# Lusin sequences under CH and under Martin's Axiom *

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**Abstract**

Assuming the continuum hypothesis there is an inseparable sequence of length $\omega_1$ that contains no Lusin subsequence, while if Martin's Axiom and $\neg CH$ is assumed then every inseparable sequence (of length $\omega_1$) is a union of countably many Lusin subsequences.

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# 1 Preface

We first fix some notations and definitions. The set of natural numbers is denoted $\omega$, and for $A, B \subseteq \omega$ we write $A \subseteq^* B$ iff $A \setminus B$ is finite, and $A \perp B$ iff $A \cap B$ is finite (almost inclusion, almost disjointness). Let $\mathcal{A} = \langle A_\zeta \mid \zeta \in \omega_1 \rangle$ be a sequence of pairwise almost disjoint, infinite subsets of $\omega$. So $A_\zeta \subset \omega$ and $A_{\zeta_1} \perp A_{\zeta_1}$ for $\zeta_1 \neq \zeta_2$. We say that $B \subseteq \omega$ *separates* $\mathcal{A}$ if $\{\xi \in \omega_1 \mid A_\xi \subseteq^* B\}$ and $\{\xi \in \omega_1 \mid A_\xi \subseteq^* \omega \setminus B\}$ are both uncountable. If no $B$ separates $\mathcal{A}$ then $\mathcal{A}$ is said to be *inseparable*. That is $\mathcal{A}$ is inseparable if it is an almost disjoint family of infinite subsets of $\omega$ such that there is no $B \subset \omega$ for which

$$(\exists^{\aleph_1} A \in \mathcal{A})(A \subseteq^* B) \ \&\ (\exists^{\aleph_1} A \in \mathcal{A})(A \subseteq^* \omega \setminus B).$$

An inseparable family of size $\aleph_1$ can be constructed in $ZFC$ alone (Lusin [1], cited by [2]). We say that $\mathcal{A}$ is a *Lusin* sequence if for every $i < \omega_1$ and $n \in \omega$

$$\{j < i \mid A_i \cap A_j \subseteq n\} \text{ is finite.}$$

A seemingly stronger property is the following. We say that $\mathcal{A}$ is a *Lusin** family if for every $i < \omega_1$ and $n \in \omega$

$$\{j < i : \ |A_i \cap A_j| < n\}$$

is finite.

It is not difficult to prove that every Lusin sequence is inseparable, and Lusin constructed a Lusin sequence in *ZFC*. Is this the only way to build inseparable families? The answer depends on set-theoretical assumptions as the following two results show (obtained by the first and second author respectively).

**Theorem 1.1** *(1) $CH$ implies that there is an inseparable family which contains no Lusin subsequence. (2) "Martin's Axiom+$\neg CH$" implies that every inseparable sequence is a countable union of Lusin* sequences.*

# 2 Proofs

## 2.1 $CH$ gives an inseparable non-Lusin sequence

Assume the continuum hypothesis ($CH$) throughout this subsection. We shall define a sequence $\mathcal{A} = \langle A_\alpha \mid \alpha \in \omega_1 \rangle$ of almost disjoint subsets of $\omega$

which is inseparable by virtue of the following property denoted **P**.

For every infinite $X \subseteq \omega$ one of the following three possibilities holds:

**P1** $X$ is finitely covered by $\mathcal{A}$ (which means that for some finite set $u \subset \omega_1$ $X \subseteq^* \bigcup\{A_\alpha \mid \alpha \in u\}$.

**P2** $\omega \setminus X$ is finitely covered by $\mathcal{A}$.

**P3** For some $\alpha_0 < \omega_1$ for all $\alpha_0 \leq \alpha < \omega_1$ $X$ splits $A_\alpha$ (which means that both $X \cap A_\alpha$ and $A_\alpha \setminus X$ are infinite).

It is quite obvious that if $\mathcal{A}$ satisfies this property then it is inseparable, and so we describe the construction assuming $CH$ of a sequence that satisfies property **P**, but does not contain any Lusin subsequence.

Let $\langle X_\xi \mid \xi \in \omega_1\rangle$ be an enumeration of all infinite subsets of $\omega$, and let $\langle e_i \mid i \in \omega_1\rangle$ be an enumeration of all countable subsets of $\omega_1$ of order-type a limit ordinal. The sequence $\mathcal{A} = \langle A_\alpha \mid \alpha \in \omega_1\rangle$ is defined by induction on $\alpha$. First $\langle A_i \mid i \in \omega\rangle$ are defined as some almost disjoint family of infinite subsets of $\omega$. Each $A_\alpha$, for $\alpha \geq \omega$, is required to satisfy the following three conditions.

**C1** $A_\alpha \subseteq \omega$ is infinite and $A_\beta \perp A_\alpha$ for all $\beta < \alpha$.

**C2** For every $\xi < \alpha$ one of the following possibilities holds:

- **p1** $X_\xi$ is finitely covered by $\langle A_\beta \mid \beta < \alpha\rangle$, or
- **p2** $\omega \setminus X_\xi$ is finitely covered by $\langle A_\beta \mid \beta < \alpha\rangle$, or else
- **p3** $X_\xi$ splits $A_\alpha$ (i.e. both $X_\xi \cap A_\alpha$ and $A_\alpha \setminus X_\xi$ are infinite).

**C3** For every $i < \alpha$ such that $e_i \subseteq \alpha$ there are two possibilities:

1. For some $m \in \omega$ $A_\alpha \cap A_\xi \subseteq m+1$ for an infinite number of indices $\xi \in e_i$. (This is the "good" possibility.)
2. For some $m \in A_\alpha$ there is $\xi_0 < \sup(e_i)$ such that for every $\xi$, $\xi_0 < \xi \in e_i$,
$$\min(A_\xi \setminus m+1) < \min(A_\alpha \setminus m+1).$$
Or, equivalently, if $n$ is the first member of $A_\alpha$ above $m$ then $A_\xi \cap (m,n) \neq \emptyset$.

If we succeed then **C1** and **C2** clearly imply that $\mathcal{A}$ is pairwise almost disjoint and inseparable. We are going to show that **C3** implies that $\mathcal{A}$ contains no Lusin subsequences. Suppose that $L = \langle A_i \mid i \in I\rangle$ is a subsequence of $\mathcal{A}$, where $I \subseteq \omega_1$ is uncountable . We want to find some $\alpha \in I$ and $m \in \omega$ for which $\{\xi \in I \cap \alpha \mid A_\alpha \cap A_\xi \subseteq m\}$ is infinite.

Consider the structure on $\omega \cup I$ with predicates for $\omega$, $I$, $\in$, and the binary relation $m \in A_i$ (for $m \in \omega$, $i \in I$). Let $e \subseteq I$ be the universe of a countable elementary substructure. Then $e = e_i$ for some $i \in \omega_1$. Let $\alpha \in I$ be any ordinal such that $\alpha > i$ and $\alpha > \sup(e)$. We want to prove that possibility **C3**(1) holds for $\alpha$. This shows that $L$ is not a Lusin sequence. Suppose instead that **C3**(2) holds. Then there are $m \in A_\alpha$ and $\xi_0 < \sup(e)$ as in **C3**(2). Namely, if $n$ is the first member of $A_\alpha$ above $m$ then

$$A_\xi \cap (m, n) \neq \emptyset \tag{1}$$

for every $\xi > \xi_0$ in $e$. However, since $e_i$ is an elementary substructure, we actually have (1) for every $\xi_0 < \xi$ in $I$. But this is clearly impossible for $\xi = \alpha$ itself!

Having shown the usefulness of the three conditions **C1**–**C3**, we return now to the inductive construction. At the $\alpha$-th stage of this construction, to construct $A_\alpha$, it is convenient to define a poset $P = (P, \leq)$ and a countable collection of dense subsets of $P$, and then to define a filter $G \subseteq P$ such that $G$ intersects each of the dense sets in the countable collection. With this we shall define $A_\alpha = \bigcup\{a \mid \exists E\ (a, E) \in G\}$, and $A_\alpha$ will satisfy all three conditions because of the choice of the dense sets. In this fashion one does not have to over specify the construction.

A condition $p = (a, E) \in P$ consists of:

1. A finite set $a \subseteq \omega$ (which will grow to become $A_\alpha$).
2. A finite set $E \subseteq \alpha$ ($p$ promises that $A_\beta \cap A_\alpha = A_\beta \cap a$ for all $\beta \in E$.

Following the tradition that $p_1 \leq p_2$ means that $p_2$ gives more information than $p_1$, the partial order on $P$ is defined by

$$(a_1, E_1) \geq (a_0, E_0) \text{ iff } E_0 \subseteq E_1 \text{ and } a_1 \text{ is an end-extension of } a_0$$
$$\text{such that, for every } \xi \in E_0,\ (a_1 \setminus a_0) \cap A_\xi = \emptyset.$$

We say that an end-extension $a_1$ of $a_0$ "respects $E$" (where $E \subseteq \omega_1$ is finite) if $(a_1 \setminus a_0) \cap A_\beta = \emptyset$ for every $\beta \in E$. So $(a_1, E_1)$ extends $(a_0, E_0)$ if and only if $E_0 \subseteq E_1$ and $a_1$ is an end-extension of $a_0$ that respects $E_0$.

Now we shall define a countable collection of dense subsets of $P$. First, to ensure that $A_\alpha$ is infinite, for every $k \in \omega$ and $p \in P$ observe that there is an extension $(a', E')$ of $p$ with $k < \sup(a')$. Then to ensure that $A_\alpha \perp A_\beta$ for all $\beta < \alpha$ observe that $(a, E \cup \{\beta\})$ extends $(a, E)$. These dense sets take care of **C1**.

For every $X \subseteq \omega$ such that neither $X$ nor $\omega \setminus X$ are finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$, and for every $k \in \omega$, define $D_{X,k} \subset P$ by:

$$D_{X,k} = \{(a, E) \in P \mid \text{both } a \cap X \text{ and } a \setminus X \text{ contain } \geq k \text{ members } \}.$$

**Claim 2.1** *$D_{X,k}$ is dense (open) in $P$.*

**Proof.** Take any $(a_0, E_0) \in P$. Since neither $X$ nor its complement are $\subseteq^*$-included in $A = \bigcup\{A_\beta \mid \beta \in E_0\}$, both $X \setminus A$ and $(\omega \setminus X) \setminus A$ are infinite. We can find an end-extension $a_1$ of $a_0$ such that $(a_1 \setminus a_0) \cap A = \emptyset$ and both $a_1 \cap X$ and $a_1 \setminus X$ contain $\geq k$ members. Thus $(a_0, E_0) < (a_1, E_0) \in D_{X,k}$. $\dashv$

So add to the countable list of dense sets all sets $D_{X_\xi,k}$ for $k \in \omega$ and $\xi < \alpha$ such that neither $X_\xi$ nor $\omega \setminus X_\xi$ are finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$. This ensures **C2**.

The main issue of the proof is to take care of **C3**. What dense sets will do the job? Fix $e = e_i$ for any $i < \alpha$ such that $e \subseteq \alpha$. We say that a condition $p = (a, E) \in P$ is of type (a) for $e$ if for some $m \in a$ the following holds.

For every end-extension $a'$ of $a$ that respects $E$ and for every $\xi_0 \in e$ there is some $\xi \in e$, $\xi_0 \leq \xi$, such that $A_\xi \cap a' \subseteq m + 1$. (2)

If $p$ is of type (a) then the least $m \in a$ that satisfies (2) is denoted $m_p$. Observe that if $p$ is of type (a) then any extension of $p$ is also of type (a) (and with the same $m$).

We say that $p = (a, E) \in P$ is of type (b) for $e$ if there are two adjacent members of $a$, $m$ and $n$ (i.e. $m, n \in a$ and $(m, n) \cap a = \emptyset$) such that for some $\xi_0 \in e$ for every $\xi_0 \leq \xi \in e$ $A_\xi \cap (m, n) \neq \emptyset$.

**Claim 2.2** *Any condition in $P$ has an extension of type (a) or an extension of type (b).*

**Proof.** Given $p = (a, E)$ let $m = \max(a)$. Is $p$ of type (a) by virtue of $m$? If yes, we are done, and if not then there are

1. $a'$ an end-extension of $a$, respecting $E$, and

2. $\xi_0 \in e$,

such that for every $\xi \in e$ with $\xi_0 \leq \xi$, $A_\xi \cap a' \setminus m+1 \neq \emptyset$. Let $n > \max a'$ be such that $n \notin \bigcup\{A_\beta \mid \beta \in E\}$ and consider the condition $p' = (a \cup \{n\}, E)$ extending $p$. Then for every $\xi_0 \leq \xi \in e$, $A_\xi \cap (m, n) \neq \emptyset$. That is, $p'$ is of type (b).$\dashv$

For every $\xi_0 \in e$ define $D_{\xi_0,e}$ by $p = (a, E) \in D_{\xi_0,e}$ iff either $p$ is of type (b) or $p$ is of type (a) and there exists some $\xi \in e \cap E$ above $\xi_0$ with $A_\xi \cap a \subseteq m+1$ (where $m = m_p$).

**Claim 2.3** *$D_{\xi_0,e}$ is dense in $P$.*

**Proof.** Suppose $p_0 \in P$ is given. If $p_0$ is extendible into a condition of type (b) then we are done. Otherwise there is $p_1 = (a_1, E_1) \geq p_0$ of type (a). By the definition of type (a), there is some $\xi \in e$, with $\xi_0 \leq \xi$ such that $A_\xi \cap a_1 \subseteq m+1$. Hence $(a_1, E_1 \cup \{\xi\}) \in D_{\xi_0,e}$ is as required. $\dashv$

Add to the countable list of dense sets all sets $D_{\xi_0,e}$ where $e = e_i$ for some $i < \alpha$ such that $e_i \subseteq \alpha$ and $\xi_0 \in e_i$. We claim that if $A_\alpha$ is defined from a filter $G$ that intersects all the above dense sets, then condition **C3** is ensured. Given $i < \alpha$ such that $e_i = e \subseteq \alpha$, we ask if there is $(a, E) \in G$ of type (b) for $e$. If yes, then possibility **C3**(2) holds for $A_\alpha$.

So we assume that $G$ contains no condition of type (b) for $e$. Since any two conditions in $G$ are compatible, it follows that if $p, q \in G$ are of type (a), then $m_p = m_q$. Let $m$ denote this common value. We claim that there is an unbounded set of $\xi \in e$ such that $A_\xi \cap A_\alpha \subseteq m+1$. To see this, consider any $\xi_0 \in e$ and pick $p = (a, E) \in D_{\xi_0,e} \cap G$. Then $p$ is of type (a) and there is $\xi \in E \cap e$ above $\xi_0$ with $A_\xi \cap a \subseteq m+1$. But then $A_\xi \cap A_\alpha \subseteq m+1$ follows.

## 2.2 Martin's Axiom: Inseparable $\Rightarrow$ contains a Lusin* subsequence

Assume Martin's Axiom $+2^{\aleph_0} > \aleph_0$. Let $\mathcal{A} = \langle A_\zeta \mid \zeta \in \omega_1 \rangle$ be an inseparable sequence of length $\omega_1$ (any length below the continuum works). Define the following poset:

$$Q = \{(u, n) \mid u \subseteq \omega_1 \text{ is finite and } n < \omega\}$$

ordered by

$$(u_1, n_1) \leq (u_2, n_2) \text{ iff } u_1 \subseteq u_2 \;\&\; n_1 \leq n_2 \;\&\; \forall i \in u_1\, \forall j \in u_2 \setminus u_1,\; j < i \Rightarrow \mid A_i \cap A_j \mid > n_1. \quad (3)$$

This relation is easily shown to be transitive. We intend to prove that $Q$ is a c.c.c poset, and that for every $\alpha < \omega_1$ and $k < \omega$ the set $D_{\alpha,k}$ of $(u, n)$ in $Q$ for which $\sup(u) > \alpha$ and $n > k$ is dense. So if $G \subset Q$ is a filter provided by Martin's Axiom which intersects each of these dense sets, then $U = \bigcup\{u \mid \exists n(u, n) \in G\}$ is uncountable and $\langle A_\alpha \mid \alpha \in U\rangle$ is a Lusin* sequence. Because if $i \in U$ and $k < \omega$ then

$$\{A_j \mid j \in U \cap i \text{ and } |A_i \cap A_j| \leq k\}$$

is finite by the following argument. For some $(u, n) \in G$, $i \in u$ and $n \geq k$. This implies that $|A_i \cap A_j| > k$ for every $j < i$ such that $j \in U \setminus u$.

The full result, concerning the decomposition of $\mathcal{A}$ into countably many Lusin* subsequences, follows from the fact that (under Martin's Axiom) if $Q$ is a c.c.c poset, then $Q$ is a countable union of filters (each intersects the required dense sets). (Consider the finite support product of $\omega$ copies of $Q$, and remember that $|Q| = \aleph_1$.)

It is easy to see that if $(u, n) \in Q$ and $v$ is any end-extension of $u$ then $(u, n) \leq (v, n)$. Also, if $n \leq m$ then $(u, n) \leq (u, m)$. This shows that the required sets $D_{\alpha,k}$ are indeed dense in $Q$, and so the main point of the proof is to show that $Q$ satisfies the countable chain condition.

**Lemma 2.4** *$Q$ satisfies the c.c.c.*

**Proof**. Let $\langle (u_\zeta, n_\zeta) \mid \zeta \in \omega_1\rangle$ be an $\omega_1$-sequence of conditions in $Q$. We may assume that for some fixed $n$ and $k$, $n = n_\zeta$ and $k = |u_\zeta|$ for all $\zeta \in \omega_1$, and that the sets $u_\zeta$ form a $\Delta$ system. That is, for some finite $c_0 \subset \omega_1$ $c_0 = u_{\zeta_1} \cap u_{\zeta_2}$ for all $\zeta_1 \neq \zeta_2$ and $\max(u_{\zeta_1}) < \min(u_{\zeta_2} \setminus c_0)$ for $\zeta_1 < \zeta_2$.

We want to find $\zeta_1 < \zeta_2$ such that $(u_{\zeta_1} \cup u_{\zeta_2}, n)$ extends both $(u_{\zeta_1}, n)$ and $(u_{\zeta_2}, n)$. It is evident that $(u_{\zeta_1} \cup u_{\zeta_2}, n)$ extends $(u_{\zeta_1}, n)$ (the lower part) but the problem is the possibility that for some $i \in u_{\zeta_2}$ and $j \in u_{\zeta_1} \setminus c_0$ $|A_i \cap A_j| \leq n$.

We shall find two uncountable sets $K, L \subseteq \omega_1$ such that for every $\zeta_1 \in K$ and $\zeta_2 \in L$, $(u_{\zeta_1}, n)$ and $(u_{\zeta_2}, n)$ are compatible. We start with $K_0 = L_0 = \omega_1$, and define $K_{i+1} \subseteq K_i$ and $L_{i+1} \subseteq L_i$ by induction, considering in turn each pair $0 \leq i, j \leq |u_\zeta \setminus c_0|$ (any $\zeta$ can be taken, as these sets have all the same size). The definition of $K_i$ and $L_i$ depends on a finite parameter set, and it is convenient to have a countable model in which the definition is carried on. So let $M \prec \langle H_{\aleph_1}, \mathcal{A}, Q, \{(u_\zeta, n_\zeta) : \zeta \in \omega_1\}\rangle$ be a countable elementary submodel (where $H_{\aleph_1}$ is the collection of all sets that are hereditarily countable). The following lemma is used.

**Lemma 2.5** *Let $U, V \in M$ be two uncountable subsets of $\omega_1$ and $n < \omega$. There are uncountable subsets $U_1 \subseteq U$ and $V_1 \subseteq V$ (definable in $M$) such that for every $\zeta \in U_1$ and $\xi \in V_1$, $|A_\zeta \cap A_\xi| > n$ (and hence $(\{\zeta\}, n)$ and $(\{\xi\}, n)$ are compatible in $Q$).*

It should be obvious how successive applications of the lemma yield the c.c.c., and so we turn to the proof of the lemma. Let $\delta = \omega_1 \cap M$ be the set of countable ordinals in our countable structure $M$.
**Case 1**: for some $\zeta \in U \setminus \delta$ and $\xi \in V \setminus \delta$ $|A_\zeta \cap A_\xi| > n$. In this case pick a finite $X \subset A_\zeta \cap A_\xi$ with $|X| > n$, and let $U_1 = \{i \in U \mid X \subset A_i\}$, $V_1 = \{j \in V \mid X \subset A_j\}$. Both $U_1$ and $V_1$ are uncountable (for if $U_1$ is countable then it would be included in $M$, but $A_\zeta$ shows that this is not the case).
**Case 2**: not Case 1. So, for every $\zeta \in U \setminus \delta$ and $\xi \in V \setminus \delta$, $|A_\zeta \cap A_\xi| \leq n$. Let $0 \leq m_0 \leq n$ be the maximal size of some intersection $F = A_\zeta \cap A_\xi$ for indices $\zeta$ and $\xi$ as above. Then $U_1 = \{i \in U \mid F \subset A_i\}$ and $V_1 = \{j \in V \mid F \subset A_j\}$ are uncountable and for $i \in U_1 \setminus \delta$ and $j \in V_1 \setminus \delta$, $A_i \cap A_j = F$ (by maximality of $|F|$). So the set $B = \bigcup\{A_i \mid i \in U_1\}$ separates $\mathcal{A}$ (as $B \cap A_j = F$ for every $j \in V_1$) which is a contradiction.

# References


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