphi \text{ a model of } T\]

\[(b) \ a^\varphi_\eta \in R^M_\varphi \text{ and } \eta \neq \nu \Rightarrow a^\varphi_\eta \neq a^\varphi_\nu\]

\[(c) \text{ for every first order formula } \varphi(x_0, \ldots, x_{n-1}) \in L_{\omega,\omega}(\tau(T)) \text{ and } \zeta(0), \ldots, \zeta(n-1) < \zeta(\ast) \text{ ordinals, there is } k < \omega \text{ such that: if } \eta_0, \ldots, \eta_{n-1} \in \omega_2, \nu_0, \ldots, \nu_{n-1} \in \omega_2 \text{ and } (\eta_\ell \upharpoonright k : \ell < n) \text{ is with no repetitions and } \eta_\ell \upharpoonright k = \nu_\ell \upharpoonright k \text{ then } M^\varphi \models \varphi(a^\zeta_\eta_0, \ldots, a^\zeta_{\eta(n-1)}) \equiv \varphi(a^\zeta_\eta_0, \ldots, a^\zeta_{\eta(n-1)})\]
(δ) if σ(x₀, . . . , xₜ₋₁) is a term of τ(T) and ζ(0), . . . , ζ(n − 1) ≤ ζ(*) and p ∈ Γ
then for some k < ω for any p₀, . . . , p(n − 1) ∈ ℱ(2) pairwise distinct there is
\bar{φ}(x) ∈ p(x) such that:

(*) ρ ∈ η ∈ ω² ⇒ M⁺ \models \neg \bar{φ}(a⁰ₜ, . . . , aⁿₜ₋₁)

(i.e. this is our way to omit the types in Γ)

(φ) if ζ(0), . . . , ζ(n − 1) < ζ(*), σ(x₀, . . . , xₜ₋₁) is a term of τ(T), and m < n, then
for some k < ω, we have

(*) if η₀, . . . , ηₜ₋₁ ∈ ω², and ν₀, . . . , νₜ₋₁ ∈ ω² and ηₜ ↾ k = νₜ ↾ k and
\langle ηₜ ↾ k : ℓ < ω \rangle is without repetitions and ζ(ℓ) < ζ(m) ⇒ ηₜ = νₜ then
M⁺ \models Qζ(νₜ(k)(a⁰ₜ, . . . , aₜ₋₁(n−1))) and Qζ(m)(σ(a⁰ₜ(0), . . . , aₜ₋₁(n−1)) ⇒
σ(a⁰ₜ(0), . . . , aₜ₋₁(n−1)) = σ(a⁰ₜ(0), . . . , aₜ₋₁(n−1))).

Now choose Y₂ ⊆ ω² of cardinality λ₂⁺ and let M* be the τ(M)-reduce of the
Skolem hull in M⁺ of \{ a₀ₕ : ζ < ζ(*) and η ∈ Y₂ \}. This is a model as required. □

**Conclusion 2.13.** If V₀ \models GCH, V = V₀ for some c.c.c. forcing notion P
then e.g.

(+) if \langle β, \tau, \sigma \rangle < 2^{β₀}, \langle α₀, α₁, α₂, α₃ ⟩ ⇒ \langle β, \tau, \sigma \rangle (see [Sh:8]) i.e., letting
λ₁ = \langle (α₀, α₁), (α₁, α₂), (α₂, α₃) \rangle and
\lambda = \langle (α₀, α₁), (α₁, α₂), (α₂, α₃) \rangle,
for any countable first order T, if every finite T' ⊆ T has a \lambda⁺-model then
T has a \lambda⁺-model

(**) \langle κ_* : ξ ≤ ζ(*) \rangle \rightarrow \langle κ_* : ξ ≤ ζ(*) \rangle if \bigwedge \zeta_{κ_*} ≤ κ_{ξ+1} and κ₀ ≤ κ₁ ≤ \cdots ≤
κ_{ζ(*)} ≤ 2^{κ₀} (and versions like 2.11(1)).

**Proof.** By 2.10 if \lambda = \langle (κₐ, κ_* : ξ < ζ(*)) \rangle, λₐ ≥ \bigwedge \kappa(κₐ) then Prn(\lambda) (really
λₐ ≥ \bigwedge \kappa(κₐ) for k depending on n only suffices, see [EHMR84]). Now ccc forcing
preserves this and now apply 2.11. Similarly we can use \theta⁺-cc forcing P and deal
with cardinals in the interval (\theta, 2^\theta) in V^P. □

**Remark 2.14.** We can say parallel things for the compactness of (\exists λ), for λ singular
≤ 2^{κ₀} (or \theta + \kappa(T) < \lambda ≤ number of \theta-branched of \mathscr{T}, e.g. we get the parallel of
2.13.

In more details, if V₀ = V^P, P satisfies the \theta⁺-c.c. then

(*) in V^P, for any singular \lambda ∈ (\theta, 2^\theta) such that V₀ \models “\lambda is strong limit” we have

⊗ the class \{ (λ, <, Rₙ, . . . : ζ < θ) : Rₙ an n₂, relation, (λ, <) is \lambda-like \} of
models is \theta-compact, and we can axiomatize it.

There are also consistent counterexamples, see [Sh:532].

The point of proving (*) is:

⊗ for a vocabulary \tau of cardinality ≤ \theta, letting T^k be a first order theory
with Skolem functions τ(T^k) (but the Skolem functions are new), then
TFAE for a first order T ⊆ I_{\omega, \theta}(\tau)

(a) T has a \lambda-like model
(b) the following is consistent: $T \cup T^k \cup \{ \sigma(\ldots, x_{n(\ell)}^{n(\ell)}, y_{n(\ell)}\ldots)_{\ell<k} = \\
\sigma(\ldots, x_{n(\ell)}^{n(\ell)}, y_{n(\ell)}\ldots)_{\ell<k} \lor \sigma(\ldots, x_{n(\ell)}^{n(\ell)}, y_{n(\ell)}\ldots) > y_n : n(\ell) < \omega, \eta \in \omega^2,$ and for some $j < \omega$, $(\eta_{\ell} \restriction j : \ell < k)$ is with no repetition, $\eta_{\ell} \restriction j = \nu_{\ell} \restriction j, n(\ell) < n \Rightarrow \eta_{\ell} = \nu_{\ell}$ $\cup \{ \sigma(\ldots, x_{n(\ell)}^{n(\ell)}, y_{n(\ell)}\ldots) < y_n : n(\ell) < n$ and $\eta_{\ell} \in \omega^2 \}.}$
§ 3. Finer Analysis of Square Existence

Definition 3.1. 1) For an $\omega$-sequence $\mathcal{T} = (T_n : n < \omega)$ of $(2,2)$-trees, we define a function $\text{degsq}(\cdot)$ (square degree).

Its domain is $\text{pfap} = \text{pfap}_\mathcal{T} = \{(u,g) : (\exists n)(u \in [\omega]^n, g$ is a 2-place function from $u$ to $\omega)$ and its values are ordinals (or $\infty$ or $-1$). For this we define the truth value of “$\text{degsq}(u,g) \geq \alpha$” by induction on the ordinal $\alpha$.

Case 1: $\alpha = -1$

$$\text{degsq}(u,g) \geq -1 \iff (u,g) \in \text{pfap}_\mathcal{T} \text{ and } \eta, \nu \in u \Rightarrow (\eta, \nu) \in T_g(\eta,\nu).$$

Case 2: $\alpha$ is limit

$$\text{degsq}(u,g) \geq \alpha \iff \text{degsq}(u,g) \geq \beta \text{ for every } \beta < \alpha.$$ 

Case 3: $\alpha = \beta + 1$

$$\text{degsq}(u,g) \geq \alpha \iff \text{for every } \rho^* \in u, \text{ for some } m, u^* \subseteq [m]2, g^* \text{ and functions } h_0, h_1, \text{ we have:}$$

$$h_i : u \rightarrow u^*;$$

$$(\forall \eta \in u)\rho < h_1(\eta)$$

$$(\forall \eta \in u)[h_0(\eta) = h_1(\eta) \Rightarrow \eta \neq \rho^*]$$

$$u^* = \text{Rang}(h_0) \cup \text{Rang}(h_1)$$

$g^*$ is a 2-place function from $u^*$ to $\omega$

$$g^*(h_i(\eta), h_i(\nu)) = g(\eta, \nu) \text{ for } i < 2 \text{ and } \eta, \nu \in u$$

$$\text{degsq}(u^*, g^*) \geq \beta \text{ (so } (u^*, g^*)\alpha \in \text{pfap}).$$

2) We define $\text{degsq}(u,g) = \alpha \iff \text{for every ordinal } \beta, \text{degsq}(u,g) \geq \beta \Leftrightarrow \beta \leq \alpha \text{ (so } \alpha = -1, \alpha = \infty \text{ are legal values).}$

3) We define $\text{degsq}(\mathcal{T}) = \bigcup\{\text{degsq}(u,g) + 1 : (u,g) \in \text{pfap}_\mathcal{T}\}.$

Claim 3.2. Assume $\mathcal{T}$ is an $\omega$-sequence of $(2,2)$-trees.

1) For every $(u,g) \in \text{pfap}_\mathcal{T}, \text{degsq}(u,g)$ is an ordinal, $\infty$ or $-1$. Any automorphism $\text{F}$ of $(\omega^2, <)$ preserves this (it acts on $\mathcal{T}$ too, i.e.

$$\text{degsq}(u,g) = \text{degsq}(\mathcal{F}(T_n) : n < \omega)(\mathcal{F}(u), g \circ \mathcal{F}^{-1}).$$

2) $\text{degsq}(\mathcal{T}) = \infty \iff \text{degsq}(\mathcal{T}) \geq \omega_1 \iff \text{there is a perfect square contained in } \bigcup_{n < \omega} \text{lim}(T_n) \text{ iff for some ccc forcing notion } \mathcal{P}, \models \mathcal{P} \text{ " } \bigcup_{n < \omega} \text{lim}(T_n) \text{ contains a } \lambda^{\omega_1}(\mathcal{N}_0)\text{-square" (so those properties are absolute).}$

3) If $\text{degsq}(\mathcal{T}) = \alpha(\ast) < \omega_1$ then $\bigcup_{n < \omega} \text{lim}(T_n)$ contains no $\lambda^{\omega_1}(\mathcal{N}_0)\text{-square}.$

4) For each $\alpha(\ast) < \omega_1$ there is an $\omega$-sequence of $(2,2)$-trees $\mathcal{T} = (T_n : n < \omega)$ with $\text{degsq}(\mathcal{T}) = \alpha(\ast)$.

5) If $\mathcal{T} = (T_n : n < \omega)$ is a sequence of $(2,2)$-trees $\text{then the existence of an } \aleph_1\text{-square in } \bigcup_{n < \omega} \text{lim}(T_n) \text{ is absolute.}$

6) Moreover for $\alpha(\ast) < \omega_1$ we have: if $\mu < \lambda^{\omega_1}(\mathcal{N}_0), A, B$ disjoint subsets of $\omega^2 \times \omega^2$ of cardinality $\leq \mu$, then some c.c.c. forcing notion $\mathcal{P}$ adds $\mathcal{T}$ as in (4) (i.e. an $\omega$-sequence of $(2,2)$-trees $\mathcal{T} = (T_n : n < \omega)$ with $\text{degsq}(\mathcal{T}) = \alpha(\ast)$) such that: $A \subseteq \bigcup_{n < \omega} \text{lim}(f(T_n)), B \cap \bigcup_{n < \omega} \text{lim}(f(T_n)) = \emptyset.$

Proof. Easy to prove. E.g.
3) Let $\lambda = \lambda_{\alpha(\omega)+1}(\aleph_0)$ and assume $\{\eta_i : i < \lambda\} \subseteq \omega^2$, $[i < j \Rightarrow \eta_i \neq \eta_j]$ and $(\eta_i, \eta_j) \in \bigcup n \lim(T_n)$ and let $(\eta_i, \eta_j) \in \lim(T_{\eta_i, \eta_j})$. For $(u, f) \in \text{pfap}_T$, $u = \{\nu_0, \ldots, \nu_{k-1}\}$ (with no repetition, $<_{\alpha}$-increasing) let $R_{(u, f)} = \{\alpha_0, \ldots, \alpha_{k-1} : \alpha_0 < \lambda$ and $\nu_\ell \in \alpha_\ell$ for $\ell < k$ and $f(\nu_\ell, \nu_m) = g(\eta_\alpha, \eta_m)$ for $\ell, m < k\}$. Let $M = (\lambda, R_{(u, f)}, f, g) \in \text{pfap}_T$. Clearly if we have $\alpha_0, \ldots, \alpha_{k-1} < \lambda$ and $n$ such that $\langle \eta_\alpha \mid n : \ell < k \rangle$ is with no repetition, $g(\eta_{\alpha_1}, \eta_{\alpha_2}) = f(\eta_{\alpha_1} \mathbin{\cup} n, \eta_{\alpha_2} \mathbin{\cup} n)$ then $R_{(u, f)}(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ and we can then prove

$$\text{rk}(\{\alpha_0, \ldots, \alpha_{k-1}\}, M) \leq \text{degsq}(u, f)$$

(by induction on the left ordinal). But $M$ is a model with countable vocabulary and cardinality $\lambda = \lambda_{\alpha(\omega)+1}(\aleph_0)$. Hence by the definition of $\lambda_{\alpha(\omega)+1}$ we have $\text{rk}(M) \geq \alpha(\omega) + 1$, so $\alpha(\omega) + 1 \leq \text{rk}(M) \leq \text{degsq}(T) \leq \alpha(\omega)$ (by previous sentence, earlier sentence and a hypothesis respectively). Contradiction.

4) Let $W = \{\eta : \eta$ is a (strictly) decreasing sequence of ordinals, possibly empty\}.

We choose by induction on $i < \omega$, $n_i$ and an indexed set $\langle (u^i, f^i, \alpha^i_0) : x \in X_i \rangle$ such that:

(a) $n_i < \omega$, $n_0 = 0$, $n_i < n_{i+1}$
(b) $X_i$ finite including $\bigcup X_{j<i}$
(c) for $x \in X_i$, $u^i_x \subseteq n^2: f^i_x$ a two place function from $u^i_x$ to $\omega$ and $\alpha^i_x \in u^i_x$
(d) for some $x \in X_i$, $u^i_x = \{0_n\}$
(e) $h_i$ is a function from $X_i$ into $W$ and $h_i \subseteq h_{i+1}$
(f) $|X_0| = 1$, and $h_0$ is constantly $()$
(g) if $x \in X_i$ then: $\alpha^{i+1}_x = \alpha^i_x$ and the function $\nu \mapsto \nu \mathbin{\cup} n_i$ is one to one from $u^{i+1}_x$ onto $u^i_x$ and $\nu \in u^{i+1}_x \Rightarrow \nu \mathbin{\cup} n_i = 0_{n_i(n_{i+1})}$ and $\eta, \nu \in u^{i+1}_x \Rightarrow f^{i+1}_x(\eta, \nu) = f^i_x(\eta \mathbin{\cup} n_i, \nu \mathbin{\cup} n_i)$
(h) for some $x = x_i \in X_i$, $\beta = h_i \in w_{i+1} \cap \alpha^{i+1}_x$ and $\rho^\prime = \rho^i_x$ in $u^i_x$ and $\Upsilon_i \in W$ such that $h_i(x_i) \mathbin{\cup} \Upsilon_i$, we can find $y = y_i$ such that:

(a) $X_{i+1} = X_i \cup \{y_i\}$, $y_i \notin X_i$
(b) $\alpha^{i+1}_y = \beta$
(c) the function $\nu \mapsto \nu \mathbin{\cup} n_i$ is a function from $u^{i+1}_y$ onto $u^i_x$, almost one to one: $\rho^\prime$ has exactly two predecessors say $\rho^y_1, \rho^y_2$ and any other $\rho \in u^i_x \setminus \{\rho^y\}$ has exactly one predecessor in $u^{i+1}_y$
(d) for $\nu, \eta \in u^{i+1}_y$ if $(\nu, \eta) \neq (\rho^y_1, \rho^y_2)$ and $(\nu, \eta) \neq (\rho^y_2, \rho^y_2)$ then $f^{i+1}_y(\eta, \nu) = f^i_y(\eta \mathbin{\cup} n_i, \nu \mathbin{\cup} n_i)$
(e) $h_{i+1} = h_i \cup \{y_i, \Upsilon_i\}$
(f) if $x_1, x_2 \in X_i$ and $u^i_{x_1} \cap u^i_{x_2} \notin \emptyset$ then $u^i_{x_1} \cap u^i_{x_2} = \{0_n\}$
(g) if $x \in X_i$, $\beta = w_i \cap \alpha^i_x$, $\rho^\prime = u^i_x$ and $h(x) \mathbin{\cup} \Upsilon \in W$, then for some $j \in (i, \omega)$ we have $x_j = x$, $\beta_j = \beta$, $\rho^\prime \subseteq \rho^j_x$ and $\Upsilon_j = \Upsilon$
(h) the numbers $f^{i+1}_y(\rho^y_1, \rho^y_2)$, $f^{i+1}_y(\rho^y_2, \rho^y_2)$ are distinct and do not belong to $\bigcup \{\text{Rng}(f^i_x) : x \in X_i\}$.
There is no problem to carry the definition.
We then let

\[ T_n = \{(\eta, \nu) : \text{ for some } i < \omega \text{ and } x \in X_i \text{ and } \eta', \nu' \in u^i_x \text{ we have } f^i_2(\eta', \nu') = n \text{ and for some } \ell \leq n, \text{ we have } (\eta, \nu) = (\eta' \upharpoonright \ell, \nu' \upharpoonright \ell)\} \]

and \( \tilde{T} = (T_n : n < \omega) \). Now it is straight to compute the rank.

5) By the completeness theorem for \( L_{\omega_1, \omega}(Q) \) (see Keisler [Kei71])

6) By the proof of 1.13.

\( \Box \) \{1.10\}

now we turn to \( \kappa \)-Souslin sets.

**Definition 3.3.** Let \( T \) be a \((2, 2, \kappa)\)-tree. Let \( \text{set}(T) \) be the set of all pairs \( (u, f) \) such that \((\exists n = n(u, f)) [u \subseteq \omega^\kappa \text{ and } f : u \times u \to \omega \kappa \text{ and } \eta, \nu \in u \Rightarrow (\eta, \nu, f(\eta, \nu)) \in T] \).

We want to define \( \text{degs}_T(x) \) for \( x \in \text{set}(T) \). For this we define by induction on the ordinal \( \alpha \) when \( \text{degs}_T(x) \geq \alpha \).

**Case 1:** \( \alpha = -1 \)

\( \text{degs}_T(u, f) \geq \alpha \iff (u, f) \in \text{set}(T) \).

**Case 2:** \( \alpha \) limit

\( \text{degs}_T(u, f) \geq \alpha \iff \text{degs}_T(u, f) \geq \beta \text{ for every } \beta < \alpha \).

**Case 3:** \( \alpha = \beta + 1 \)

\( \text{degs}_T(u, f) \geq \alpha \iff \text{ for every } \eta^* \in u, \text{ for some } m > n(u, f) \) there are \((u^*, f^*) \in \text{set}(T) \text{ and functions } h_0, h_1 \text{ such that } \text{degs}_T(u^*, f^*) \geq \beta \) and:

(i) \( n(u^*, f^*) = m \)

(ii) \( h_i \) is a function from \( u \) to \( \omega^2 \)

(iii) \( \eta \upharpoonright h_i(\eta) \) for \( i < 2 \)

(iv) \( \eta \in u \text{ we have } h_0(\eta) \neq h_1(\eta) \iff \eta = \eta^* \)

(v) \( \text{ for } \eta \neq \eta^* \in u, \text{ for some } m > n(u, f) \) there are \( f(\eta, \eta_2) < f^*(h_i(\eta_1), h_i(\eta_2)) \)

(vi) \( \text{ for } \eta \in u^* \text{ we have } f(\eta[n], \eta[n]) < f^*(\eta, \eta) \)

Lastly, \( \text{degs}_T(u, f) = \alpha \iff \text{ for every ordinal } \beta \text{ we have } [\beta \leq \alpha \iff \text{degs}_T(u, f) \geq \beta] \).

Also let \( \text{degs}(T) = \text{degs}_T(\{\}, \{<, >\}) \).

**Claim 3.4.** 1) For a \((2, 2, \kappa)\)-tree \( T \), for \((u, f) \in \text{set}(T) \), \( \text{degs}_T(u, f) \) is an ordinal or infinity or \( = -1 \). And similarly \( \text{degs}(T) \). All are absolute. Also \( \text{degs}(T) \geq \kappa^+ \) implies \( \text{degs}(T) = \infty \) and similarly for \( \text{degs}_T(u, f) \).

2) \( \text{degs}(T) = \infty \iff \#^P \text{ “prj lim}(T) = \{ (\eta, \nu) \in \omega^{2 \times 2} : \text{ for some } \rho \in \omega^\kappa, \bigwedge_{n<\omega} (\eta[n], \nu[n], \rho[n]) \in T \} \) contains a perfect square” for every forcing notion \( P \) including a trivial one i.e. \( V^P = V \iff \#^P \text{ “prj lim}(T) \) contains a \( 2^\lambda \)-square” for the forcing notion \( P \) which is adding \( \lambda \) Cohen reals for \( \lambda = \lambda_\omega(\kappa) \) some \( \lambda \) for some \( P, \#^P \text{ “prj lim}(T) \) contains \( \lambda < (\aleph_0) \text{-square”} \).
3) If $\alpha(*) = \text{degsq}(T) < \kappa^+$, then $\text{prj lim}(T)$ does not contain a $\lambda_{\alpha(*)+1}(\kappa)$-square.

Proof. 1) Easy.
2) Assume $\text{degsq}(T) = \infty$, and note that $\alpha^* = \{\text{degsq}_T(u, f) : (u, f) \in \text{set}(T)\}$ and $\text{degsq}_T(u, f) < \infty \} \setminus \{\infty\}$ is an ordinal so $(u, f) \in \text{set}(T)$ and $\text{degsq}_T(u, f) \geq \alpha^* \Rightarrow \text{degsq}_T(u, f) = \infty$ (in fact any ordinal $\alpha \geq \sup \{\text{degsq}_T(u, f) + 1 : (u, f) \in \text{set}(T)\}$ will do). Let set $\infty(T) = \{(u, f) \in \text{set}(T) : \text{degsq}_T(u, f) = \infty\}$.

Now

$\ast_1$ there is $(u, f) \in \text{set}(T)$

$\ast_2$ for every $(u, f) \in \text{set}(T)$ and $\rho \in u$ we can find $(u^+, f^+) \in \text{set}(T)$

and for $e = 1, 2, h_e : u \to u^+$ such that $(\forall \eta \in u)(\eta \triangleleft h_\eta(\eta), (\forall \eta, \nu \in u)(f(\eta, \nu) \triangleleft f^+(h_\eta(\eta), h_\nu(\nu)), (\forall \eta \in u)[h_1(\eta) = h_2(\eta) \iff \eta = \rho]$. [Why? As degsq$(u, f) = \infty$ it is $\geq \alpha^* + 1$ so by the definition we can find $(u^+, f^+)$, $h_1, h_2$ as above but only with $\text{degsq}_T(u^+, f^+) \geq \alpha^*$, but this implies $\text{degsq}_T(u^+, f^+) = \infty$.]

$\ast_3$ for every $(u, f) \in \text{set}(T)$ with $u = \{\eta_\rho : \rho \in n^2\} \subseteq (n_i)2$ (no repetition) we can find $n_2 > n_1$ and $(u^+, f^+) \in \text{set}(T)$ with $u^+ = \{\eta_\rho : \rho \in n_i+2\} \subseteq (n_i)2$ (no repetitions) such that

(i) $\rho \in n_i+2 \Rightarrow \eta_{\rho_1|n} \triangleleft \eta_\rho$

(ii) for $\rho_1, \rho_2 \in n_i+2$, $\rho_1 \neq \rho_2 \mid n \Rightarrow f(\eta_{\rho_1|n}, \eta_{\rho_2|n}) \triangleleft f^+(\eta_{\rho_1}, \eta_{\rho_2})$

(iii) for $\rho \in n_i+2$, $f(\eta_{\rho|n}, \eta_\rho) \triangleleft f(\eta_\rho, \eta_\rho)$.

[Why? Repeat $\ast_2$ $2^n$ times.]

So we can find $\{\eta_\rho : \rho \in n^2\}, f_n$ by induction on $n$ such that $\{\eta_\rho : \rho \in n^2\}$ is with no repetition, $\text{degsq}_T(\{\eta_\rho : \rho \in n^2\}, f_n) = \infty$, and for each $n$ clauses (i), (ii), (iii) of $\ast_3$ hold, i.e. $\rho_1, \rho_2 \in n_i+2$, $\rho_1 \neq \rho_2 \mid n \Rightarrow f_n(\eta_{\rho_1|n}, \eta_{\rho_2|n}) \triangleleft f_{n+1}(\eta_{\rho_1}, \eta_{\rho_2})$ and for $\rho \in n_i+2$ we have $f_n(\eta_{\rho|n}, \eta_\rho) \triangleleft f_{n+1}(\eta_\rho, \eta_\rho)$ and of course $\{\eta_\rho : \rho \in n^2\} \subseteq (n_k)2$, $k_0 < k_{n+1} < \omega$.

So we have proved that the first clause implies the second (about the forcing: the degsq$(T) = \infty$ is absolute so holds also in $\mathbb{V}^\mathbb{P}$ for any forcing notion $\mathbb{P}$). Trivially the second clause implies the third and fourth.

So assume the third clause and we shall prove the first. By $1.8$ $\lambda_2^\kappa(*) (\kappa)$ is well defined (e.g. $\leq \Delta_\kappa$), but $\lambda_2^\kappa(*) = \lambda_2^\kappa(\kappa)$ by $1.5(3)$, let $\mathbb{P}$ be the forcing notion adding $\lambda_2^\kappa(*)$ Cohen reals. By $1.1(2)$ in $\mathbb{V}^\mathbb{P}$, $\kappa^\kappa(*) \leq \lambda_2^\kappa(*) \leq 2^{n_0}$, and so there are pairwise disjoint $\eta_i \in \omega^2$ for $i < \kappa_\kappa(*)$ such that $(\eta_i, \eta_j) \in \text{prj lim}(T)$ for $i, j < \kappa_\kappa(*)$.

Lastly, we prove forcing implies first in $\mathbb{V}^\mathbb{P}$. By part (3) of the claim proved below we get for every $\alpha < \kappa^+$, as $\kappa_\alpha(*) \leq \lambda_{\kappa(*)}(\kappa)$ that $\neg[\alpha = \text{degsq}(T)]$. Hence $\text{degsq}(T) \geq \kappa^+$, but by part (1) this implies $\text{degsq}(T) = \infty$.  

3) Just like the proof of $3.2(3)$.

\[\square\]

\{3.2\}

We shall prove in [Sh:532]

\{3.5\}

Claim 3.5. Assume $\alpha(*) < \kappa^+$ and $\lambda < \lambda_{\alpha(*)}(\kappa)$.

1) For some c.c.c. forcing notion $\mathbb{P}$, in $\mathbb{V}^\mathbb{P}$ there is a $\kappa$-Souslin subset $A = \text{prj lim}(T)$ (where $T$ is a $(2, 2, \kappa)$-tree) such that:

\[(*) \text{ A contains a } \lambda\text{-square but } \text{degsq}(T) \leq \alpha(*)\]
2) For given \( B \subseteq (\omega_2 \times \omega_2)^V \) of cardinality \( \leq \lambda \) we can replace (*) by

\[(\ast) \quad A \cap (\omega_2 \times \omega_2)^V = B \text{ but } \degq(T) \leq \alpha(\ast). \]

\begin{equation}
\text{(3.5A)}
\end{equation}

**Remark 3.6.** The following says in fact that “colouring of pairs is enough”: say for the Hanf number of \( L_{\omega_1, \omega} \) below the continuum, for clarification see 3.8.

\begin{equation}
\text{(3.6A)}
\end{equation}

**Claim 3.7.** [MA] Assume \( \lambda < 2^{\aleph_0} \) and \( \alpha(\ast) < \omega_1 \) is a limit ordinal, \( \lambda < \mu := \lambda_{\alpha(\ast)}(\aleph_0) \). Then for some symmetric 2-place function \( F \) from \( \lambda \) to \( \omega \) we have:

\( (\ast)_{\mu, F} \) for no two place (symmetric) function \( F' \) from \( \mu \) to \( \omega \) do we have:

\( (** \quad \text{for every } n < \omega, \text{ and pairwise distinct } \beta_0, \ldots, \beta_{n+1-1} < \mu \text{ there are pairwise distinct } \alpha_0, \ldots, \alpha_{n-1} < \lambda \text{ such that} \)

\[ \bigwedge_{k < \ell < n} F'(\beta_k, \beta_\ell) = F(\alpha_k, \alpha_\ell). \]

\begin{equation}
\text{(3.6A)}
\end{equation}

**Remark 3.8.** 1) This is close to Gilchrist Shelah [GeSh:491].

2) The proof of 3.7 says that letting \( R_n = \{ (\alpha, \beta) : F(\alpha, \beta) = n \} \), and \( N = (\lambda, R_n)_{n < \omega} \), we have \( \rk(N) < \alpha(\ast) \).

3) So 3.7 improves §2 by saying that the examples for the Hanf number of \( L_{\omega_1, \omega} \) below the continuum being large can be very simple, speaking only on “finite patterns” of colouring pairs by countably many colours.

**Proof.** Let \( M \) be a model of cardinality \( \lambda \) with a countable vocabulary and \( \rk^1(M) < \alpha(\ast) \). Without loss of generality it has the universe \( \lambda \), has the relation \( < \) and individual constants \( c_\alpha \) for \( \alpha \leq \omega \). Let \( k^M, \varphi^M \) be as in the proof of 1.13.

Let \( \mathbf{P} \) be the set of triples \((u, f, w)\) such that:

\begin{enumerate}
\item \( u \) is a finite subset of \( \lambda \)
\item \( f \) is a symmetric two place function from \( u \) to \( \omega \)
\item \( w \) is a family of nonempty subsets of \( u \)
\end{enumerate}

such that

\begin{enumerate}[\( (d) \)]
\item if \( \alpha \in u \) then \( \{ \alpha \} \in w \)
\item if \( w = \{ \alpha_0, \ldots, \alpha_{n-1} \} \subseteq w \) (the increasing enumeration), \( k = k^M(w), \)
\( \alpha \in u \setminus w \) and \( (\forall \ell)[\ell < n \wedge \ell \neq k \Rightarrow f(\alpha_\ell, \alpha) = f(\alpha_\ell, \alpha_k)], \) then \( w \cup \{ \alpha \} \)
\( \) belongs to \( w, (\forall m \neq k)(\alpha < \alpha_m \Leftrightarrow \alpha_k < \alpha_m) \) and \( k = k^M(w \cup \{ \alpha \}) \setminus \{ \alpha \} \)
\item and \( M \models \varphi^M(w)[\alpha_0, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_{n-1}] \)
\item if \( w^i = \{ \alpha_0^i, \ldots, \alpha^i_{n-1} \} \subseteq u \) (increasing enumeration, so with no repetition), \( i = 0, 1, \) and \( (\forall \ell < k < n)[f(\alpha^i_\ell, \alpha^i_k) = f(\alpha^i_\ell, \alpha^i_k)] \) then \( w^0 \in w \Leftrightarrow w^1 \in w \) and if \( w^i \in w \) then \( \varphi^M(w^0) = \varphi^M(w^1), k^M(w^0) = k^M(w^1) \) and \( \rk^1(w^0, M) = \rk^1(w^1, M). \)
\end{enumerate}

The order is the natural one.

It is easy to check that:

\( \oplus_1 \) \( \mathbf{P} \) satisfies the c.c.c.

\( \oplus_2 \) for every \( \alpha < \lambda, \mathcal{L}_\alpha = \{ (u, f, w) \in \mathbf{P} : \alpha \in u \} \) is dense.
Hence there is a directed graph $G \subseteq \mathcal{P}$ not disjoint from $\mathcal{L}_\alpha$ for every $\alpha < \lambda$. Let $F = \bigcup\{ f : \text{for some } u, w \text{ we have } (u, f, w) \in G \}$. We shall show that it is as required. Clearly $F$ is a symmetric two place function from $\lambda$ to $\omega$; so the only thing that can fail is if there is a symmetric two place function $F'$ from $\mu$ to $\omega$ such that $(\star\star)$ of 3.7 holds. By the compactness theorem for propositional logic, there is a linear order $<^*$ of $\mu$ such that

$(\star')$ for every $n < \omega$ and $\beta_0 <^* \cdots <^* \beta_{n-1}$ from $\mu$ there are $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$ such that $\bigwedge_{k < \ell < n} F'((\beta_k, \beta_\ell)) = F((\alpha_k, \alpha_\ell))$.

Let

$$W = \bigcup \{ w : \text{for some } u, f \text{ we have } (u, f, w) \in G \}. $$

Let $N = (\lambda, R_n)_{n<\omega}$ where $R_n = \{ (\alpha, \beta) : F(\alpha, \beta) = n \}$, so $rk^1(w, N)$ for $w \in [\lambda]^*$. \[\{3.1\}\]

We define a forcing $\mathcal{P}$ (again it can be proved by induction on $rk^1(w, M)$ using the same “witness” $k^M$).

Lastly

$$\bigwedge_{k < \ell < m} F'(\alpha_k, \alpha_\ell) = F(\beta_k, \beta_\ell) \text{ and } (\beta_0, \ldots, \beta_{m-1}) \in W, \text{ then } rk^1((\alpha_0, \ldots, \alpha_{m-1}), N) \leq rk^1((\beta_0, \ldots, \beta_{m-1}), N)$$

(again it can be proved by induction on $rk^1((\beta_0, \ldots, \beta_{m-1}), N)$, the choice of $N'$ and our assumption toward contradiction that $(\star\star)$ from the claim holds). Now by \[\{3.1\}\] (and \[\{3.2\}\]) we have $rk^1(N') \leq rk^1(N) < \alpha(\ast)$ but this contradicts $\|N'\| = \mu = \lambda_{\alpha(\ast)}(8_0)$. \[\{3.7\}\]

Claim 3.9. If $\lambda, \mu, F$ are as in 3.7 (i.e. $(\star)_{\lambda, \mu, F}$ holds) then some Borel set $B \subseteq \omega^2$ has a $\lambda$-square but no $\mu$-square.

\[\{3.7\}\]

Remark 3.10. 1) The converse holds too, of course.

2) Instead of $\lambda$ we can use “all $\lambda < \mu$”.

Proof. Without loss of generality of $(\lambda) > \aleph_0$ (otherwise combine $\omega$ examples). Let $F$ be a symmetric two place function from $\lambda$ to $\omega$ such that $(\star)_{\lambda, \mu, F}$. For simplicity let $f^* : \omega \to \omega$ be such that $\forall n \exists m f^*(m) = n$. We define a forcing notion $\mathcal{P}$ as in 1.13 except that we require in addition for $p \in \mathcal{P}$:

\[\{1.10\}\]
\( \otimes_1 f^*(g^p(\alpha, \beta)) = F(\alpha, \beta) \)

\( \otimes_2 \) If \( \alpha' \neq \beta', \alpha'' \neq \beta'' \) are from \( u^p \), \( k < \omega, \eta'_{\alpha', k} \mid k = \eta''_{\alpha', k} \neq \eta''_{\beta', k} \mid k = \eta''_{\beta', k} \) both not constantly 1 then \( F(\alpha', \beta') = F(\alpha'', \beta'') \)

\( \otimes_3 \) If \( \eta, \nu \in n[p] \), then for at most one \( m < \omega \), \( \eta, \nu \in n[p] \), then for some \( \mu < \lambda \), \( \exists \alpha, \beta \), such that

\( \bigwedge_{k < \ell < n} f^*(h(k, \ell)) = F(\alpha_k, \alpha_\ell) \).

We have

\( \otimes_1 \) if \( \alpha < \beta < \lambda \), there is a unique \( n < \omega \) such that \( (\eta_\alpha, \eta_\beta) \in \text{lim } T_n \).

Thus \( \bigcup_{n \in \omega} T_n \) contains a \( \lambda \)-square. In proving that it does not contain a \( \mu \)-square we apply \((*)_{\lambda, \mu, F}\). For this the crucial fact is:

\( \otimes_2 \) if \( n < \omega, \eta_0, \ldots, \eta_{n-1} \in \#2 \) are distinct, \( (\eta_k, \eta_\ell) \in \text{lim}(T_{h(k, \ell)}) \), then for some pairwise distinct \( \alpha_0, \ldots, \alpha_{n-1} < \lambda \),

\[ \bigwedge_{k < \ell < n} f^*(h(k, \ell)) = F(\alpha_k, \alpha_\ell). \]

Instead of “\( F^3 \) is dense” it is enough to show

\( \otimes_3 \) for some \( p \in P, p \vdash \text{“the number of } \alpha < \lambda \text{ such that for some } q \in G P \alpha \in u^q \text{ is } \lambda” \).

\( \square \)

\textbf{Claim 3.11 (MA)}. Assume \( \aleph_0 < \lambda < 2^{\aleph_0} \). Then the following are equivalent:

\( (a) \) For some symmetric 2-place functions \( F_\mu \) from \( \mu \) to \( \omega \) (for \( \mu < \lambda \)) we have

\( (*_{F_\mu, \mu < \lambda}) \) for no two place function \( F' \) from \( \lambda \) to \( \omega \) do we have:

\( (**) \) for every \( n < \omega \) and pairwise distinct \( \beta_0, \ldots, \beta_{n-1} < \lambda \) there are \( \mu < \lambda \) and pairwise distinct \( \alpha_0, \ldots, \alpha_{n-1} < \mu \) such that

\[ \bigwedge_{k < \ell < n} F'(\beta_k, \beta_\ell) = F_\mu(\alpha_k, \alpha_\ell). \]

\( (b) \) Some Borel subset of \( ^\omega 2 \times ^\omega 2 \) contains a \( \mu \)-square \( \text{iff } \mu < \lambda \) (in fact \( B \) is an \( F_{\sigma} \) set).
§ 4. Rectangles

Simpler than squares are rectangles: subsets of $\beth 2 \times \beth 2$ of the form $X_0 \times X_1$, so a pair of cardinals characterize them: $(\lambda_0, \lambda_1)$ where $\lambda_\ell = |X_\ell|$. So we would like to define ranks and cardinals which characterize their existence just as $\text{rk}^\ell(w, M; \kappa)$, $\lambda^*_\ell(\kappa), \text{degq}_\ell(-), \text{degq}_\ell(-)$ have done for squares. Though the demands are weaker, the formulation is more cumbersome: we have to have two "kinds" of variables one corresponding to $\lambda_0$ one to $\lambda_1$. So the models have two distinguished predicates, $R_0, R_1$ (corresponding to $X_0, X_1$ respectively) and in the definition of rank we connect only elements from distinct sides (in fact in §33944 we already concentrate on two place relations explaining not much is lost). This is very natural as except for inequality nothing connects two members of $X_0$ or two members of $X_1$.

\begin{enumerate}
\item \textbf{Definition 4.1.} We shall define $\text{Prk}^\ell_\omega(\lambda_1, \lambda_2; < \kappa, \theta_0, \theta_1), \text{rrk}^\ell((w_1, w_2), M, \lambda_1, \lambda_2; \kappa, \theta_0, \theta_1)$ as in 1.1 (but $w_\ell \in [R^M_\ell]^*$ and $|R^M_\ell| = \lambda_\ell$), replacing $\text{rk}$ by $\text{rrk}$ etc. Let $\lambda = (\lambda_1, \lambda_2)$, $\bar{w} = (w_1, w_2)$.
\begin{enumerate}
\item For $\ell < 6$, and cardinals $\bar{\lambda} = (\lambda_1, \lambda_2), \lambda_1, \lambda_2 \geq \kappa$ and $\bar{\theta} = (\theta_0, \theta_1), \theta_0 \leq \theta_1 < \lambda_1, \lambda_2$ and an ordinal $\alpha$ let $\text{Prk}^\ell_\omega(\lambda; < \kappa, \bar{\theta})$ mean that: for every model $M$ with vocabulary of cardinality $\leq \theta_0$, such that $\bigwedge_{i=1}^2 |R_i^M| = \lambda_i, R_i^M \cap R_j^M = 0, F^M$ is a two place function with range included in $\theta_1 = Q^M$, we have $\text{rrk}^\ell(M; < \kappa) \geq \alpha$ (defined below).

Let $\text{NP rk}^\ell_\omega(\lambda; < \kappa, \bar{\theta})$ be the negation. Instead of $< \kappa^+$ we may write $\kappa$; if $\kappa = \theta_0^+$ we may omit $\theta_0$; if $\theta_0 = \varnothing_0, \kappa = \aleph_1$, we may omit them. We may write $\theta_0, \theta_1$ instead $\bar{\theta} = (\theta_0, \theta_1)$ and similarly for $\bar{\lambda}$.

Lastly, we let $\lambda \text{rk}^\ell_\omega(\lambda; \lambda; < \kappa, \bar{\theta}) = \min\{\lambda : \text{Prk}^\ell_\omega(\lambda, \lambda; < \kappa, \bar{\theta})\}$.
\item For a model $M, \text{rrk}^\ell(\lambda; \lambda; < \kappa) = \sup\{\text{rk}^{\ell+}(\bar{w}, M; < \kappa) \mid 1 : \bar{w} = (w_1, w_2) \text{ where } w_i \subseteq R^M_i \text{ are finite non empty for } i = 1, 2 \text{ and } (\exists c \subseteq Q^M)(\forall a, b)[a \in w_1 \text{ and } b \in w_2 \Rightarrow F(a, b) = c]\}$ where $\text{rk}^\ell$ is as defined in part (3) below.
\item For a model $M$, and $\bar{w} \in [M]^\delta = \{\bar{w} : \bar{w} = (u_1, u_2), u_i \subseteq R^M_i \text{ are finite nonempty and } (\exists c)(\forall a, b)[a \in u_1 \text{ and } b \in u_2 \Rightarrow F(a, b) = c]\}$ we shall define the truth value of $\text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$ by induction on the ordinal $\alpha$ (for $\ell = 0, 1, \kappa$ can be omitted). If we write $w$ instead of $w_1, w_2$ we mean $w_1 = w \cap R^M_1, w_2 = w \cap R^M_2$ (here $R^M_1 \cap R^M_2 = 0$ helps).

Then we can note:
\begin{align*}
(*)_0 & \quad \alpha \leq \beta \text{ and } \text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \beta \Rightarrow \text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \alpha \\
(*)_1 & \quad \text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \delta (\delta \text{ limit}) \iff \bigwedge_{\alpha < \delta} \text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \alpha \\
(*)_2 & \quad \text{rrk}^\ell(\bar{w}, M; < \kappa) \geq 0 \iff \bar{w} \in [M]^\delta.
\end{align*}
\end{enumerate}
\end{enumerate}

So we can define $\text{rrk}^\ell(\bar{w}, M; < \kappa) = \alpha$ for the maximal $\alpha$ such that $\text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$, and $\infty$ if this holds for every $\alpha$ (and $-1$ if $\text{rrk}^\ell(\bar{w}, M; < \kappa) \notin \{\}$). Now the inductive definition of $\text{rrk}^\ell(\bar{w}, M; < \kappa) \geq \alpha$ was already done above for $\alpha = 0$ and $\alpha$ limit, so for $\alpha = \beta + 1$ we let
\begin{align*}
\text{Let } w = w_1 \cup w_2, & \quad n = |w|, \quad w = \{a_0, \ldots, a_{n-1}\}, \\
\text{we have: } & \quad \forall k < n \text{ and quantifier free formula } \phi(x_0, \ldots, x_{n-1}) = \bigwedge_{i<j} x_i \neq x_j \\
& \quad \bigwedge_{i < j} \{R_1(x_i) \land R_2(x_j) \land \phi_i,j(x_i, x_j) : R_1(a_i) \text{ and } R_2(a_j)\}.
\end{align*}

(In the vocabulary of \(M\)) for which \(M \models \phi[a_0, \ldots, a_{n-1}]\) we have:

**Case 1:** \(\ell = 1\).

There are \(a^0_m \in M\) for \(m < n, i < 2\) such that:

- (a) \(\text{rk}^k(\{a^i_m : i < 2, m < n\}, M ; < \kappa) \geq \beta\),
- (b) \(M \models \phi[a^i_0, \ldots, a^i_{n-1}]\) (for \(i = 1, 2\), so there is no repetition in \(a^i_0, \ldots, a^i_{n-1}\) and \(a^i_m \in R^M_i \iff a_m \in R^M_i\) for \(j = 1, 2\))
- (c) \(a^i_k \neq a^i_j\) but if \(m < n\) and \((a_m \in R^M_i \iff a_k \notin R^M_i)\) then \(a^0_m = a^0_k\)
- (d) if \(a_m \in R^M_1, a_m \in R^M_2\) then for any \(i, j \in \{1, 2\}\) we have \(F^M(a^i_m, a^j_m) = F^M(a^i_m, a^j_m)\).

**Case 2:** \(\ell = 0\)

As for \(l = 1\) but in addition:

- (c) \(\bigwedge_m a_m = a^0_m\).

**Case 3:** \(\ell = 3\)

The definition is like case 1 but \(i < \kappa\); i.e. there are \(a^i_m \in M\) for \(m < n, i < \kappa\) such that:

- (a) for \(i < j < \kappa\) we have \(\text{rk}^k(\{a^i_m, a^j_m : m < n\}, M ; < \kappa) \geq \beta\)
- (b) \(M \models \phi[a^i_0, \ldots, a^i_{n-1}]\) (for \(i < \kappa\); so there are no repetitions in \(a^i_0, \ldots, a^i_{n-1}\))
- (c) for \(i < j < \kappa, a^i_k \neq a^i_j\) but if \(m < n\) and \((a_m \in R^M_1 \iff a_k \notin R^M_1)\) then \(a^i_m = a^i_k\)
- (d) if \(a_m \in R^M_1, a_m \in R^M_2\) then for any \(i, j \in \{1, 2\}\) we have \(F^M(a^i_m, a^j_m) = F^M(a^i_m, a^j_m)\).

**Case 4:** \(\ell = 2\)

Like case 3 but in addition:

- (c) \(a_m = a^0_m\) for \(m < n\).

**Case 5:** \(\ell = 5\)

Like case 3 except that we replace clause (a) by:

- (a) for every function \(H\), \(\text{Dom}(H) = \kappa, |\text{Rang}(H)| < \kappa\) for some \(i < j < \kappa\) we have \(H(i) = H(j)\) and \(\text{rk}^k(\{a^i_m, a^j_m : m < n\}, M ; < \kappa) \geq \beta\).

**Case 6:** \(\ell = 4\)

Like case 4 using clause (a) instead (a).

- 4) For \(M\) as above and \(c \in Q^M\) we define \(\text{rk}^k(M, c ; < \kappa)\) as \(\sup \{\text{rk}^k(\bar{w}, M ; < \kappa) + 1 : \bar{w} \in [M]^0 \land (\forall a \in w_1)(\forall b \in w_2)[F(a, b) = c]\}\).
- 5) Let \(\text{Prd}^k_\alpha(\lambda, \kappa, \bar{\theta})\) mean \(\text{rk}^k(M, c ; < \kappa, \bar{\theta}) \geq \alpha\) for every \(M\) for some \(c \in M\) when \(M\) is such that \(|R^M_1| = \lambda_1, |R^M_2| = \lambda_2, |\tau(M)| \leq \theta_0, F^M : R^M_1 \times R^M_2 \rightarrow Q^M\),
\(|Q^M| = \theta_1\). Let NPrrd\(^\alpha\)(\(\lambda, \kappa, \theta\)) mean its negation and \(\lambda rhd\(^\alpha\)(\(\kappa, \theta\))\) be the minimal \(\lambda\) such that Prd\(^\alpha\)(\(\lambda, \lambda, \kappa, \theta\)).

\[\{4.1a\}\]

**Remark 4.2.** The reader may wonder why in addition to Prk we use the variant Prd. The point is that for the existence of the rectangle \(X_1 \times X_2\) with \(F \cap (X_1 \times X_2)\) constantly \(c^*\), this constant plays a special role. So in our main claim 4.6, to get a model as there, we need to choose it, one out of \(\theta_1\), but the other choices are out of \(\kappa\). So though the difference between the two variants is small (see 4.5 below) we actually prefer the Prd version.

\[\{4.2\}\]

**Claim 4.3.** The parallels of 1.2 (+statements in 1.1), also ??, 1.5(2), 1.6, 1.10 hold.

\[\{4.3\}\]

**Claim 4.4.**
1) If \(w_i \in [R^M]\) for \(i = 1, 2\) then \(\omega \times rkk\((w_1, w_2, M; \kappa) \geq rkk\((w_1 \cup w_2, M; \kappa)\).
2) If \(R_1^M = R_2^M\) (abuse of the notation) then \(rkk\(M; \kappa\) \geq rkk\(M; \kappa\).
3) If \(\lambda_1 = \lambda_2 = \lambda\) then Prd\((\lambda, \kappa) \Rightarrow Prd\((\lambda_1, \lambda_2; \kappa, \kappa)\).

\[\{4.3a\}\]

**Claim 4.5.** \(\lambda rhd\(^\alpha\)(\(\kappa, \theta\)) = \lambda rhd\(^\alpha\)(\(\kappa, \theta\))\) if \(\alpha\) is a successor ordinal or \(cf(\alpha) > \theta\).

\[\{4.4\}\]

**Claim 4.6.** Assume \(\kappa \leq \theta < \lambda_1, \lambda_2\). Then the following are equivalent:
1) Prd\(^\alpha\)(\(\lambda_1, \lambda_2; \kappa, \theta\).
2) Assume \(M\) is a model with a countable vocabulary, \(|R^M| = \lambda_\ell\) for \(\ell = 1, 2\), \(P^M = \kappa, Q^M = \theta\), and \(F^M\) a two-place function (really just \(F\)\((R_1^M \times R_2^M)\) interests us) and the range of \(F\)\((R_1^M \times R_2^M)\) is included in \(Q^M\) and \(G\) is a function from \([R^M] \times [R_2^M]^\ast\) to \(P^M\). Then we can find \(\tau(M)\)-models \(M_0, N\) and elements, \(c^*, a_\eta, b_\eta\) (for \(\eta \in \omega^2\)) such that:

(i) \(N\) is a model with the vocabulary of \(M\) (but functions may be interpreted as partial ones, i.e. as relations)
(ii) \(a_\eta \in N^M, b_\eta \in N^M\) are pairwise distinct and \(F^N(a_\eta, b_\eta) = c^*(\in N)\)
(iii) \(M_0\) countable, \(M_0 \subseteq M\), \(c^* \in Q^M, M_0\) is the closure of \((M_0 \cap P^M) \cup \{c^*\}\) in \(M\), in fact for some \(M_0' \prec M\) we have \(M_0 = closure of P^M_0 \cup \{c^*\}\), \(c^* \in M_0'\)
(iv) \(M_0 \subseteq N, P^M_0 = P^N\)
(v) \(|N| = \{\sigma(a_\eta, b_\eta, d) : \sigma\ is a \(\tau(M)\)-term, \(\eta \in \omega^2, \nu \in \omega^2 and d \subseteq M_0\}\)
(vi) for \(\{\eta_\ell : \ell < \ell(*)\}, \{\nu_m : m < m(*)\} \subseteq \omega^2\) (both without repetitions non-empty) there is \(d^\ell \in P^M_0\) such that if \(d \subseteq P^M_0\) and quantifier free formulas \(\varphi_{\ell, m}\) are such that \(N \models \bigwedge_{\ell < \ell(*), m < m(*)} \varphi_{\ell, m}[a_\eta, b_\eta, d]\), then for some \(\{a_\ell : \ell < \ell(*)\}, \{b_\ell : \ell < \ell(*)\}\) we have \(M \models \bigwedge_{\ell < \ell(*)} \varphi_{\ell, m}[a_\ell, b_\ell, d]\) and \(G\((a_\ell : \ell < \ell(*)\), \(\{b_\ell : m < m(*)\}\) = d^\ell\)
(vii) for every quantifier free first order formula \(\varphi = \varphi(x, y, z_0, \ldots) \in L(\tau(M))\) and \(d_0, \ldots, d_0_i \in M_0\) there is \(k < \omega\) such that: for every \(\eta_1, \eta_2, \nu_1, \nu_2 \in \omega^2\) such that \(\eta_1 \models k = \eta_2 \models k, \nu_1 \models k = \nu_2 \models k\) we have
\[N \models \varphi[a_{\eta_1}, b_{\nu_1}, d_1, \ldots] = \varphi[a_{\eta_2}, b_{\nu_2}, d_1, \ldots].\]

Moreover
Proof. \((B)^- \Rightarrow (A)\).

Toward contradiction assume \(\text{NPrrd}_{\alpha^+}(\lambda_1, \lambda_2; \kappa)\) hence there is a model \(M'\) witnessing it, so \(\mathcal{V}(M') \leq \kappa\). So \(c \in QM' \Rightarrow \text{rk}(M', c; \kappa) < \kappa^+\) (note that Prrd was defined by cases of \(\text{rk}(M, c, \kappa)\)).

Let \(\{\varphi_i(x, y) : i < \kappa\}\) list the quantifier free formulas in \(L_{\omega, \omega}(\mathcal{V}(M'))\) with free variables \(x, y\). Let \(\{u_i : i < \kappa\}\) list the finite subsets of \(\kappa\). For \(c \in QM'\) and \(a_0, \ldots, a_{\ell(i)-1} \in R_1^M, b_0, \ldots, b_{\ell(m)-1} \in R_2^M\) \((\bar{a} = (a_0, \ldots, a_{\ell(i)-1}), \bar{b} = (b_0, \ldots, b_{\ell(m)-1})\) and for notation let \(a_m + n = b_0\)

\[\alpha_{c, \bar{a}, \bar{b}} = \text{rk}^1(\{a_0, \ldots, a_{\ell(i)-1}\}, \{b_0, \ldots, b_{\ell(m)-1}\}, M', c; \kappa),\]

and \(k_{c, \bar{a}, \bar{b}}, \varphi_{c, \bar{a}, \bar{b}}\) be witnesses for \(\text{rk}^1(\{\bar{a}, \bar{b}\}, M', c; \kappa, \theta) \neq \alpha_{c, \bar{a}, \bar{b}} + 1\). Let \(i(c, \bar{a}, \bar{b}) < \kappa\) be such that \(\varphi_{c, \bar{a}, \bar{b}}\) is a conjunction of formulas of the form \(\varphi_j(x, y_m)\) for \(j \in u_{i(c, \bar{a}, \bar{b})} \).

We define \(M\): the universe is \(\{M'\}\), the function \(F^{M'}\), relations \(R_1^{M'}, R_2^{M'}, Q^{M'}, P^{M'}\), the pairing function on ordinals,

\[R_n = \{(i, a, b) : a \in R_1^M, b \in R_2^M \text{ and if } |u_i| > n \text{ then } M \models \varphi_j[a, b] \text{ where } j \text{ is the } n\text{-th member of } u_i\}\]

and let \(H_c\) be one to one from \(\omega \times \text{rk}(M', c; \kappa, \theta) \times \kappa\) into \(\kappa\); we define the function \(G : G_{c, \bar{a}, \bar{b}} = H(G_{c, \bar{a}, \bar{b}}, G_{c, \bar{a}, \bar{b}}, G_{c, \bar{a}, \bar{b}}) = H(k_{c, \bar{a}, \bar{b}}, \alpha_{c, \bar{a}, \bar{b}}, i_{c, \bar{a}, \bar{b}})\).

Now we can apply statement \((B)\)-of 4.6 which we are assuming and get \(M_0, N, c^*, \alpha_{\eta, \bar{a}, \bar{b}}\) (for \(\eta \in \omega\)) satisfying clauses \((i)\)---\((vii)\) there. So \(c^* \in M_0 \subseteq M' \cap N\), so \(\beta' = \text{rk}(M', c; \kappa)\) satisfies \(\beta' < \infty\), even \(< \kappa^*\). Clearly \(\text{rk}^1(N, c^*; \kappa) \leq \beta'\).

Consider all sequences \(\langle \langle \nu_m : m < \ell(\bar{s}) \rangle, \nu_m : m < m(\bar{s}) \rangle, d_1, d_2, \varphi_{\bar{s}, \bar{m}} : \ell < \ell(\bar{s}), m < m(\bar{s}) \rangle, \langle a_{\bar{s}} : \ell < \ell(\bar{s}), m < m(\bar{s}) \rangle, \langle b_{\bar{s}} : m < m(\bar{s}) \rangle \rangle\) which are as in clause \((vi)\) of \((B)\).

Among those tuples choose one with \(\alpha^* = \text{rk}^1(\{a_{\bar{s}} : \ell < \ell(\bar{s})\}, \{b_{\bar{s}} : m < m(\bar{s})\}, N, c^*; \kappa)\) minimal. Let this rank not being \(\geq \alpha^* + 1\) be exemplified by \(\varphi\) and \(k < \ell(\bar{s}) + m(\bar{s})\), so by symmetry without loss of generality \(k < \ell(\bar{s})\).

Choose \(k^* < \omega\) large enough for clause \((vii)\) of \((B)\) for all formulas \(\varphi(x, y)\) appearing in \(\{\varphi_j(x, y) : j \in u_{i_{c, \bar{a}, \bar{b}}}\}\) where \(i_{c, \bar{a}, \bar{b}} = G_{c, \bar{a}, \bar{b}}(\bar{a}, \bar{b})\) and \(\eta \cup k^* : \ell < \ell(\bar{s}), m < m(\bar{s})\) and with no repetition. Choose \(\eta_{\ell(s)} \in 2^\omega \setminus \{\eta_k\}\) such that \(\eta_{\ell(s)} \cup k^* : m < m(\bar{s})\) and \(\eta' = \langle \nu_m : m < m(\bar{s}) \rangle, \varphi_{\bar{s}, m}(x_{\bar{s}, m}, x_{\bar{m}_s}, d_1, \bar{d}) \rangle\).

By the choice of \(\varphi\), these clearly satisfy \(\text{rk}^1(\{a'_{\bar{s}} : \ell < \ell(\bar{s})\}, \{b'_{\bar{m}_s} : m < m(\bar{s})\}, N, c^*; \kappa, \bar{d}) < \text{rk}^1(\{a_{\bar{s}} : \ell < \ell(\bar{s})\}, \{b_{\bar{m}_s} : m < m(\bar{s})\}, N, c^*; \kappa, \bar{d}) = \alpha^*\), but by this we easily contradict the choice of \(\alpha^*\) as minimal.

\((A) \Rightarrow (B)\)
As in the proof of 2.5, 2.1 (choosing a fixed c).

(B) ⇒ (B')

Trivial.

Discussion 4.7. 1) When applying 4.6 (1) ⇒ (2), or 2.5 we can use M which is an expansion of \( (\mathcal{H}(\chi), \in, <^\ast) \) by Skolem functions, \( P^M = \kappa, \chi \) large enough, so for \( \eta, \nu \in \mu^2, \mathcal{N}_{<^\mu} \) is a model of ZFC, not well founded but with standard \( \omega \) and more: its \( \{ i : i < \kappa \} \) is a part of the true \( \kappa \). In [Sh:532] we will have as in 2.1 \( M_0 \subseteq N, M_0 < M, M_0 < N_{\eta,\nu} \), and if \( \eta, \nu \models "\varphi(\eta, \nu), \nu \) are (essentially) in \( \mu^2, \varphi \in M_0 \) a \( \kappa \)-Souslin relation", then \( V \models \varphi(\eta, \nu) \).

2) We can give a rank to subsets of \( \lambda_1 \times \lambda_2 \) and have parallel theorems.

Claim 4.8. If \( \varphi \subseteq \mu^2 \times \nu^2 \) is \( \bigvee_{i < \theta} \varphi_i \), each \( \varphi_i \) is \( \kappa \)-Souslin, \( \varphi \) contains a \( (\lambda_1, \lambda_2) \)-rectangle, and \( \text{Pr} \mu_{<^\ast+1}(\lambda_1, \lambda_2; \theta) \), then \( \varphi \) contains a perfect rectangle.

Proof. Let \( \varphi_i(\eta, \nu) = (\exists \rho)((\eta, \nu, \rho) \in \text{lim}(T_\theta)) \) where \( T_\theta \) is a \( (2, 2, \kappa) \)-tree. Let \( M \) be \( (\mathcal{H}(\chi), \in, <^\ast, T_\theta, T_1, T_2, \mathcal{R}_1, \mathcal{R}_2, Q, n, \mathcal{N}_{\rho}) \) expanded by Skolem functions, where \( Q^M = \theta \) and choosing \( \eta, \nu \in \mu \) pairwise distinct, \( \nu_j \in \mu \) for \( j < \lambda \) pairwise distinct, \( R^M_1 = \{ \eta : i < \lambda_1 \} \), \( R^M_2 = \{ \nu : j < \lambda_2 \} \) and let \( T, h \) be functions such that \( (\eta, \nu, \varphi(\eta, \nu)) \in \text{lim}(T_{\eta,\nu}) \) for \( \lambda, \eta, \nu \). Let \( \mathcal{N}, M, c, a_\eta \) (for \( \eta \in \mu^2 \), \( b_\eta \) (for \( \eta \in \mu \)) be as in clause (B) of 4.6. Now \( M \) has elimination of quantifiers, so there are quantifier free formulas \( \varphi_\mu(x) \) saying (in \( M \)) that \( x \in R_\theta \) and \( x(y) = 1 \), and \( H_n(x, y) \) such that \( x \in R^M_\theta \), each \( (\eta, \nu) \in \mu^2 \), \( \varphi(\eta, \nu) \in \text{lim}(M) \).

Now \( N : = \{ \sigma^1_\eta : \eta \in \mu^2 \} \), \( B : = \{ \sigma^2_\eta : \eta \in \mu \} \) are perfect and for \( \eta, \nu \in \mu \), \( \sigma^1_\eta, \sigma^2_\eta \in \text{lim}(M) \) hence \( A \times B \) is a perfect rectangle inside \( \text{prj}(\text{lim}(T)) \).

Fact 4.9. 1) Assume that \( \varphi \subseteq \mu^2 \times \nu^2 \) is \( \theta_1 \)-Souslin, \( \kappa < \theta_1 \), \( \theta = \text{cf} (\mathcal{S}_{\leq \kappa}(\theta_1), \subseteq) \). Then \( \varphi \) can be represented as \( \bigvee_{i < \kappa^+} \varphi_i \), each \( \varphi_i \) is \( \kappa \)-Borel (i.e. can be obtained from clopen sets by unions and intersections of size \( \leq \kappa \)).

Proof. 1) Easy. 2) If \( \varphi \) is co-\( \kappa \)-Souslin, then it can be represented as \( \bigvee_{i < \kappa^+} \varphi_i \), each \( \varphi_i \) is \( \kappa \)-Borel.

Conclusion 4.10. If \( \varphi \) is an \( \mathcal{N}_\kappa \)-Souslin subset of \( \mu^2 \times \nu^2 \) containing a \( (\mathcal{L}_{\omega_1}(\mathcal{N}_\kappa), \mathcal{L}_{\omega_1}(\mathcal{N}_\kappa)) \)-rectangle, then it contains a perfect rectangle. (Note: \( \mathcal{N}_\kappa \) can replaced by \( \kappa \) if \( \text{cf}(\mathcal{S}_{\leq \kappa}(\kappa), \subseteq) = \kappa \), e.g. \( \mathcal{N}_\kappa \), by [Sh:g, Ch.IX, §4].)

Conclusion 4.11. 1) For \( \ell < 6 \), \( \text{Pr} \mu_{\mathcal{P}_\kappa}(\kappa^+, (2^\kappa)^+; \kappa) \).

2) If \( \mathcal{V} = \mathcal{V}_\kappa^\mathcal{P} \), \( \mathcal{P} \models \text{c.c.c.} \) and \( \mathcal{V}_\kappa \models \text{GCH} \), then \( \text{Pr} \mathcal{P}_{\mathcal{N}_\kappa}(\mathcal{N}_\kappa, \mathcal{N}_\kappa) \).

Proof. 1) For a model \( M \), letting \( (\lambda_1, \lambda_2) = (\kappa^+, (2^\kappa)^+) \), choose for \( m = 1, 2 \) \( \bar{a}_m^2 \) a nonempty sequence from \( R^M_\kappa \) for \( i < \lambda_m, \{ \bar{a}_m^i : i < \lambda_m \} \) pairwise disjoint. For
(i, j) ∈ λ1 × λ2 let βi,j = rkk(a1, a2, M) with witnesses $k^M(a1, a2, M)$ for $\neg rkk^1(a1, a2, M) > βi,j$. Similarly, there is $B_1 ⊆ \lambda_1$, $|B_1| = \lambda_1(= \kappa^+)$ such that for $j = \min(B_2)$ the values

$$k^M(a1, a2, M), \varphi^M(a1, a2)$$

are the same for all $i ∈ B_1$; but they do not depend on $j ∈ B_2$ either. So for $(i, j) ∈ B_1 × B_2$ we have $k^M(a1, a2) = k^*$, $\varphi^M(a1, a2) = \varphi^*$. Let $k^*$ “speak” on $a1$, for definiteness only. Choose distinct $i_\zeta$ in $B_1$ (for $\zeta < \kappa^+$). Without loss of generality $rkk^1(a1, a2, M) ≤ rkk^1(a1, a2, M)$.

Now $a1$ give contradiction to $rkk^1(a1, a2, M) \nleq \betai,j$.

2) This can be proved directly (or see [Sh:532] through preservation by c.c.c. forcing notion of rank which are relations of rkc similarly to 1.10.

Remark 4.12. 1) If $T$ is an $(\omega, \omega)$-tree and $A × B ⊆ \lim(T)$, with $A, B ⊆ \omega$ uncountable (or just not scattered) then $\lim(T)$ contains a perfect rectangle. Instead $\lim(T)$ (i.e. a closed set) we can use countable intersection of open sets. The proof is just like 1.17.

2) We can define a rank for $(2, 2, \kappa)$-trees measuring whether $\proj \lim(T) \subseteq \omega_2 × \omega_2$ contains a perfect rectangle, and similarly for $(\omega, \omega)$-tree $T$ measuring whether $\lim(T)$ contains a perfect rectangle. We then have theorems parallel to those of §1. See below and in [Sh:532].

* * *

The use of $\omega$ below is just notational change.

Definition 4.13. For $T$ a $(\omega, \omega)$-tree we define a function $degc_T$ (rectangle degree). Its domain is $rcr(T): = \{(u_1, u_2): \text{for some } \ell < \omega, u_1, u_2 \text{ are finite nonempty subsets of } \omega \text{ and } g \text{ a function from } u_1 \times u_2 \text{ to } \omega \text{ such that } (\eta_0, \eta_1) ∈ T \text{ for } \eta_i ∈ u_i\}$. Its value is an ordinal $degc_T(u_1, u_2)$ (or $-1$ or $\infty$). For this we define the truth value of $degc_T(u_1, u_2) ≥ \alpha$ by induction on the ordinal $\alpha$.

Case 1: $\alpha = -1$

$degc_T(u_1, u_2) ≥ -1$ if $(u_1, u_2)$ is in $rcr(T)$.

Case 2: $\alpha$ limit

$degc_T(u_1, u_2) ≥ \alpha$ if $degc_T(u_1, u_2) ≥ \beta$ for every $\beta < \alpha$.

Case 3: $\alpha = \beta + 1$

$degc_T(u_1, u_2) ≥ \alpha$ if $\ell \in \{1, 2\}$, $\eta^* ∈ u_k$ we can find $\ell(\eta^*) < \omega$, and functions $h_0, h_1$ such that: Dom $(h_1) = u_1 \cup u_2$, $\eta \in u_1 \cup u_2 ⇒ \eta \notin h_i(\eta) ∈ \ell(\eta^*) \omega$ such that $h_0(\eta^*) \neq h_1(\eta^*)$, $\eta \in u_k ⇒ h_0(\eta) = h_1(\eta)$ and letting $u_i^1 = \text{Rang}(h_0[u_i] \cup \text{Rang}(h_1[u_i])$ we have $degc_T(u_0, u_1^1) ≥ \beta$.

Lastly define: $degc_T(u_0, u_1) = \alpha$ if $\bigwedge_\beta[degc_T(u_0, u_1) ≥ \beta ⇔ \alpha ≥ \beta]$ (\alpha an ordinal or $\infty$).
Also \( \text{degrc}(T) = \text{degrc}_T(\{\}, \{\}) \).

\[ \tag{4.11} \]

**Claim 4.14.** Assume \( T \) is in an \((\omega, \omega)\)-tree.

1. For every \((u_0, u_1) \in \text{rcpr}(T), \) \( \text{degcr}(u_0, u_1) \) is an ordinal or \( \infty \) or \(-1\); if \( f \) is an automorphism of \((\omega^\varnothing, \varnothing)\), then \( \text{degcr}_f(u_0, u_1) = \text{degcr}_f(f(u_0), f(u_1)) \).

2. \( \text{drc}(T) = \infty \) if \( f \) there is a perfect rectangle in \( T \) iff \( \text{degcr}_T(u_0, u_1) \geq \omega_1 \) for some \((u_0, u_1)\) (so those statements are absolute).

3. If \( \text{drc}(T) = \alpha(\ast) < \omega_1 \), then \( \text{lim}(T) \) contains no \( (\text{Arc}_{\alpha(\ast) + 1}(\aleph_0), \text{Arc}_{\alpha(\ast) + 1}(\aleph_0)) \)-rectangle.

4. If \( T = (T_n : n < \omega) \) is a sequence of \((\omega, \omega)\)-trees and \( \text{degcr}(T_n) \leq \alpha(\ast) \), and \( A = \bigcup_{n < \omega} \text{lim}(T_n) \) then \( A \) contains no \( (\text{Arc}_{\alpha(\ast) + 1}(\aleph_0), \text{Arc}_{\alpha(\ast) + 1}(\aleph_0)) \)-rectangle.

5. In part 4. we can replace \( \omega \) by any infinite cardinal \( \theta \).

**Proof.** 1), 2), 3) Left to the reader.

4) Follows from part 5).

5) Let \( \lambda = \text{Arc}_{\alpha(\ast) + 1}(\theta) \), and let \( T = (T_i : i < \theta) \), \( \text{degcr}(T_i) \leq \alpha(\ast) \) and \( A = \bigcup_{i < \theta} \text{lim}(T_i) \). Let \( \eta_\alpha : \alpha < \lambda \times \{\nu_\beta : \beta < \lambda\} \subseteq A \) where \( \alpha < \beta \Rightarrow \eta_\alpha \neq \eta_\beta \) and \( \nu_\alpha \neq \nu_\beta \), and for simplicity \( \{\eta_\alpha : \alpha < \lambda\} \cap \{\nu_\beta : \beta < \lambda\} = \emptyset \).

We define a model \( M \), with universe \( \mathcal{H}(\mathbb{R}^+(\aleph_0)^+) \) and relation: all those definable in \( \mathcal{H}(\mathbb{R}^+(\aleph_0)^+) \), \( e, c^*, R_1, R_2, g, T, i < \theta \) where \( R_1^M = \{\eta_\alpha : \alpha < \lambda\}, R_2^M = \{\nu_\beta : \beta < \lambda\}, \) \( g(\eta_\alpha, \nu_\beta) = \min\{i : (\eta_\alpha, \nu_\beta) \in \text{lim}(T_i)\} \).

Next we prove

\[(\ast) \text{ if } w_\ell \in [R_\ell^M] \text{ for } \ell = 1, 2, \text{ then} \]

\[ \text{rkk}(w_1, w_2, M) \leq \min\{ \text{degcr}_{T_i}(\{\eta_\alpha \mid k: \alpha \in u_1\}, \{\nu_\beta \mid k: \beta \in u_2\}) : 1 \subseteq u_1, 2 \subseteq u_2, u_1 \neq 0, u_2 \neq 0, k < \omega \text{ and } \langle \eta_\alpha \mid k: \alpha \in u_1\rangle \text{ is with no repetitions, and } \langle \nu_\beta \mid k: \beta \in u_2\rangle \text{ is with no repetitions}\}. \]

We prove \((\ast)\) by induction on the left side of the inequality. Now by the definitions we are done.

\[ \tag{4.15} \]

**Claim 4.15.**

1. For each \( \alpha(\ast) < \omega_1 \), there is an \( \omega \)-sequence \( T = (T_n : n < \omega) \) of \((\omega, \omega)\)-trees such that:

   \[(\alpha) \text{ for every } \mu < \text{Arc}_{\alpha(\ast)}(\aleph_0), \text{ some c.c.c. forcing notion adds a } (\mu, \mu) \text{-rectangle to } \bigcup_{n < \omega} \text{lim}(T_n), \]

   \[(\beta) \text{ degcr}(T_n) = \alpha(\ast). \]

2. If \( \text{NP} \text{r}_{\alpha(\ast)}(\lambda_1, \lambda_2; \aleph_0) \) then for some \((\omega, \omega)\)-tree \( T \): some c.c.c. forcing notion adds a \((\lambda_1, \lambda_2)\)-rectangle to \( \text{lim}(T) \) such that \( \alpha(\ast) = \text{degcr}(T) \) (consequently, if \( \text{Prk}_{\alpha(\ast)}(\lambda, \lambda; \aleph_0) \) then there is no \((\lambda, \lambda)\)-rectangle in \( \text{lim}(T) \)).

3. Moreover, we can have for the tree \( T \) of (4): if \( \mu < \text{Arc}_{\alpha(\ast)}(\aleph_0) \), \( A, B \) disjoint subsets of \( \omega^2 \) of cardinality \( \leq \mu \), then some c.c.c. forcing notion \( P \), adds an automorphism \( f \) of \((\omega^2, \varnothing)\) such that: \( A \subseteq \lim^*[f(T)], B \cap \lim^*[f(T)] = \emptyset \) (the \( \lim^* \) means closure under finite changes).
Proof. 1) We define the forcing for part 2, and delay the others to [Sh:532].

2) It is enough to do it for successor $\alpha(*)$, say $\beta(*) + 1$. It is like 1.13; we will give the basic definition and the new points. Let $M$ be a model as in Definition 4.1, $|R^M_1| = \lambda^*_e, \text{rkk}^1(M) < \alpha(*)$ so $\text{rkk}^1(M) \leq \beta(*)$. We assume that $R^M_1, R^M_2$ are disjoint sets of ordinals. For non empty $\bar{a}_1 \subseteq R^M_1$ ($l < 2$; no repetition inside $\bar{a}_1$), let $\phi^M(\bar{a}_1, \bar{a}_2), k^M(\bar{a}_1, \bar{a}_2) \in \text{Rang}(\bar{a}_1) \cup \text{Rang}(\bar{a}_2)$ be witnesses to the value of $\text{rkk}^1(\bar{a}_1, \bar{a}_2, M)$ which is $< \alpha(*)$.

We define the forcing notion $P$: a condition $p$ consists of:

1) $\dot{u}^ρ = (u^ρ_0, u^ρ_1)$ and $u^ρ_2 = u_ρ[p]$ is a finite subset of $R^M_ϕ$ for $ϕ < 2$,

2) $n^ρ = n[ρ] < ω$ and $η^ρ_2 = η_2[ρ] \in n[ρ]ω$ for $α ∈ u^ρ_2$ such that $α \neq β ∈ u^ρ_2$ ⇒ $η^ρ_2, η^ρ_0 \neq η^ρ_0$,

3) $0 < m^ρ < ω$ and $t^ρ_1 ⊆ \bigcup \{ω^i × ω : i ≤ n^ρ\}$ closed under initial segments and such that the $ι$-maximal elements have the length $n^ρ$ and $(i) ∈ t^ρ_1,

4) the domain of $f^ρ$ is $\{u = (u_1, u_2) :$ for some $l = l(u_1, u_2) \leq n$ and $m = m(u_1, u_2) < m^ρ$ we have $u_ρ ≤ t^ρ_1$ $\cap$ $l$ and if $α_1 ∈ [u_1, α_2] ∈ [u_2, η^ρ_2] \cap l \in u_1, η^ρ_2, l ≤ u_2$ then $g^ρ(α_1, α_2) = m$ and $F^ρ(τ) = (f^ρ_0(τ), f^ρ_1(τ), f^ρ_2(τ)) ∈ α(*) × (u_1, u_2) × L_{α(ω)}(τ(M))$,

5) a function $g^ρ : u^ρ_1 × u^ρ_2 → \{0, ..., m - 1\}, m^ρ < ω$,

6) $t^ρ_1 \cap (n^ρ)ω = \{η^ρ_1, η^ρ_0 : α ∈ u^ρ_1, β ∈ u^ρ_2$ and $m = g^ρ(α, β)\}$

7) if $\emptyset \neq u_ρ \subseteq t^ρ_1 \cap l(ω, f^ρ(u_1, u_2) = (β^*, ρ^*, ϕ^*), l ≤ l(*) ≤ n^ρ$, for $i = 0, 1$ a function $e_i ϕ$ has the domain $u_ρ$, $[\forall l](ρ ∈ u_ρ ⇒ ρ ∈ e_i ϕ(ρ) ∈ t^ρ_1 \cap l(ω)]$, $[ρ^* ∈ u_ρ ⇒ e_0, ϕ(ρ) = e_1, ϕ(ρ)]$, $\{ρ^* ∈ u_ρ ⇒ e_0, ϕ(ρ^*) = e_1, ϕ(ρ^*)\}$

and $F^ρ(e_0,1(u_1) \cup e_1,1(u_1), e_0,2(u_2) \cup e_1,2(u_2)) = (β^*, ρ^*, ϕ^*)$ (so well defined), then $β^* < β^*$,

8) if $l ≤ n^ρ$, for $ϕ = 1, 2$ we have $u_ρ \subseteq u^ρ_2$ are non empty, the sequence $\langle η^ρ_0[l : α ∈ u_ρ]⟩$ is with no repetition, and $u^ρ_2 = \{η^ρ_0[l : α ∈ u_ρ]\}$ and $F^ρ(u_1, u_2)$ is well defined, then

$f^ρ_0(u_1, u_2) = \phi^M(u_1, u_2), f^ρ_0(u_1, u_2) = η^ρ_0[u, l$,

where $α$ is $k^M(u_1, u_2)$ and

$f^ρ_0(u_1, u_2) = \text{rkk}(u_1, u_2, M)$,

9) if $(u_1, u_2) ∈ \text{Dom}(f^ρ)$ then there are $l, u_1, u_2$ as above,

10) if $(η, η_2) ∈ t^ρ_1 \cap (n^ρ × n^ρ)$ then for some $α_1 ∈ u^ρ_1, α_2 ∈ u^ρ_2$ we have $g^ρ(α_1, α_2) = m$ and $η_1 ≤ η^ρ_0, η_2 ≤ η^ρ_2$.

Remark 4.16. We can generalize 4.13, 4.14, ?? to Souslin relations.

References


\[\text{Sh:e} \] ______, *Non-structure theory*, vol. accepted, Oxford University Press.


**Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 9190401, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA**

*E-mail address: shelah@math.huji.ac.il*

*URL: http://shelah.logic.at*