

# SUCCESSORS OF SINGULAR CARDINALS AND COLORING THEOREMS I

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ABSTRACT. We investigate the existence of strong colorings on successors of singular cardinals. This work continues Section 2 of [1], but now our emphasis is on finding colorings of pairs of ordinals, rather than colorings of finite sets of ordinals.

## 1. INTRODUCTION

The theme of this paper is that strong coloring theorems hold at successors of singular cardinals of uncountable cofinality, except possibly in the case where the singular cardinal is a limit of regular cardinals that are Jonsson in a strong sense.

Our general framework is that  $\lambda = \mu^+$ , where  $\mu$  is singular of uncountable cofinality. We will be searching for colorings of pairs of ordinals  $< \lambda$  that exhibit quite complicated behaviour. The following definition (taken from [2]) explains what “complicated” means in the previous sentence.

**Definition 1.1.** Let  $\lambda$  be an infinite cardinal, and suppose  $\kappa + \theta \leq \mu \leq \lambda$ .  $\text{Pr}_1(\lambda, \mu, \kappa, \theta)$  means that there is a symmetric two-place function  $c$  from  $\lambda$  to  $\kappa$  such that if  $\xi < \theta$  and for  $i < \mu$ ,  $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$  is a strictly increasing sequence of ordinals  $< \lambda$  with all  $\alpha_{i,\zeta}$ ’s distinct, then for every  $\gamma < \kappa$  there are  $i < j < \mu$  such that

$$(1.1) \quad \zeta_1 < \xi \text{ and } \zeta_2 < \xi \implies c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2}) = \gamma.$$

Just as in [1], one of our main tools is a game that measures how “Jonsson” a given cardinal is.

Recall that a cardinal  $\lambda$  is a Jonsson cardinal if for every  $c : [\lambda]^{<\omega} \rightarrow \lambda$ , we can find a subset  $I \subseteq \lambda$  of cardinality  $\lambda$  such that the range of  $c \upharpoonright I$  is a proper subset of  $\lambda$ . A reader seeking more background should investigate [4] and [3] in [5].

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**Definition 1.2.** Assume  $\mu \leq \lambda$  are cardinals,  $\gamma$  is an ordinal,  $n \leq \omega$ , and  $J$  is an ideal on  $\lambda$ . We define the game  $\text{Gm}_J^n[\lambda, \mu, \gamma]$  as follows:

A play lasts  $\gamma$  moves.

In the  $\alpha^{\text{th}}$  move, the first player chooses a function  $F_\alpha : [\lambda]^{<n} \rightarrow \mu$ , and the second player responds by choosing (if possible) a subset  $A_\alpha \subseteq \lambda$  such that

- $A_\alpha \subseteq \bigcap_{\beta < \alpha} A_\beta$
- $A_\alpha \in J^+$
- $\text{ran}(F_\alpha \upharpoonright [A_\alpha]^{<n})$  is a proper subset of  $\mu$ .

The second player loses if he has no legal move for some  $\alpha < \gamma$ , and he wins otherwise.

In the previous definition, if  $J = J_\lambda^{\text{bd}}$  then we may omit it. Note that it causes no harm if we use a set  $E$  of cardinality  $\lambda$  instead of  $\lambda$  itself; in this case, we write  $\text{Gm}_J^n[E, \mu, \gamma]$ .

Note that  $\lambda$  is a Jonsson cardinal if and only if Player I does not have a winning strategy in the game  $\text{Gm}^\omega[\lambda, \lambda, 1]$ . One may view the lack of a winning strategy for Player I in games of longer length as a strong version of Jonsson-ness or a weak version of measurability — if  $\lambda$  is measurable, then Player II can make sure her moves are elements of some  $\lambda$ -complete ultrafilter.

The following claim investigates how the existence of winning strategies is affected by modifications to the game; the proof is left to the reader.

**Claim 1.3.**

- (1) If  $\mu' \leq \mu$  and the first player has a winning strategy in  $\text{Gm}_J^n[\lambda, \mu, \gamma]$ , then she has a winning strategy in  $\text{Gm}_J^n[\lambda, \mu', \gamma]$ .

- (2) Suppose we weaken the demand on the second player to

$$(1.2) \quad “(\exists \zeta < \lambda)[\text{ran}(F_\alpha \upharpoonright [A_\alpha \setminus \zeta]^{<n}) \text{ is a proper subset of } \mu].”$$

If  $\text{cf}(\lambda) \geq \gamma$  and  $J \supseteq J_\lambda^{\text{bd}}$ , then the first player has a winning strategy in the revised game if and only if she has a winning strategy in the original game.

- (3) If  $J$  is  $\gamma$ -complete, then the same applies to the case where we weaken the demand on the second player to

$$(1.3) \quad “(\exists Y \in J)[\text{ran}(F_\alpha \upharpoonright [A_\alpha \setminus Y]^{<n}) \text{ is a proper subset of } \mu].”$$

- (4) We can allow the second player to pass, i.e., to let  $A_\alpha = \bigcap_{\beta < \alpha} A_\beta$  (even if this is not a legal move) as long as we declare that the

second player loses if the order-type of the set of moves where he did not pass is  $< \gamma$ .

- (5) If Player I has a winning strategy in  $\text{Gm}_J^n[\lambda, \mu, \gamma]$  for every  $\mu < \mu^*$  where  $\mu^*$  is singular and  $\gamma > \text{cf}(\mu^*)$  is regular, then Player I has a winning strategy in  $\text{Gm}_J^n[\lambda, \mu^*, \gamma]$ . We can weaken the requirement that  $\gamma$  is regular and instead require that  $\text{cf}(\gamma) > \text{cf}(\mu^*)$  and  $\omega^\gamma = \gamma$ .

In Section 2 of [1], the existence of winning strategies for Player I in variants of the game is investigated. We will prove one such result here; the reader should look in [1] for others.

**Claim 1.4.** If  $2^x < \lambda < \beth_{(2^x)^+}(\chi)$  then Player I has a winning strategy in  $\text{Gm}^\omega[\lambda, \chi, (2^x)^+]$ .

*Proof.* At a stage  $i$ , Player I will select a function  $F_i : [\lambda]^{<\omega} \rightarrow \chi$  coding the Skolem functions of some model  $M_i$ .

For the initial move, we let the model  $M_0$  have universe  $\lambda$ , and include in our language all relations on  $\lambda$  and all functions from  $\lambda$  to  $\lambda$  of any finite arity that are first order definable in the structure  $\langle H(\lambda^+), \in, <_{\lambda^+}^* \rangle$  with the parameters  $\chi$  and  $\lambda$ .

For subsequent moves,  $M_i$  is an expansion of  $M_0$  with universe  $\lambda$  that has all relations on  $\lambda$  and all functions from  $\lambda$  to  $\lambda$  of any finite arity that are first order definable in the structure  $\langle H(\lambda^+), \in, <_{\lambda^+}^* \rangle$  from the parameters  $\chi, \lambda, M_0$ , and  $\langle A_j : j < i \rangle$ .

To obtain the function  $F_i$ , we let  $\langle F_n^i : n < \omega \rangle$  list the Skolem functions of  $M_i$  in such a way that  $F_n^i$  has  $m_i(n) \leq n$  places. Let  $h : \omega \rightarrow \omega$  be such that for all  $n$ ,  $h(n) \leq n$  and  $h^{-1}(\{n\})$  is infinite. We then define

$$(1.4) \quad F_i(u) = \begin{cases} F_{h(|u|)}^i(\{\alpha \in u : |u \cap \alpha| < m_i(n)\}) & \text{if this is } < \chi \\ 0 & \text{otherwise} \end{cases}$$

The point of doing this is that whenever Player II chooses  $A_i$ , we know that  $\text{ran}(F_i \upharpoonright [A_i]^{<\omega})$  will look like the result of intersecting an elementary submodel of  $M_i$  with  $\chi$ ; in particular, this range will be closed under the functions from  $M_i$ .

Note that  $M_0$  (and all expansions of it) has definable Skolem functions and so for any  $i$  and  $A \subseteq \lambda$ , the Skolem hull of  $A$  in  $M_i$  (denoted by  $\text{Sk}_{M_i}(A)$ ) is well-defined.

Let  $\langle (F_i, A_i) : i < (2^x)^+ \rangle$  be a play of the game in which Player I uses this strategy (with  $M_i$  the model corresponding to  $F_i$ ). For each  $i$ , define

$$(1.5) \quad \alpha_i = \min\{\alpha : |\text{Sk}_{M_0}(A_i) \cap \beth_\alpha(\chi)| > \chi\}.$$

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By the choice of  $M_0$  and  $M_i$ , clearly  $\alpha(i)$  is a successor ordinal or a limit ordinal of cofinality  $\chi^+$ , and

$$(1.6) \quad |\text{Sk}_{M_0}(A_i) \cap \beth_{\alpha_i}(\chi)| \leq 2^\chi.$$

Since  $A_i \subseteq A_j$  for  $i > j$ , we know the sequence  $\langle \alpha_i : i < (2^\chi)^+ \rangle$  is non-decreasing. Furthermore, for each  $i$  we know

$$(1.7) \quad \alpha_i < \min\{\beta : \lambda \leq \beth_\beta(\chi)\} < (2^\chi)^+.$$

This means that the sequence  $\langle \alpha_i : i < (2^\chi)^+ \rangle$  is eventually constant, say with value  $\alpha^*$ . Let  $i^*$  be the least ordinal  $< (2^\chi)^+$  such that  $\alpha_i = \alpha^*$  for  $i \geq i^*$ .

**Proposition 1.5.** If  $i^* \leq i < (2^\chi)^+$ , then  $\text{Sk}_{M_0}(A_{i+1}) \cap \beth_{\alpha^*}(\chi)$  is a proper subset of  $\text{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$ .

*Proof.* Note that  $i^*$ ,  $\alpha^*$ , and  $\beth_{\alpha^*}(\chi)$  are all elements of  $M_{i+1}$  as they are definable in  $\langle H(\lambda^+), \in, <_{\lambda^+} \rangle$  from the parameters  $M_0$  and  $\langle A_j : j \leq i \rangle$ . Furthermore,

$$(1.8) \quad \gamma^* := \min\{\gamma < \lambda : |\text{Sk}_{M_0}(A_i) \cap \gamma| = \chi\}$$

is also definable in  $M_{i+1}$  (and  $< (2^\chi)^+$ ). Thus the language of  $M_{i+1}$  includes a bijection between  $\text{Sk}_{M_0}(A_i) \cap \gamma^*$  and  $\chi$ .

If Player I has not won the game at this stage, after Player I selects  $A_{i+1}$  we will be able to find an ordinal  $\beta < \chi$  such that  $\beta \notin \text{ran}(F_{i+1} \upharpoonright [A_{i+1}]^{<\omega})$ . By definition of  $h$ , we know  $\beta' := h^{-1}(\beta)$  is an element of  $\text{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$ . However,  $\beta'$  is not an element of  $\text{Sk}_{M_{i+1}}(A_{i+1})$  – since  $F_{i+1}$  codes the Skolem functions of  $M_{i+1}$ , the range of  $F_{i+1} \upharpoonright [A_{i+1}]^{<\omega}$  is  $\text{Sk}_{M_{i+1}}(A_{i+1}) \cap \chi$ . Since  $\text{Sk}_{M_{i+1}}(A_{i+1})$  is closed under  $h$ , this contradicts our choice of  $\beta$ . Since  $\text{Sk}_{M_0}(A_{i+1}) \subseteq \text{Sk}_{M_{i+1}}(A_{i+1})$ , we have established the proposition.  $\square$

Note that the preceding proposition finishes the proof of the claim — if play of the game continues for all  $(2^\chi)^+$  steps, then  $\langle \text{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi) : i < (2^\chi)^+ \rangle$  is a strictly decreasing family of subsets of  $\text{Sk}_{M_0}(A_{i^*})$ , contradicting (1.6).  $\square$

## 2. CLUB-GUESSING TECHNOLOGY

In this section, we prove that if  $\lambda = \mu^+$ , where  $\mu$  is singular, then under certain circumstances we can find a complicated “library” of colorings of smaller cardinals. In the next section, we will use this library of colorings to get a complicated coloring of  $\lambda$ .

The basics of club-guessing are explained in [4], but we will take a few minutes to recall some of the definitions.

Let us recall that if  $S$  is a stationary subset of  $\lambda$ , then an  $S$ -club system is a sequence  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  such that for (limit)  $\delta \in S$ ,  $C_\delta$  is closed unbounded in  $\delta$ .

In this section, we will be concerned with the case where  $\lambda$  is the successor of a singular cardinal, i.e.,  $\lambda = \mu^+$  where  $\text{cf}(\mu) < \mu$ . In this context, if  $\bar{C}$  is an  $S$ -club system, then for  $\delta \in S$  we define an ideal  $J_\delta^{b[\mu]}$  on  $C_\delta$  by  $A \in J_\delta^{b[\mu]}$  if and only if  $A \subseteq C_\delta$ , and for some  $\theta < \mu$  and  $\gamma < \delta$ ,

$$\beta \in A \cap \text{nacc}(C_\delta) \Rightarrow [\beta < \gamma \text{ or } \text{cf}(\beta) < \theta].$$

Note that it is a bit easier to understand the definition of  $J_\delta^{b[\mu]}$  by looking at the contrapositive — a subset  $A$  of  $C_\delta$  is “large”, i.e., not in  $J_\delta^{b[\mu]}$ , if and only if  $A \cap \text{nacc}(C_\delta)$  is cofinal in  $\delta$ , and the cofinalities of members of any end segment of  $A \cap \text{nacc}(C_\delta)$  are unbounded below  $\mu$ .

**Claim 2.1.** Let  $\lambda = \mu^+$ , where  $\mu$  is a singular cardinal of cofinality  $\kappa < \mu$ . Let  $S \subseteq \lambda$  be stationary, and assume that  $\sup\{\text{cf}(\delta) : \delta \in S\} = \mu^* < \mu$ . Let  $\bar{C}$  be an  $S$ -club system, and for each  $\delta \in S$ , let  $J_\delta$  be the ideal  $J_\delta^{b[\mu]}$ . Let  $\langle \kappa_i : i < \kappa \rangle$  be a non-decreasing sequence of cardinals such that

$$(2.1) \quad \kappa^* = \sum_{i < \kappa} \kappa_i \leq \mu,$$

and let  $\gamma^* < \mu$ .

Assume we are given a  $\lambda$ -club system  $\bar{e}$  and a sequence of ideals  $\bar{I} = \langle I_\alpha : \alpha < \lambda \rangle$  such that

- (1)  $I_\alpha$  is an ideal on  $e_\alpha$  extending  $J_{e_\alpha}^{\text{bd}}$
- (2) if  $\delta \in S$ , then for each  $i < \kappa$ ,

$$\{\alpha \in \text{nacc}(C_\delta) : \text{Player I wins } \text{Gm}_{I_\alpha}^\omega[e_\alpha, \kappa_i, \gamma^*]\} = \text{nacc}(C_\delta) \pmod{J_\delta}$$

- (3) for any club  $E \subseteq \lambda$ , for stationarily many  $\delta \in S$ ,

$$\{\alpha \in \text{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta,$$

where

$$B_0[E, e_\alpha] = \{\beta \in \text{nacc}(e_\alpha) : E \text{ meets the interval } (\sup(\beta \cap e_\alpha), \beta)\}.$$

Then there is a function  $h : \lambda \rightarrow (\kappa + 1)$  and a sequence

$$\bar{F} = \langle F_\delta : \delta < \lambda, \delta \text{ a limit} \rangle$$

such that

$$\textcircled{*}_1 F_\delta : [e_\delta]^{<\omega} \longrightarrow \kappa_{h(\delta)} \text{ (where } \kappa^* := \kappa_\kappa \text{)}$$

and

- $\otimes_2$  for every club  $E \subseteq \lambda$ , for each  $i < \kappa$  there are stationarily many  $\delta \in S$  such that the set of  $\beta \in \text{nacc}(C_\delta)$  satisfying the following
- $h(\beta) \geq i$
  - $B_0[E, e_\beta] \notin I_\beta$
  - for all  $\gamma < \beta$ ,  $\kappa_{h(\beta)} \subseteq \text{ran}(F_\beta \upharpoonright [B_0[E, e_\beta] \setminus \gamma]^{<\omega})$  is *not* in  $J_\delta$ .

Now admittedly the previous claim is quite a lot to digest, so we will take a little time to illuminate the basic situation we have in mind.

**Claim 2.2.** The assumptions of Claim 2.1 are satisfied if

- (1)  $\lambda = \mu^+$  where  $\kappa = \text{cf}(\mu) < \mu$
- (2)  $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$
- (3)  $\delta \in S \rightarrow |\delta| = \mu$  (i.e.,  $S \subseteq \lambda \setminus \mu$ )
- (4)  $\bar{C}$  is an  $S$ -club system
- (5)  $\bar{J} = \langle J_\delta : \delta \in S \rangle$  where  $J_\delta = J_{C_\delta}^{b[\mu]}$
- (6)  $\text{id}_p(\bar{C}, \bar{J})$  is a proper ideal
- (7)  $\langle \kappa_i : i < \kappa \rangle$  is a non-decreasing sequence of cardinals with supremum  $\kappa^* \leq \mu$
- (8)  $\gamma^* < \mu$ , and for each  $i < \kappa$ , Player I wins the game  $\text{Gm}^\omega[\theta, \kappa_i, \gamma^*]$  for all large enough regular  $\theta < \mu$
- (9)  $\bar{e}$  is a  $\lambda$ -club system such that  $|e_\beta| < \mu$
- (10) for  $\alpha < \lambda$ ,  $I_\alpha = J_{e_\alpha}^{bd}$

*Proof of Claim 2.2.* We need only check items (2) and (3) in the statement of Claim 2.1 — everything else is trivially satisfied. Concerning (2), given  $\delta \in S$  and  $i < \kappa$ , we need to show

$$\{\alpha \in \text{nacc}(C_\delta) : \text{Player I wins } \text{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]\} = \text{nacc}(C_\delta) \pmod{J_\delta}.$$

Let  $A$  consist of those  $\alpha \in \text{nacc}(C_\delta)$  for which Player I does not win the game  $\text{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]$ . By our assumptions, there is a  $\theta < \mu$  such that  $|e_\alpha| < \theta$  for all  $\alpha \in A$ , and therefore  $A$  is in the ideal  $J_{C_\delta}^{b[\mu]} = J_\delta$  and we have what we need.

Concerning (3), given  $E \subseteq \lambda$  club, we must find stationarily many  $\delta \in S$  such that

$$\{\alpha \in \text{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta.$$

Let  $E' = \{\xi \in E : \text{otp}(E \cap \xi) = \xi \text{ and } \mu \text{ divides } \xi\}$ . Clearly  $E'$  is a closed unbounded subset of  $E$ , and since  $\text{id}_p(\bar{C}, \bar{J})$  is a proper ideal,

the set

$$S^* := \{\delta \in S \cap E' : E' \cap \text{nacc}(C_\delta) \notin J_\delta\}$$

is stationary.

Fix  $\delta \in S^*$ , and suppose we are given  $\theta < \mu$  and  $\xi < \delta$ . Since  $E' \cap \text{nacc}(C_\delta) \notin J_\delta$ , we can find  $\alpha \in E' \cap \text{nacc}(C_\delta)$  such that  $\alpha > \max\{\xi, \mu\}$  and  $\text{cf}(\alpha) > \theta$ . Since the order-type of  $E \cap \alpha$  is  $\alpha \geq \mu > |e_\alpha|$ , we know that  $B_0[E, e_\alpha]$  is unbounded in  $e_\alpha$  hence a member of  $I_\alpha$ . This shows that the set of such  $\alpha$  is in  $J_\delta^+$ , as required.  $\square$

Now we return to the proof of Claim 2.1.

*Proof of Claim 2.1.* Let  $\sigma = \text{cf}(\sigma)$  be a regular cardinal  $< \mu$  that is greater than  $\mu^*$  and  $\gamma^*$ . For each limit  $\beta < \lambda$ , if there is an  $i \leq \kappa$  such that Player I wins the version of  $\text{Gm}_{I_\beta}^\omega[e_\beta, \kappa_i, \sigma^+]$  where we allow Player II to pass, then we let  $h(\beta)$  be the maximal such  $i$  — note that  $i$  exists by (5) of Claim 1.3 — and let  $\text{Str}_\beta$  be a strategy that witnesses this.

Note that since  $\gamma^* < \sigma^+$  and  $J_\delta = J_\delta^{b[\mu]}$  for  $\delta \in S$ , we have that for  $\delta \in S$  and  $i < \kappa$  that

$$\{\beta \in \text{nacc}(C_\delta) : \text{Str}_\beta \text{ is defined and } i \leq h(\beta)\} = \text{nacc}(C_\delta) \pmod{J_\delta}.$$

We will make  $\sigma^+$  attempts to build  $\bar{F}$  witnessing the conclusion. In stage  $\zeta < \sigma^+$ , we assume that our prior work has furnished us with a decreasing sequence  $\langle E_\xi : \xi < \zeta \rangle$  of clubs in  $\lambda$ , and, for each  $\beta < \lambda$  where  $\text{Str}_\beta$  is defined, an initial segment  $\langle F_\beta^\xi, A_\beta^\xi : \xi < \zeta \rangle$  of a play of  $\text{Gm}_{I_\beta}^\omega[e_\beta, \kappa_{h(\beta)}^*, \sigma^+]$  in which Player I uses  $\text{Str}_\beta$ . (Note that our convention is that if Player II chooses to pass at a stage, we let  $A_\beta^\xi$  be undefined.)

For each such  $\beta$ , let  $F_\beta^\zeta : [e_\beta]^{<\omega} \rightarrow \kappa_{h(\beta)}$  be given by  $\text{Str}_\beta$ , and for those  $\beta$  for which  $\text{Str}_\beta$  is undefined, we let  $F_\beta^\zeta$  be any such function. Now if  $\langle F_\beta^\zeta : \beta < \lambda \rangle := \bar{F}^\zeta$  is as required then we are done. Otherwise, there is a club  $E' \subseteq \lambda$  and  $i_\zeta < \kappa$  exemplifying the failure of  $\bar{F}^\zeta$ , and without loss of generality,

$$(2.2) \quad (\forall \delta \in S) [B_{i_\zeta}[E'_\zeta, C_\delta, \bar{I}, \bar{e}, \bar{F}^\zeta] \in J_\delta.$$

Now let  $E_\zeta = \text{acc}(E'_\zeta \cap \bigcap_{\xi < \zeta} E_\xi)$ . For each  $\beta$  where  $\text{Str}_\beta$  is defined, we let Player II respond to  $F_\beta^\zeta$  by playing the set  $B_0[E_\zeta, e_\beta]$  if it is a legal move, otherwise we let him pass. We then proceed to stage  $\zeta + 1$ .

Assuming that this construction continues for all  $\sigma^+$  stages, we will arrive at a contradiction. Let  $E = \bigcap_{\zeta < \sigma^+} E_\zeta$ . By assumption (3) there

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is a  $\delta(*) \in S$  for which

$$A_1 := \{\beta \in \text{nacc}(C_{\delta(*)}) : B_0[E, e_\beta] \notin I_\beta\} \notin J_{\delta(*)}.$$

By assumption (2), we have

$$A_2 := \{\beta \in A_1 : \text{Str}_\beta \text{ is defined}\} \notin J_{\delta(*)}.$$

For  $\beta \in A_2$ , look at the play  $\langle F_\beta^\zeta, A_\beta^\zeta : \zeta < \sigma^+ \rangle$ . Since Player I wins, there is a  $\zeta_\beta < \sigma^+$  such that Player II passed at stage  $\zeta$  for all  $\zeta \geq \zeta_\beta$ . Since  $\sigma > \mu^*$  and  $J_{\delta(*)}$  is  $\mu^*$ -based, for some  $\zeta^* < \sigma^+$ ,

$$A_3 = \{\beta \in A_1 : \text{Str}_\beta \text{ is defined and } \zeta_\beta \leq \zeta^*\} \notin J_{\delta(*)}.$$

Now  $E_{\zeta^*}$  was defined so that for some  $i_{\zeta^*}$ , for all  $\delta \in S$ ,

$$(2.3) \quad B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}] \in J_\delta,$$

but (again by assumption (2))

$$A_4 = \{\beta \in A_1 : \text{Str}_\beta \text{ is defined, } \zeta_\beta \leq \zeta^*, \text{ and } i_{\zeta^*} \leq h(\beta)\} \notin J_{\delta(*)}.$$

For  $\beta \in A_4$ , we know that at stage  $\zeta^*$  of our play of  $\text{Gm}_{I_\beta}^\omega[e_\beta, \kappa_{h(\beta)}, \sigma^+]$  the set  $B_0[E_{\zeta^*}, e_\beta]$  was not a legal move. Since our sequence of clubs is decreasing, we know that  $B_0[E_{\zeta^*}, e_\beta]$  is a subset of  $B_0[E_\xi, e_\beta]$  for all  $\xi < \zeta^*$ , so we have

$$B_0[E_{\zeta^*}, e_\beta] \subseteq \bigcap_{\xi < \zeta^*} A_\beta^\xi.$$

Since  $\beta \in A_1$ , we know that  $B_0[E_{\zeta^*}, e_\beta] \notin I_\beta$ . Thus the reason for  $B_0[E_{\zeta^*}, e_\beta]$  being an illegal move must be that for all  $\gamma < \beta$ ,

$$\kappa_{h(\beta)}^* \subseteq \text{ran}(F_\beta^{\zeta^*} \upharpoonright [B_0[E_{\zeta^*}, e_\beta] \setminus \gamma]^{<\omega}).$$

All of these facts combine to tell us that  $\beta \in B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}]$ , and thus

$$A_4 \subseteq B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}^*, \bar{F}^{\zeta^*}] \notin J_{\delta(*)},$$

contradicting (2.3).  $\square$

The proofs in this section (and the next) can be considerably simplified if we are willing to restrict ourselves to the case  $\kappa^* < \mu$ , as we can dispense with the sequence  $\langle \kappa_i : i < \kappa \rangle$ .



3. BUILDING THE COLORING

We now come to the main point of this paper; we dedicate this section and the next to proving the following theorem.

**Theorem 1.** *Assume  $\lambda = \mu^+$ , where  $\mu$  is a singular cardinal of uncountable cofinality, say  $\aleph_0 < \kappa = \text{cf}(\mu) < \mu$ . Assume  $\langle \kappa_i : i < \kappa \rangle$  is non-decreasing with supremum  $\kappa^* \leq \mu$ , and there is a  $\gamma^* < \mu$  such that for each  $i$ , for every large enough regular  $\theta < \mu$ , Player I has a winning strategy in the game  $\text{Gm}^\omega[\theta, \kappa_i, \gamma^*]$ . Then  $\text{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$  holds.*

Let  $\langle S_i : i < \kappa \rangle$  be a sequence of pairwise disjoint stationary subsets of  $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ . For  $i < \kappa$ , let  $\bar{C}^i$  be an  $S_i$ -club system such that

- $\lambda \notin \text{id}_p(\bar{C}^i, \bar{J}^i)$ , where  $\bar{J}^i = \langle J_{C_\delta^i}^{b[\mu]} : \delta \in S_i \rangle$
- for  $\delta \in S_i$ ,  $\text{otp}(C_\delta^i) = \text{cf}(\delta) = \kappa = \text{cf}(\mu)$

Such ladder systems can be found by Claim 2.6 (and Remark 2.6A (6)) of [2] — for the second statement to hold, we need that  $\mu$  has uncountable cofinality.

**Claim 3.1.** There is a  $\lambda$ -club system  $\bar{e}$  such that  $|e_\beta| \leq \text{cf}(\beta) + \text{cf}(\mu)$ , and  $\bar{e}$  “swallows” each  $\bar{C}^i$ , i.e., if  $\delta \in S_i \cap (e_\beta \cup \{\beta\})$ , then  $C_\delta^i \subseteq e_\beta$ .

*Proof.* Let  $S = \cup_{i < \kappa} S_i$ , and let  $\beta < \lambda$  be a limit ordinal. Let  $e_\beta^0$  be a closed cofinal subset of  $\beta$  of order-type  $\text{cf}(\beta)$ . We will construct the required ladder  $e_\beta$  in  $\omega$ -stages, with  $e_\beta^n$  denoting the result of the first  $n$  stages of our procedure. The construction is straightforward, but it is worthwhile to note that we need to use the fact that each member of  $S$  has uncountable cofinality.

Given  $e_\beta^n$ , let us define

$$(3.1) \quad B_n = S \cap (e_\beta^n \cup \{\beta\}).$$

Now we let  $e_\beta^{n+1}$  be the closure in  $\beta$  of

$$(3.2) \quad e_\beta^n \cup \bigcup \{C_\delta : \delta \in B_n\}.$$

Note that  $|e_\beta^{n+1}| \leq \text{cf}(\mu) + \text{cf}(\beta)$  as  $|C_\delta| = \text{cf}(\mu) = \kappa$  for each  $\delta \in S$ . Finally, we let  $e_\beta$  be the closure of  $\cup_{n < \omega} e_\beta^n$  in  $\beta$ .

Clearly  $|e_\beta| \leq \text{cf}(\mu) + \text{cf}(\beta)$ . Also, since each element of  $S$  has uncountable cofinality, if  $\delta \in S \cap e_\beta$ , then there is an  $n$  such that  $\delta \in e_\beta^n$ , and therefore

$$(3.3) \quad C_\delta \subseteq e_\beta^{n+1} \subseteq e_\beta,$$

as required. □

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For each  $i < \kappa$ , there are  $h_i$  and  $\bar{F}^i = \langle F_\delta^i : \delta < \lambda, \delta \text{ limit} \rangle$  as in the conclusion of Claim 2.1 applied to  $\bar{C}^i$  and  $\bar{e}$ ; note that we satisfy the assumptions of Claim 2.1 by way of Claim 2.2.

Let  $\langle \lambda_i : i < \kappa \rangle$  be a strictly increasing sequence of regular cardinals  $> \kappa$  and cofinal in  $\mu$  such that

$$(3.4) \quad \lambda = \text{tcf}\left(\prod_{i < \kappa} \lambda_i / J_\kappa^{\text{bd}}\right),$$

and let  $\langle f_\alpha : \alpha < \lambda \rangle$  exemplify this. Finally, let  $h_0^* : \kappa \rightarrow \omega$  and  $h_1^* : \kappa \rightarrow \kappa$  be such that

$$(3.5) \quad (\forall n)(\forall i < \kappa)(\exists^\kappa j < \kappa)[h_0^*(j) = n \text{ and } h_1^*(j) = i].$$

Before we can define our coloring, we must recall some of the terminology of [2].

**Definition 3.2.** Let  $0 < \alpha < \beta < \lambda$ , and define

$$\gamma(\alpha, \beta) = \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

We also define (by induction on  $\ell$ )

$$\gamma_0(\alpha, \beta) = \beta,$$

$$\gamma_{\ell+1}(\alpha, \beta) = \gamma(\alpha, \gamma_\ell(\alpha, \beta)) \text{ (if defined).}$$

We let  $k(\alpha, \beta)$  be the first  $\ell$  for which  $\gamma_\ell(\alpha, \beta) = \alpha$ . The sequence  $\langle \gamma_i(\alpha, \beta) : i \leq k(\alpha, \beta) \rangle$  will be referred to as the *walk from  $\beta$  to  $\alpha$  along the ladder system  $\bar{e}$* .

We now define the coloring  $c$  that will witness  $\text{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$ . Recall that  $c$  must be a symmetric two-place function from  $\lambda$  to  $\kappa^*$ .

Given  $\alpha < \beta$ , we let  $i = i(\alpha, \beta)$  be the maximal  $j < \kappa$  such that  $f_\beta(j) < f_\alpha(j)$  (if such an  $j$  exists). Next, we walk from  $\beta$  down to  $\alpha$  along  $\bar{e}$  until we reach an ordinal  $\nu(\alpha, \beta)$  such that

$$f_\alpha(i) < f_{\nu(\alpha, \beta)}(i),$$

(again, if such an ordinal exists.) After this, we walk along  $\bar{e}$  from  $\alpha$  toward the ordinal  $\max(\alpha \cap e_{\nu(\alpha, \beta)})$  until we reach an ordinal  $\eta(\alpha, \beta)$  for which

$$f_{\nu(\alpha, \beta)}(i) < f_{\eta(\alpha, \beta)}(i).$$

The idea now is to look at how the ladders  $e_{\nu(\alpha, \beta)}$  and  $e_{\eta(\alpha, \beta)}$  intertwine. Let us make a temporary definition by calling an ordinal  $\xi \in e_{\nu(\alpha, \beta)}$  *relevant* if  $e_{\eta(\alpha, \beta)}$  meets the interval  $(\sup(\xi \cap e_{\nu(\alpha, \beta)}), \xi)$ .

If it makes sense, we let  $w(\alpha, \beta) \subseteq e_{\nu(\alpha, \beta)}$  be the last  $h_0^*(i(\alpha, \beta))$  relevant ordinals in  $e_{\nu(\alpha, \beta)}$  (so we need that the relevant ordinals have order-type  $\gamma + h_0^*(i(\alpha, \beta))$  for some  $\gamma$ ).

Finally, we define our coloring by

$$(3.6) \quad c(\alpha, \beta) = F_{\nu(\alpha, \beta)}^{h_1^*(i(\alpha, \beta))}(w(\alpha, \beta)).$$

If the attempt to define  $c(\alpha, \beta)$  breaks down at some point for some specific  $\alpha < \beta$ , then we set  $c(\alpha, \beta) = 0$ .

We now prove that this coloring works, so suppose  $\langle t_\alpha : \alpha < \lambda \rangle$  are pairwise disjoint subsets of  $\lambda$  such that  $|t_\alpha| = \theta_1 < \kappa$  and  $j^* < \kappa^*$ , and without loss of generality  $\alpha < \min t_\alpha$  and  $\theta_1 \geq \omega$ . We need to find  $\delta_0$  and  $\delta_1$  such that

$$(3.7) \quad \alpha \in t_{\delta_0} \text{ and } \beta \in t_{\delta_1} \Rightarrow \alpha < \beta \text{ and } c(\alpha, \beta) = j^*.$$

Let  $j_1$  be the least  $j$  such that  $j^* < \kappa_j$ , and let  $S, \bar{C}$ , and  $\bar{F}$  denote  $S_{j_1}, \bar{C}^{j_1}$ , and  $\bar{F}^{j_1}$  respectively.

Given  $\delta < \lambda$ , we define the *envelope of  $t_\delta$*  (denoted  $\text{env}(t_\delta)$ ) by the formula

$$(3.8) \quad \text{env}(t_\delta) = \bigcup_{\zeta \in t_\delta} \{\gamma_\ell(\delta, \zeta) : \ell \leq k(\delta, \zeta)\}.$$

The envelope of  $t_\delta$  is the set of all ordinals obtained by walking down to  $\delta$  from some  $\zeta \in t_\delta$  using the ladder system  $\bar{e}$ . This makes sense as we have arranged that  $\delta < \min t_\delta$ . Note also that  $|\text{env}(t_\delta)| \leq |t_\delta| = \theta_1$ .

Next we define functions  $g_\delta^{\min}$  and  $g_\delta^{\max}$  in  $\prod_{i < \kappa} \lambda_i$  by

$$(3.9) \quad g_\delta^{\min}(i) = \min\{f_\gamma(i) : \gamma \in \text{env}(t_\delta)\},$$

and

$$(3.10) \quad g_\delta^{\max}(i) = \sup\{f_\gamma(i) + 1 : \gamma \in \text{env}(t_\delta)\}.$$

Note that  $g_\delta^{\max}$  is well-defined as we assume that  $\kappa < \min\{\lambda_i : i < \kappa\}$ .

The following claim is quite easy, and the proof is left to the reader.

**Claim 3.3.**

- (1)  $f_\delta = J_{\kappa^{\text{bd}}} g_\delta^{\min}$
- (2)  $g_\delta^{\min}(i) \leq g_\delta^{\max}(i)$  for all  $i < \kappa$
- (3) There is a  $\delta' > \delta$  such that  $g_\delta^{\max} \leq J_{\kappa^{\text{bd}}} g_{\delta'}^{\min}$ .

Now let  $\chi^* = (2^\lambda)^+$ , and let  $\langle M_\alpha : \alpha < \lambda \rangle$  be a sequence of elementary submodels of  $\langle H(\chi^*), \in, <_{\chi^*} \rangle$  that is increasing and continuous in  $\alpha$  and such that each  $M_\alpha \cap \lambda$  is an ordinal,  $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$ , and  $\langle f_\alpha : \alpha < \lambda \rangle, g, c, \bar{e}, S, \bar{C}, \langle t_\alpha : \alpha < \lambda \rangle$  all belong to  $M_0$ . Note that  $\mu + 1 \subseteq M_0$ .

The set  $E = \{\alpha < \lambda : M_\alpha \cap \lambda = \alpha\}$  is closed unbounded in  $\lambda$ , and furthermore,

$$(3.11) \quad \alpha < \delta \in E \Rightarrow \sup t_\alpha < \delta.$$

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By the choice of  $\bar{C}$  and  $\bar{F}$ , for some  $\delta \in S \cap E$  we have the set

$$(3.12) \quad A = \{\beta \in \text{nacc}(C_\delta) : (\forall \gamma < \beta) \text{ran}(F_\beta \upharpoonright [B_0[E, e_\beta] \setminus \gamma]^{<\omega}) \supseteq \kappa_{j_1}\}$$

is not in  $J_{C_\delta}^{b[\mu]}$ .

Note that  $A \subseteq \text{acc}(E)$ , as  $B_0[E, e_\beta]$  is unbounded in  $\beta$  for  $\beta \in A$ . For  $\beta \in t_\delta$ , if  $\ell < k(\delta, \beta)$  then  $e_{\gamma_\ell(\delta, \beta)} \cap \delta$  is bounded in  $\delta$ , and since it is closed it has a well-defined maximum. Since  $|t_\delta| < \kappa = \text{cf}(\delta)$ , this means the ordinal

$$\gamma^\otimes := \sup\{\max[e_{\gamma_\ell(\delta, \beta)} \cap \delta] : \beta \in t_\delta \text{ and } \ell < k(\delta, \beta)\}$$

is strictly less than  $\delta$ .

For  $\beta \in t_\delta$ , let us define

$$(3.13) \quad A_\beta := \{\beta' \in A : (\exists \ell \leq k(\beta, \delta))[\text{cf}(\beta') \leq |e_{\gamma_\ell(\delta, \beta)}|]\}.$$

Since the cardinality of each ladder in  $\bar{e}$  is less than  $\mu$ , each set  $A_\beta$  is an element of  $J_{C_\delta}^{b[\mu]}$ . The ideal  $J_{C_\delta}^{b[\mu]}$  is  $\kappa$ -complete, so the fact that  $|t_\delta| < \kappa$  and  $k(\beta, \delta)$  is finite for each  $\beta \in t_\delta$  together imply that

$$(3.14) \quad \bigcup_{\beta \in t_\delta} A_\beta \in J_{C_\delta}^{b[\mu]}.$$

By the definition of  $A$  and our choice of  $\delta$ , this means it is possible to choose  $\beta^* \in A \setminus (\gamma^\otimes + 1)$  that is not in any  $A_\beta$ , i.e.,

$$(3.15) \quad \beta \in t_\delta \text{ and } \ell < k(\delta, \beta) \implies \text{cf}(\beta^*) > |e_{\gamma_\ell(\delta, \beta)}|.$$

**Claim 3.4.**

- (1) If  $\epsilon \in t_\delta$ , and  $\ell = k(\delta, \epsilon) - 1$ , then  $\beta^* \in \text{nacc}(e_{\gamma_\ell(\delta, \epsilon)})$ .
- (2) If  $\epsilon \in t_\delta$  and  $\gamma^\otimes < \gamma' \leq \beta^*$ , then
  - $\gamma_\ell(\delta, \epsilon) = \gamma_\ell(\gamma', \epsilon)$  for  $\ell < k(\delta, \epsilon)$ , and
  - $\gamma_{k(\delta, \epsilon)}(\gamma', \epsilon) = \beta^*$

*Proof.* For the first clause, note that  $\delta$  is an element of  $e_{\gamma_\ell(\delta, \epsilon)}$  and hence by our choice of  $\bar{e}$ ,  $C_\delta \subseteq e_{\gamma_\ell(\delta, \epsilon)}$ . Thus  $\beta^* \in e_{\gamma_\ell(\delta, \epsilon)}$ , and since  $\text{cf}(\beta^*) > |e_{\gamma_\ell(\delta, \epsilon)}|$ , we know that  $\beta^*$  cannot be an accumulation point of  $e_{\gamma_\ell(\delta, \epsilon)}$ .

The first part of the second statement follows because of the definition of  $\gamma^\otimes$ . As far as the second part of the second statement goes, it is best visualized as follows:

We walk down the ladder system  $\bar{e}$  from  $\epsilon$  to  $\gamma'$ , we eventually hit a ladder that contains  $\delta$  — this happens at stage  $k(\delta, \epsilon) - 1$ . Since  $C_\delta$  is a subset of this ladder, the next step in our walk from  $\epsilon$  to  $\gamma'$  must be down to  $\beta^*$  because  $\gamma^\otimes < \gamma' < \beta^*$ .  $\square$

We can visualize the preceding claim in the following manner:  $\beta^*$  is chosen so that for all sufficiently large  $\gamma' < \beta^*$ , all the walks from some element of  $t_\delta$  to  $\gamma'$  are funnelled through  $\beta^*$  —  $\beta^*$  acts as a bottleneck. This will be key when want to prove that our coloring works.

Since  $\beta^* \in A$ , we can choose a finite increasing sequence  $\xi_0 < \xi_1 < \dots < \xi_n$  of ordinals in  $\text{acc}(E) \cap \text{nacc}(e_{\beta^*}) \setminus (\gamma^\otimes + 1)$  such that  $F_{\beta^*}^{j_1}(\{\xi_0, \dots, \xi_n\}) = j^*$ , the color we are aiming for.

For each  $\ell \leq n$ , we can find  $\zeta_\ell \in E \setminus (\gamma^\otimes + 1)$  such that

$$\sup(e_{\beta^*} \cap \xi_\ell) < \zeta_\ell < \xi_\ell.$$

Now we let  $\phi(x_0, y_0, x_1, y_1, \dots, x_n, y_n, z_0, z_1)$  be the formula (with parameters  $\gamma^\otimes, f, \langle \lambda_i : i < \kappa \rangle, \bar{C}, \bar{e}, \langle t_\alpha : \alpha < \lambda \rangle, h, h_0, j^*$ ) that describes our current situation with  $x_\ell, y_\ell$  standing for  $\zeta_\ell, \xi_\ell$ , and  $z_0, z_1$  standing for  $\beta^*, \delta$ , i.e.,  $\phi$  states

- $\gamma^\otimes < x_0 < y_0 < \dots < x_n < y_n < z_0 < z_1$  are ordinals  $< \lambda$
- $z_1 \in S$  and  $z_0 \in \text{nacc}(C_{z_1})$
- $\gamma^\otimes = \sup\{\max[e_{\gamma_\ell(z_1, \zeta)} \cap z_1] : \ell < k(z_1, \zeta) \text{ and } \zeta \in t_{z_1}\}$
- $z_0 \in \text{nacc}(e_{\gamma_{k(z_1, \epsilon)}(z_1, \epsilon)})$  for all  $\epsilon \in t_{z_1}$
- $F_{z_0}^{j_1}(\{y_0, \dots, y_n\}) = j^*$

Now clearly we have

$$(3.16) \quad H(\chi) \models \phi[\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, \beta^*, \delta].$$

Recall that all the parameters needed in  $\phi$  are in  $M_0$ , except possibly for  $\gamma^\otimes$ , so the model  $M_{\gamma^\otimes+1}$  contains all the parameters we need. Also,  $\{\zeta_0, \xi_0, \dots, \zeta_n, \xi_n\} \in M_{\beta^*}$ ,  $\beta^* \in M_\delta \setminus M_{\beta^*}$ , and since  $\delta \in \lambda \setminus M_\delta$ , we have (recalling that  $\exists^* z < \lambda$  means “for unboundedly many  $z < \lambda$ ”)

$$(3.17) \quad M_\delta \models (\exists^* z_1 < \lambda) \phi(\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, \beta^*, z_1).$$

Therefore, this formula is true in  $H(\chi)$  because of elementarity. Similarly, we have

$$H(\chi) \models (\exists^* z_0 < \lambda) (\exists^* z_1 < \lambda) \phi(\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, z_0, z_1).$$

Now each of the intervals  $[\gamma^\otimes + 1, \zeta_0), [\zeta_0, \xi_0), \dots$ , contains a member of  $E$ , so (by the definition of  $E$ ) similar considerations give us

$$H(\chi) \models (\exists^* x_0 < \lambda) \dots (\exists^* y_n < \lambda) (\exists^* z_0 < \lambda) (\exists^* z_1 < \lambda) \phi(x_0, y_0, \dots, z_0, z_1).$$

Now we can choose (in order)

$$(3.18) \quad \zeta_0^a < \zeta_0^b < \xi_0^a < \zeta_1^a < \xi_0^b < \zeta_1^b < \dots < \zeta_n^a < \xi_{n-1}^b < \zeta_n^b < \xi_n^a$$

such that

$$(3.19) \quad (\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^a, \dots, \xi_{n-1}^a, \zeta_n^a, \xi_n^a, z_0, z_1)],$$

and

$$(3.20) \quad (\exists^* y_n < \lambda)(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^b, \dots, \xi_{n-1}^b, \zeta_n^b, y_n, z_0, z_1)],$$

Our goal is to show that for all sufficiently large  $i < \kappa$ , it is possible to choose objects  $\beta^a$ ,  $\delta^a$ ,  $\xi_n^b$ ,  $\beta^b$ , and  $\delta^b$  such that

- (1)  $\zeta_n^b < \beta^a < \delta^a < \min(t_{\delta^a}) \leq \max(t_{\delta^a}) < \xi_n^b < \beta^b < \delta^b$
- (2)  $\phi(\zeta_0^a, \dots, \xi_n^a, \beta^a, \delta^a)$
- (3)  $\phi(\zeta_0^b, \dots, \xi_n^b, \beta^b, \delta^b)$
- (4) for all  $\epsilon \in \text{env}(t_{\delta^a})$ ,  $g_{\delta^a}^{\min} \upharpoonright [i, \kappa] \leq f_\epsilon \upharpoonright [i, \kappa] \leq g_{\delta^a}^{\max} \upharpoonright [i, \kappa]$
- (5) for all  $\epsilon \in \text{env}(t_{\delta^b})$ ,  $g_{\delta^b}^{\min} \upharpoonright [i, \kappa] \leq f_\epsilon \upharpoonright [i, \kappa] \leq g_{\delta^b}^{\max} \upharpoonright [i, \kappa]$
- (6)  $g_{\delta^b}^{\max}(i) < g_{\delta^a}^{\min}(i) \leq g_{\delta^a}^{\max}(i) < f_{\beta^b}(i) < f_{\beta^a}(i)$
- (7)  $g_{\delta^a}^{\max} \upharpoonright [i+1, \kappa] < g_{\delta^b}^{\min} \upharpoonright [i+1, \kappa]$

**Table 1**

**Claim 3.5.** If for all sufficiently large  $i < \kappa$  it is possible to find objects satisfying the requirements of Table 1, then we can find  $\delta^a < \delta^b$  such that  $c(\epsilon^a, \epsilon^b) = j^*$  for all  $\epsilon^a \in t_{\delta^a}$  and  $\epsilon^b \in t_{\delta^b}$ .

*Proof.* Let us choose  $i^* < \kappa$  such that

- suitable objects (as above) can be found, and
- $h_1^*(i^*) = j_1$  and  $h_0^*(i^*) = n$

Choose  $\epsilon^a \in t_{\delta^a}$  and  $\epsilon^b \in t_{\delta^b}$ ; we verify that  $c(\epsilon^a, \epsilon^b) = j^*$ .

*Subclaim 1.*  $i(\epsilon^a, \epsilon^b) = i^*$ .

*Proof.* Immediate by (4)-(7) in the table. □

*Subclaim 2.*  $\nu(\epsilon^a, \epsilon^b) = \beta^b$ .

*Proof.* Note that  $\gamma^\otimes < \epsilon^a < \beta^b$ . Clause (3) of the table implies that the assumptions of Claim 3.4 hold. Thus by Claim 3.4, for  $\ell < k(\delta^b, \epsilon^b)$  we have

$$\gamma_\ell(\epsilon^a, \epsilon^b) = \gamma_\ell(\delta^b, \epsilon^b),$$

hence  $\gamma_\ell(\epsilon^a, \epsilon^b) \in \text{env}(t_{\delta^b})$  and (by (6) of the table and the definitions involved)

$$(3.21) \quad f_{\gamma_\ell(\epsilon^a, \epsilon^b)}(i^*) \leq g_{\delta^b}^{\max}(i^*) < g_{\delta^a}^{\min}(i^*) \leq f_{\epsilon^a}(i^*).$$

For  $\ell = k(\delta^b, \epsilon^b)$ , Claim 3.4 tells us

$$\gamma_\ell(\epsilon^a, \epsilon^b) = \beta^b,$$

and we have arranged that

$$(3.22) \quad f_{\epsilon^a}(i^*) \leq g_{\delta^a}^{\max}(i^*) < f_{\beta^b}(i^*).$$

This establishes  $\beta^b = \nu(\epsilon^a, \epsilon^b)$ . □

*Subclaim 3.*  $\eta(\epsilon^a, \epsilon^b) = \beta^a$ .

*Proof.* Let  $\alpha = \max(e_{\beta^b} \cap \epsilon^a)$ . We have arranged that

$$\zeta_n^b < \beta^a < \delta^a < \epsilon^a < \xi_n^b$$

and  $\gamma^\otimes < \max(e_{\beta^b} \cap \delta^a)$ , hence  $\gamma^\otimes < \alpha < \beta^a$ . For  $\ell < k(\delta^a, \epsilon^a)$ , Claim 3.4 implies

$$\gamma_\ell(\alpha, \epsilon^a) = \gamma_\ell(\delta^a, \epsilon^a) \in \text{env}(t_{\delta^a}).$$

By our choice of  $i^*$ , we have

$$(3.23) \quad f_{\gamma_\ell(\alpha, \epsilon^a)}(i^*) \leq g_{\delta^a}^{\max}(i^*) < f_{\beta^b}(i^*).$$

For  $\ell = k(\delta^a, \epsilon^a)$ , Claim 3.4 implies  $\gamma_\ell(\alpha, \epsilon^a) = \beta^a$ , and we have ensured

$$(3.24) \quad f_{\beta^b}(i^*) < f_{\beta^a}(i^*).$$

Thus  $\beta^a$  is the first ordinal  $\eta$  in the walk from  $\epsilon^a$  to  $\max(e_{\beta^b} \cap \epsilon^a)$  for which  $f_\eta(i^*) > f_{\beta^b}(i^*)$ , and therefore  $\eta(\epsilon^a, \epsilon^b) = \beta^a$ . □

*Subclaim 4.*  $w(\epsilon^a, \epsilon^b) = \{\zeta_0^b, \dots, \zeta_n^b\}$ .

*Proof.* Our previous subclaims imply that an ordinal  $\xi \in e_{\beta^b}$  is relevant if and only if the ladder  $e_{\beta^a}$  meets the interval  $(\sup(e_{\beta^b} \cap \xi), \xi)$ . Since  $h_0^*(i^*) = n + 1$ , we know that  $w(\epsilon^a, \epsilon^b)$  consists of the last  $n + 1$  relevant ordinals in  $e_{\beta^b}$ .

For  $i \leq n$ , clearly  $\xi_i^b \in e_{\beta^b}$  and  $\sup(\xi_i^b \cap e_{\beta^b}) \leq \zeta_n^b$ . We have made sure that  $e_{\beta^a} \cap (\zeta_i^b, \xi_i^b) \neq \emptyset$  (for example,  $\xi_i^a$  is an element in this intersection) and so each  $\xi_i^b$  is relevant.

Since  $\beta^a < \zeta_n^b$ , it is clear that there are no relevant ordinals larger than  $\xi_n^b$ .

Given  $i < n$ , if  $\xi \in e_{\beta^b} \cap (\xi_i^b, \xi_{i+1}^b)$ , then

$$\xi_i^b \leq \sup(\xi \cap e_{\beta^b}) \leq \xi \leq \xi_{i+1}^b.$$

Since  $\zeta_{i+1}^a < \xi_i^b < \zeta_{i+1}^b < \xi_{i+1}^a$ , it follows that

$$[\sup(\xi \cap e_{\beta^b}), \xi] \subseteq [\zeta_{i+1}^a, \xi_{i+1}^a],$$

and so  $\xi$  is not relevant. Thus  $\{\xi_0^b, \dots, \xi_n^b\}$  are the last  $n + 1$  relevant elements of  $e_{\beta^b}$ , as was required.  $\square$

To finish the proof of Claim 3.5, we note that as  $h_1^*(i^*) = j^*$ , we have

$$(3.25) \quad c(\epsilon^a, \epsilon^b) = F_{\beta^b}^{j_1}(\{\xi_0^b, \dots, \xi_n^b\}) = j^*.$$

$\square$

#### 4. FINDING THE REQUIRED ORDINALS

The whole of this section will be occupied with showing that for all sufficiently large  $i < \kappa$ , it is possible to find objects satisfying the requirements of Table 1.

We begin with some notation intended to simplify the presentation.

- $\phi^a(z_0, z_1)$  abbreviates the formula  $\phi(\zeta_0^a, \dots, \xi_n^a, z_0, z_1)$
- $\phi^b(y_n, z_0, z_1)$  abbreviates the formula  $\phi(\zeta_0^b, \zeta_n^b, y_n, z_0, z_1)$
- For  $i < \kappa$ ,  $\psi(i, z_1)$  abbreviates the formula

$$(4.1) \quad (\forall \epsilon \in \text{env}(t_{z_1})) [g_{z_1}^{\min} \upharpoonright [i, \kappa] \leq f_\epsilon \upharpoonright [i, \kappa] \leq g_{z_1}^{\max} \upharpoonright [i, \kappa]]$$

We have arranged things so that the sentence

$$(4.2) \quad (\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\phi^a(z_0^a, z_1^a) \wedge \phi^b(y_n^b, z_0^b, z_1^b)]$$

holds.

There are far too many alternations of quantifiers in the above formula for most people to deal with comfortably; the best way to view them is as a single quantifier that asserts the existence of a tree of 5-tuples with the property that every node of the tree has  $\lambda$  successors, and every branch through the tree gives us five objects satisfying  $\phi^a \wedge \phi^b$ .

Let  $\Phi(i, z_0^a, \dots, z_1^b)$  abbreviate the formula

$$\phi^a(z_0^a, z_1^a) \wedge \phi^b(y_n^b, z_0^b, z_1^b) \wedge \psi(i, z_1^a) \wedge \psi(i, z_1^b) \\ \wedge \left( g_{z_1^a}^{\max} \upharpoonright [i + 1, \kappa] < g_{z_1^b}^{\min} \upharpoonright [i + 1, \kappa] \right).$$



By pruning the tree so that every branch through it is a strictly increasing 5-tuple, we get

$$(4.3) \quad (\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, z_0^a, \dots, z_1^b)].$$

We now make a rather *ad hoc* definition of another quantifier in an attempt to make the arguments that follow a little bit clearer. Given  $i < \kappa$ , let the quantifier  $\exists^{*,i} z_0^b < \lambda$  mean that not only are there unboundedly many  $z_0^b$ 's below  $\lambda$  satisfying whatever property, but also that for each  $\alpha < \lambda_i$ , we can find unboundedly many suitable  $z_0^b$ 's for which  $f_{z_0^b}(i)$  is greater than  $\alpha$ .

**Claim 4.1.** If we choose  $\beta^a < \delta^a < \xi_n^b$  such that

$$(4.4) \quad (\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)],$$

then

$$(4.5) \quad (\forall^* i < \kappa)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)].$$

*Proof.* Suppose that we have  $\beta^a < \delta^a < \xi_n^b$  such that (4.4) holds but (4.5) fails. Then there is an unbounded  $I \subseteq \kappa$  such that for each  $i \in I$ ,

$$(4.6) \quad \neg(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)].$$

In (4.4), we can move the quantifier " $\forall^* i < \kappa$ " past the quantifiers to its left, i.e.,

$$(4.7) \quad (\forall^* i < \kappa)(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)],$$

so without loss of generality, for all  $i \in I$ ,

$$(4.8) \quad (\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)].$$

Since (4.6) holds for all  $i \in I$ , it must be the case that for each  $i \in I$ , there is a value  $g(i) < \lambda_i$  such that for all sufficiently large  $\beta < \lambda$ , if

$$(4.9) \quad (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta, z_1^b)],$$

then

$$(4.10) \quad f_\beta(i) \leq g(i).$$

Since  $\{f_\alpha : \alpha < \lambda\}$  witnesses that the true cofinality of  $\prod_{i < \kappa} \lambda_i$  is  $\lambda$ , we know

$$(4.11) \quad (\forall^* x < \lambda)(\forall^* i \in I)[g(i) < f_x(i)].$$

When we combine this with (4.4), we see that it is possible to choose  $\beta^b < \lambda$  such that

$$(4.12) \quad (\forall^* i \in I)[g(i) < f_{\beta^b}(i)],$$

and

$$(4.13) \quad (\exists^* z_1^b < \lambda)(\forall^* j < \kappa)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)].$$

(Note that we have quietly used the fact that  $|I| < \lambda = \text{cf}(\lambda)$  to get a  $\beta^b$  that is “large enough” so that (4.9) implies (4.10) for all  $i \in I$  for this particular  $\beta^b$ .) This last equation implies

$$(\forall^* j < \kappa)(\exists^* z_1^b < \lambda)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)],$$

so it is possible to choose  $i \in I$  large enough so that

$$g(i) < f_{\beta^b}(i)$$

and

$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)].$$

This is a contradiction, as (4.9) holds for our choice of  $i$  and  $\beta = \beta^b$ , yet (4.10) fails.  $\square$

Notice that an immediate corollary of the preceding claim is

$$(4.14) \quad (\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)].$$

**Claim 4.2.** If  $\beta^a < \lambda$  is chosen so that

$$(4.15) \quad (\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)],$$

then

$$(\forall^* i < \kappa)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda)[\psi' \wedge \psi'']$$

where

$$\psi' := g_{z_1^a}^{\max}(i) < v,$$

and

$$\psi'' := (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \left[ v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \right].$$

*Proof.* In (4.15), we can move the quantifier “ $(\forall^* i < \kappa)$ ” past the other quantifiers to its left, so

$$(4.16) \quad (\forall^* i < \kappa)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]$$

holds. The claim will be established if we show that for each  $i < \kappa$  for which

$$(4.17) \quad (\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]$$

holds, it is possible to find  $v < \lambda_i$  such that

$$(4.18) \quad (\exists^* z_1^a < \lambda) \left[ g_{z_1^a}^{\max}(i) < v \text{ and } (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \left[ v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \right] \right].$$

Despite the lengths of the formulas involved, this is not that hard to accomplish. Since  $\lambda_i < \lambda = \text{cf}(\lambda)$ , we can find  $v < \lambda_i$  such that

$$(\exists^* z_1^a < \lambda) \left[ g_{z_1^a}^{\max}(i) < v \text{ and } (\exists^* y_n^b < \lambda)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \right],$$

and now the result follows from of the definition of “ $\exists^{*,i} z_1^b < \lambda$ ”.  $\square$

Thus there are unboundedly many  $z_0^a < \lambda$  for which there is a function  $g \in \prod_{i < \kappa} \lambda_i$  such that for all sufficiently large  $i < \kappa$ ,

$$(4.19) \quad (\exists^* z_1^a < \lambda) \left[ g_{z_1^a}^{\max}(i) \leq g(i) \text{ and } (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \left[ g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)] \right] \right].$$

Now this is logically equivalent to the statement

$$(4.20) \quad (\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \left[ g_{z_1^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)] \right].$$

Suppose we are given a particular  $z_0^a < \lambda$  for which a function  $g$  as above can be found, and let us fix  $i < \kappa$  “large enough” so that (4.19) holds. Also fix ordinals  $\delta^a < \lambda$  and  $\xi_n^b < \lambda$  that serve as suitable  $z_1^a$  and  $y_n^b$ . Just to be clear, this means that for these choices we have

$$(\exists^* z_0^b < \lambda) \left[ g_{\delta^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)] \right].$$

Since  $\lambda_i < \lambda = \text{cf}(\lambda)$ , there must be some value  $w$  satisfying

$$(\exists^* z_0^b < \lambda) \left[ g(i) < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)] \right].$$

This implies for our particular  $\beta^a, g, i, \delta^a$ , and  $\xi_n^b$  that

$$(4.21) \quad (\forall^* w < \lambda_i)(\exists^* z_0^b < \lambda) \left[ g_{\delta^a}^{\max}(i) \leq g(i) < f_{z_1^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, y_n^b, z_0^b, z_1^b)] \right].$$

Since  $\lambda_i < \lambda = \text{cf}(\lambda)$ , the quantifier  $(\forall^* w < \lambda_i)$  can move to the left past the quantifiers  $(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$ . This tells us that for our  $\beta^a$  and  $g$ ,

$$(4.22) \quad (\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \\ [g_{z_1^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) < w \text{ and} \\ (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]]].$$

When we put all this together, we end up with the statement

$$(4.23) \quad (\exists^* z_0^a < \lambda)(\forall^* i < \kappa)(\exists v < \lambda_i)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda) \\ (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < w \\ \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]]].$$

Since both  $\kappa$  and  $\lambda_i$  are less than  $\lambda = \text{cf}(\lambda)$ , we can move some quantifiers around and achieve

$$(4.24) \quad (\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda) \\ (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < w \\ \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]]].$$

Thus there is a function  $h \in \prod_{i < \kappa} \lambda_i$  such that

$$(4.25) \quad (\forall^* i < \kappa)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda) \\ (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < h(i) \\ \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]]].$$

After all this work, it is finally time to prove that we can select objects  $\beta^a < \delta^a < \xi_n^b < \beta^b < \delta^b$  that satisfy all of our requirements.

Clearly, for every unbounded  $\Lambda \subseteq \lambda$ ,

$$(\exists i < \kappa)(\exists^* x \in \Lambda)(h \upharpoonright [i, \kappa) < f_x \upharpoonright [i, \kappa)).$$

Thus we can choose  $i^* < \kappa$  such that  $h_1^*(i^*) = j_1$  and  $h_0^*(i^*) = n$ , and

$$(\exists^* z_0^a < \lambda) \left[ h \upharpoonright [i^*, \kappa) < f_{z_0^a} \upharpoonright [i^*, \kappa) \text{ and } (\exists v < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \right. \\ (\exists^* z_0^b < \lambda)[g_{z_1^a}^{\max}(i^*) \leq v < f_{z_0^b}(i^*) < h(i^*) \text{ and} \\ \left. (\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \dots, z_1^b)]] \right].$$

So now we choose  $\beta^a$  such that  $h(i^*) < f_{\beta^a}(i^*)$  and for some  $\alpha < \lambda_{i^*}$ ,

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)[g_{z_1^a}^{\max}(i^*) \leq \alpha < f_{z_0^b}(i^*) < h(i^*) \text{ and} \\ (\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \dots, z_1^b)]] .$$

Now we choose  $\delta^a$ ,  $\xi_n^b$ ,  $\beta^b$ , and  $\delta^b$  such that

- $\beta^a < \delta^a < \xi_n^b < \beta^b$
- $g_{\delta^a}^{\max}(i^*) \leq \alpha < f_{\beta^b}(i^*) < h(i^*) < f_{\beta^a}(i^*)$
- $\Phi(i^*, \beta^a, \delta^a, \xi_n^b, \beta^b, \delta^b)$

It is straightforward to check that these objects satisfy all the requirements listed in Table 1, so by Claim 3.5, we are done.

### 5. CONCLUSIONS

In this final section, we will deduce some conclusions in a few concrete cases.

**Theorem 2.** *If  $\mu$  is a singular cardinal of uncountable cofinality that is not a limit of regular Jonsson cardinals, then  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$  holds.*

*Proof.* The proof of this theorem occurs in two stages—we first show that  $\text{Pr}_1(\mu^+, \mu^+, \mu, \text{cf}(\mu))$  holds, and then we show that this result can be upgraded to obtain  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ .

Let  $\mu$  be as hypothesized, and let us define  $\lambda = \mu^+$  and  $\kappa = \text{cf}(\mu)$ .

**Claim 5.1.**  $\text{Pr}_1(\lambda, \lambda, \mu, \kappa)$  holds.

*Proof.* Let  $\langle \kappa_i : i < \kappa \rangle$  be a strictly increasing continuous sequence cofinal in  $\mu$ . Let  $S \subseteq \{\delta \in [\mu, \lambda) : \text{cf}(\delta) = \kappa\}$  be stationary. Standard club-guessing results tell us that there is an  $S$ -club system  $\bar{C}$  such that  $\text{id}_p(\bar{C}, \bar{J})$  is a proper ideal, where  $J_\delta$  is the ideal  $J_{C_\delta}^{b[\mu]}$  for  $\delta \in S$ , and furthermore, satisfying  $|C_\delta| = \kappa$ . (Note that this last requires that  $\kappa = \text{cf}(\mu)$  is uncountable.)

At this point, we have satisfied all of the assumptions of Claim 2.2 except possibly for clause (8). It suffices to show that for each  $i < \kappa$ , for all sufficiently large regular  $\theta < \mu$ , Player I has a winning strategy in the game  $\text{Gm}^\omega[\theta, \kappa_i, 1]$ . Since  $\mu$  is not a limit of regular Jonsson cardinals, it follows that for all sufficiently large regular  $\theta < \mu$ , Player I has a winning strategy in  $\text{Gm}^\omega[\theta, \theta, 1]$ . This implies, by Lemma 1.3 (1), that for all sufficiently large regular  $\theta$ , Player I has a winning strategy in  $\text{Gm}^\omega[\theta, \kappa_i, 1]$ , and so clause (8) of Claim 2.2 is satisfied.  $\square$

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To finish the proof of Theorem 2, it remains to show that we can increase the number of colors from  $\mu$  to  $\lambda = \mu^+$  — we need  $\text{Pr}_1(\lambda, \lambda, \lambda, \kappa)$  instead of  $\text{Pr}_1(\lambda, \lambda, \mu, \kappa)$ .

**Lemma 5.2.** There is a coloring  $c_1 : [\lambda]^2 \rightarrow \lambda$  such that whenever we are given

- $\theta < \kappa$ ,
- $\langle t_\alpha : \alpha < \lambda \rangle$  a sequence of pairwise disjoint elements of  $[\lambda]^\theta$ ,
- $\zeta_\alpha \in t_\alpha$  for  $\alpha < \lambda$ , and
- $\Upsilon < \lambda$ ,

we can find  $\alpha < \beta$  such that  $t_\alpha \subseteq \min(t_\beta)$  and

$$(5.1) \quad (\forall \zeta \in t_\alpha)[c_1(\zeta, \zeta_\beta) = \Upsilon].$$

*Proof.* Let  $c : [\lambda]^2 \rightarrow \mu$  be a coloring that witnesses  $\text{Pr}_1(\lambda, \lambda, \mu, \kappa)$ . For each  $\alpha < \lambda$ , let  $g_\alpha$  be a one-to-one function from  $\alpha$  into  $\mu$ . We define

$$(5.2) \quad c_1(\alpha, \beta) = g_\beta^{-1}(c(\alpha, \beta)).$$

Suppose now that we are given objects  $\theta$ ,  $\langle t_\alpha : \alpha < \lambda \rangle$ ,  $\langle \zeta_\alpha : \alpha < \lambda \rangle$ , and  $\Upsilon$  as in the statement of the lemma. Clearly we may assume that  $\min(t_\alpha) > \alpha$ .

For  $i < \mu$ , we define  $X_i := \{\alpha \in [\gamma, \lambda) : g_{\zeta_\alpha}(\Upsilon) = i\}$ . Since  $\lambda$  is a regular cardinal, it is clear that there is  $i^* < \mu$  for which  $|X_{i^*}| = \lambda$ . Since  $c$  exemplifies  $\text{Pr}_1(\lambda, \lambda, \mu, \kappa)$ , for some  $\alpha < \beta$  in  $X_{i^*}$  we have  $t_\alpha \subseteq \min(t_\beta)$  and

$$(5.3) \quad (\forall \zeta \in t_\alpha)[c(\zeta, \zeta_\beta) = i^*].$$

By definition, this means

$$(5.4) \quad (\forall \zeta \in t_\alpha)[c_1(\zeta, \zeta_\beta) = g^{-1}(c(\alpha, \beta)) = g^{-1}(i^*) = \Upsilon],$$

hence  $\alpha$  and  $\beta$  are as required. □

To continue the proof of Theorem 2, we define a coloring  $c_2 : [\lambda]^2 \rightarrow \lambda$  by

$$(5.5) \quad c_2(\alpha, \beta) = c_1(\alpha, \nu(\alpha, \beta)),$$

where  $\nu(\alpha, \beta)$  is as in the proof of Theorem 1.

It remains to check that  $c_2$  witnesses  $\text{Pr}_1(\lambda, \lambda, \lambda, \kappa)$ . Toward this end, suppose we are given  $\theta < \kappa$ ,  $\langle t_\alpha : \alpha < \lambda \rangle$  a sequence of pairwise disjoint members of  $[\lambda]^\theta$ , and  $\Upsilon < \lambda$ . We need to find  $\delta^a$  and  $\delta^b$  less than  $\lambda$  such that

$$(5.6) \quad \epsilon^a \in t_{\delta^a} \wedge \epsilon^b \in t_{\delta^b} \implies c_2(\epsilon^a, \epsilon^b) = \Upsilon.$$

**Lemma 5.3.** There is a stationary set of  $\gamma_1 < \lambda$  such that for some  $\gamma_0 < \gamma_1$  and  $\beta \in [\gamma_1, \lambda)$ , if  $\gamma_0 \leq \alpha < \gamma_1$ , then the function  $\nu$  is constant on  $t_\alpha \times t_\beta$ .

*Proof.* Let  $E$  be an arbitrary closed unbounded subset of  $\lambda$ , and let  $W$  be the set of ordinals  $< \lambda$  satisfying the properties of  $\gamma_1$ . In the proof of Theorem 1, without loss of generality we can have  $E \in M_0$ . This means that the ordinal  $\beta^*$  found in the course of that proof will be in  $E$ , so we finish by observing that  $\beta^* \in W$ .  $\square$

An application of Fodor’s Lemma gives us a single ordinal  $\gamma_0$  and a stationary  $W' \subseteq W$  such that for all  $\gamma \in W'$ , there is a  $\beta_\gamma \in [\gamma, \lambda)$  such that for all  $\alpha \in [\gamma_0, \gamma)$ ,  $\nu \upharpoonright (t_\alpha \times t_{\beta_\gamma})$  is constant.

Using properties of the coloring  $c_1$ , we can find  $\alpha$  and  $\gamma$  such that

- $\gamma_0 \leq \alpha < \lambda$
- $\gamma \in W' \setminus (\sup(t_\alpha) + 1)$ , and
- $\zeta \in t_\alpha \implies c_1(\zeta, \gamma) = \Upsilon$ .

Now given  $\epsilon^a \in t_\alpha$  and  $\epsilon^b \in t_{\beta_\gamma}$ , we find

$$(5.7) \quad c_2(\epsilon^a, \epsilon^b) = c_1(\epsilon^a, \gamma) = \Upsilon,$$

and therefore  $c_2$  exemplifies  $\text{Pr}(\lambda, \lambda, \lambda, \kappa)$ .  $\square$

Theorem 2 strengthens results in [1] as clearly  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$  implies that  $\mu^+$  has a Jonsson algebra (i.e.,  $\mu^+$  is not a Jonsson cardinal). The question of whether the successor of a singular cardinal can be a Jonsson cardinal is a well-known open question.

We note that many of the results from Section 2 of [1] dealing with the existence of winning strategies for Player I in  $\text{Gm}^\omega[\lambda, \mu, \gamma]$  can be combined with Theorem 1 to give new results. For example, we have the following result from [1].

**Proposition 5.4.** If  $\tau \leq 2^\kappa$  but  $(\forall \theta < \kappa)[2^\theta < \tau]$ , then Player I has a winning strategy in the game  $\text{Gm}^\omega(\tau, \kappa, \kappa^+)$ .

*Proof.* See Claim 2.3(1) and Claim 2.4(1) of [1].  $\square$

Armed with this, the following claim is straightforward.

**Claim 5.5.** Let  $\mu$  be a singular cardinal of uncountable cofinality. Further assume that  $\chi$  is a cardinal such that  $2^{<\chi} \leq \mu < 2^\chi$ . Then  $\text{Pr}_1(\mu^+, \mu^+, \chi, \text{cf}(\mu))$  holds.

*Proof.* If  $2^{<\chi} < \mu$ , then Claims 2.3(1) and 2.4(1) of [1] imply that for every sufficiently large  $\theta < \mu$ , Player I has a winning strategy in the game  $\text{Gm}^\omega(\theta, \chi, \chi^+)$ .

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If  $\mu = 2^{<\chi}$ , then  $\text{cf}(\mu) = \text{cf}(\chi)$ . Let  $\langle \kappa_i : i < \text{cf}(\mu) \rangle$  be a strictly increasing continuous sequence of cardinals cofinal in  $\chi$ . Given  $i < \text{cf}(\mu)$ , we claim that for all sufficiently large regular  $\tau < \mu$ , Player I has a winning strategy in  $\text{Gm}^\omega(\tau, \kappa_i, \chi)$ . Once we have established this,  $\text{Pr}_1(\mu^+, \mu^+, \chi, \text{cf}(\mu))$  follows by Theorem 1.

Given  $\tau = \text{cf}(\tau)$  satisfying  $2^{\kappa_i} < \tau < \mu$ , let  $\eta$  be the least cardinal such that  $\tau \leq 2^\eta$ . Clearly  $\kappa_i < \eta < \chi$ . By Proposition 5.4, Player I wins the game  $\text{Gm}^\omega(\tau, \eta, \eta^+)$ . This implies (since  $\eta^+ < \chi$  and  $\kappa_i < \eta$ ) that Player I wins the game  $\text{Gm}^\omega(\tau, \kappa_i, \chi)$  as required.  $\square$

We can also use Claim 1.4 to prove similar results. For example we have the following.

**Claim 5.6.** Let  $\mu$  be a singular cardinal of uncountable cofinality. Further assume that  $\chi < \mu$  satisfies  $2^\chi < \mu < \beth_{(2^\chi)^+}(\chi)$ . Then  $\text{Pr}_1(\mu^+, \mu^+, \chi, \text{cf}(\mu))$  holds.

*Proof.* Again, the main point is that for all sufficiently large regular  $\theta < \mu$ , Player I has a winning strategy in the game  $\text{Gm}^\omega[\theta, \chi, (2^\chi)^+]$ . This follows immediately from Claim 1.4. Since  $(2^\chi)^+ < \mu$ , Theorem 1 is applicable.  $\square$

In a sequel to this paper, we will address the situation where  $\lambda$  is the successor of a singular cardinal of countable cofinality. Similar results hold, but the combinatorics involved are trickier.

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