The distributivity numbers of finite products of $\mathcal{P}(\omega)$/fin

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ABSTRACT: Generalizing [ShSp], for every $n < \omega$ we construct a ZFC-model where the distributivity number of r.o.$(\mathcal{P}(\omega)/\text{fin})^{n+1}$, $\mathfrak{h}(n+1)$, is smaller than the one of r.o.$(\mathcal{P}(\omega)/\text{fin})^n$. This answers an old problem of Balcar, Pelant and Simon (see [BaPeSi]). We also show that both of Laver and Miller forcing collapse the continuum to $\mathfrak{h}(n)$ for every $n < \omega$, hence by the first result, consistently they collapse it below $\mathfrak{h}(n)$.

Introduction

For λ a cardinal let $\mathfrak{h}(\lambda)$ be the least cardinal κ for which r.o.$(\mathcal{P}(\omega)/\text{fin})^\lambda$ is not $\kappa-$distributive, where by $(\mathcal{P}(\omega)/\text{fin})^\lambda$ we mean the (full) $\lambda-$product of $\mathcal{P}(\omega)$/fin in the forcing sense; so $f \in (\mathcal{P}(\omega)/\text{fin})^\lambda$ if and only if $f : \lambda \to \mathcal{P}(\omega)/\text{fin} \setminus \{0\}$, and the ordering is coordinatewise.

In [ShSp] the consistency of $\mathfrak{h}(2) < \mathfrak{h}$ with ZFC has been proved, which provided a (partial) answer to a question of Balcar, Pelant and Simon in [BaPeSi]. This inequality holds in a model obtained by forcing with a countable support iteration of Mathias forcing over a model of GCH. The proof is long and difficult. The following are the key properties of Mathias forcing (M.f.) which are essential to the proof (see [ShSp] or below for precise definitions):

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(1) M.f. factors into a σ-closed and a σ-centered forcing.
(2) M.f. is Suslin-proper which means that, firstly, it is simply definable, and, secondly, it permits generic conditions over every countable model of ZF^-.
(3) Every infinite subset of a Mathias real is also a Mathias real.
(4) Mathias forcing does not change the cofinality of any cardinal from above $\mathfrak{h}$ to below $\mathfrak{h}$.
(5) Mathias forcing has the pure decision property and it has the Laver property.

In this paper we present a forcing Q^n, where $0 < n < \omega$, which is an n-dimensional version of M.f. which satisfies all the analogues of the five key properties of M.f. In this paper we only prove these. Once this has been done the proof of [ShSp] can be generalized in a straightforward way, to prove the following:

Theorem. *Suppose $V \models ZFC + GCH$. If P is a countable support iteration of Q^n of length ω_2 and G is P-generic over V, then $V[G] \models \mathfrak{h}(n+1) = \omega_1 \wedge \mathfrak{h}(n) = \omega_2$.*

Besides the fact that the consistency of $\mathfrak{h}(n+1) < \mathfrak{h}(n)$ was an open problem in [BaPeSi], our motivation for working on it was that in [GoReShSp] it was shown that both of Laver and Miller forcing collapse the continuum to $\mathfrak{h}$. Moreover, using ideas from [GoJoSp] and [GoReShSp] it can be proved that these forcings do not collapse $\mathfrak{c}$ below $\mathfrak{h}(\omega)$. We do not know whether they do collapse it to $\mathfrak{h}(\omega)$. But in §2 we show that they collapse it to $\mathfrak{h}(n)$, for every $n < \omega$. Combining this with the first result we conclude that for every $n < \omega$, consistently Laver and Miller forcing collapse $\mathfrak{c}$ strictly below $\mathfrak{h}(n)$.

The reader should have a copy of [ShSp] at hand. We do not repeat all the definitions from [ShSp] here. Notions as Ramsey ultrafilter, Rudin-Keisler ordering, Suslin-proper are explained there and references are given.

1. The forcing

Definition 1.1. Suppose that $D_0, \ldots, D_{n-1}$ are ultrafilters on ω. The game $G(D_0, \ldots, D_{n-1})$ is defined as follows: In his mth move player I chooses $\langle A_0, \ldots, A_{n-1}\rangle \in D_0 \times \ldots \times D_{n-1}$ and player II responds playing $k_m \in A_{m \bmod n}$. Finally player II wins if and only if for every $i < n$, $\{k_j : j = i \bmod n\} \in D_i$ holds.

Lemma 1.2. *Suppose $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent. Let $\langle m(l) : l < \omega\rangle$ be an increasing sequence of integers. There exists a subsequence $\langle m(l_j) : j < \omega\rangle$ and sets $Z_i \in D_i$, $i < n$, such that:*
(1) $l_{j+1} - l_j \geq 2$, for all $j < \omega$,
(2) *$Z_i \subseteq \bigcup_{j = i \bmod n} [m(l_j), m(l_{j+1}))$, for all $i < n$,*
(3) *$Z_i \cap [m(l_j), m(l_{j+1}))$ has precisely one member, for every $i < n$ and $j = i \bmod n$.*

PROOF: For $j < 3$, $k < \omega$ define:

$$I_{j,k} = \bigcup_{s=(2n-1)(3k+j)}^{(2n-1)(3k+j+1)-1} [m_s, m_{s+1}),$$

$$J_j = \bigcup_{k<\omega} I_{j,k}.$$

As the D_i are Ramsey ultrafilters, there exist $X_i \in D_i$ such that for every $i < n$:

(a) $X_i \subseteq J_j$ for some $j < 3$,

(b) if $X_i \subseteq J_j$, then $X_i \cap I_{j,k}$ contains precisely one member, for every $k < \omega$.

Next we want to find $Y_i \in D_i$, $Y_i \subseteq X_i$, such that for every distinct $i, i' < n$, Z_i and $Z_{i'}$ do not meet any adjacent intervals $I_{j,k}$.

Define $h : X_0 \to X_1$ as follows. Suppose $X_0 \subseteq J_j$. For every $k < \omega$, h maps the unique element of $X_0 \cap I_{j,k}$ to the unique element of X_1 which belongs to either $I_{j,k}$ or to one of the two intervals of the form $I_{j',k'}$ which are adjacent to $I_{j,k}$ (note that these are $I_{2,k-1}, I_{1,k}$ if $j = 0$, or $I_{0,k}, I_{2,k}$ if $j = 1$, or $I_{1,k}, I_{0,k+1}$ if $j = 2$). As h does not witness that D_0, D_1 are RK-equivalent, there exist $X'_i \in D_i$, $X'_i \subseteq X_i$ ($i < 2$) such that $h[X'_0] \cap X'_1 = \emptyset$. Note that if $n = 2$, we can let $Y_i = X'_i$. Otherwise we repeat this procedure, starting from X'_0 and X_2, and get X''_0 and X'_2. We repeat it again, starting from X'_1 and X'_2, and get X''_1 and X''_2. If $n = 3$ we are done. Otherwise we continue similarly. After finitely many steps we obtain Y_i as desired.

By definition of $I_{j,k}$ it is now easy to add more elements to each Y_i in order to get Z_i as in the Lemma. The "worst" case is that some Y_i contains integers $s < t$ such that $(s,t) \cap Y_u = \emptyset$ for all $u < n$. By construction there is some $I_{j,k} \subseteq (s,t)$. For every $u < n-1$ pick

$$x_u \in [m((2n-1)(3k+j) + 2u + 1), m((2n-1)(3k+j) + 2u + 2))$$

and add x_u to $Y_{i+u+1 \bmod n}$. The other cases are similar. □

Corollary 1.3. *Suppose $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent. Then in the game $G(D_0, \ldots, D_{n-1})$ player I does not have a winning strategy.*

PROOF: Suppose σ is a strategy for player I. For every $m < \omega, i < n$ let $\mathcal{A}_i^m \subseteq D_i$ be the set of all ith coordinates of moves of player I in an initial segment of length at most $2m + 1$ of a play in which player I follows σ and player II plays only members of m.

As the D_i are p−points and each $\mathcal{A}_i^m$ is finite, there exist $X_i \in D_i$ such that $\forall m \forall i < n \forall A \in \mathcal{A}_i^m (X_i \subseteq^* A)$. Moreover we may clearly find a strictly increasing sequence $\langle m(l) : l < \omega \rangle$ such that $m(0) = 0$ and for all $l < \omega$:

$$\forall i < n \forall A \in \mathcal{A}_i^{m(l)} (X_i \subseteq A \cup m(l+1) \wedge X_i \cap [m(l), m(l+1)) \neq \emptyset).$$

Applying Lemma 1.2, we obtain a subsequence $\langle m(l_j) : j < \omega \rangle$ and sets $Z_i \in D_i$.

Now let in his jth move player II play k_j, where k_j is the unique member of $[m(l_j), m(l_{j+1})) \cap X_{j \bmod n} \cap Z_{j \bmod n}$ if it exists, or otherwise is any member of $[m(l_j), m(l_{j+1})) \cap X_{j \bmod n}$ (note that this intersection is nonempty by definition of $m(l_{j+1})$. Then this play is consistent with σ, moreover $X_i \cap Z_i \subseteq \{k_j : j = i \bmod n\}$ for every $i < n$, and hence it is won by player II. Consequently σ could not have been a winning strategy for player I. □

Definition 1.3. Let $n < \omega$ be fixed. The forcing Q (really Q^n) is defined as follows: Its members are $(w, \bar{A}) \in [\omega]^{<\omega} \times [\omega]^\omega$. If $\langle k_j : j < \omega\rangle$ is the increasing enumeration of $\bar{A}$ we let $\bar{A}_i = \{k_j : j = i \bmod n\}$ for $i < n$, and if $\langle l_j : j < m\rangle$ is the increasing enumeration of w then let $w_i = \{l_j : j = i \bmod n\}$, for $i < n$.

Let $(w, \bar{A}) \leq (v, \bar{B})$ if and only if $w \cap (\max(v) + 1) = v$, $w_i \setminus v_i \subseteq \bar{B}_i$ and $\bar{A}_i \subseteq \bar{B}_i$, for every $i < n$.

If $p \in Q$, then $w^p, w_i^p, \bar{A}^p, \bar{A}_i^p$ have the obvious meaning. We write $p \leq^0 q$ and say "p is a pure extension of q" if $p \leq q$ and $w^p = w^q$.

If $D_0, \ldots, D_{n-1}$ are ultrafilters on ω, let $Q(D_0, \ldots, D_{n-1})$ denote the subordering of Q containing only those $(w, \bar{A}) \in Q$ with the property $\bar{A}_i \in D_i$, for every $i < n$.

Lemma 1.4. *The forcing Q is equivalent to $(\mathcal{P}(\omega)/fin)^n * Q(\dot{G}_0, \ldots \dot{G}_{n-1})$, where $(\dot{G}_0, \ldots, \dot{G}_{n-1})$ is the canonical name for the generic object added by $(\mathcal{P}(\omega)/fin)^n$, which consists of n pairwise not RK-equivalent Ramsey ultrafilters.*

Proof: Clearly $(\mathcal{P}(\omega)/\mathrm{fin})^n$ is $\sigma-$closed and hence does not add reals. Moreover, members $\langle x_0, \ldots, x_{n-1}\rangle \in (\mathcal{P}(\omega)/\mathrm{fin})^n$ with the property that if $\bar{A} = \bigcup\{x_i : i < n\}$, then $x_i = \bar{A}_i$ for every $i < n$ are dense. Hence the map $(w, \bar{A}) \mapsto (\langle \bar{A}_0, \ldots, \bar{A}_{n-1}\rangle, (w, \bar{A}))$ is a dense embedding of the respective forcings.

That $\dot{G}_0, \ldots, \dot{G}_{n-1}$ are $((\mathcal{P}(\omega)/\mathrm{fin})^n-$forced to be) pairwise not RK-equivalent Ramsey ultrafilters follows by an easy genericity argument and again the fact that no new reals are added. □

Notation. We will usually abbreviate the decomposition of Q from Lemma 1.4. by writing $Q = Q' * Q''$. So members of Q' are $\bar{A}, \bar{B} \in [\omega]^\omega$ ordered by $\bar{A}_i \subseteq \bar{B}_i$ for all $i < n$; Q'' is $Q(\dot{G}_0, \ldots, \dot{G}_{n-1})$. If G is a $Q-$generic filter, by $G' * \dot{G}''$ we denote its decomposition according to $Q = Q' * \dot{Q}''$, and we write $G' = (G'_0, \ldots, G'_{n-1})$.

Definition 1.5. Let $I \subseteq Q(D_0, \ldots, D_{n-1})$ be open dense. We define a rank function rk_I on $[\omega]^{<\omega}$ as follows. Let $\mathrm{rk}_I(w) = 0$ if and only if $(w, \bar{A}) \in I$ for some $\bar{A}$. Let $\mathrm{rk}_I(w) = \alpha$ if and only if α is minimal such that there exists $A \in D_{|w| \bmod n}$ with the property that for every $k \in A$, $\mathrm{rk}_I(w \cup \{k\}) = \beta$ for some $\beta < \alpha$. Let $\mathrm{rk}_I(w) = \infty$ if for no ordinal α, $\mathrm{rk}_I(w) = \alpha$.

Lemma 1.6. *If $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent and $I \subseteq Q(D_0, \ldots, D_{n-1})$ is open dense, then for every $w \in [\omega]^{<\omega}$, $rk_I(w) \neq \infty$.*

Proof: Suppose we had $\mathrm{rk}_I(w) = \infty$ for some w. We define a strategy σ for player I in $G(D_0, \ldots, D_{n-1})$ as follows: $\sigma(\emptyset) = \langle A_0, \ldots, A_{n-1}\rangle \in D_0 \times \ldots \times D_{n-1}$ such that for every $k \in A_{|w| \bmod n}$, $\mathrm{rk}_I(w \cup \{k\}) = \infty$. This choice is possible by assumption and as the D_i are ultrafilters. In general, suppose that σ has been defined for plays of length $2m$ such that whenever $k_0, \ldots, k_{m-1}$ are moves of player II which are consistent with σ, then $k_0 < k_1 \ldots < k_{m-1}$ and for every $\{k_{i_0} < \ldots < k_{i_{l-1}}\} \subseteq \{k_0, \ldots, k_{m-1}\}$ with $i_j = j \bmod n, j < l$, we have $\mathrm{rk}_I(w \cup \{k_{i_0}, \ldots, k_{i_{l-1}}\}) = \infty$. Let S be the set of all $\{k_{i_0} < \ldots < k_{i_{l-1}}\} \subseteq \{k_0, \ldots, k_{m-1}\}$ with $i_j = j \bmod n, j < l$, and $l = m \bmod n$. As $D_{|w|+m \bmod n}$ is an ultrafilter, by induction hypothesis we have that, letting

$$A_{|w|+m \bmod n} = \{k > k_{m-1} : \forall s \in S \quad \mathrm{rk}_I(w \cup s \cup \{k\}) = \infty\},$$

$A_{|w|+m \mathrm{mod} n} \in D_{|w|+m \mathrm{mod} n}$. For $i \neq |w| + m \mathrm{mod} n$, choose $A_i \in D_i$ arbitrarily, and define

$$\sigma\langle k_0, \ldots k_{m-1}\rangle = \langle A_0, \ldots, A_{n-1}\rangle.$$

Since by Lemma 1.2. σ is not a winning strategy for player I, there exist $k_0 < \ldots < k_m < \ldots$ which are moves of player II consistent with σ, such that, letting $\bar{A} = \{k_m : m < \omega\}$, we have $(w, \bar{A}) \in Q(D_0, \ldots, D_{n-1})$. By construction we have that for every $(v, \bar{B}) \leq (w, \bar{A})$, $\mathrm{rk}_I(v) = \infty$. This contradicts the assumption that I is dense.

Definition 1.7. Let $p \in Q$. A set of the form $w^p \cup \{k_{|w|} < k_{|w|+1} < \ldots\} \in [\omega]^\omega$ is called a *branch* of p if and only if $\max(w^p) < k_{|w|}$ and $\{k_j : j = i \mathrm{mod} n\} \subseteq \bar{A}^p_i$, for every $i < n$. A set $F \subseteq [\omega]^{<\omega}$ is called a *front* in p if for every $w \in F$, $(w, \bar{A}^p) \leq p$ and for every branch B of p, $B \cap m \in F$ for some $m < \omega$.

Lemma 1.8. *Suppose $D_0, \ldots, D_{n-1}$ are pairwise not RK-equivalent Ramsey ultrafilters. Suppose $p \in Q(D_0, \ldots, D_{n-1})$ and $\langle I_m : m < \omega\rangle$ is a family of open dense sets in $Q(D_0, \ldots, D_{n-1})$. There exists $q \in Q(D_0, \ldots, D_{n-1})$, $q \leq^0 p$, such that for every m, $\{w \in [\omega]^{<\omega} : (w, \bar{A}^q) \in I_m \wedge (w, \bar{A}^q) \leq q\}$ is a front in q.*

Proof: First we prove it in the case $I_m = I$ for all $m < \omega$, by induction on $\mathrm{rk}_I(w^p)$. We define a strategy σ for player I in $G(D_0, \ldots, D_{n-1})$ as follows. Generally we require that

$$\sigma\langle k_0, \ldots, k_r\rangle_i \subseteq \sigma\langle k_0, \ldots, k_s\rangle_i$$

for every $s < r$ and $i < n$, where $\sigma\langle k_0, \ldots, k_r\rangle_i$ is the ith coordinate of $\sigma\langle k_0, \ldots, k_r\rangle$. We also require that σ ensures that the moves of II are increasing (see the proof of 1.7). Define $\sigma(\emptyset) = \langle A_0, \ldots, A_{n-1}\rangle$ such that for every $k \in A_{|w^p| \mathrm{mod} n}$, $\mathrm{rk}_I(w^p \cup \{k\}) < \mathrm{rk}_I(w^p)$.

Suppose now that σ has been defined for plays of length $2m$, and let $\langle k_0, \ldots, k_{m-1}$ be moves of II, consistent with σ. The interesting case is that $m - 1 = 0 \mathrm{mod} n$. Let us assume this first. By definition of $\sigma(\emptyset)$ and the general requirement on σ we conclude $\mathrm{rk}_I(w^p \cup \{k_{m-1}\}) < \mathrm{rk}_I(w^p)$. By induction hypothesis there exists $\langle A_0, \ldots, A_{n-1}\rangle \in D_0 \times \ldots \times D_{n-1}$ such that, letting $\bar{A} = \bigcup_{i<n} A_i$, we have $(w^p, \bar{A}) \leq p$ and

$$\{v \in [\omega]^{<\omega} : (v, \bar{A}) \in I \wedge (v, \bar{A}) \leq (w^p \cup \{k_{m-1}\}, \bar{A})\}$$

is a front in $(w^p \cup \{k_{m-1}\}, \bar{A})$. We shrink $\bar{A}$ such that, letting

$$\sigma\langle k_0, \ldots, k_{m-1}\rangle = \langle A_0, \ldots, A_{n-1}\rangle,$$

the general requirements on σ above are satisfied.

In the case that $m - 1 \neq 0 \mathrm{mod} n$, define $\sigma\langle k_0, \ldots, k_{m-1}\rangle$ arbitrarily, but consistent with the rules and the general reqirements above.

Let $\bar{A} = \{k_i : i < \omega\}$ be moves of player II witnessing that σ is not a winning strategy. Let $q = (w^p, \bar{A})$. Let $B = w^p \cup \{l_{|w^p|} < l_{|w^p|+1} < \ldots\}$ be a branch of q. Hence $l_{|w^p|} = k_j$ for some $j = 0 \mathrm{mod} n$. Then $w^p \cup \{k_j\} \cup \{l_{|w^p|+1}, l_{|w^p|+2}, \ldots\}$ is a branch of $(w^p \cup \{k_j\}, \sigma\langle k_0, \ldots, k_j\rangle)$. By definition of σ there exists m such that $(B \cap m, \sigma\langle k_0, \ldots, k_j\rangle) \in I$. As $(B \cap m, \bar{A}) \leq (B \cap m, \sigma\langle k_0, \ldots, k_j\rangle)$ and I is open we are done.

For the general case where we have infinitely many I_m, we make a diagonalization, using the first part of the present proof. Define a strategy σ for player I satisfying the same general requirements as in the first part as follows. Let $\sigma(\emptyset) = \langle A_0, \dots, A_{n-1} \rangle$ such that, letting $\bar{A} = \bigcup\{A_i : i < n\}$, $(w^p, \bar{A}) \leq^0 p$ and it satisfies the conclusion of the Lemma for I_0. In general, let $\sigma\langle k_0, \dots, k_{m-1} \rangle = \langle A_0, \dots, A_{n-1} \rangle$ such that, letting $\bar{A} = \bigcup\{A_i : i < n\}$, for every $v \subseteq \{k_i : i < m\}$ and $j \leq m$, $(w^p \cup v, \bar{A}) \leq^0 (w^p \cup v, \bar{A}^p)$ and it satisfies the conclusion of the Lemma for I_j (In fact we don't have to consider all such v here, but it does not hurt doing it). If then $\bar{A} = \{k_i : i < \omega\}$ are moves of player II witnessing that σ is not a winning strategy for I, similarly as in the first part it can be verified that $q = (w^p, \bar{A})$ is as desired. □

Corollary 1.9. *Let $D_0, \dots, D_{n-1}$ be pairwise not RK-equivalent Ramsey ultrafilters. Suppose $\bar{A} \in [\omega]^\omega$ is such that for every $i < n$ and $X \in D_i$, $\bar{A}_i \subseteq^* X$. Then $\bar{A}$ is $Q(D_0, \dots, D_{n-1})-$generic over V.*

Proof: Let $I \subseteq Q(D_0, \dots, D_{n-1})$ be open dense. Let $w \in [\omega]^{<\omega}$. It is easy to see that the set

$$I_w = \{(v, \bar{B}) \in Q(D_0, \dots, D_{n-1}) : (w \cup [v \setminus \min\{k \in v_{|w| \mathrm{mod} n} : k > \max(w)\}], \bar{B}) \in I\}$$

is open dense. If we apply Lemma 1.8. to $p = (\emptyset, \omega, \dots, \omega)$ and the countably many open dense sets I_w where $w \in [\omega]^{<\omega}$, we obtain $q = (\emptyset, \bar{B})$. Let $\langle a_i : i < \omega \rangle$ be the increasing enumeration of $\bar{A}$. Choose m large enough so that for each $i < n$, $\bar{A}_i \setminus \{a_j : j < mn\} \subseteq \bar{B}_i$. Let $w = \{a_j : j < mn\}$. By construction, there exists $v \subseteq \bar{A} \cap \bar{B} \setminus (a_{mn-1} + 1)$ such that $(v, \bar{B}) \in I_w$ and $w \cup v = \bar{A} \cap k$, for some $k < \omega$. Hence $(w \cup v, \bar{B}) \in I$, and so the filter on $Q(D_0, \dots, D_{n-1})$ determined by $\bar{A}$ intersects I. As I was arbitrary, we are done. □

An immediate consequence of Lemma 1.4. and Corollary 1.9. is the following.

Corollary 1.10. *Suppose $\bar{A} \in [\omega]^\omega$ is $Q-$generic over V, and $\bar{B} \in [\omega]^\omega$ is such that $\bar{B}_i \subseteq \bar{A}_i$ for every $i < n$. Then $\bar{B}$ is $Q-$generic over V as well.*

Remember that a forcing is called Suslin, if its underlying set is an analytic set of reals and its order and incompatibility relations are analytic subsets of the plane. A forcing P is called Suslin-proper if it is Suslin and for every countable transitive model $(N, \in)$ of ZF$^-$ which contains the real coding P and for every $p \in P \cap N$, there exists a $(N, P)-$generic condition extending p. See [JuSh] for the theory of Suslin-proper forcing and [ShSp] for its properties which are relevant here.

Corollary 1.11. *The forcing Q is Suslin-proper.*

Proof: It is trivial to note that Q is Suslin, without parameter in its definition. Let $(N, \in)$ be a countable model of ZFC$^-$, and let $p \in Q \cap N$. Without loss of generality, $|w^p| = 0 \mathrm{mod} n$. Let $\bar{A} \in [\omega]^\omega \cap V$ be $Q-$generic over N such that p belongs to its generic filter. Hence $w^p_i \subseteq \bar{A}_i \subseteq w^p_i \cup (\bar{A}^p_i \setminus (\max(w^p) + 1))$ for all $i < n$. But if $q = (w^p, \bar{A})$, then clearly $q \leq^0 p$ and q is $(N, Q)-$generic, as every $\bar{B} \in [\omega]^\omega$ which is $Q-$generic over V and contains q in its generic filter is a subset of $\bar{A}$ and hence $Q \cap N-$generic over N by Corollary 1.10. applied in N. □

The following is an immediate consequence of Corollary 1.11.

Corollary 1.12. *If $p \in Q$ and $\langle \tau_n : n < \omega \rangle$ are $Q-$names for members of V, there exist $q \in Q$, $q \leq^0 p$ and $\langle X_n : n < \omega \rangle$ such that $X_n \in V \cap [V]^\omega$ and $q \Vdash_Q \forall n (\tau_n \in X_n)$.*

Corollary 1.13. *Forcing with Q does not change the cofinality of any cardinal λ with $cf(\lambda) \geq \mathfrak{h}(n)$ to a cardinal below λ.*

PROOF: Suppose there were a cardinal $\kappa < \mathfrak{h}(n)$ and a $Q-$name $\dot{f}$ for a cofinal function from κ to λ. Working in V and using Corollary 1.12., for every $\alpha < \kappa$ we may construct a maximal antichain $\langle p^\alpha_\beta : \beta < \mathfrak{c}\rangle$ in Q and $\langle X^\alpha_\beta : \beta < \mathfrak{c}\rangle$ such that for all $\beta < \mathfrak{c}$, $w^{p^\alpha_\beta} = \emptyset$, $X^\alpha_\beta \in [V]^\omega \cap V$ and $p^\alpha_\beta \Vdash_Q \dot{f}(\alpha) \in X^\alpha_\beta$.

Then clearly $\mathcal{A}_\alpha = \langle\langle \bar{A}_i^{p^\alpha_\beta} : i < n\rangle : \beta < \mathfrak{c}\rangle$ is a maximal antichain in $(\mathcal{P}(\omega)/\mathrm{fin})^n$. By $\kappa < \mathfrak{h}(n)$, $\langle \mathcal{A}_\alpha : \alpha < \kappa\rangle$ has a refinement, say $\mathcal{A}$. Choose $\langle \bar{A}_i : i < n\rangle \in \mathcal{A}$. Let $\bar{A} = \bigcup\{\bar{A}_i : i < n\}$. We may assume that the $\bar{A}_i$ also have the meaning from Definition 1.3. with respect to $\bar{A}$. For each $\alpha < \kappa$ there exists $\beta(\alpha)$ such that $\langle \bar{A}_i : i < n\rangle \leq_{(\mathcal{P}(\omega)/fin)^n} \langle \bar{A}_i^{p^\alpha_{\beta(\alpha)}} : i < n\rangle$. Then clearly

$$(\emptyset, \bar{A}) \Vdash_Q \mathrm{range}(\dot{f}) \subseteq \bigcup\{X^\alpha_{\beta(\alpha)} : \alpha < \kappa\}\,.$$

But as $\mathrm{cf}(\lambda) \geq \mathfrak{h}(n)$ and $\kappa < \mathfrak{h}(n)$, we have a contradiction. □

Lemma 1.14. *Suppose $D_0, \ldots, D_{n-1}$ are pairwise not RK-equivalent Ramsey ultrafilters. Then $Q(D_0, \ldots, D_{n-1})$ has the pure decision property (for finite disjunctions), i.e. given a $Q(D_0, \ldots, D_{n-1})-$name τ for a member of $\{0,1\}$ and $p \in Q(D_0, \ldots, D_{n-1})$, there exist $q \in Q(D_0, \ldots, D_{n-1})$ and $i \in \{0,1\}$ such that $q \leq^0 p$ and $q \Vdash_{Q(D_0,\ldots,D_{n-1})} \tau = i$.*

PROOF: The set $I = \{r \in Q(D_0, \ldots, D_{n-1}) : r \text{ decides } \tau\}$ is open dense. By a similar induction on rk_I as in the proof of Lemma 1.7. we may find $q \in Q(D_0, \ldots, D_{n-1})$, $q \leq^0 p$, such that for every $q' \leq q$, if q' decides τ then $(w^{q'}, \bar{A}^q)$ decides τ. Now again by incuction on rk_I we may assume that for every $k \in \bar{A}^q_{|w^q| \mathrm{mod} n}$, $(w^q \cup \{k\}, \bar{A}^q)$ satisfies the conclusion of the Lemma, and hence by the construction of q, $(w^q \cup \{k\}, \bar{A}^q)$ decides τ. But then clearly a pure extension of q decides τ, and hence q does. □

Lemma 1.15. *Lemma 1.14 holds if $Q(D_0, \ldots, D_{n-1})$ is replaced by Q.*

PROOF: Suppose $p \in Q$, τ is a $Q-$name and $p \Vdash_Q \tau \in \{0,1\}$. As $\bar{A}^p \Vdash_{Q'}$ "$p \in Q(\dot{G}_0, \ldots, \dot{G}_{n-1})$", by Lemma 1.14 there exists a $Q'-$name $\dot{\bar{A}}$ such that

$$\bar{A}^p \Vdash_{Q'} \text{"}(w^p, \dot{\bar{A}}) \in Q'' \wedge (w^p, \dot{\bar{A}}) \leq p \wedge (w^p, \dot{\bar{A}}) \text{ decides } \tau\text{"}\,.$$

As Q' does not add reals there exist $\bar{A}_1, \bar{A}_2 \in [\omega]^\omega \cap V$ such that $\bar{A}_1 \subseteq \bar{A}^p$ and $\bar{A}_1 \Vdash_{Q'} \dot{\bar{A}} = \bar{A}_2$. Letting $\bar{B} = \bar{A}_1 \cap \bar{A}_2$ we conclude $(w^p, \bar{B}) \in Q$, $(w^p, \bar{B}) \leq^0 p$ and $(w^p, \bar{B})$ decides τ. □

The rest of this section is devoted to the proof that if the forcing Q is iterated with countable supports, then in the resulting model $\mathrm{cov}(\mathcal{M}) = \omega_1$, where $\mathcal{M}$ is the ideal of meagre subsets of the real line, and $\mathrm{cov}(\mathcal{M})$ is the least number of meagre sets needed to cover the real line. Hence for every $n < \omega$, we obtain the consistency of $\mathrm{cov}(\mathcal{M}) < \mathfrak{h}(n)$.

Definition 1.16. A forcing P is said to have the *Laver property* if for every $P-$name $\dot{f}$ for a member of ${}^\omega\omega$, $g \in {}^\omega\omega \cap V$ and $p \in P$, if

$$p \Vdash_P \forall n < \omega(\dot{f}(n) < g(n))\,,$$

then there exist $H : \omega \to [\omega]^{<\omega}$ and $q \in P$ such that $H \in V$, $\forall n < \omega(|H(n)| \leq 2^n)$, $q \leq p$ and

$$q \Vdash_P \forall n < \omega(\dot{f}(n) \in H(n)) .$$

It is not difficult to see that a forcing with the Laver property does not add Cohen reals. Moreover, by [Shb, 2.12., p.207] the Laver property is preserved by a countable support iteration of proper forcings. See also [Go, 6.33., p.349] for a more accessible proof.

Lemma 1.17. *The Forcing Q has the Laver property.*

Suppose $\dot{f}$ is a $Q-$name for a member of ${}^{\omega}\omega$ and $g \in {}^{\omega}\omega \cap V$ such that $p \Vdash_Q \forall n < \omega(\dot{f}(n) < g(n))$. We shall define $q \leq^0 p$ and $\langle H(i) : i < \omega \rangle$ such that $|H(i)| \leq 2^i$ and $q \Vdash_Q \forall i(\dot{f}(i) \in H(i))$. We may assume $|w^p| = 0 \mathrm{mod} n$ and $\min(\bar{A}^p) > \max(w^p)$.

By Lemma 1.14 choose $q_0 \leq^0 p$ and K^0 such that $q_0 \Vdash_Q \dot{f}(0) = K^0$, and let $H(0) = \{K^0\}$.

Suppose $q_i \leq^0 p$, $\langle H(j) : j \leq i \rangle$ have been constructed and let a^i be the set of the first $i+1$ members of $\bar{A}^{q_i}$. Let $\langle v^k : k < k^* \rangle$ list all subsets v of a^i such that $v_l \subseteq (a^i)_l$, for every $l < n$ (see Definition 1.3.). Then clearly $k^* \leq 2^{i+1}$. By Lemma 1.14 we may shrink $\bar{A}^{q_i}$ k^* times so to obtain $\bar{A}$ and $\langle K^{i+1}_k : k < k^* \rangle$ such that for every $k < k^*$, $(w^{q_i} \cup v^k, \bar{A}) \Vdash_Q \dot{f}(i+1) = K^{i+1}_k$. Without loss of generality, $\min(\bar{A}) > \max(a^i)$. Let q_{i+1} be defined by $w^{q_{i+1}} = w^p$ and $\bar{A}^{q_{i+1}} = a^i \cup \bar{A}'$, where $\bar{A}'$ is $\bar{A}$ without its first $(i+1)\mathrm{mod} n$ members. Let $H(i+1) = \{K^{i+1}_k : k < k^*\}$. Then $q^{i+1} \Vdash_Q \dot{f}(i+1) \in H(i+1)$. Finally let q be defined by $w^q = w^p$ and $\bar{A}^q = \bigcup\{a^i : i < \omega\}$. Then q and $\langle H(i) : i < \omega \rangle$ is as desired. □

As explained above, from Lemma 1.17 and Shelah's preservation theorem it follows that if P is a countable support iteration of Q and G is $P-$generic over V, then in $V[G]$ no real is Cohen over V; equivalently, the meagre sets in V cover all the reals of $V[G]$. Now starting with V satisfying CH we obtain the following theorem.

Theorem 1.18. *For every $n < \omega$, the inequality* $\mathrm{cov}(\mathcal{M}) < \mathfrak{h}(n)$ *is consistent with ZFC.*

2. Both of Laver and Miller forcing collapse the continuum below each $\mathfrak{h}(n)$

Definition 2.1. Let $p \subseteq {}^{<\omega}\omega$ be a tree. For any $\eta \in p$ let $\mathrm{succ}_\eta(p) = \{n < \omega : \eta\hat{\ }\langle n \rangle \in p\}$. We say that p has a stem and denote it stem(p), if there is $\eta \in p$ such that $|\mathrm{succ}_\eta(p)| \geq 2$ and for every $\nu \subset \eta$, $|\mathrm{succ}_\nu(p)| = 1$. Clearly stem$(p)$ is uniquely determined, if it exists. If p has a stem, by p^- we denote the set $\{\eta \in p : \mathrm{stem}(p) \subseteq \eta\}$. We say that p is a Laver tree if p has a stem and for every $\eta \in p^-$, $\mathrm{succ}_\eta(p)$ is infinite. We say that p is superperfect if for every $\eta \in p$ there exists $\nu \in p$ with $\eta \subseteq \nu$ and $|\mathrm{succ}_\nu(p)| = \omega$. By $\mathbb{L}$ we denote the set of all Laver trees, ordered by reverse inclusion. By $\mathbb{M}$ we denote the set of all superperfect trees, ordered by reverse inclusion. $\mathbb{L}, \mathbb{M}$ is usually called Laver, Miller forcing, respectively.

Theorem 2.2. *Suppose that G is $\mathbb{L}-$generic or $\mathbb{M}-$generic over V. Then in $V[G]$, $|\mathfrak{c}^V| = |\mathfrak{h}(n)|^V$.*

Proof: Completely similarly as in [BaPeSi] for the case $n = 1$, a base tree T for $(\mathcal{P}(\omega)/\mathrm{fin})^n$ of height $\mathfrak{h}(n)$ can be constructed. I.e.

(1) $T \subseteq (\mathcal{P}(\omega)/\mathrm{fin})^n$ is dense;

(2) $(T, \supseteq^*)$ is a tree of height $\mathfrak{h}(n)$;

(3) each level T_α, $\alpha < \mathfrak{h}(n)$, is a maximal antichain in $(\mathcal{P}(\omega)/\mathrm{fin})^n$;

(4) every member of T has 2^ω immediate successors.

It follows easily that, firstly, every chain in T of length of countable cofinality has an upper bound, and secondly, every member of T has an extension in T_α for arbitrarily large $\alpha < \mathfrak{h}(n)$.

Using T, we will define a $\mathbb{L}-$name for a map from $\mathfrak{h}(n)$ onto $\mathfrak{c}$. For $p \in \mathbb{L}$ and $\{\eta_0, \dots, \eta_{n-1}\} \in [p^-]^n$, let $\bar{A}^p_{\{\eta_i : i<n\}} = \langle \mathrm{succ}_{\eta_i}(p) : i < n\rangle$.

By induction on $\alpha < \mathfrak{c}$ we will construct $(p_\alpha, \delta_\alpha, \gamma_\alpha) \in \mathbb{L} \times \mathfrak{h}(n) \times \mathfrak{c}$ such that the following clauses hold:

(5) if $\{\eta_0, \dots, \eta_{n-1}\} \in [p_\alpha]^n$, then $\bar{A}^{p_\alpha}_{\{\eta_i : i<\omega\}} \in T_{\delta_\alpha}$;

(6) if $\beta < \alpha$, $\delta_\beta = \delta_\alpha$, $\{\eta_0, \dots, \eta_{n-1}\} \in [p^-_\alpha]^n \cap [p^-_\beta]^n$, then $\bar{A}^{p_\alpha}_{\{\eta_i : i<n\}}$, $\bar{A}^{p_\beta}_{\{\eta_i : i<n\}}$ are incompatible in $(\mathcal{P}(\omega)/\mathrm{fin})^n$;

(7) if $p \in \mathbb{L}$, $\gamma < \mathfrak{c}$, then for some $\alpha < \mathfrak{c}$, every extension of p_α is compatible with p and $\gamma_\alpha = \gamma$.

At stage α, by a suitable bookkeeping we are given $\gamma < \mathfrak{c}$, $p \in \mathbb{L}$, and have to find δ_α, p_α such that (5), (6), (7) hold. For $\eta \in p^-$ let $B_\eta = \mathrm{succ}_\eta(p)$; for $\eta \in {}^{<\omega}\omega \setminus p^-$, $B_\eta = \omega$. Let $\langle \{\eta^i_0, \dots, \eta^i_{n-1}\} : i < \omega\rangle$ list $[{}^{<\omega}\omega]^n$ such that every member is listed $\aleph_0$ times.

Inductively we define $\langle \xi_i : i < \omega\rangle$ and $\langle B^\rho_\eta : \eta \in {}^{<\omega}\omega, \rho \in {}^{<\omega}2\rangle$ such that

(8) $B^\rho_\eta \in [\omega]^\omega$ and $\langle \xi_i : i < \omega\rangle$ is a strictly increasing sequence of ordinals below $\mathfrak{h}(n)$;

(9) $B^\emptyset_\eta = B_\eta$;

(10) for every $i < \omega$, the map $\rho \mapsto \langle B^\rho_{\eta^i_0}, \dots, B^\rho_{\eta^i_{n-1}}\rangle$ is one-to-one from ${}^{i+1}2$ into T_{ξ_i};

(11) for every $i < k$, for every $\rho \in {}^{k+1}2$, $B^\rho_\eta \subseteq^* B^{\rho\restriction i+1}_\eta \subseteq^* B^\emptyset_\eta$;

Suppose that at stage i of the construction, $\langle \xi_j : j < i\rangle$ and $\langle B^\rho_\eta : \eta \in \{\eta^j_0, \dots, \eta^j_{n-1} : j < i\}, \rho \in {}^{\leq i}2\rangle$ have been constructed.

For $\eta \in \{\eta^i_0, \dots, \eta^i_{n-1}\}$ and $\rho \in {}^{\leq i}2$, if B^ρ_η is not yet defined, there is no problem to choose it such that (8) and (11) hold. Next by the properties of T it is easy to find ξ_i and B^ρ_η, for every $\rho \in {}^{i+1}2$ and $\eta \in \{\eta^i_0, \dots, \eta^i_{n-1}\}$ such that (8), (9), (10), (11) hold up to i.

By the remark following the properties of T, letting $\delta_\alpha = \sup\{\xi_i : i < \omega\}$, for every $\eta \in {}^{<\omega}\omega$ and $\rho \in {}^\omega 2$, there exists $B^\rho_\eta \in [\omega]^\omega$ such that

(12) for all $i < \omega$, $B^\rho_\eta \subseteq^* B^{\rho\restriction i}_\eta$;

(13) for all $\{\eta_0, \dots, \eta_{n-1}\} \in [{}^{<\omega}\omega]^n$, $\langle B^\rho_{\eta_0}, \dots, B^\rho_{\eta_{n-1}}\rangle \in T_{\delta_\alpha}$.

For $\rho \in {}^\omega 2$ let $p^\rho \in \mathbb{L}$ be defined by stem$(p^\rho)=$ stem(p_α)

$\forall\eta\in(p^\rho)^-(\mathrm{succ}_\eta(p^\rho)=B^\rho_\eta)$.

It is easy to see that every extension of p^ρ is compatible with p_α. Moreover, if $\{\eta_0,\dots,\eta_{n-1}\}\in[(p^\rho)^-]$, then $\bar{A}^{p^\rho}_{\{\eta_i:i<n\}}\in T_{\delta_\alpha}$ by construction. Hence we have to find $\rho\in{}^\omega 2$ such that, letting $p_\alpha=p^\rho$, (6) holds. Note that for every $\{\eta_0,\dots,\eta_{n-1}\}\in[{}^{<\omega}\omega]^n$ and $\beta<\alpha$ with $\delta_\beta=\delta_\alpha$ and $\{\eta_0,\dots,\eta_{n-1}\}\in[p^-_\beta]^n$ there exists at most one $\rho\in{}^\omega 2$ such that $\{\eta_0,\dots,\eta_{n-1}\}\in[(p^\rho)^-]^n$ and $\bar{A}^{p^\rho}_{\{\eta_i:i<n\}}$, $\bar{A}^{p_\beta}_{\{\eta_i:i<n\}}$ are compatible in $(\mathcal{P}(\omega)/\mathrm{fin})^n$. In fact, by construction and as T_{δ_α} is an antichain, either $\bar{A}^{p^\rho}_{\{\eta_i:i<n\}}=\bar{A}^{p_\beta}_{\{\eta_i:i<n\}}$ or they are incompatible; and moreover for $\rho\neq\sigma$, $\bar{A}^{p^\rho}_{\{\eta_i:i<n\}}$, $\bar{A}^{p^\sigma}_{\{\eta_i:i<n\}}$ are incompatible. Hence, as $\aleph_0\cdot|\alpha|<\mathfrak{c}$ we may certainly find ρ such that, letting $p_\alpha=p^\rho$ and $\gamma_\alpha=\gamma$, (5), (6), (7) hold.

But now it is easy to define an $\mathbb{L}-$name $\dot f$ for a function ¿from $\mathfrak{h}(n)$ to $\mathfrak{c}$ such that for every $\alpha<\mathfrak{c}$, $p_\alpha\Vdash_{\mathbb{L}}\dot f(\delta_\alpha)=\gamma_\alpha$. By (7) we conclude $\Vdash_{\mathbb{L}}$ "$\dot f:\mathfrak{h}(n)^V\to\mathfrak{c}^V$ is onto".

A similar argument works for Miller forcing. □

Combining Theorem 2.2 with Con$(\mathfrak{h}(n+1)<\mathfrak{h}(n))$ from §1 we obtain the following:

Corollary 2.3. *For every $n<\omega$, it is consistent that both of Laver and Miller forcing collapse the continuum (strictly) below $\mathfrak{h}(n)$.*

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