ALMOST FREE GROUPS BELOW THE FIRST FIX POINT EXIST

SH523

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§1 λ-Free Does Not Imply Free for λ < First Fix Point

1.1 Definition. $NPT(\lambda, \kappa)$ means that there is a family $\{A_i : i < \lambda\}$ of subsets of $\lambda, |A_i| \le \kappa$, the family has no transversal but for $\alpha < \lambda$ we have $\{A_i : i < \alpha\}$ has a transversal. Let $PT(\lambda, \kappa)$ be $\neg NPT(\lambda, \kappa)$.
Now by [Sh:161], $NPT(\lambda, \aleph_0)$ is equivalent to the existence of λ-free not λ^+-free abelian group. Also "canonical" examples of "incompactness" are given there.

Magidor Shelah [MgSh:204] proved

1.2 Theorem (ZFC). *If λ is smaller than the first $\alpha = \aleph_\alpha > \aleph_0$ then there is a λ-free not free abelian group.*

This was done by providing an induction step from incompactness in κ, $\aleph_0 < \kappa = cf(\kappa) < \aleph_\kappa$ to incompactness in some $\lambda \ge \aleph_\kappa$, in fact $\aleph_{\kappa+1}$.
However, subsequently, to answer questions of Eklof and Mekler (in ?, ?) and of Magidor and Foreman (in ?), we do more.

Explanation of the History: Generally on the history of this problem see the book of Eklof and Mekler [EM]. *The existence of such group for $\lambda = \aleph_{\omega+1}$ is proved in* [Sh:108] *assuming $\aleph_\omega$ strong limit or just $2^{\aleph_0} < \aleph_\omega$; this is based on investigating $I[\aleph_{\omega+1}]$ (where for a regular uncountable λ,*
$I[\lambda] = \{S \subseteq \lambda$: for some club E of λ and $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ we have $a_\alpha \subseteq \alpha$, $otp(a_\alpha) < \alpha$ and $\alpha \in S \cap E \Rightarrow \alpha = \sup(a_\alpha)\}$), see on it [Sh:g, AG §1]. *In the earlier version of* [MgSh:204], *for $\lambda > cf(\lambda) = \kappa$ when $2^\kappa < \lambda$ we replace a division to cases defined from properties $I[\lambda]$ by one defined from $\bar{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle <_I$-increasing cofinal in $\prod_{i<\kappa} \lambda_i / I$.*
It is shown there that the set of bad (for $\bar{f}$) ordinals $< \aleph_{\kappa+1}$ is "small", (but assuming something like $2^\kappa < \lambda$, then an example in λ is gotten from one on κ^+. In the present version of [MgSh:204] *this was eliminated by showing: for good $\delta < \lambda$ there is a club of good ordinals $< \delta$ (for $\bar{f}$); and similarly for bad (we ignore ordinals of small cofinality).*

In this section we represent analyses from [MgSh:204] needed in §2; it is organized around the proof of

1.3 Lemma. *If $NPT(\kappa, \theta)$, and $\kappa \ge \theta = cf(\theta)$ then $NPT(\kappa^{+(\kappa+1)}, \theta)$.*

Remark. 1) Similarly NPT_I, see on it [Sh:355, §6] (for I an ideal on θ extending J^{bd}_θ).

Till end of the proof of 1.2 we make:

1.4 Convention. κ is a fixed regular cardinal. λ is a cardinal $> \kappa$ of cofinality κ.
(For proving 1.2, we need just $\lambda = \aleph_\kappa$, but latter we will be interested in other values too). J denotes ideals on κ extending $J^{bd}_\kappa = \{A \subseteq \kappa : A \text{ bounded}\}$.
D, E denotes filters on κ containing the co-bounded subsets of λ.

The rest of the section proves 1.3.

1.5 Claim. *Suppose* $\lambda = \kappa^{+\kappa}$, $\mu = \lambda^+; J = J^{bd}_\kappa$, *then there are* $\langle \lambda_i : i < \kappa \rangle$, $\langle f_\alpha : \alpha < \mu \rangle$ *such that the following holds:*

$(*)(a)$ $\langle \lambda_i : i < \kappa \rangle$ *strictly increasing sequence of regular cardinals* $> \kappa^+$ *with limit* λ,

(b) *for* $\alpha < \mu, f_\alpha \in \Pi\lambda_i, [\alpha < \beta < \mu \Rightarrow f_\alpha <_J f_\beta]$,

(c) $\left(\forall f \in \prod\limits_{i<\kappa} \lambda_i\right) (\exists \alpha < \mu)[f <_J f_\alpha]$,

(d) $\mu = cf(\mu) > \lambda$ *(what is trivial here),*

(e) J *is an ideal on* κ *containing* J^{bd}_κ.

Proof. By [Sh:355, 1.5].

* * *

For awhile (till we finish proving ?) we shall assume $(*)$ of 1.5.

1.6 Context. $(*)$ of 1.5.

Let us quote [Sh:355, 1.2], for the reader's convenience.

1.7Claim. *Assume* $cf(\delta) > \kappa^+, I$ *is an ideal on* κ *and suppose* $\langle f_\alpha : \alpha < \delta \rangle$ *is a* $<_I$*-increasing sequence of members of* ${}^\kappa Ord$. *Then exactly one of the following holds:*

(i) *for some ultrafilter* D *on* κ *disjoint from* I *we have:*

$(*)_D$ *there are sets* $s_i \subseteq Ord, |s_i| \leq \kappa$ *and* $\langle \alpha_\zeta : \zeta < cf(\delta) \rangle$ *increasing continuous with limit* δ, *such that for each* $\zeta < cf(\delta)$ *for some* $h_\zeta \in \prod\limits_{i<\kappa} s_i$ *we have:*

$f_{\alpha_\zeta}/D < h_\zeta/D < f_{\alpha_{\zeta+1}}/D$

(ii) $(**)_I$ *some* $f \in {}^\kappa Ord$ *is a* $<_I$*-eub of* $\langle f_\alpha : \alpha < \delta \rangle$, *(i.e.* f *satisfies* $(\alpha) + (\beta)$ *below) and* (γ) *holds:*

(α) *for* $\alpha < \delta, f_\alpha <_I f$

(β) *if* $g \in {}^\kappa Ord, g <_I f$ *then for some* $\alpha, g <_I f_\alpha$ *and*

(γ) $cf[f(i)] > \kappa$ *for* $i < \kappa$

(iii) *condition* (i) *fails and*

$(***)_I$ *for some unbounded* $A \subseteq \delta$ *and* $t_\alpha \subseteq \kappa$ *for* $\alpha \in A$ *and* $g \in {}^\kappa Ord$ *we have:*

(α) *for* $\alpha < \beta$ *in* $A, t_\beta \backslash t_\alpha \in I$ *but* $t_\alpha \backslash t_\beta \notin I$ *(i.e.* $\langle t_\alpha/I : \alpha \in A \rangle$ *is strictly decreasing in* $\mathcal{P}(\kappa)/I$*)*

(β) $t_\alpha = \{i < \kappa : f_\alpha(i) \leq g(i)\}$.

1.8Claim. *1) For $\langle f_\alpha : \alpha < \delta \rangle, \kappa, I$ as in the hypothesis of Claim 1.3 such that $cf(\delta) > \kappa^+$, the following are equivalent:*

(a) there are $A \subseteq \delta$ unbounded, and $s_\alpha \in I$ for $\alpha \in A$ such that: $\langle f_\alpha(i) : \alpha \in A, i \in \kappa \backslash s_\alpha \rangle$ is strictly increasing in α for each $i < \kappa$

(b) $(**)_I$ of Claim 1.3 holds for some f and

$$\{i < \kappa : cf[f(i)] \neq \ cf(\delta)\} \in I$$

(note: f/I is unique as a $<_I$-lub of $\langle f_\alpha : \alpha < \delta \rangle$).

2) If $cf(\delta) > \ gen(I)$, in clause (a) above without loss of generality $s_\alpha = s$ for $\alpha \in A$; if $I = J^{bd}_\kappa$ without loss of generality $s_\alpha = s = [i(*), \kappa)$ for some fixed $i(*)$.

1.9 Definition. We define (for $\bar f$ and J, κ, λ, μ):
1) S^{gd} is the set of $\bar f$-good ordinals. An ordinal δ is weakly $\bar f$-good if δ is a limit ordinal, $\delta < \ell g(\bar f)$ is of cofinality $> \kappa$ but $< \lambda$ such that for some $t \subseteq \kappa$, $t \notin J$ and for some unbounded $A \subseteq \delta$ and $\langle s_\alpha : \alpha \in A \rangle$ such that $s_\alpha \in J$ and for each $i < \kappa$, $\langle f_\alpha(i) : \alpha \in A, i \in t \backslash s_\alpha \rangle$ is strictly increasing (we may say δ is $\bar f$- good by t); so $\bar f \restriction \delta$ has a $<_J$-eub (see below) $f^*_\delta, f^*_\delta(i) = \sup\{f_\alpha(i) : \alpha \in A \text{ and } i \in s_\alpha\}$. An ordinal δ is $\bar f$-good if we could have $t = \kappa$.
2) S^{bd}, a set of bad points for $\bar f$, is the set of limit ordinals $\delta < \ell g(\bar f)$ of cofinality $> \kappa$ but $< \lambda$ such that $\bar f \restriction \delta$ satisfies $(**)_J$ of Claim [Sh:355, 1.2] - for some f^*_δ (i.e. $\bar f \restriction \delta$ has as a $<_J -eub \, f^*_\delta$ (i.e. $<_J$-exact upper bound, this means (here): $\alpha < \delta \Rightarrow f_\alpha <_J f_\delta$ and $g <_J f_\delta \Rightarrow \bigvee_{\alpha<\delta} g <_J f_\delta$) such that $\{i : \text{cf}[f^*_\delta(i)] \leq \kappa\} \in J$), but $\delta \notin S^{gd}$; without loss of generality $f^*_\delta \in \prod_{i<\kappa} \lambda_i$ (as $f^*_\delta \leq_J f_\delta$).
3) S^{ch}, a set of δ chaotic points for $\bar f$, is the set of limit ordinals $\delta < \ell g(\bar f)$ of cofinality $> \kappa$ but $< \lambda$ such that:

(a) $\bar f \restriction \delta$ satisfies $(*)_{D_\delta}$ of Claim [Sh:345, 1.2] i.e. for some ultrafilter D_δ on κ disjoint to J and $\langle a_i : i < \kappa \rangle$, a_i a set of $\leq \kappa$ ordinals, for every $i < \delta$ for some $j < \delta$ (but $> i$) and $g \in \prod_{i<\kappa} a_i$ we have $f_i/D < g/D < f_j/D$

or

(b) $\bar f \restriction \delta$ satisfies $(***)_I$ of Claim [Sh:355, 1.2(iii)]; (i.e. for some A, g we have $A \subseteq \delta$ unbounded, $g \in {}^\kappa Ord$ and $\langle \{i : f_\alpha(i) \leq g(i)\}/J : \alpha \in A \rangle$ is strictly decreasing in $\mathcal{P}(\kappa)/J$. (Hence $\text{cf}(\delta) \leq 2^\kappa$ and if J is a maximal ideal, this does not occur).

4) $S^{wg}(\bar f)$ is the set of weakly good $\delta < \mu$ satisfying $\delta \notin S^{ch}$; S^{vbd} is the set $S^{bd} \backslash S^{wgd} \backslash S^{ch}$.

1.10 Remark. 1) For the specific $J = J^{bd}_\kappa$ we could redefine S^{gd} as follows: for some unbounded $A \subseteq \delta$ and $i(*) < \kappa$ for $\alpha \in A$, for each $i \in (i(*), \kappa)$ $\langle f_\alpha(i) : \alpha \in A \rangle$ is strictly increasing. We use: J is generated by a family of $< cf(\delta)$ subsets of κ.
2) More accurately, we should have written $S^{gd}[\bar f]$ or even $S^{gd}_J[\bar f, \lambda]$, (as κ can be

reconstructed), similarly for bd and ch.
3) gd, bd, ch stands for good, bad, chaotic, respectively.

1.11 Fact. 1) $\langle S^{gd}, S^{bd}, S^{ch}\rangle$ is a partition of $\{\delta < \ell g(\bar f) : \kappa < cf(\delta) < \lambda\}$. Also $\langle S^{wgd}, S^{vbd}, S^{ch}\rangle$ is a partition of that set.
2) $\delta \in S^{gd} \cup S^{bd}$ iff $\bar f \restriction \delta$ satisfies $(*)_I$ of Claim [Sh:355, 1.2], for $I = J$; i.e. $\bar f \restriction \delta$ has a $<_J$-eub.
3) $S^{gd} \subseteq S^{wg}$ and $S^{vbd} \subseteq S^{bd}$.

Proof. If $\delta \in S^{gd}$, <u>then</u> $\bar f \restriction \delta$ satisfies $(*)_I$ of Claim [Sh:355, 1.2] (for $I = J$), by 1.9. Now Part 2) holds by the definition of S^{bd}. Now Part (1) is immediate by Claim [Sh:345, 1.2] (and Definition 1.9).
2), 3) Easy, too. $\square_{1.11}$

1.12 Claim. *1) If $\delta \in S^{bd}$ <u>then</u> letting $\lambda^\delta_i = cf[f^*_\delta(i)]$ (see 1.9(2)), we have:*

(i) *$\{i : \lambda^\delta_i \le \kappa$ or $\lambda^\delta_i > cf(\delta)\} \in J$ (so without loss of generality it is empty)*
(ii) *$\lambda^\delta_i < \lambda_i$ and*
(iii) *$(\prod\limits_{i<\kappa} \lambda^\delta_i, <_J)$ has true cofinality $cf(\delta)$ and*
(iv) *for no λ' is $\{i : \lambda^\delta_i \ne \lambda'\} \in J$*
(v) *If δ is not weakly good, then for every λ' we have $\{i : \lambda^\delta_i = \lambda'\} \in J$*
(vi) for every $\lambda' \ne cf(\delta)$ we have $\{i : \lambda^\delta_i = \lambda'\} \in J$.

2) If J is θ-complete, $\lambda = \chi^{+\zeta}$, $\zeta \le \theta$ a limit ordinal and $\kappa \le \chi$, <u>then</u>:

(i) $\delta \in S^{gd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ne cf(\delta)\} \in J$.
(ii) $\delta \in S^{vd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ge \chi\} \in J$.
(iii) $\delta \in S^{wg} \Rightarrow \{i < \kappa : \lambda^\delta_i \ge \chi, \lambda^\delta_i \ne cf(\delta)\} \in J$.
(iv) $cf(\delta) \ge \chi$ & $\delta \in S^{wg} \backslash S^{gd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ge \chi$ and $\lambda^\delta_i = cf(\delta)\} \in J^+$.

3) If J is θ-complete, $\zeta \le \theta$ (a limit ordinal) and $\lambda = \kappa^{+\zeta}$ then $S^{bd} = \emptyset$.

Proof. 1) The first phrase holds as f^*_δ is $<_J$-exact upper bound of $\bar f \restriction \delta$ (see Claim [Sh:355, 1.2]). For the rest see [Sh:355, 1.6].
2) Suppose $\delta \in S^{bd}$ is not weakly good and $t =: \{i : \lambda^\delta_i > \chi\}$ is not in J; as $\delta \in S^{bd}$ trivially $\{i : \lambda^\delta_i > \text{cf}(\delta)\} \in J$, so $t_1 =: \{i : \chi < \lambda^\delta_i \le \text{cf}(\delta)\}$ is not in J. As $\text{cf}(\delta) < \chi^{+\zeta}, \zeta \le \theta$, the set $\{\sigma : \chi \le \sigma = \text{cf}(\sigma) < \chi^{+\zeta}\}$ has $< |\zeta| \le \theta$ members but J is θ-complete, hence for some $s \subseteq t_1$, $s \notin J$ and $\langle \lambda^\delta_i : i \in s\rangle$ is constant, say λ^*, now also $(\prod\limits_{i\in s} \lambda^\delta_i, <_J)$ has true cofinality $\text{cf}(\delta)$ (by 1.12(1)(iii)), but this implies δ is weakly $\bar f$-good by "$\Leftarrow$" of [Sh:355, 1.6(1)]. So we have proved (ii). For (i), (iii), (iv) the proof is similar. (In fact for clause (i) the assumption of part (2) is not needed).
3) If $\delta \in S^{bd}$, by 1.12(2)(i) we have $\{i < \kappa : \lambda^\delta_i > \kappa\} \in J$, so one of the clauses of Definition 1.9(2) fails by (ii), (iii) of part (2) of our Claim, so $\delta \notin S^{bd}$, contradiction. $\square_{1.12}$

1.13 Claim. *If $\delta \in S^{gd}$, then for some club C of δ we have:*

$$\alpha \in C \ \ \& \ \ cf(\alpha) > \kappa \Rightarrow \alpha \in S^{gd}.$$

Similarly for S^{wgd} (by t).

Proof. Straightforward by its definition.

1.14 Claim. *If $\delta \in S^{ch}$, then for some club C of δ*

$$\alpha \in C \ \ \& \ \ cf(\alpha) > \kappa \Rightarrow \alpha \in S^{ch}.$$

Proof. If $\delta \in S^{ch}$, then
Case a: For some ultrafilter D on κ disjoint to J (say D_δ) and club C of δ, there are $\langle g_\alpha : \alpha \in C\rangle$ such that:

(a) for $\alpha < \beta$ in C we have $f_\alpha/D < g_\alpha/D < f_\beta/D$
(b) for each $i < \kappa$ the set $s_i = \{g_\alpha(i) : \alpha \in C\}$ has power $\leq \kappa$.

Now for every $\delta' \in C$ such that $\delta' = \sup(\delta' \cap C)$ and $cf(\delta') > \kappa$ clearly $\langle g_\alpha : \alpha \in C \cap \delta'\rangle$, $C \cap \delta'$, $\langle s_i : i < \kappa\rangle$ witness that $\langle f_\alpha : \alpha < \delta'\rangle$ also satisfies $(*)_D$ of [Sh:355, Claim 1.2], hence $\delta' \in S^{ch}$.
Case b: (Not case (a)). (See 1.9(3)). Also trivial proof. $\square_{1.14}$

1.15 Claim. *For every regular $\chi, \kappa < \chi < \lambda, S^{gd}_\chi =: \{\delta \in S^{gd} : cf(\delta) = \chi\}$ is stationary.*

Proof. Suppose C^* is a club of μ. We choose by induction on $\zeta < \chi$, α_ζ, g_ζ, s_ζ such that:

(1) $\alpha_\zeta \in C^*$
(2) $\bigcup_{\xi<\zeta} \alpha_\xi < \alpha_\zeta$
(3) $g_\zeta \in \prod_{i<\kappa} \lambda_i$, $g_\zeta(i) =: \sup \{f_{\alpha_\xi}(i) + 1 : \xi < \zeta\}$
(4) $s_\zeta \in J$
(5) $i \in \kappa \backslash s_\zeta \Rightarrow g_\zeta(i) < f_{\alpha_\zeta}(i)$.

In stage ζ, first define g_ζ by (3) then find $\beta_\zeta < \mu$ such that $g_\zeta <_J f_{\beta_\zeta}$ (possible as $\langle \lambda_j : j < \kappa\rangle$ is the $<_J$-eub of $\langle f_\alpha : \alpha < \mu\rangle$) then choose α_ζ, $\beta_\zeta \cup \bigcup_{\xi<\zeta} \alpha_\xi < \alpha_\zeta \in C^*$ (possible as $|\zeta| < \lambda$, C^* unbounded in μ). Lastly choose s_ζ to satisfy (5) by the definition of $<_J$. Clearly (1)-(5) holds.

Let $\delta = \bigcup_{\zeta<\mu} \alpha_\zeta$, clearly $\delta \in C^*$ (being closed) and $cf(\delta) = \chi$ and clearly $\delta \in S^{gd}_\chi$. [Note: if $gen(J) < \chi$, as $\chi = \text{cf}(\chi)$ for some $t \in J$, $\{\zeta < \mu : s_\zeta \subseteq t\}$ is unbounded in χ. Also it is natural to deduce it from the existence of a stationary $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \chi\}, S \in I[\mu]$ (see [Sh:420, §1])].

We shall not use

1.16 Lemma. *Suppose* $\lambda = \kappa^{+\kappa}, \langle \lambda_i : i < \kappa \rangle, \mu, J$ *are as in* $(*)$ *of 1.5 and in addition* $S^{bd} = \emptyset$. *Let* $C^* = \{\delta < \lambda^+ : \delta$ *divisible by* $\lambda^2\}$. *Then we can find* $\langle \alpha^\delta_{i,j} : \delta \in C^* \cap S^{gd}, i < cf(\delta), j < \kappa \rangle$ *such that:*

(a) $\alpha^\delta_{i,j} < \delta$ *(and, if you want,* $\langle \alpha^\delta_{i,j} : i < cf(\delta) \rangle$ *is increasing continuous with limit* $\delta, \langle \alpha^\delta_{i,j} : j < \kappa \rangle$ *increasing with* j*).*

(b) *for every* $\delta(*) < \lambda^+$ *we can find* $\langle C^\delta, s^\delta_i : \delta \in \delta(*) \cap C^* \cap S^{gd}, i < cf(\delta) \rangle$ *such that:*

  (α) C^δ *a club of* $cf(\delta)$

  (β) $s^\delta_i \in J$

  (γ) $\langle \{\alpha^\delta_{i,j} : i \in C^\delta, j \notin s^\delta_i\} : \delta \in \delta(*) \cap C^* \cap S^{gd} \rangle$ *are pairwise disjoint.*

This we refer to [MgSh:204], anyhow see proof of ?.

§2 Nicely Compact Set and Non-Reflection

This section answers the following two questions.
One question of Foreman and Magidor asks for consistency of the following reflection principle:

$(*)_\lambda$ <u>if</u> $S \subseteq \mathcal{S}_{<\aleph_2}(\lambda)$ is a stationary set (of $\mathcal{S}_{<\aleph_2}(\lambda)$, not of λ), each $a \in S$ is ω-closed (i.e. $\mathrm{cf}(\delta) = \aleph_0$ & $\delta = \sup(\delta \cap a) < \lambda \Rightarrow \delta \in a$) <u>then</u> for some $B \subseteq \lambda$, $|B| = \aleph_2$ and $S \cap \mathcal{S}_{<\aleph_2}(B)$ is stationary.

We shall show that it is very hard to get them (in a way defeating the application they have in mind: $\theta < \aleph_2$ follows from large cardinals). E.g. if μ is strong limit singular of cofinality $\aleph_0$, $2^\mu = \mu^+$ then $(*)_{\mu^+}$ is false.

Another question of Mekler and Eklof, improve somewhat [MgSh:204] showing in particular that if we have gotten in the (regular) λ, κ examples for incompactness then we can get one for $\lambda^{+(\kappa+1)}$.

To do this (see below) if $\lambda > \kappa^{+\kappa}$, we need a counterexample for λ, which without loss of generality is as in [Sh:161] canonical examples in which only cardinals in which examples exist appear (see below). For this we define when a set K of cardinals is nicely incompact and prove that the family of such sets is closed under the required operation.

2.1 Definition. A set K of regular cardinals is called nicely incompact if:

(a) for each $\lambda \in K$ there is a λ-free not λ^+-free abelian group or $\lambda = \aleph_0$

(b) <u>if</u> λ is in $K, \kappa_1, \cdots, \kappa_m \in K \cap \lambda$ <u>then</u> there is an example $\langle s^\ell_\eta : \eta \in \mathfrak{S}_f, \ell < m\rangle$ of incompactness (see [Sh:161, §3] or [Sh:521], we assume knowledge of e.g. [Sh:521, AP]) such that:

$$(\lambda_{<>} = \lambda \text{ and) for some } \langle k_1, \cdots, k_m\rangle \text{ we have}$$
$$\eta \in S \ \ \& \ \ \ell g(\eta) = k_\ell \Rightarrow \lambda_\eta = \kappa_\ell.$$

2.2 Claim. *1) If K is nicely incompact, $\lambda = \max(K), \lambda$ regular <u>then</u> $K \cup \{\lambda^+\}$ is nicely incompact.*
2) Assume $K = \{\lambda_i : i < \delta\}$ increasing and $j < \delta \Rightarrow \{\lambda_i : i \leq j\}$ nicely incompact. <u>Then</u> K is nicely incompact.
3) $\{\aleph_0\}$ is nicely incompact.

Proof. Easy.

Remark. If proof of 2.2(1) is not clear - read proof of ? and throw away most (use $\{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ here instead $\{\delta \in A_g : \mathrm{cf}(\delta) = \lambda'\}$ there.

2.3 Theorem. *If K is nicely incompact, $\chi, \kappa \in K, K$ has no last element and $\chi^{+\kappa} \geq$ sup (K) then $K \cup \{\chi^{+(\kappa+1)}\}$ is nicely incompact.*

Remark. By the closure properties 2.2(1), 2.2(2), 2.3 (and as a starting point 2.2(3): $\{\aleph_0\}$ being nicely incompact) we get a family of $\aleph_0$ cardinals which probably is the minimal set of incompactness.

Proof. If $\chi < \kappa = \aleph_\kappa$, then $\chi^{+(\kappa+1)} = \kappa^+$ and the desired conclusion holds is by Claim 2.2(1).

If $\chi < \kappa < \aleph_\kappa$ then $\chi^{+\kappa} = (\kappa^+)^\kappa = \aleph_\kappa$ so without loss of generality $\chi = \kappa^+$.

So without loss of generality $\chi \geq \kappa^+, \lambda = \chi^{+\kappa}, \mu = \lambda^+, \lambda > \kappa$ and J, $\langle \lambda_i : i < \kappa \rangle, \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be as in $(*)$ of 1.5 and for simplicity $f_{\alpha_1}(i_1) = f_{\alpha_2}(i_2) \Rightarrow i_1 = i_2$, e.g. $f_\alpha(i) > \sum_{j<i} \lambda_j$; and let S^{gd}, S^{bd}, S^{ch} be from Definition 1.9. Remember that by 1.13, 1.14 we have

$(*)$ if $x \in \{gd, wgd, ch\}, \delta \in S^x$ then for some club E of δ we have: $\alpha \in E$ & $\mathrm{cf}(\alpha) \geq \kappa^+ \Rightarrow \alpha \in S^x$.

In the definition of "K is nicely incompact" we have to check only for $\mu = \lambda^+$. Let $m < \omega$ and $\kappa_1, \ldots, \kappa_m \in K \cap \lambda^+$. So $\langle \kappa_1, \ldots, \kappa_m \rangle \in K \cap \chi^{+\kappa}$, without loss of generality $\kappa_1 < \kappa_2 < \ldots < \kappa_m$.
Let $\lambda' \in K$ be such that $\kappa < \lambda, \chi < \lambda'$ and $\kappa_m < \lambda'$ (there is such a λ' as K has no last member).

Let $S = \{\delta \in S^{gd}(\bar{f}) : \mathrm{cf}(\delta) = \lambda'\}$ (by 1.15 it is stationary) and for $\delta \in S$ choose $E_\delta \subseteq \delta$ an unbounded subset of order type λ'.

Let $E_\delta = \{\alpha^\delta_\zeta : \zeta < \lambda'\}$ (increasing).
Now choose an example for $\lambda' : \langle s^\ell_\eta : \eta \in \mathfrak{S}'_f$ and $\ell < n(*) \rangle, \mathfrak{S}'$ a λ'-set (see [Sh:161, §3] or [Sh:521, AP]), such that for some sequence $\langle k_1, \ldots, k_n, k \rangle$ of natural number $\leq lg(\eta)$ for $\eta \in \mathfrak{S}'$

$$\eta \in \mathfrak{S}' \ \& \ \ell g(\eta) = k_i \Rightarrow \lambda_\eta = \kappa_\ell$$
$$\eta \in \mathfrak{S}' \ \& \ \ell g(\eta) = k \Rightarrow \lambda_\eta = \kappa.$$

(possible by Definition 2.1).
Lastly, let

$$\mathfrak{S} = \{<>\} \cup \left\{ < \delta > \hat{} \eta : \delta \in S \subseteq \lambda^+ \text{ and } \eta \in \mathfrak{S}' \right\}.$$

For $\eta \in \mathfrak{S}_f$ let (denoting $\delta = \eta(0)$ which necessarily is in S and letting $\eta = \langle \delta \rangle \hat{} \nu$ so $\nu = \eta \restriction [1, \ell g(\eta))$):

$$s^{\ell+1}_\eta = (s^\ell_\nu \times \{\delta\}) \text{ if } \ell < n(*), \text{ and } s^0_\eta = \left\{ \langle f_\delta(\eta(k)), \eta \restriction [1, \ell g(\eta)), \alpha^\delta_{\eta(k)} \rangle : k < \omega \right\}.$$

$$A_\eta = \cup \{s^\ell_\eta : \ell \leq n(*)\}$$

First note that $\langle A_\eta : \eta \in \mathfrak{S}_f \rangle$ has the right form hence has no one-to-one choice function (see [Sh:161, §3] or [Sh:521, AP]).

Secondly, to prove every subfamily of cardinality $< \mu$ has a one-to-one choice function it is enough to prove ? below. It applies with λ', μ here standing for χ, μ^* there. The no reflection is by ? below.

2.4 Observation. Let $\bar{\lambda}, J, \mu, \lambda$ be as in $(*)$ of 1.5.
1) If J is θ-complete, $\lambda = \chi^{+\zeta}, \zeta \le \theta$ a limit ordinal, $\mathrm{cf}(\chi) > \kappa$, then:

(i) $\delta \in S^{gd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ne \mathrm{cf}(\delta)\} \in J$
(ii) $\delta \in S^{vbd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ne \mathrm{cf}(\delta)\} \in J$
(iii) $\delta \in S^{wg} \Rightarrow \{i : \lambda^\delta_i \ge \chi, \lambda^\delta_i \ne \mathrm{cf}(\delta)\} \in J$
(iv) $\mathrm{cf}(\delta) \ge \chi \ \& \ \delta \in S^{wg} \backslash S^{gd} \Rightarrow \{i < \kappa : \lambda^\delta_i \ge \chi \text{ and } \lambda^\delta_i = \mathrm{cf}(\delta)\} \in J^+$.

2) If J is θ-complete, $\zeta \le \theta$ (a limit ordinal) and $\lambda = \chi^{+\zeta}$ then for no $\delta \in S^{bd}$ do we have: $\{\alpha \in S^{gd} : \alpha < \delta, \mathrm{cf}(\alpha) \ge \chi\}$ is stationary.

Proof. 1) Like 1.12.
2) Let δ be a counterexample, let f^*_δ be a $<_J$-eub of $\bar{f} \restriction \delta$ and by part (1) without loss of generality $s^* = \{i < \kappa : \lambda^i_\delta = \mathrm{cf}[f^*_\delta(i)] < \chi\}\} \in J^+$. Choose $E_i \subseteq f^*_\delta(i)$ a club of order type λ^i_δ.
Choose $\langle \alpha_\zeta : \zeta < \mathrm{cf}(\delta) \rangle$ is a strictly increasing sequence of ordinals with limit δ. Now we can choose by induction on $\zeta < \mathrm{cf}(\delta)$ a pair (β_ζ, g_ζ) such that:

(a) $g_\zeta \in \prod\limits_{i<\kappa} E_i$ such that
$\varepsilon < \zeta \Rightarrow f_{\beta_\varepsilon} \restriction s^* <_J g_\zeta \restriction s^*$
(b) $\beta_\zeta \in \left(\bigcup\limits_{\varepsilon<\zeta} \beta_\varepsilon \cup \alpha_\zeta, \delta \right)$ is such that $g_\zeta \restriction s^* <_J f_{\beta_\zeta} \restriction s^*$.

So if δ is a counterexample, then for some $\zeta < \mathrm{cf}(\delta)$ we have: $\mathrm{cf}(\zeta) \ge \chi$ and $\beta^* = \bigcup\limits_{\varepsilon<\zeta} \beta_\varepsilon \in S^{gd}$. Hence there is an increasing sequence $\langle \gamma_j : j < \mathrm{cf}(\beta^*) \rangle$ of ordinals with limit β^*, and $t_j \in J$ for $j < \mathrm{cf}(\beta^*)$ such that for each $i, \langle f_{\gamma_j}(i) : j < \mathrm{cf}(\beta^*), i \notin t_j \rangle$ is strictly increasing. For each $i \in s^*$ there is $j(i) < \mathrm{cf}(\beta^*)$ such that all the ordinals $\{f_{\gamma_j}(i) : j(i) \le j < \mathrm{cf}(\beta^*) \text{ and } i \notin t_j\}$ realize the same Dedekind cut of E_i. Let $j(*) = \sup\limits_{i<\kappa} j(i) < \mathrm{cf}(\beta^*)$ (as $\mathrm{cf}(\beta^*) \ge \mathrm{cf}(\chi) > \kappa$). Now the contradiction should be clear. $\square_{2.4}$

2.5 Remark. 1) Alternatively, in the proof of 2.3 redefine $S^{ch}(\bar{f})$ replacing "$|s_i| \le \kappa$" by "$|s_i| \le \chi$". We call it $S^{ch}_\chi(\bar{f})$ (similarly the others).
2) Note: if λ belongs to some nicely incompact K^*, then there is a finite nicely incompact $K \subseteq K^*$ to which λ belongs.

2.6 Claim. *In the context of* $(*)$ *of 1.5, suppose* $\kappa < \chi = cf(\chi) < \mu^* \leq \mu$ *and* $S \subseteq \{\delta \in S^{gd} : cf(\delta) = \chi\}$ *is stationary but reflect in no* $\delta \in S^{bd}$ *of cofinality* $< \mu^*$ *and* $f_{\alpha_1}(i_1) = f_{\alpha_2}(i_2) \Rightarrow i_1 = i_2$ *(e.g.* $f_\alpha(i) > \sup\limits_{j<i}\lambda_j$*). For* $\delta \in S$ *let* E_δ *be an unbounded subset of* δ*. Let* $A_\delta = E_\delta \times (Rang\ f_\delta) \subseteq \delta \times \lambda$*. Then* $\langle A_\delta : \delta \in S\rangle$ *has no one-to-one choice function but is* μ^**-free in the following sense:*

$(*)$ *if* $B \subseteq S$, $|B| < \mu^*$ *then we can find* $\alpha_\delta < \delta$ *and* $s_\delta \in J$ *for* $\delta \in B$ *such that the sets* $\langle (E_\delta \backslash \alpha_\delta) \times\ Rang\ (f_\delta \restriction (\kappa \backslash s_\delta) : \delta \in B\rangle$ *are pairwise disjoint.*

Remark. 1) When we use it in 2.3, $\mu^* = \mu$ and the non-reflection is guaranteed by 2.4.
2) If we want to replace A_δ by a subset of δ, assume every $\delta \in S$ is divisible by λ^2, and every $\alpha \in E_\delta$ is divisible by λ, and let

$$A_\delta = \{\alpha^\delta_\zeta + f_\delta(i) : i < \kappa, \theta < \chi\} \text{ where } S_\delta = \{\alpha^\delta_\zeta : \zeta < \chi\} \text{ (increasing)} .$$

3) If $J = J^{bd}_\kappa$ it may look nicer to assume $\langle \text{Rang } f_\alpha \restriction [i,\kappa) : \alpha \in E_\delta\rangle$ (for suitable $i = i_\delta < \kappa$) are pairwise disjoint (not hard to get as $\delta \in S^{gd}$). Then use $f_\alpha \restriction [i,\kappa)$ only.
4) It seems reasonable to demand (and it causes no problem) $i < \kappa\ \&\ \alpha < \mu \Rightarrow f_\alpha(i) > \sup\{f_\beta(j) : j < i, \beta < \mu\}$.

Proof. This is proved by induction on $\cup\{\delta + 1 : \delta \in B\}$.
Without loss of generality $B \neq \emptyset$.

Case 1. B has a last element $\delta(*)$.

By the induction hypothesis there are $\langle (\alpha^1_\delta, s^1_\delta) : \delta \in B\backslash\{\delta(*)\}\rangle$ as required for $B\backslash\{\delta(*)\}$. For each $\delta \in B \cap \delta(*)$ we have $t_\delta = \{i < \kappa : f_\delta(i) \geq f_{\delta(*)}(i)\} \in J$. Let us define for $\delta \in B$:

$$\alpha_\delta = \begin{cases} \alpha^1_\delta & \underline{\text{if}} \quad \delta \in B, \delta \neq \delta(*) \\ 0 & \underline{\text{if}} \quad \delta = \delta(*) \end{cases}$$

$$s_\delta = \begin{cases} s_\delta \cup t_\delta & \underline{\text{if}} \quad \delta \in B, \delta \neq \delta(*) \\ \emptyset & \underline{\text{if}} \quad \delta = \delta(*). \end{cases}$$

Case 2. B has no last element, $\delta(*) = \sup(B)$ has cofinality $\leq \chi$.

Let $\langle \gamma_\zeta : \zeta < cf\delta(*)\rangle$ be increasing continuous sequence with limit $\delta(*), \gamma_0 = 0$ and $\text{cf}(\gamma_\zeta) \neq \chi$, hence $\gamma_\zeta \notin S$. For each ζ, apply the induction hypothesis to $B_\zeta =: B \cap (\gamma_\zeta, \gamma_{\zeta+1})$ and get $\langle (\alpha^\zeta_\delta, s_\delta) : \delta \in B_\zeta\rangle$.

We let for $\delta \in B$ (note: B is the disjoint union of B_ζ for $\zeta < \chi$):

$$\alpha_\delta = \text{Max}\{\alpha^\zeta_\delta, \gamma_\zeta\} \text{ for } \delta \in B_\zeta$$

$$s_\delta = s^\zeta_\delta \text{ for } \delta \in B_\zeta.$$

Easy to check that they are as required.

Case 3. B has no last element, $\delta(*) =$ sup(B) has cofinality $> \chi$ and S^{gd} (hence S) does not reflect in $\delta(*)$.
Same proof as Case 2.

Case 4. B has no last element, $\delta(*) =$ sup(B) has cofinality $> \chi$ and S^{gd} reflects in $\delta(*)$.

As $|B| < \mu^*$, clearly $cf(\delta(*)) < \mu^*$ hence (by assumption) $\delta(*) \notin S^{bd}$. Also $\delta(*) \notin S^{ch}$ (by 1.14 as: $S^{gd} \cap \delta(*)$ is stationary and $S^{gd} \cap S^{ch} = \emptyset$ (which holds by 1.11)). So $\delta(*) \in S^{gd}$, hence we can find $\langle \gamma_{\zeta+1} : \zeta < \text{cf}(\delta(*)) \rangle$ increasing with limit $\delta(*)$, and $s_\zeta \in J$ for $\zeta < \text{cf}(\delta(*))$ be such that:

$(*)$ $i < \kappa \ \& \ \zeta < \xi < \text{cf}(\delta(*)) \ \& \ i \notin s_\zeta \cup s_\xi \Rightarrow f_{\gamma_\zeta}(i) < f_{\gamma_\xi}(i)$
$(**)$ $\gamma_\zeta \in S \Rightarrow \text{cf}(\zeta) = \chi$.

For limit $\zeta < \text{cf}(\delta(*))$ let $\gamma_\zeta =: \bigcup_{\xi<\zeta} \gamma_{\xi+1}$ and let $\gamma_0 = 0$.

Now if $\zeta < \text{cf}(\delta(*))$ is limit and $\gamma_\zeta \in S$ (so $\text{cf}(\zeta) = \chi$), let

$$t^0_\zeta =: \{i : \text{ for arbitrarily large } \xi < \zeta, i \in \kappa \backslash s_{\xi+1} \text{ and } f_{\gamma_{\xi+1}}(i) \geq f_{\gamma_\zeta}(i)\}$$

now t^0_ζ is in J [otherwise, for each $i \in t^0_\zeta$ let $Y^i_\zeta = \{\xi < \zeta : i \in \kappa \backslash s_{\xi+1}$ and $f_{\gamma_{\xi+1}}(i) \geq f_{\gamma_\zeta}(i)\}$ by the choice of $\langle s_{\xi+1} : \xi < \text{cf}(\delta(*)) \rangle$, clearly

$$\xi_1 < \xi_2 < \zeta \ \& \ \xi_1 \in Y^i_\zeta \ \& \ i \in \kappa \backslash s_{\xi_2} \Rightarrow \xi_2 \in Y^i_\zeta.$$

Let $\xi(i) \leq \zeta$ be $\text{Min}(Y^i_\zeta \cup \{\zeta\})$, and $\xi(*) = \text{Min}\{\xi(i) : i < \kappa, \xi(i) < \zeta\}$, now $\xi(*) < \zeta$ as $\text{cf}(\zeta) = \chi > \kappa$; and look at what occurs for $\xi = \xi(*) + 1$].

Let $t^1_\xi = \{i : \text{ for every large enough } \xi < \zeta, i \notin s_{\xi+1}\}$. Again easily $t^1_\xi \in J$. Lastly let $t^2_\xi = \{i : f_{\gamma_\xi}(i) \geq f_{\gamma_{\xi+1}}(i)\}$ now as $\gamma_\xi < \gamma_{\xi+1}$ clearly $t^2_\xi \in J$ and $t^3_\zeta = s_{\zeta+1} \in J$.

Now for each $\zeta < \text{cf}(\delta(*)), B_\zeta =: B \cap (\gamma_\zeta, \gamma_{\zeta+1})$ satisfies the induction hypothesis, hence we have $\langle (\alpha^\zeta_\delta, s^\zeta_\delta) : \delta \in B_\zeta \rangle$ as required.

Notice that B is partitioned to B_ζ (for $\zeta < \text{cf}(\delta(*))$ and $\{\gamma_\zeta : \zeta \text{ limit } < \text{cf}(\delta(*))\} \cap S$. We define

$$\alpha_\delta = \begin{cases} \text{Max}\{\alpha^\zeta_\delta, \gamma_\zeta\} & \underline{\text{if}} \quad \delta \in B_\zeta, \ \zeta < cf(\delta(*)) \\ 0 & \underline{\text{if}} \quad \delta \in \{\gamma_\zeta : \zeta \text{ limit }\} \cap S \end{cases}$$

for $\zeta < \text{cf}(\delta(*))$ and $\delta \in B_\zeta$ let

$$t_{\zeta,\delta} = s_\zeta \cup s_{\zeta+1} \cup \{i < \kappa : \neg[f_{\gamma_\zeta}(i) < f_\delta(i) < f_{\gamma_{\zeta+1}}(i)]\} \in J$$

$$s_\delta = \begin{cases} s^\zeta_\delta \cup t_{\delta,\zeta} & \underline{\text{if}} \quad \delta \in B_\zeta,\, \zeta < \text{ cf}(\delta(*)) \\ t^0_\zeta \cup t^1_\zeta \cup t^2_\zeta \cup t^3_\zeta & \underline{\text{if}} \quad \delta = \gamma_\zeta,\, \zeta < \text{ cf}(\delta(*)) \text{ limit },\, \delta \in B(\subseteq S) \end{cases}$$

It is easy to check that $\langle(\alpha_\delta, s_\delta) : \delta \in B\rangle$ is as required: let $\delta_1, \delta_2 \in B$ be distinct, without loss of generality $\delta_1 < \delta_2$, now

Case 1: If $\delta_1, \delta_2 \in B_\zeta$ for some ζ; then use the choice of $\langle\alpha^\zeta_{\delta_1}, s^\zeta_{\delta_1}\rangle, \langle\alpha^\zeta_{\delta_2}, s^\zeta_{\delta_2}\rangle$.

Case 2: If $\delta_1 \in B_{\zeta_1}, \delta_2 \in B_{\zeta_2}, \zeta_1 \neq \zeta_2$ then note $\alpha_{\delta_2} \geq \gamma_{\zeta_2}$.

Case 3: If $\delta_1, \delta_2 \in \{\gamma_\zeta : \zeta < \text{ cf}(\delta(*))\} \cap S$, then let $\delta_1 = \gamma_{\zeta_1}, \delta_2 = \gamma_{\zeta_2}$ hence by $(**)$ we have $\zeta_1 + 1 < \zeta_2$, so $i \in \kappa \backslash t^2_{\zeta_1} \Rightarrow f_{\gamma_{\zeta_1}}(i) < f_{\gamma_{\zeta_1}+1}(i)$ and $i \in \kappa \backslash t^3_\zeta \backslash t^0_\zeta \backslash t^1_\zeta \Rightarrow f_{\gamma_{\zeta_1}+1}(i) < f_{\gamma_{\zeta_2}}(i)$ so the conclusion is easy.

Case 4: If $\delta_2 \in B_{\zeta_2}, \delta_1 \in \{\gamma_\zeta : \zeta < \text{ cf}(\delta(*))\} \cap S$ then use $\alpha_{\delta_2} \geq \gamma_{\zeta_2}$.

Case 5: If $\delta_1 \in B_{\zeta_1}, \delta_2 = \gamma_{\zeta_2} \in \{\gamma_\zeta : \zeta < \text{ cf}(\delta(*))\} \cap S$ then $\zeta_2 \geq \zeta_1$ hence by $(**)$ we have $\zeta_2 > \zeta_1 + 1$, hence $i \in \kappa \backslash s_{\zeta_1+1} \backslash t^0_{\zeta_2} \backslash t^1_{\zeta_2} \Rightarrow f_{\gamma_{\zeta_1}+1}(i) < f_{\gamma_{\zeta_2}}(i)$ hence $i \in \kappa \backslash s_{\delta_1} \backslash s_{\delta_2} \Rightarrow i \in \kappa \backslash t_{\zeta,\delta} \backslash s_{\zeta_1+1} \backslash t^0_{\zeta_2} \backslash t^1_{\zeta_2} \Rightarrow f_{\delta_1}(i) < f_{\gamma_{\zeta_1}+1}(i) < f_{\delta_2}(i)$. $\square_{2.6}$

2.7 Conclusion. Suppose $\text{cf}(\lambda) = \kappa < \lambda, \kappa < \chi = \text{ cf}(\chi) < \lambda$. Then there is a stationary $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \chi\}$ and $\langle A_\delta : \delta \in S\rangle$ such that:

- (α) $A_\delta = \left\{\alpha^\delta_{i,\zeta} : i < \chi, \zeta < \kappa\right\} \subseteq \delta$
- (β) $\alpha^{\delta^1}_{i_1,\zeta_1} = \alpha^{\delta^2}_{i_2,\zeta_2} \Rightarrow i_1 = i_2 \ \& \ \zeta_1 = \zeta_2$
- (γ) for each $\delta \in S$, $\zeta < \kappa$ we have $\bigwedge\limits_\gamma$ [the sequence $\langle\alpha_{i,\zeta} : i < \chi\rangle$ is increasing with limit δ]
- (δ) $\langle A_\delta : \delta \in S\rangle$ is $\chi^{+\kappa+1}$-free in the strong sense of 2.6; i.e. if $E \subseteq S, |E| \leq \chi^{+\kappa}$ then we can find $\langle\langle i_\delta, \zeta_\delta\rangle : \delta \in E\rangle$ such that:
  - (a) $i_\delta < \chi$, $\zeta_\delta < \kappa$
  - (b) the sets $A'_\delta = \{\alpha^\delta_{i,\zeta} : \zeta_\delta < \zeta < \kappa \text{ and } i_\delta < i < \chi\}$ (for $\delta \in E$) are pairwise disjoint.

Proof. Combine the previous claims.
(Note: λ necessarily is $\geq \chi^{+\kappa+1}$ but can be strictly bigger).

2.8 Conclusion. 1) If $\lambda > \text{ cf}(\lambda) = \kappa, \kappa < \chi = \text{ cf}(\chi) < \lambda$, then we can find a stationary $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \chi\}$ and $A_\delta \subseteq \delta = \text{ sup}(A_\delta), |A_\delta| = \chi$ for $\delta \in S$, such that:

- $(*)$ if $B \subseteq S$, $|B| \leq \chi^{+\kappa}$ then $\{A_\delta : \delta \in B\}$ has a one-to-one choice function.

2) If in (1) $\diamondsuit_S$ holds, without loss of generality $\{A_\delta : \delta \in S\}$ is a stationary subset of $\mathcal{S}_{<\chi}(\lambda^+)$ each A_δ is ω-closed (and even $(< \chi)$-closed).
3) $\diamondsuit_S$ holds if $2^\lambda = \lambda^+ \ \& \ \lambda \geq \beth_\omega$ (also if $\lambda < \beth_\omega$ this is normally O.K.).
4) Condition $(*)$ of (1) implies that for no $A^* \subseteq \lambda$ of power $\leq \chi^{+\kappa}$, is $\{A_\delta : A_\delta \subseteq A^*\}$ a stationary subset of $\mathcal{S}_{<\chi^+}(A^*)$.

Proof. 1) By 2.7.
2) We can replace A_δ by any A'_δ, $A_\delta \subseteq A'_\delta \subseteq \delta$, $|A'_\delta| = \chi$, and preserve the conclusion of (1); by $\diamondsuit_S$ we can do it as to get stationary set.
3) By [Sh:460] (see more there).
4) Check. $\square_{2.7}$

2.9 Observation. Suppose $\mathrm{cf}(\lambda) = \kappa < \lambda$ and there is $\langle A_\alpha : \alpha < \lambda^+ \rangle$, $A_\alpha \in [\lambda]^\kappa$, such that:

$$\text{for each } \beta < \lambda^+ \text{ we can find } \langle A'_\alpha : \alpha < \beta \rangle \text{ pairwise disjoint with} \\ A'_\alpha \subseteq A_\alpha \ \& \ |A_\alpha \backslash A_\alpha| < \kappa.$$

Then:

- (A) for every regular $\chi \in (\kappa, \lambda)$ we can find $\langle \langle \alpha^\delta_{i,\zeta} : i < \chi, \zeta < \kappa \rangle : \delta < \lambda^+, \mathrm{cf}(\delta) = \chi \rangle$ as in 2.7, λ^+-free, in 2.6's sense; i.e. see clause (δ) of 2.7.
- (B) If K is a nicely incompact set, $\kappa = \max(K), \lambda \geq \sup(K)$ and then $K \cup \{\lambda^+\}$ is nicely incompact.

Proof. Trivial. For (A) just replace every ordinal of $< \lambda$ by χ ordinals; and copy them on each interval $[\lambda\alpha, \lambda\alpha + \lambda)$. Also (B) is immediate. $\square_{2.9}$

2.10 Conclusion. If $\kappa = \mathrm{cf}(\lambda) < \lambda$ and $pp_{\Gamma(\kappa)}(\lambda) > \lambda^+$ then the assumption of 2.9 hence the conclusions of 2.9 and 2.7 holds.

Proof. By [Sh:355, 1.6(2)] the assumptions of 2.9 holds.

§3 NPT is Not Necessarily Transitive

3.1 Lemma. *Assume the consistency of ZFC+ "there are two weakly compact cardinals $> \aleph_0$". Then it is consistent with ZFC that for some regular $\lambda > \mu > \aleph_0$, $NPT(\lambda,\mu) + NPT(\mu,\aleph_0)$ does not imply $NPT(\lambda,\aleph_0)$; (i.e. in some generic extension of universe V this holds provided that $V \models$ "$\aleph_0 < \kappa < \lambda$, κ and λ are weakly compact").*

3.2 Remark. We give more specific information in the three cases below.

Proof. First assume $\kappa < \mu < \lambda$ are such that:

(∗)(1) $PT(\kappa,\aleph_0)$, κ regular
(2) $NPT(\mu,\aleph_0)$, (μ regular)
(3) $PT(\lambda,\aleph_0)$, $\lambda = \lambda^{<\lambda}$, even if we add a Cohen subset to λ
(4) $\lambda \in I[\lambda]$ (see on it and references in the discussion after 2.2; if λ is strongly inaccessible or $\lambda = \chi^+$ & $\chi = \chi^{<\chi}$ this always holds).

(So all three are regular uncountable cardinals).
Shoot a non-reflecting stationary subset $\underset{\sim}{S}$ of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ by the forcing notion

$$Q_0 = \Big\{ h : h \text{ a function from } (\alpha+1) \text{ to } \{0,1\}, \alpha < \lambda, \\ [\delta \le \alpha \ \& \ h(\delta) = 1 \Rightarrow cf(\delta) = \kappa], \\ h^{-1}(\{1\}) \text{ is non-reflecting} \Big\}.$$

In V^{Q_0} no bounded subset of λ is added (so $NPT(\mu,\aleph_0)$, $PT(\kappa,\aleph_0)$ continue to hold). Also trivially $NPT(\lambda,\kappa)$ hence $NPT(\lambda,\mu)$ (as $\mu \ge \kappa$, e.g. for $\delta \in \underset{\sim}{S}$ choose $A_\delta \subseteq \delta$, $|A_\delta| = \ \mathrm{otp}(A_\delta) = \kappa$, $\sup(A_\delta) = \delta$).

So it is enough to prove $PT(\lambda,\aleph_0)$. If not, there is a λ-set $\mathfrak{S}$ and $\{A_\eta : \eta \in \mathfrak{S}_f\}$ witnessing it as in [Sh:161, §3] or see [Sh:521, AP]. As $PT(\kappa,\aleph_0)$ (also in V^Q) we have $\eta \in S \Rightarrow \lambda_\eta \ne \kappa$, hence (see [Sh:161, §3] or [Sh:521, AP])

(∗) $< i > \in S \Rightarrow cf(i) \ne \kappa$.

So if we kill the stationary set $\underset{\sim}{S}$ by a forcing notion
$\underset{\sim}{Q_1} = \{g : a$ a function for $\alpha + 1 < \lambda$ to $\{0,1\}, h^{-1}(\{1\})$ closed disjoint to $\underset{\sim}{S}\}$, then in the universe $\left(V^{Q_0}\right)^{\underset{\sim}{Q_1}}$, still $NPT(\lambda,\aleph_0)$ by the same example.
[Why? The set $S = \{i < \lambda : < i > \in \mathfrak{S}\}$ is stationary (as $\mathfrak{S}$ is a λ-set). Now (see [Sh:88, AP]) the set S is in $I[\lambda]$ in V^Q (as λ is) and is stationary and disjoint to $\underset{\sim}{S_0}$ hence is stationary also in $\left(V^{Q_0}\right)^{\underset{\sim}{Q_1}}$, and no bounded subset of λ is added.]

But $Q_0 * \underset{\sim}{Q}_1$ is just adding a Cohen subset of λ hence by $(*)(3)$ we know $\left(V^{Q_0}\right)^{*\underset{\sim}{Q}_1} \models PT(\lambda, \aleph_0)$. So we have gotten an example as required in the lemma, provided we start with a universe satisfying $(*)$. So when $(*)$ holds?

Case 1. λ supercompact, κ first measurable, choose any μ such that $\kappa < \mu < \lambda$, $\mu = \text{cf}(\mu)$, we can add to μ a set $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \aleph_0\}$ stationary non-reflecting and then make λ Laver indestructible by forcing notions not adding subsets of λ. So G.C.H. is O.K.

Case 2 (Small Cardinals). By the forcing in [MgSh:204] (but for λ regular adding λ-Cohen sets) the following case is O.K.

$$GCH + \kappa = \aleph_{\omega^2+1} + \mu = \aleph_{\omega_1+1} + \lambda = \aleph_{\omega_1+\omega^2+1}.$$

Case 3. $V = L, \kappa < \lambda$ are uncountable weakly compact cardinals in $L, \mu = \kappa^+, P$ is a μ^+-complete forcing notion making $(*)(3)$ hold without adding any subset to μ. (E.g. P is P_λ, where $\langle P_i, \underset{\sim}{Q}_j : i \leq \lambda, j < \lambda\rangle$ is an iteration with Easton support, λ_j-the jth inaccessible cardinal in L which is $> \kappa$, Q_j is the λ_j-Cohen forcing in V^{P_j}); i.e.
$Q_j = \{h : h \text{ a function from some } \alpha < \lambda_j \text{ to } \lambda_j \text{ in } V^{P_j}\}$.
Now in $V^P, \kappa, \mu, \lambda$ are as required. $\square_{3.1}$

By work of Jensen there is a non-reflecting stationary $S \subseteq \lambda$ for λ regular non weakly compact

REFERENCES.

[EM] P. Eklof and A. Mekler. *Almost free modules; Set theoretic methods.* North Holland Library, 1990.

[MgSh 204] Menachem Magidor and Saharon Shelah. When does almost free imply free? (For groups, transversal etc.). *Journal of the American Mathematical Society*, **accepted**.

[Sh 521] Saharon Shelah. If there is an exactly λ-free abelian group then there is a λ-separable one. *Journal of Symbolic Logic*, **to appear**.

[Sh 460] Saharon Shelah. The Generalized Continuum Hypothesis revisited. *Israel Journal of Mathematics*, **submitted**.

[Sh 108] Saharon Shelah. On successors of singular cardinals. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math*, pages 357–380. North-Holland, Amsterdam-New York, 1979.

[Sh 161] Saharon Shelah. Incompactness in regular cardinals. *Notre Dame Journal of Formal Logic*, **26**:195–228, 1985.

[Sh 88] Saharon Shelah. Classification of nonelementary classes. II. Abstract elementary classes. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 419–497. Springer, Berlin, 1987. Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh 345] Saharon Shelah. Products of regular cardinals and cardinal invariants of products of Boolean algebras. *Israel Journal of Mathematics*, **70**:129–187, 1990.

[Sh 420] Saharon Shelah. Advances in Cardinal Arithmetic. In *Finite and Infinite Combinatorics in Sets and Logic*, pages 355–383. Kluwer Academic Publishers, 1993. N.W. Sauer et al (eds.).

[Sh 355] Saharon Shelah. $\aleph_{\omega+1}$ has a Jonsson Algebra. In *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*, chapter II. Oxford University Press, 1994.

[Sh:g] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides.* Oxford University Press, 1994.