6, 2003

Uniformization, choice functions and well orders in the class of trees.

BY

SHMUEL LIFSCHES and SAHARON SHELAH*

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

The monadic second-order theory of trees allows quantification over elements and over arbitrary subsets. We classify the class of trees with respect to the question: does a tree T have a definable choice function (by a monadic formula with parameters)? A natural dichotomy arises where the trees that fall in the first class don't have a definable choice function and the trees in the second class have even a definable well ordering of their elements. This has a close connection to the uniformization problem.

0. Introduction

The *uniformization problem* for a theory T in a language L can be formulated as follows: Suppose $T \vdash (\forall \bar{Y})(\exists \bar{X})\phi(\bar{X},\bar{Y})$ where ϕ is an L-formula and $\bar{X},\bar{Y}$ are tuples of variables. Is there another L-formula ϕ^* such that

$$T \vdash (\forall \bar{Y})(\forall \bar{X})[\phi^*(\bar{X},\bar{Y}) \Rightarrow \phi(\bar{X},\bar{Y})] \quad \text{and} \quad T \vdash (\forall \bar{Y})(\exists! \bar{X})\phi^*(\bar{X},\bar{Y})?$$

Here $\exists!$ means "there is a unique".

The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language L can be described as the augmentation of L by a list of quantifiable set variables and by new atomic formulas $t \in X$ where t is a first order term and X is a set variable. The monadic theory of a structure $\mathcal{M}$ is the theory of $\mathcal{M}$ in the extended language where the set variables range over all subsets of $|\mathcal{M}|$ and $\in$ is the membership relation.

Given a tree T we may ask the following question: is there a sequence $\bar{P}$ of subsets of T and a formula $\varphi(x,X,\bar{Z})$ in the monadic language of trees such that

$T \models \varphi(a,A,\bar{P}) \ \Rightarrow \ [A \neq \emptyset \ \& \ a \in A] \quad T \models (\forall X)(\exists y)[X \neq \emptyset \Rightarrow \varphi(y,X,\bar{P})]$ and

$T \models \varphi(a,A,\bar{P}) \wedge \varphi(b,A,\bar{P}) \ \Rightarrow a = b$?

* The second author would like to thank the U.S.–Israel Binational Science Foundation for partially supporting this research. Publ. 539

If the answer is positive we will say that T has a (monadically) definable choice function (with parameters) and that φ defines a choice function from non-empty subsets of T. Note that if we let $\phi(x, Y)$ be the formula that says "if Y is not empty then $x \in Y$" then a negative answer to the choice function problem for T implies a negative answer to the uniformization problem for the monadic theory of T (with ϕ being a counter-example).

dealing with the choice function problem we split the class of trees into two natural parts, wild trees and tame trees and prove the following:

Theorem. *Let T be a tree. If T is wild or T embeds ${}^{\omega>}2$ then there is no definable choice function on T (by a monadic formula with parameters). If T is tame and does not embed ${}^{\omega>}2$ then there is even a definable well ordering of the elements of T by a monadic formula (with parameters) $\varphi(x, y, \bar{P})$.*

Looking at the definitions and proofs we observe that a tree is tame [wild] if and only if it's completion is tame [wild] and that the counter-examples for the choice function problem are either anti-chains or linearly ordered subsets of T. Hence we can prove:

Conclusion. *Let T be a tree and T' be it's completion. Then the following are equivalent:*
a) For some $n, l < \omega$, for every anti-chain/branch A of T there is a monadic formula $\varphi_A(x, X, \bar{P}_A)$ with quantifier depth $\leq n$ and $\leq l$ parameters from T, that defines a choice function from non empty subsets of A.
b) There is a monadic formula, with parameters, $\psi(x, y, \bar{P})$ that defines a well ordering of the elements of T.
c) There is a monadic formula, with parameters, $\psi'(x, y, \bar{P}')$ that defines a well ordering of the elements of T'.

The paper continues the work by Gurevich-Shelah ([GuSh]) who answered negatively a question by Rabin ([Ra]), by showing that the answer for the choice function problem is negative in ${}^{\omega>}2$.
The 'positive' results on the existence of a definable well ordering (§§3,5) are elementary and do not require knowledge of monadic logic. The negative results (§§2,3,4) are based on understanding of some composition theorems that hold for the monadic theory of trees. These facts are collected in §1.
More details and Historical background can be found in [Gu] and [GuSh].

1. Composition Theorems

In this section we will define partial theories and establish the technical tools that will be applied later. We will formalize composition theorems that will enable to compute the partial theory of a tree from partial theories of it's parts. Using such theorems enables to prove that if for example a dense chain does not have definable choice function then a tree with a dense branch does not have a definable choice function.

Definition 1.1. $(T, \triangleleft)$ is a *tree* if $\triangleleft$ is a partial order on T and for every $\eta \in T$, $\{\nu : \nu \triangleleft \eta\}$ is linearly ordered by $\triangleleft$.
Note, a chain $(C, <)$ and even a set without structure I is a tree.

Definition 1.2. Let T be a tree

1. $X \subseteq T$ is a *convex subset* if $\eta, \nu \in X$ and $\eta \triangleleft \sigma \triangleleft \nu \in T$ implies $\sigma \in X$. If T is a chain we use the term a *convex segment* or just a *segment*.
2. $(S, \triangleleft)$ is a *subtree* of $(T, \triangleleft)$ if $S \subseteq T$ and S is a convex subset of T.
3. $B \subseteq T$ is a *sub-branch* of T if B is convex and $\triangleleft$–linearily ordered.
4. $B \subseteq T$ is a *branch* of T if B is a maximal sub-branch of T.
5. $A \subseteq T$ is an *initial segment* of T if A is a sub-branch that is $\triangleleft$–downward closed. η is *above* an initial segment A if $\nu \in A \Rightarrow \nu \triangleleft \eta$.
6. For $\eta \in T$, $T_{\geq \eta}$ is the sub-tree $(\{\nu \in T : \eta \triangleleft \nu\}, \triangleleft)$. $T_{>\eta}$ is the sub-tree $(T_{\geq \eta} \setminus \{\eta\}\ ,\ \triangleleft)$. For $A \subseteq T$ an initial segment, $T_{\geq A}$ and $T_{>A}$ are defined naturally.
7. For $\eta \in T$ we deote by $suc(\eta)$ or $suc_T(\eta)$ the set of $\triangleleft$–immediate successors of η (which may be empty).
8. For $\eta, \nu \in T$ we denote the *intersection of η and ν in T* by $\eta \wedge \nu$. This may be a member of T or an initial segment of T, in any case the meaning of $\eta \wedge \nu \triangleleft \sigma$ is natural and $\sigma \triangleleft \eta \wedge \nu$ $[\sigma \in \eta \wedge \nu]$ is used only when $\eta \wedge \nu$ is an element [an initial segment].
9. If there is an $\eta \in T$ that satisfies $(\forall \nu \in T)[\eta \triangleleft \nu]$ we say that T has a root and denote η by $root(T)$.
10. $\eta, \nu \in T$ are *incomparable in T* if neither $\eta \triangleleft \nu$ nor $\nu \triangleleft \eta$. $X \subseteq T$ is an *anti-chain of T* if X consists of pairwise incomparable elements of T.
11. A *gap* in T is a pair (A, B) where $A \cap B = \emptyset$, $A \cup B$ is a sub-branch, A is an initial segment, (so $\eta \in A, \nu \in B \Rightarrow \eta \triangleleft \nu$), A without a $\triangleleft$-maximal element, B without a $\triangleleft$-minimal element, and for some $\sigma \in T$ for every $\eta \in A$ and $\nu \in B$ we have $\eta \triangleleft \sigma$ & ν, σ are incomparable.
12. *Filling a gap* (A, B) in T is adding a node τ to T such that $\eta \in A \Rightarrow \eta \triangleleft \tau$, $\nu \in B \Rightarrow \tau \triangleleft \nu$ and for every σ as in (11) we have $\tau \triangleleft \sigma$.

Definition 1.3. The *full binary tree* is the tree $({}^{\omega>}2\ ,\ \triangleleft)$ where for sequences $\eta, \nu \in {}^{\omega>}2$, $\eta \triangleleft \nu$ means η is an initial segment of ν.

Definition 1.4. The *monadic language of trees* L is the monadic version of the language of partial orders $\{\triangleleft\}$. Usually $\triangleleft$ means "smaller than or equal" but when we restrict ourselves to chains (linearily ordered sets) we use $<$ and $\leq$. For simplicity, we add to L the predicate $sing(X)$ saying "X is a singleton" so that we can quantify only over subsets. Note that everything that is defined in 1.2 is definable in L.

Next we define, following [Sh], the partial theories of a tree T. These are finite approximations of the monadic theory of T. $Th^n(T; \bar{P})$ is essentially the monadic theory of $(T, \bar{P}, \triangleleft)$ restricted to sentences of quantifier depth n.

Definition 1.5. For any tree T, $\bar{A} \in \mathcal{P}(T)^{lg(\bar{A})}$, and a natural number n, define by induction

$$t = Th^n(T; \bar{A}).$$

for $n = 0$:

$$t = \big\{\phi(\bar{X}) :\ \phi(\bar{X}) \in L,\ \phi(\bar{X}) \text{ quantifier free},\ T \models \phi(\bar{A})\big\}.$$

for $n = m + 1$:

$$t = \big\{Th^m(T; \bar{A}\,{}^\wedge B) :\ B \in \mathcal{P}(T)\big\}.$$

$T_{n,l}$ is the set of all formally possible $Th^n(T; \bar{P})$ where T is a tree and $l(\bar{P}) = l$.

Fact 1.6. *(A) For every formula $\psi(\bar{X}) \in L$ there is an n such that from $Th^n(T;\bar{A})$ we can effectively decide whether $T \models \psi(\bar{X})$.*
(B) If $m \geq n$ then $Th^n(T;\bar{A})$ can be effectively computed from $Th^m(T;\bar{A})$.
(C) Each $Th^n(T;\bar{A})$ is hereditarily finite, and we can effectively compute the set $T_{n,l}$ of formally possible $Th^n(T,\bar{A})$.

Next we recall the composition theorem for linear orders which states that the partial theory of a chain can be computed from the partial theories of it's convex parts. This allows us to sum partial theories formally.

Definition 1.7. If C, D are chains then $C + D$ is any chain that can be split into an initial segment isomorphic to C and a final segment isomorphic to D.
If $\langle C_i : i < \alpha \rangle$ is a sequence of chains then $\sum_{i<\alpha} C_i$ is any chain D that is the concatenation of segments D_i, such that each D_i is isomorphic to C_i.

Theorem 1.8 (composition theorem for linear orders).
(1) If $l(\bar{A}) = l(\bar{B}) = l(\bar{A}') = l(\bar{B}') = l$, and

$$Th^m(C,\bar{A}) = Th^m(C',\bar{A}') \text{ and } Th^m(D,\bar{B}) = Th^m(D',\bar{B}')$$

then

$$Th^m(C + D, A_0 \cup B_0, \dots, A_{l-1} \cup B_{l-1}) = Th^m(C' + D', A'_0 \cup B'_0, \dots, A'_{l-1} \cup B'_{l-1}).$$

(2) If $Th^m(C_i,\bar{A}_i) = Th^m(D_i,\bar{B}_i)$, $l(\bar{A}_i) = l(\bar{B}_i) = l$ for each $i < \alpha$, then

$$Th^m\Big(\sum_{i<\alpha} C_i,\ \cup_i A_{1,i}, \dots, \cup_i A_{l-1,i}\Big) = Th^m\Big(\sum_{i<\alpha} D_i,\ \cup_i B_{1,i}, \dots, \cup_i B_{l-1,i}\Big).$$

Proof. By [Sh] Theorem 2.4 (where a more general theorem is proved), or directly by induction on m.

♡

Notation 1.9.
(1) $t_1 + t_2 = t_3$ means: for some $m, l < \omega$, $t_1, t_2, t_3 \in T_{m,l}$ (remember definition 1.5) and if

$$t_1 = Th^m(C, A_0, \dots, A_{l-1}) \text{ and } t_2 = Th^m(D, B_0, \dots, B_{l-1})$$

then

$$t_3 = Th^m(C + D, A_0 \cup B_0, \dots, A_{l-1} \cup B_{l-1}).$$

By the previous theorem, the choice of C and D is immaterial.
(2) $\sum_{i<\alpha} Th^m(C_i,\bar{A}_i)$ is $Th^m(\sum_{i<\alpha} C_i,\ \cup_{i<\alpha} A_{1,i}, \dots, \cup_{i<\alpha} A_{l-1,i})$.
(3) If D is a subchain of C and $X_1, \dots, X_{l-1}$ are subsets of C then $Th^m(D, X_0, \dots, X_{l-1})$ abbreviates $Th^m(D, X_0 \cap D, \dots, X_{l-1} \cap D)$.
(4) We use abbreviations as $\bar{P} \cup \bar{Q}$, $\cup_i \bar{P}_i$ and $\bar{P} \subseteq C$. The meanings should be clear.
(5) For C a chain, $a < b \in C$ and $\bar{P} \subseteq C$ we denote by $Th^n(C;\bar{P}) \restriction_{[a,b)}$ the theory $Th^n([a,b); \bar{P} \cap [a,b))$.

The class of trees has some weaker (but sufficient for our purpose) composition theorems. First we define the composition of subtrees of the full binary tree following [GuSh] and quote the the respective composition theorem.

Definition 1.10. Let $M \subseteq {}^{\omega>}2$ be a tree. A *grafting function* on M is a function g satisfying the following conditions:
(a) $Dom(g) \subseteq M \times \{0,1\}$,
(b) if $(x,0) \in Dom(g)$ then $x\,{}^\wedge\langle 0\rangle \notin M$ and if $(x,1) \in Dom(g)$ then $x\,{}^\wedge\langle 1\rangle \notin M$,
(c) every value $g(x,d)$ of g ($d \in \{0,1\}$) is a tree $\subseteq {}^{\omega>}2$.
A *composition* of a tree M and a grafting function g is the tree

$$M \cup \big\{x\,{}^\wedge\langle d\rangle\,{}^\wedge y : (x,d) \in Dom(g), y \in g(x,d)\big\}$$

.

Theorem 1.11 (composition theorem for binary trees). *Let $M \subseteq {}^{\omega>}2$ be a tree, $N \subseteq {}^{\omega>}2$ be the composition of M and a grafting function g, $\bar{X} \subseteq N$ and $n < \omega$. Then, there is $m = m(n) < \omega$ (effectively computable from n) such that from $Th^m(M;\bar{X},\bar{L}^g(n,\bar{X}),\bar{R}^g(n,\bar{X}))$ we can effectively compute $Th^m(N;\bar{X})$ where*
$L^g_t(n,\bar{X}) := \big\{x \in M : (x,0) \in Dom(g), Th^n(g(x,0),\bar{X}) = t\big\}$
$\bar{L}^g(n,\bar{X}) := \big\{L^g_t(n,\bar{X}) : t \text{ a formally possible n} - \text{theory}\big\}$
and $\bar{R}^g(n,\bar{X})$ is defined similary by replacing $L,0$ with $R,1$.

Proof. This is theorem 2 in §2.3. of [GuSh]. The language that is used there is different from our L but all the mentioned symbols are monadically inter-definable (with some additional parameters) with our $\triangleleft$, (For example the relation "X is an immediate left successor of Y" is easily definable from $\triangleleft$ and the parameter $A := \{\eta \in {}^{\omega>}2 : (\exists\nu \in {}^{\omega>}2)[\eta = \nu\,{}^\wedge\langle 0\rangle]\}$). Thus the translation of [GuSh]'s proof is clear.

♡

The next three theorems allow us to compute a partial theory $Th^n(T;\bar{X})$ from partial theories of sub-structures of T. The proofs are by induction on n noting that $Th^0(T;\bar{P})$ can express only statements as $P_i \triangleleft P_j$, $P_i \in P_j$ and $P_i = P_j$ and that Th^{n+1} is a collection of n-theories. Everything is basically the same as in the previous case and we will not elaborate beyond that.

Theorem 1.12 (composition theorem for general successors). *Let T be a tree, $\bar{X} \subseteq T$ and $A \subseteq T$ an initial segment (i.e. linearily ordered by $\triangleleft$ and downward closed).*
For every x above A ($x \notin A$ and $y \in A \Rightarrow y \triangleleft x$) denote by $T_{A,x}$ the sub-tree
$\{y \in T : (\exists z)[z \triangleleft x \;\&\; z \triangleleft y \;\&\; z \text{ above } A]\}$.
We say that x and y are equivalent above A if x and y are above A and $T_{A,x} = T_{A,y}$ (compare with definition 4.1), finally let $\{T_i : i \in I_A\}$ list the equivalence classes above A (it's a disjoint union of sub-trees).
Then for every $n < \omega$, there is $m = m(n) < \omega$ (effectively computable from n) such that from $Th^m(T_{\le A};\bar{X})$ and $Th^m(I_A;\bar{P}^A(n,\bar{X}))$ we can effectively compute $Th^n(T;\bar{X})$ where
$T_{\le A} := \{y \in T : y \text{ not above } A\}$
$P^A_t(n,\bar{X}) := \{i \in I_A : Th^n(T_i;\bar{X}) = t\}$,
$\bar{P}^A(\bar{X}) := \{P^A_t(n,\bar{X}) : t \text{ a formally possible n} - \text{theory}\}$,
and $Th^m(I_A;\bar{P}^A(n,\bar{X}))$ is the m-theory of a set without sructure – i.e. in the monadic language of equality.
(A natural case is when for some $y \in T$ we have $A = \{z : z \triangleleft y\}$, $\{x_i : i \in I\} = suc(y)$ and $T_i = T_{\ge x_i}$).

♡

Theorem 1.13 (composition theorem for branches). *Let T be a tree, $B \subseteq T$ a branch, $\bar{X} \subseteq T$ and $n < \omega$. $(B', \triangleleft)$ is the chain that is obtained by adding nodes to fill the gaps in B – remember 1.2(12), (so B' is contained in the completion of B). And let T' be the tree obtained by replacing the branch B by B'*
Then there is $m = m(n) < \omega$ (effectively computable from n) such that from $Th^m(B'; \bar{P}^{B'}(n, \bar{X}))$ we can effectively compute $Th^n(T; \bar{X})$ where
for $\eta \in B'$, $T'^{B'}_{\geq\eta} := T'_{\geq\eta} \setminus B'$
$P^{B'}_t(n, \bar{X}) := \{\eta \in B : Th^m(T'^{B'}_{\geq\eta}; \bar{X}) = t\}$
$P^{B'}_t(n, \bar{X}) := \{P^{B'}_t(n, \bar{X}) : t \text{ a formally possible n} - \text{theory}\}$.
Moreover, if $\bar{Y} \subseteq B$ then from $Th^m(B'; \bar{P}^{B'}(n, \bar{X}), \bar{Y})$ we can effectively compute $Th^n(T; \bar{X}, \bar{Y})$.

♡

Notations 1.14. For stating the next composition theorem we need a considerable amount of notations.

Let T be a tree, by "F: ${}^{\omega>}2 \hookrightarrow T$ is an embedding" we mean F is 1-1 and for $\eta, \nu \in {}^{\omega>}2$, $\eta \triangleleft \nu \iff F(\eta) \triangleleft F(\nu)$, we also assume that T has a root and $F(root({}^{\omega>}2)) = root(T)$.

let $S \subseteq T$ be $F''({}^{\omega>}2)$, it is a tree (but not necesarily a subtree of T) that can be identified with ${}^{\omega>}2$.

For $x = F(\eta) \in S$ define x^0 $[x^1] \in S$ to be $F(\eta\,^\frown\langle 0\rangle)$ $[F(\eta\,^\frown\langle 1\rangle)]$.

For $Y \subseteq S$ an anti-chain (hence an anti-chain of T) let $Bush(Y) := \{x \in T : (\exists y \in Y)[x \triangleleft y]\}$ (it's a subtree of T) and let $Bush_S(Y) := Bush(Y) \cap S$ (it's a subtree of S).

For every $y \in S$ denote $y^0 \wedge y^1$ by y^i. It may be an element of T or an initial segment but remember the convention in 1.2(8).

For every $y \in S$ we define some subtrees of $T_{\geq y}$ (some of them may be trivial if for example $y = y^i$):

0) $T_0(y) := T_{\geq y}$.

1) $T_1(y) := \{x \in T : (\neg y^i \triangleleft x)\ \&\ (\exists z \neq y)[(z \triangleleft x) \& (y \triangleleft z \triangleleft y^i)]\}$, [These are the elements that split from the segment (y, y^i)].

2) $T_2(y) := \{x \in T : (y \triangleleft x)\ \&\ (\forall z)[(z \triangleleft y^i) \& (z \triangleleft x) \Rightarrow (z \triangleleft y)]\}$. (If y^i is an initial segment replace $z \triangleleft y^i$ with $z \in y^i$), [These are the elements that split from y but not from the segment (y, y^i)].

3) $T_3(y) := \{x \in T : (\neg y^0 \triangleleft x)\ \&\ (\exists z \neq y^i)[(z \triangleleft x) \& (y^i \triangleleft z \triangleleft y^0)]\}$. (If y^i is an initial segment replace $(\exists z \neq y^i)$ with $(\exists z)(y^i \triangleleft z)$), [These are the elements that split from the segment (y^i, y^0)].

4) $T_4(y) := \{x \in T : (\neg y^1 \triangleleft x)\ \&\ (\exists z \neq y^i)[(z \triangleleft x) \& (y^i \triangleleft z \triangleleft y^1)]\}$. (If y^i is an initial segment replace $(\exists z \neq y^i)$ with $(\exists z)(y^i \triangleleft z)$), [These are the elements that split from the segment (y^i, y^1)].

5) $T_5(y) := \{x \in T : (y^i \triangleleft x)\ \&\ (\forall z)[(z \triangleleft x) \& (z \triangleleft y^0 \vee z \triangleleft y^1) \Rightarrow (z \triangleleft y^i)]\}$. (If y^i is an initial segment replace $z \triangleleft y^i$ with $z \in y^i$), [These are the elements that split from y^i but not from the segments (y^i, y^0) and (y^i, y^1)].

6) $T_6(y) := T_{\geq y^0}$.

7) $T_7(y) := T_{\geq y^1}$.

For $y \in S$, $\bar{P} \subseteq T$, $\bar{t} = \langle t_0, t_1 \ldots, t_7\rangle$, t_i a possible n-theory, we have $y \in Q_{\bar{t}} \iff Th^n(T_0(y); \bar{P}) = t_0\ \& \ldots \&\ Th^n(T_7(y); \bar{P}) = t_7$. For $y \notin S$ we have $y \in Q_\emptyset$.

Finaly let $\bar{Q}(n, \bar{P})$ be $\langle Q_{\bar{t}} : \bar{t}$ a possible sequence of n-theories$\rangle\,^\frown\langle Q_\emptyset\rangle$

Note that every anti-chain Y is definable from $Bush_S(Y)$ and S is definable from $\bar{Q}$.

Theorem 1.15 (composition theorem for embeddings). *Following the above notations, let T be a tree and F: ${}^{\omega>}2 \hookrightarrow T$ an embedding.*

<u>Then</u> *for every* $Y \subseteq S$ *an anti-chain,* $y \in Y$, $\bar{P} \subseteq T$ *and* $n < \omega$, *there is* $m = m(n) < \omega$ *(effectively computable from* n*) such that from* $Th^m(Bush_S(Y); y, \bar{Q}(n, \bar{P}))$ *we can effectively compute* $Th^n(T; y, Y, \bar{P})$.

♡

2. Dense linear orders

Every finite set A has a definable well ordering by a formula with $|A|$ parameters. This is not the case for infinite models.

Claim 2.1. *Let* A *be an infinite set without structure. Then there is no definable choice function on* A*. Moreover, if* $|A| > 2^l$ *then no formula with* $\leq l$ *parameters defines a choice function on* A.

Proof. Let $\bar{P} \subseteq A$ and suppose $\varphi(x, X, \bar{P})$ defines a choice function on an infinite A. Let $B \subseteq A$ be an indiscernible set with respect to (belonging to) $\bar{P}$ of size ≥ 2. Then, for every $b_1, b_2 \in B$, $A \models \varphi(b_1, B, \bar{P})$ iff $A \models \varphi(b_2, B, \bar{P})$, a contradiction. The second part is clear.

♡

A chain C that embeds a dense linear order (hence the rational order $\mathbb{Q}$) does not have a definable choice function. The proof is by applying a Ramsey-like theorem for additive colourings from [Sh].

Definition 2.2. (a) A *colouring* of a chain C is a function f from the set of unordered pairs of distinct elements of C, into a finite set I of colours.
(b) The colouring f is *additive* if for $x_i < y_i < z_i \in C$ $(i = 1, 2)$,

$$[f(x_1, y_1) = f(x_2, y_2),\ f(y_1, z_1) = f(y_2, z_2)] \ \Rightarrow\ f(x_1, z_1) = f(x_2, z_2).$$

In this case a partial operation $+$ is defined on I, such that for $x < y < z \in C$, $f(x, z) = f(x, y) = f(y, z)$. (Compare with 1.9(1)).
(c) A subchain $D \subseteq C$ is *homogeneous* (for f) if there is an $i_0 \in I$ such that for every $x < y \in D$, $f(x, y) = i_0$.

Theorem 2.3. *If* f *is an additive colouring of a dense chain* C*, by a finite set* I *of colours, then there is an interval of* C *which has a dense homogeneous subset.*

Proof. This is theorem 1.3. in [Sh].

♡

Claim 2.4. *Let* $(C, <)$ *be a linear order that embeds a dense linear order. Then there is no definable choice function on* C.

Proof. Let $\bar{P} \subseteq C$ and suppose $\varphi(x, X, \bar{P})$ defines a choice function on C. Let n be so that from $Th^n(C; x, X, \bar{P})$ we know if $\varphi(x, X, \bar{P})$ holds and finaly let $D \subseteq C$ be dense (in itself). By 2.3 there is an $A \subseteq D$, dense inside an interval of D, hence in itself, homogeneous with respect to the colouring $f(a, b) = Th^n(C; \bar{P}) \restriction_{[a,b)}$, (Remember the notation 1.9(5)).
Let t^* be the constant theory $Th^n(C; \bar{P}) \restriction_{[a,b)}$ for every $a < b$ in A. Let $\mathbb{Z}$ be the set of integers and $X \subseteq A$, $X := \{x_n : n \in \mathbb{Z}\}$ be of order type $\mathbb{Z}$. Suppose our choice function picks x_m from X, i.e. $C \models \varphi(x_m, X, \bar{P})$.

We assume for simplicity of notations that $inf(X)$ and $sup(X)$ belong to C and denote $inf(X)$ by 0 and $sup(X)$ by 1. So $Th^n(C;\bar{P}) \restriction_X = Th^n(C;\bar{P}) \restriction_{(0,1)}$.
Letting t_0 be $Th^n(C;\bar{P}) \restriction_{\{x:x\leq 0\}}$, and t_1 be $Th^n(C;\bar{P}) \restriction_{\{x:x\geq 1\}}$ we get:

$$Th^n(C;\bar{P}) = t_0 + \sum_{k\in\mathbb{Z}} Th^n(C;\bar{P}) \restriction_{[x_k,x_{k+1})} + t_1 = t_0 + \sum_{k\in\mathbb{Z}} t^* + t_1$$

Now denote:
$t'_0 := Th^n(C;x_m,X,\bar{P}) \restriction_{\{x:x\leq 0\}}$ ($= Th^n(C;\emptyset,\emptyset,\bar{P}) \restriction_{\{x:x\leq 0\}}$),
$t'_1 := Th^n(C;x_m,X,\bar{P}) \restriction_{\{x:x\geq 1\}}$ ($= Th^n(C;\emptyset,\emptyset,\bar{P}) \restriction_{\{x:x\geq 1\}}$),
$t' := Th^n(C;x_l,X,\bar{P}) \restriction_{[x_k,x_{k+1})}$ for $k \neq l$, ($= Th^n(C;\emptyset,x_k,\bar{P}) \restriction_{[x_k,x_{k+1})}$) and
$t^{(l)} := Th^n(C;x_l,X,\bar{P}) \restriction_{[x_l,x_{l+1})}$ ($= Th^n(C;x_l,x_l,\bar{P}) \restriction_{[x_l,x_{l+1})}$).
Clearly t_0 determines t'_0, t_1 determines t'_1, t'_0 and t'_1 do not depend on m and t^* determines t' and $t^{(l)}$. We also have, for every $l \in \mathbb{Z}$:

$$Th^n(C;x_l,X,\bar{P}) = t'_0 + \sum_{j\in\mathbb{Z},\daleth<\lessdot} t' + t^{(l)} + \sum_{j\in\mathbb{Z},\daleth>\lessdot} t' + t_1$$

But, by homogeneity, we get for every $k,l \in \mathbb{Z}$:
1) $t^{(k)} = t^{(l)}$,
2) $Th^n(C;x_l,X,\bar{P}) \restriction_{(0,x_l)} = \sum_{j\in\mathbb{Z},\daleth<\lessdot} t' = \sum_{j\in\mathbb{Z},\daleth<\neg} t' = Th^n(C;x_k,X,\bar{P}) \restriction_{(0,x_k)}$,
3) $Th^n(C;x_l,X,\bar{P}) \restriction_{(x_l,1)} = \sum_{j\in\mathbb{Z},\daleth>\lessdot} t' = \sum_{j\in\mathbb{Z},\daleth>\neg} t' = Th^n(C;x_k,X,\bar{P}) \restriction_{(x_k,1)}$.
It follows that $Th^n(C;x_m,X,\bar{P}) = Th^n(C;x_l,X,\bar{P})$ for every $l \in \mathbb{Z}$, but φ "chooses" x_m from X, (and can be computed from Th^n) – a contradiction.

♡

3. Scattered orders

A scattered order is a linear order that does not embed a dense order. We will define Hdeg, the Hausdorff degree of scattered chains, and show that a scattered chain $(C,<_C)$ has a definable well ordering if $Hdeg(C) < \omega$ and that $Hdeg(C) \geq \omega \Rightarrow$ there is no definable choice function on C.

Definition 3.1. We define by recursion the Hausdorff degree of a scattered chain $(C,<_C)$:
$Hdeg(C) = 0$ iff C is finite
$Hdeg(C) = \alpha$ iff $\wedge_{\beta<\alpha} Hdeg(C) \neq \beta$ and $C = \sum_{i\in I} C_i$ where I is well ordered or inversely well ordered and for every $i \in I$, $\vee_{\beta<\alpha} Hdeg(C_i) = \beta$.
$Hdeg(C) \geq \delta$ iff $(\forall\alpha<\delta)(Hdeg(C) > \alpha)$ (δ limit).

Claim 3.2. *(1) Let C be a scattered chain with $Hdeg(C) = \alpha$, C' the completion of C and $D \subseteq C'$. Then C' and D are scattered and $Hdeg(D) \leq Hdeg(C') = \alpha$.*
(2) Let C be a scattered chain. $Hdeg(C)$ is well defined (i.e. it is an ordinal α).

Proof. (1) By induction on α.
(2) By [Ha].

♡

Claim 3.3. *Let C be a scattered chain with $Hdeg(C) = n$. Then there are $\bar{P} \subseteq C$, $lg(\bar{P}) = n-1$, and a formula (depending on n only) $\varphi_n(x,y,\bar{P})$ that defines a well ordering of C.*

Proof. By induction on $n = Hdeg(C)$:
$n \leq 1$: $Hdeg(C) \leq 1$ implies $(C, <_C)$ is well ordered or inversely well ordered. A well ordering of C is easily definable from $<_C$.
$Hdeg(C) = n+1$: Suppose $C = \sum_{i \in I} C_i$ and each C_i is of Hausdorff degree n. By the induction hypothesis there are a formula $\varphi_n(x, y, \bar{Z})$ and a sequence $\langle \bar{P}^i : i \in I \rangle$ with $\bar{P}^i \subseteq C_i$, $\bar{P}^i = \langle P^i_1, \ldots, P^i_{n-1} \rangle$ such that $\varphi_n(x, y, \bar{P}^i)$ defines a well ordering of C_i.
Let for $0 < k < n$, $P_k := \cup_{i \in I} P^i_k$ (we may assume that the union is disjoint) and $P_n := \cup \{C_i : i \text{ even}\}$.
We will define an equivalence relation $\sim$ by $x \sim y$ iff $\bigwedge_i (x \in C_i \Leftrightarrow y \in C_i)$.
$\sim$ and $[x]$, (the equivalence class of an element x), are easily definable from P_n and $<_C$. We can also decide from P_n if I is well or inversely well ordered (by looking at subsets of C consisted of nonequivalent elements) and define $<'$ to be $<$ if I is well ordered and the inverse of $<$ if not. $\varphi_{n+1}(x, y, P_1, \ldots, P_n)$ will be defined by:

$$\varphi_{n+1}(x, y, \bar{P}) \Leftrightarrow \left[x \not\sim y \;\&\; x <' y\right] \vee \left[x \sim y \;\&\; \varphi_n(x, y, P_1 \cap [x], \ldots, P_{n-1} \cap [x])\right]$$

$\varphi_{n+1}(x, y, \bar{P})$ well orders C.

♡

Next we prove that a scattered orders of infinite $Hdeg$ don't have a definable choice function (hence a well ordering).

Definition 3.4. We define for every $n < \omega$ a model $\mathcal{M}^n$ in the language consisted of a binary relation $<^n$:
a) The universe of $\mathcal{M}^n$, which will be denoted by M^n, is the tree ${}^{n\geq}\omega$.
b) Let, for every $\eta \in {}^{n\geq}\omega$, $<_\eta$ be a linear ordering of $suc(\eta) := \{\eta^\wedge\langle k \rangle : k < \omega\}$ such that if $lev(\eta)$ is even then $k < l \Rightarrow \eta^\wedge\langle k \rangle <_\eta \eta^\wedge\langle l \rangle$, and if $lev(\eta)$ is odd then $k < l \Rightarrow \eta^\wedge\langle l \rangle <_\eta \eta^\wedge\langle k \rangle$.
(So $<_\eta$ orders $suc(\eta)$ with order type ω if η is in an even level and with order type ω^* if η is in an odd level).
c) $<^n$ is the lexicographic order induced by the orders $<_\eta$ of immediate successors.
$(M^n, <^n)$ is hence a chain. Note, the 'usual' partial order $\triangleleft$ on ${}^{n\geq}\omega$ (being an initial segment), is not definable in $\mathcal{M}^n$.

Definition 3.5. We define by induction the scattered chains C_n and C^*_n:
$C_1 := \omega$, $\quad C^*_1 := \omega^*$,
$C_2 := \sum_{i \in \omega} \omega^*$, $\quad C^*_2 := \sum_{i \in \omega^*} \omega$,
and in general:
$C_n := \sum_{i \in \omega} C^*_n$, $\quad C^*_n := \sum_{i \in \omega^*} C_n$.

Definition 3.6. $f : \mathcal{M}^n \hookrightarrow C$ is an embedding of $\mathcal{M}^n$ in a scatterd chain $(C, <_C)$ if f is 1–1 and $\sigma <^n \tau \Rightarrow f(\sigma) <_C f(\tau)$

Fact 3.7. *Let C be a scattered chain with $Hdeg(C) \geq n+1$. Then there is an embedding $f : \mathcal{M}^n \hookrightarrow C$.*

Proof. *Clearly the following hold:*
(α) *For a scattered chain C: $Hdeg(C) = n \Rightarrow [C_n \subseteq C$ or $C^*_n \subseteq C]$.*
(β) $\mathcal{M}^n \subseteq \mathcal{M}^{n+1}$
(γ) *There is an embedding $g : \mathcal{M}^n \hookrightarrow C_n$.*

Now assume $Hdeg(C) = n+1$ and use (α). In the case $C_{n+1} \subseteq C$ we have by (γ) an embedding $g\colon \mathcal{M}^{n+1} \hookrightarrow C$ and by (β) an embedding $f\colon \mathcal{M}^n \hookrightarrow C$. In the case $C^*_{n+1} \subseteq C$ we have, by the definition of C^*_{n+1}, $C_n \subseteq C^*_{n+1}$ and by (γ) an embedding $f\colon \mathcal{M}^n \hookrightarrow C$.

♡

Conclusion 3.8. *Let C be a scattered chain with $Hdeg(C) \geq \omega$. Then, for every $n < \omega$ there is an embedding of $\mathcal{M}^n$ into C.*

♡

Lemma 3.9. *If C is scattered and $Hdeg(C) \geq \omega$ then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on C.*

Proof. Assume towards a contradiction that there is $\bar{P} \subseteq C$, $lg(\bar{P}) = l$ and $\varphi(x, X, \bar{P})$ defines a choice function on C. Let m be so that from $Th^m(C; x, X, \bar{P})$ we can decide if $C \models \varphi(x, X, \bar{P})$. As in the proof of 2.4 it is enough to find an $B \subseteq C$, of order type $\mathbb{Z}$, homogeneous with respect to the colouring $f(a,b) = Th^m(C; \bar{P}) \restriction_{[a,b)}$. Let

$$n > |\{Th^m(D; \bar{Q}) : D \text{ a chain },\ \bar{Q} \subseteq D,\ l(\bar{Q}) = l\}| = |T_{m,l}|$$

and $f\colon \mathcal{M}^n \hookrightarrow C$ be an embedding. Let $T \subseteq C$ be the image of f and we will identify T with ${}^{n\geq}\omega$ and the submodel $(T, <_C) \subseteq (C, <_C)$ with the model $({}^{n\geq}\omega, <^n)$

Notation: We will write $<$ instead of $<_C$ and it's restriction $<^n$. Given $\eta < \nu \in {}^{n\geq}\omega = T$ we will write $Th^m[\eta, \nu)$ instead of $Th^m(C; \bar{P}) \restriction_{[\eta,\nu)}$. $T_{\geq\eta}$ and $T_{>\eta}$ are the usual subsets of ${}^{n\geq}\omega = T$

We will begin to thin out the tree $T = {}^{n\geq}\omega$, in order to obtain a quite homogeneous subtree $A \subseteq T$ going down with the levels. Arriving to a node η, we will have defined $A_{\geq\nu}$ for every $\nu \in suc(\eta)$ and will define $A_{\geq\eta}$ by thinning out $suc(\eta)$ to a set B_η and taking $\{\eta\} \cup \{A_{\leq\nu} : \nu \in B_\eta\}$. $A_{\geq\eta}$ will satisfy the following:

$$(*)\quad [\sigma < \tau \in A_{\geq\eta},\ lev(\sigma) = lev(\tau)] \ \Rightarrow\ Th^m[\sigma, \tau) \text{ depends only on } lev(\sigma \wedge \tau)$$

Assume w.l.o.g that n is odd.
<u>Step 1</u>: for every $\eta \in {}^{n\geq}\omega$ with $lev(\eta) = n-1$ pick out an infinite set $B_\eta \subseteq \omega$ such that

$$k < l \in B_\eta \ \Rightarrow\ Th^m(C; \bar{P}) \restriction_{[\eta\,\hat{}\,\langle k\rangle, \eta\,\hat{}\,\langle l\rangle)} = t_\eta$$

(note that $k < l < \omega \Rightarrow \eta\,\hat{}\,\langle k\rangle < \eta\,\hat{}\,\langle l\rangle$), let k_η be the second element of B_η. Let $A_{\geq\eta}$ be $\{\eta\} \cup \{\eta\,\hat{}\,\langle k\rangle : k_\eta \leq k \in B_\eta\}$, this is a subtree of T and $(*)$ clearly holds.
<u>Step 2</u>: Given $\nu \in {}^{n\geq}\omega$ with $lev(\nu) = n-2$ we have defined B_σ, k_σ and $A_{\geq\sigma}$ for every $\sigma \in suc(\nu)$. Pick out an infinite $B^0_\nu \subseteq \omega$ so that $(*)$ will hold for $lev(\sigma) = lev(\tau) = n-1$, $lev(\sigma \wedge \tau) = n-2$ i.e.

$$k > l \in B^0_\nu \ \Rightarrow\ Th^m[\nu\,\hat{}\,\langle k\rangle, \nu\,\hat{}\,\langle l\rangle) = t_\nu$$

($suc(\nu)$ are ordered as ω^*). Thin out B^0_ν to an infinite $B^1_\nu \subseteq \omega$ so that $(*)$ will hold for $lev(\sigma) = lev(\tau) = n$, $lev(\sigma \wedge \tau) = n-2$ i.e.

$$k > l \in B^1_\nu,\ \sigma = \nu\,\hat{}\,\langle k\rangle,\ \tau = \nu\,\hat{}\,\langle l\rangle \ \Rightarrow\ Th^m[\sigma\,\hat{}\,\langle k_\sigma\rangle, \tau\,\hat{}\,\langle k_\tau\rangle) \text{ is constant}$$

Why does it suffice to look only at e.g. $\sigma\,\hat{}\,\langle k_\sigma\rangle$? because by the choice of t_σ and A_σ we have $t_\sigma + t_\sigma = t_\sigma$ hence for every $\eta < \sigma \in T$ we can break the paths $[\eta, \sigma\,\hat{}\,\langle k_\sigma\rangle)$ and $[\eta, \sigma\,\hat{}\,\langle l\rangle)$, for $l \in A_\sigma$,

into three parts: first from η to σ then from σ to it's 'first' successor in A_σ, and then to $\sigma{}^\frown\langle k_\sigma\rangle$ or $\sigma{}^\frown\langle l\rangle$ (this is why we chose k_σ to be the second element of A_σ), but adding the last theory does not change the sum hence $Th^m[\eta, \sigma{}^\frown\langle k_\sigma\rangle) = Th^m[\eta, \sigma{}^\frown\langle l\rangle)$ for every $l \in A_\sigma$. By a similar argument we can show that for every $l \in A_\sigma$ we have $Th^m[\sigma{}^\frown\langle k_\sigma\rangle, \eta) = Th^m[\sigma{}^\frown\langle l\rangle, \eta)$.
Next, thin out B^1_ν to get B_ν so that $(*)$ will hold for $lev(\sigma) = lev(\tau) = n$, $lev(\sigma \wedge \tau) = n-1$ i.e.

$$k \in B_\nu,\ \sigma = \nu{}^\frown\langle k\rangle \ \Rightarrow\ t_\sigma \text{ is constant}$$

let k_ν to be the second element of B_ν. Define the subtree $A_{\geq\nu}$ to be $\{\nu\} \cup \{A_{\nu{}^\frown\langle k\rangle} : k_\nu \leq k \in B_\nu\}$. Clearly $A_{\geq\nu}$ satisfies $(*)$.
Step $n-1$: we have reached $\langle e\rangle$, the root of ${}^{n\geq}\omega$. $B^0_e, B^1_e, \ldots, B_e^{(n-1)\cdot(n-2)} = B_e$ are defined as before, taking care of $(*)$ for all the possibilities of the form $lev(\sigma) = lev(\tau) = k$, $lev(\sigma \wedge \tau) = l$ (some thinning outs are not necessary as they have been taken care of in previous steps), k_e, t_e and $A_{\geq e} = A$ are defined as well.
Final Step: By our construction, for every $\eta < \nu$ in A, with $lev(\eta) = lev(\nu)$, $Th^m(C; \bar{P}) \restriction_{[\eta,\nu)}$ depends only on $lev(\eta \wedge \nu)$ and we define t_k by:

$$t_k := Th^m[\eta, \nu) \text{ where } \eta < \nu, lev(\eta) = lev(\nu) = n,\ lev(\eta \wedge \nu) = n-k$$

By our choice of n we have some $k < l \leq n$ with $t_k = t_l$. Let's show how to get a suitable homogeneous subset B of $T\ (C)$ from this.
Example 1. $t_1 = t_2$
Pick $\eta \in A$ with $lev(\eta) = n-2$. The successors of η in A have order type ω^* and for every successor of η in A, it's successors have order type ω. Define:

$$B_1 := \left\{\eta{}^\frown\langle l\rangle{}^\frown\langle k_{\eta{}^\frown\langle l\rangle}\rangle \ :\ l \in A_\eta, l > k_\eta\right\}$$

and

$$B_2 := \left\{\eta{}^\frown\langle k_\eta\rangle{}^\frown\langle k\rangle \ :\ k \in A_{\eta{}^\frown\langle k_\eta\rangle}\right\}$$

and let $B = B_1 \cup B_2$.
Clearly B_1 has order type ω^*, B_2 has order type ω and B has order type $\mathbb{Z}$. Moreover, for every $\sigma < \tau \in B_1$ we have $Th^m[\sigma, \tau) = t_1$ (since $lev(\sigma \wedge \tau) = n-1$) and for every $\sigma < \tau \in B$ with $\tau \in B_2$ we have $Th^m[\sigma, \tau) = t_2$ (since $lev(\sigma \wedge \tau) = n-2$). By $t_1 = t_2$ we conclude:

$$\forall(\sigma < \tau \in B)\left[Th^m(C; \bar{P}) \restriction_{[\sigma,\tau)} = t_1\right].$$

Finding a homogeneous subset of C of order type $\mathbb{Z}$, we can proceed as in claim 2.2 to get a contradiction to "$\varphi(x, X, \bar{P})$ defines a choice function on C".
Example 2. $t_2 = t_3$
Pick $\eta \in A$ with $lev(\eta) = n-3$. The successors of η in A have order type ω and for every successor of η in A, it's successors have order type ω^*. Let $\sigma := \eta{}^\frown\langle k_\eta\rangle$ $(lev(\sigma) = n-2)$, for $l > k_\eta \in A_\eta$ $\eta_l := \eta{}^\frown\langle l\rangle$ $(lev(\eta_l) = n-2)$ and $\sigma_l := \eta_l{}^\frown\langle k_{\eta_l}\rangle$ $(lev(\sigma_l) = n-1)$. Define:

$$B_1 := \left\{\sigma_l{}^\frown\langle k_{\sigma_l}\rangle \ :\ l \in A_\eta, l > k_\eta\right\}$$

(B_1 has order type ω^*). To define B_2 we let, for $l \in A_\sigma$, $\tau_l := \sigma{}^\frown\langle l\rangle$ (τ_l are extensions of η and σ with $lev(\tau_l) = n-1$) and then extend each τ_l to a ρ_l defined by $\rho_l := \tau_l{}^\frown\langle k_{\tau_l}\rangle$. So

$$B_2 := \left\{\rho_l \ :\ l \in A_\sigma\right\}$$

and it has order type ω. $B := B_1 \cup B_2$ has order type $\mathbb{Z}$ and we can easily check that for every $\nu_1 < \nu_2 \in B$ we have $Th^m[\nu_1, \nu_2) = t_3$ (as $\nu_1 \wedge \nu_2 = \eta$ so $lev(\nu_1 \wedge \nu_2) = n-3$) and for every $\nu_1 < \nu_2 \in B_2$ we have $Th^m[\nu_1, \nu_2) = t_2$ (as $\nu_1 \wedge \nu_2 = \sigma$ so $lev(\nu_1 \wedge \nu_2) = n-2$). By $t_2 = t_3$ we conclude:

$$\forall(\sigma < \tau \in B)\big[Th^m(C;\bar{P}) \restriction_{[\sigma,\tau)} = t_2\big].$$

and we proceed as before.
What we did in both examples can be described as follows: we fixed a node $\eta \in A$ and a successor σ of η, we extended the other successors of η and the successors of σ in a "canonical" way, (ν is extended to $\nu{}^\wedge\langle k_\nu\rangle$) to nodes of level n. The result is a homogeneous subset of C of order type $\mathbb{Z}$.
General case. $l+1 < r$, $t_l = t_r$
Let $\sigma, \tau \in A$ be such that $lev(\sigma) = lev(\tau) = n$ and $lev(\sigma \wedge \tau) = n-r$, so $Th^m[\sigma,\tau) = t_r$. Then find $\rho \in A$ with $\sigma < \rho < \tau$, $lev(\rho) = n$, $lev(\sigma \wedge \rho) = n-(l+1)$ and $lev(\rho \wedge \tau) = n-r$. What we get is the following equation:

$$t_r \;=\; Th^m[\sigma,\tau) \;=\; Th^m[\sigma,\rho) + Th^m[\rho,\tau) \;=\; t_{l+1} + t_r$$

but $t_r = t_l$ hence

$$(*)\ t_l = t_{l+1} + t_l$$

Imitate this computation: let $\sigma, \tau \in A$ be such that $lev(\sigma) = lev(\tau) = n$ and $lev(\sigma \wedge \tau) = n-(l+1)$, so $Th^m[\sigma,\tau) = t_r$ and find $\rho \in A$ with $\sigma < \rho < \tau$, $lev(\rho) = n$, $lev(\sigma \wedge \rho) = n-(l+1)$ and $lev(\rho \wedge \tau) = n-l$. What we get is the following equation:

$$t_{l+1} \;=\; Th^m[\sigma,\tau) \;=\; Th^m[\sigma,\rho) + Th^m[\rho,\tau) \;=\; t_{l+1} + t_l$$

hence

$$(**)\ t_{l+1} = t_{l+1} + t_l$$

Combining $(*)$ and $(**)$ we get $t_{l+1} = t_l$. Now proceed as in example 1 (if l is odd) or as in example 2 (if l is even) by taking "canonical extensions" of successors to get the required homogeneous subset B of order type $\mathbb{Z}$.

♡

Conclusion 3.10. *For every $m, l < \omega$ there is an $n < \omega$ such that if C is a scattered chain and $Hdeg(C) \geq n+1$ then C does not have a definable choice function by a formula with quantifier depth $\leq m$ and with $\leq l$ parameters.*

Proof. Let n be larger than $|T_{m,l}|$. Now if $Hdeg(C) \geq n+1$ then we can embed ${}^{n\geq}\omega$ into C and immitate the previous proof.

♡

4. Wild trees

Intuitively, wild trees are trees that have a large amount of splitting (4.2(1)(i)) or have 'complicated' branches (4.2(1)(ii)(iii)), the next two definitions state this formally. Wild trees don't have a definable choice function (4.6).

Definition 4.1. Let $(T, \lhd)$ be a tree
(1) If A is an initial segment of T then $top(A)$ is $\{x \in T : (\forall t \in A)[t \lhd x]\}$. (It's a tree).
(2) Let A be an initial segment of T then the binary relation $\sim_A^0$ on $T \setminus A$ is defined by

$$x \sim_A^0 y \iff (\forall t \in A)[t \lhd x \equiv t \lhd y]$$

(It's an equivalence relation that says "x and y 'break' A in the same place").
(3) Let A be an initial segment of T then the binary relation $\sim_A^1$ on $T \setminus A$ is defined by

$$x \sim_A^1 y \iff [x \sim_A^0 y] \ \& \ (\exists z)[z \lhd x \ \& \ z \lhd y \ \& \ z \sim_A^0 x]$$

(It's an equivalence relation that divides – for every initial segment $B \subseteq A$ – $top(B)/\sim_B^0$ into disjoint subtrees).

Definition 4.2. (1) A tree T is called *wild* if either
(i) $sup\{|top(A)/\sim_A^1| : A \subseteq T$ an initial segment$\} \geq \aleph_0$ or
(ii) There is a branch $B \subseteq T$ and an embedding $f : \mathbb{Q} \to \mathbb{B}$ or
(iii) All the branches of T are scattered linear orders but $sup\{Hdeg(B) : B$ a branch of $T\} \geq \omega$.
(2) A tree T is *tame for* (n^*, k^*) if the value in (i) is $\leq n^*$, (ii) does not hold and the value in (iii) is $\leq k^*$
(3) A tree T is *tame* if T is tame for (n^*, k^*) for some $n^*, k^* \leq \omega$.

Claim 4.3. *If T is a wild tree and (1)(i) of 4.2 holds then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

Proof. We will use the composition theorem for general successors 1.12.
Suppose $\varphi(x, X, \bar{P})$ defines a choice function on T and $Th^n(T; x, X, \bar{P})$ computes φ. For an initial segment $A \subseteq T$ let $top(A)/sim_A^1 = \{T_i : i \in I_A\}$, by our assumption, for every $l < \omega$ there is an initial segment $A \subseteq T$ such that $|I_A| > l$. Choose a large enough l (see below) and a corresponding A and for every $i \in I_A$ pick $x_i \in T_i$.
If l is larger than the number of possible theories ($= |T_{n,l(\bar{P})}|$) then there are $i \neq j \in I_A$ such that $Th^n(T_i; x_i, \bar{P}) = Th^n(T_j; x_j, \bar{P})$ and let's assume that we have chosen such an l. Now let $\bar{R}_1 = \{x_i\} \cup \{x_i, x_j\} \cup \bar{P}$ and $\bar{R}_2 = \{x_j\} \cup \{x_i, x_j\} \cup \bar{P}$. Apply 1.12: clearly

$$Th^m(T_{\leq A}; \bar{R}_1) = Th^m(T_{\leq A}; \bar{R}_2) = Th^m(T_{\leq A}; \emptyset, \emptyset, \bar{P})$$

and easily

$$Th^m(I_A; \bar{Q}^A(n, \bar{R}_1) = Th^m(I_A; \bar{Q}^A(n, \bar{R}_2)$$

but by 1.12 these theories determine $Th^n(T; x_i, \{x_i, x_j\}, \bar{P})$ and $Th^n(T; x_j, \{x_i, x_j\}, \bar{P})$ hence

$$T \models \varphi(x_i, \{x_i, x_j\}, \bar{P}) \iff T \models \varphi(x_j, \{x_i, x_j\}, \bar{P})$$

a contradiction.

♡

Claim 4.4. *If T is a wild tree and (1)(ii) of 4.2 holds then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

Proof. Let $B \subseteq T$ be a branch that embeds $\mathbb{Q}$. We will apply 1.13 and "translate" the choice function on T to a choice function on B but by 2.4 there is no definable choice function on B.
So assume that $\varphi(x, X, \bar{P})$ defines a choice function on T and is determined by $Th^n(T; x, X, \bar{P})$. By 1.13 there is an $m < \omega$, a chain B' with $(B, \triangleleft) \subseteq (B', \triangleleft)$ and a sequence of parameters $\bar{Q} \subseteq B'$ such that from $Th^m(B'; \bar{Q})$ we can compute $Th^n(T; \bar{P})$. Define, for $\eta \triangleleft \nu \in B$, $f(\eta, \nu) = Th^m(B'; \bar{Q}) \restriction_{[\eta,\nu)}$. f is an additive colouring hence by 2.3 there is $X = \{\eta_i\}_{i \in \mathbb{Z}}$, of order type $\mathbb{Z}$, homogeneous with respect to f. As in the proof of 2.4 we have:

$$i, j \in \mathbb{Z} \Rightarrow Th^m(B'; \eta_i, X, \bar{Q}) = Th^m(B'; \eta_j, X, \bar{Q})$$

and (by the 'moreover' clause in 1.13) this implies

$$i, j \in \mathbb{Z} \Rightarrow Th^n(T; \eta_i, X, \bar{P}) = Th^n(T; \eta_j, X, \bar{P}).$$

Hence

$$i, j \in \mathbb{Z} \Rightarrow [T \models \varphi(\eta_i, X, \bar{P}) \iff T \models \varphi(\eta_j, X, \bar{P})]$$

and this contradicts "φ chooses an element from X".

♡

Claim 4.5. *If T is a wild tree and (1)(iii) of 4.1 holds then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

Proof. Similar to the previous proof.
By (1)(*iii*) for every $m < \omega$ there is a branch $B \subseteq T$ with $Hdeg(B) > m$. Use 1.13, 3.10 and the proof of 3.9 to find, for a suitable branch B, a homogeneous subset that contradicts the assumption that $\varphi(x, X, \bar{P})$ defines a choice function on T.
The details are left to the reader.

♡

We conclude

Theorem 4.6. *T is a wild tree $\Rightarrow$ T does not have a monadically definable choice function. Moreover, every candidate fails to choose from either linearily ordered subsets (4.4, 4.5) or anti-chains (4.3).*

♡

5. Tame trees

By [GuSh] ${}^{\omega>}2$ does not have a definable choice function. To know if a tame tree T has a definable choice function we just have to ask if there is an embedding of $f\colon {}^{\omega>}2 \hookrightarrow T$. If such an embedding exists we use [GuSh] to show that T does not have one, if not, T has even a definable well ordering.

Claim 5.1. *Let T be a tree and $F\colon {}^{\omega>}2 \hookrightarrow T$ be a tree embedding. Then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

Proof. We will use [GuSh] 1.15 and the notations of 1.14. First, we may assume w.l.o.g that T has a root (adding a root will not effect the existence of a choice function) and that $F(root({}^{\omega>}2)) = root(T)$. Now apply the proof in §5 of [GuSh]. From the proof there we learn that for every $\bar{Q} \subseteq {}^{\omega>}2$ and $m < \omega$ there is an infinite anti-chain $Y \subseteq {}^{\omega>}2$ such that for every $y \in Y$ there is $y^* \neq y \in Y$ with $Th^m(Bush_{{}^{\omega>}2}(Y); y, \bar{Q}) = Th^m(Bush_{{}^{\omega>}2}(Y); y^*, Y, \bar{Q})$. In our context ($F''({}^{\omega>}2) = S \subseteq T$) the result has the form:

(∗) for every $\bar{Q} \subseteq S$ and $m < \omega$ there is an infinite anti-chain $Y \subseteq S$ such that for every $y \in Y$ there is $y^* \neq y \in Y$ with $Th^m(Bush_S(Y); y, \bar{Q}) = Th^m(Bush_S(Y); y^*, \bar{Q})$.

Let $\varphi(x, X, \bar{P})$ be a candidate for a definition of a choice function on T and suppose $Th^n(T; x, X, \bar{P})$ decides φ. Let $m < \omega$ and $\bar{Q} = \bar{Q}(n, \bar{P})$ be as in 1.15 and $Y \subseteq S$ be the anti-chain from (∗). Suppose $T \models \varphi(y, Y, \bar{P})$, by (∗) we have $y^* \in Y$ as in there. Now $Th^m(Bush_S(Y); y, \bar{Q}) = Th^m(Bush_S(Y); y^*, \bar{Q})$ and by 1.15

$$Th^n(T; y, Y, \bar{P}) = Th^n(T; y^*, Y, \bar{P})$$

hence

$$T \models \varphi(y, Y, \bar{P}) \iff T \models \varphi(y^*, Y, \bar{P})$$

hence φ fails to define a choice function on T.

♡

Definition 5.2. Let T be a tree. For $\eta \in T$ we define by recursion a rank function $rk(\eta)$ by:
$rk(\eta) \geq \alpha + 1 \iff$ there are $\nu_1, \nu_2 \in T$ with $\eta \triangleleft \nu_1$ and $\eta \triangleleft \nu_2$ such that ν_1, ν_2 are incomparable in T and $rk(\nu_1), rk(\nu_2) \geq \alpha$
If $rk(\eta)$ is not defined we stipulate $rk(\eta) = \infty$.

Fact 5.3. *(1)* $\eta \triangleleft \nu \in T \Rightarrow rk(\nu) \leq rk(\eta)$ *where* $\leq$ *has the obvious meaning.*
(2) ${}^{\omega>}2$ *is not embeddable in a tree* $T \iff$ *for every* $\eta \in T$, $rk(\eta) \neq \infty$

Lemma 5.4. *Let T be a tame tree. If ${}^{\omega>}2$ is not embeddable in T then there are $\bar{Q} \subseteq T$ and a monadic formula $\varphi(x, y, \bar{Q})$ that defines a well ordering of T.*

Proof. Assume T is (n^*, k^*) tame, recall definitions 4.1 and 4.2 and remember that for every $x \in T$, $rk(x)$ is well defined (i.e. $< \infty$). We will partition T into a disjoint union of sub-branches, indexed by the nodes of a well founded tree Γ and reduce the problem of a well ordering of T to a problem of a well ordering of Γ.

Step 1. Define by induction on α a set $\Gamma_\alpha \subseteq {}^{\alpha}Ord$ (this is a our set of indices), for every $\eta \in \Gamma_\alpha$ define a tree $T_\eta \subseteq T$ and a branch $A_\eta \subseteq T_\eta$.
$\alpha = 0$: Γ_0 is $\{\langle\rangle\}$, $T_{\langle\rangle}$ is T and $A_{\langle\rangle}$ is a branch (i.e. a maximal linearily ordered subset) of T.
$\alpha = 1$: Look at $(T \setminus A_{\langle\rangle}) / \sim^1_{A_{\langle\rangle}}$, it's a disjoint union of trees and name it $\langle T_{\langle i\rangle} : i < i^* \rangle$, let $\Gamma_1 := \{\langle i\rangle : i < i^*\}$ and for every $\langle i\rangle \in \Gamma_1$ let $A_{\langle i\rangle}$ be a branch of $T_{\langle i\rangle}$.
$\alpha = \beta + 1$: For $\eta \in \Gamma_\beta$ denote $(T_\eta \setminus A_\eta) / \sim^1_{A_\eta}$ by $\{T_{\eta\hat{}\langle i\rangle} : i < i_\eta\}$, let $\Gamma_\alpha = \{\eta\hat{}\langle i\rangle : \eta \in \Gamma_\beta,\ i < i_\eta\}$ and choose $A_{\eta\hat{}\langle i\rangle}$ to be a branch of $A_{\eta\hat{}\langle i\rangle}$.
α limit: Let $\Gamma_\alpha = \{\eta \in {}^{\alpha}Ord : \wedge_{\beta<\alpha}\eta\restriction_\beta \in \Gamma_\beta,\ \wedge_{\beta<\alpha} T_{\eta\restriction_\beta} \neq \emptyset\}$, let for $\eta \in \Gamma_\alpha$ $T_\eta = \cap_{\beta<\alpha} T_{\eta\restriction_\beta}$ and A_η a branch of T_η. (T_η may be empty).
Now, at some stage $\alpha \leq |T|^+$ we have $\Gamma_\alpha = \emptyset$ and let $\Gamma = \cup_{\beta<\alpha}\Gamma_\beta$. Clearly $\{A_\eta : \eta \in \Gamma\}$ is a partition of T into disjoint sub-branches.
Notation: having two trees T and Γ, to avoid confusion, we use x, y, s, t for nodes of T and η, ν, σ for nodes of Γ.

Step 2. We want to show that $\Gamma_\omega = \emptyset$ hence Γ is a well founded tree. Note that we made no restrictions on the choice of the A_η's and we add one now in order to make the above statement true. Let $\eta{}^\frown\langle i\rangle \in \Gamma$ define $A_{\eta,i}$ to be the sub-branch $\{t \in A_\eta : (\forall s \in A_{\eta{}^\frown\langle i\rangle})[rk(t) \le rk(s)]\}$ and $\gamma_{\eta,i}$ to be $rk(t)$ for some $t \in A_{\eta,i}$. By 5.5(1) and the inexistence of a stricly decreasing sequence of ordinals, $A_{\eta,i} \ne \emptyset$ and $\gamma_{\eta,i}$ is well defined. Note also that $s \in A_{\eta{}^\frown\langle i\rangle} \Rightarrow rk(s) \le \gamma_{\eta,i}$.

Proviso: For every $\eta \in \Gamma$ and $i < i_\eta$ the sub-branch $A_{\eta{}^\frown\langle i\rangle}$ contains every $s \in T_{\eta{}^\frown\langle i\rangle}$ with $rk(s) = \gamma_{\eta,i}$.

Following this we claim: "Γ does not contain an infinite, stricly increasing sequence". Otherwise let $\{\eta_i\}_{i<\omega}$ be one, and choose $s_n \in A_{\eta_n,\eta_{n+1}(n)}$ (so $s_n \in A_{\eta_n}$). Clearly $rk(s_n) \ge rk(s_{n+1})$ and by the proviso we get

$$rk(s_n) = rk(s_{n+1}) \Rightarrow rk(s_{n+1}) > rk(s_{n+2})$$

therefore $\{rk(s_n)\}_{n<\omega}$ contains an infinite, stricly decreasing sequence of ordinals which is absurd.

Step 3. Next we want to make "x and y belong to the same A_η" definable.

For each $\eta \in \Gamma$ choose $s_\eta \in A_\eta$, and let $Q \subseteq T$ be the set of representatives. Let $h{:}T \to \{d_0, \dots, d_{n^*-1}\}$ be a colouring that satisfies: $h\restriction_{A_{\langle\rangle}} = d_0$ and for every $\eta{}^\frown\langle i\rangle \in \Gamma$, $h\restriction_{A_{\eta{}^\frown\langle i\rangle}}$ is constant and, when $j < i$ and $s_{\eta{}^\frown\langle j\rangle} \sim^0_{A_\eta} s_{\eta{}^\frown\langle j\rangle}$ we have $h\restriction_{A_{\eta{}^\frown\langle i\rangle}} \ne h\restriction_{A_{\eta{}^\frown\langle j\rangle}}$. This can be done as T is (n^*, d^*) tame.

Using the parameters $D_0, \dots, D_{n^*-1}$ ($x \in D_i$ iff $h(x) = d_i$), we can define $\vee_\eta x, y \in A_\eta$ by "x, y are comparable and the sub-branch $[x, y]$ (or $[y, x]$) has a constant colour".

Step 4. As every A_η has Hausdorff degree at most k^*, we can define a well ordering of it using parameters $P^\eta_1, \dots, P^\eta_{k^*}$ and by taking $\bar{P}$ to be the (disjoint) union of the $\bar{P}^\eta$'s we can define a partial ordering on T which well orders every A_η.

By our construction $\eta \triangleleft \nu$ if and only if there is an element in A_ν that 'breaks' A_η i.e. is above a proper initial segment of A_η. (Caution, if T does not have a root this may not be the case for $\langle\rangle$ and a $< n^*$ number of $\langle i\rangle$'s and we may need parameters for expressing that). Therefore, as by step 3 "being in the same A_η" is definable, we can define a partial order on the sub-branches A_η (or the representatives s_η) by $\eta \triangleleft \nu \Rightarrow A_\eta \le A_\nu$.

Next, note that "ν is an immediate successor of η in Γ" is definable as a relation between s_ν and s_η hence the set $A^+_\eta := A_\eta \cup \{s_{\eta{}^\frown\langle i\rangle}\}$ is definable from s_η. Now the order on A_η induces an order on $\{s_{\eta{}^\frown\langle i\rangle}/\sim^0_{A_\eta}\}$ which is can be embedded in the complition of A_η hence has Hdeg$\le k^*$. Using additional parameters $Q^\eta_1, \dots, Q^\eta_{k^*}$, we have a definable well ordering on $\{s_{\eta{}^\frown\langle i\rangle}/\sim^0_{A_\eta}\}$. As for the ordering on each $\sim^1_{A_\eta}$ equivalence class (finite with $\le n^*$ elements), define it by their colours (i.e. the element with the smaller colour is the smaller according to the order).

Using $\bar{D}$, $\bar{P}$, Q and $\bar{Q} = \cup_\eta \bar{Q}^\eta$ we can define a partial ordering which well orders each A^+_η in such a way that every $x \in A_\eta$ is smaller then every $s_{\eta{}^\frown\langle i\rangle}$.

Summing up we can define (using the above parameters) a partial order on subsets of T that well orders each A_η, orders sub-branches A_η, A_ν when the indices are comparable in Γ and well orders all the "immediate successors" sub-branches of a sub-branch A_η.

Step 5. The well ordering of T will be defined by $x < y \iff$

a) x and y belong to the same A_η and $x < y$ by the well order on A_η; or

b) $x \in A_\eta$, $y \in A_\nu$ and $\eta \triangleleft \nu$; or

c) $x \in A_\eta$, $y \in A_\nu$, $\sigma = \eta \wedge \nu$ in Γ (defined as a relation between sub-branches), $\sigma{}^\frown\langle i\rangle \triangleleft \eta$, $\sigma{}^\frown\langle j\rangle \triangleleft \nu$ and $s_{\sigma{}^\frown\langle i\rangle} < s_{\sigma{}^\frown\langle j\rangle}$ in the order of A^+_σ.

Note, that $<$ is a linear order on T and every A_η is a convex and well ordered sub-chain. Moreover $<$ is a linear order on Γ and the order on the s_η's is isomorphic to a lexicographic order on Γ.

Why is the above (which is clearly definable with our parameters) a well order? Because of the above note and because a lexicographic ordering of a well founded tree is a well order, provided that immediate successors are well ordered. In detail, assume $X = \{x_i\}_{i<\omega}$ is a stricly decreasing sequence of elements of T. Let η_i be the unique node in Γ such that $x_i \in A_{\eta_i}$ and by the above note w.l.o.g $i \neq j \Rightarrow \eta_i \neq \eta_j$. By the well foundedness of Γ and clause (b) we may also assume w.l.o.g that the η_i's form an anti-chain in Γ. Look at $\nu_i := \eta_1 \wedge \eta_i$ which is constant for infinitely many i's and w.l.o.g equals to ν for every i. Ask:

$(*)$ is there is an infinite $B \subseteq \omega$ such that $i, j \in B \Rightarrow x_i \sim^0_{A_\nu} x_j$?

If this occurs we have $\nu_1 \neq \nu$ with $\nu \triangleleft \nu_1$ such that for some infinite $B' \subseteq B \subseteq \omega$ we have $i \in B' \Rightarrow \nu_1 \triangleleft \eta_i$. (use the fact that $\sim^1_{A_\nu}$ is finite). W.l.o.g $B' = \omega$ and we may ask if $(*)$ holds for ν_1. Eventually, since Γ does not have an infinite branch, we will have a negative answer to $(*)$. We can conclude that w.l.o.g there is $\nu \in \Gamma$ such that $i \neq j \Rightarrow x_i \not\sim^0_{A_\nu} x_j$ i.e. the x_i's "break" A_ν in "different places".

Define now ν_i to be the unique immediate successor of ν such that $\nu_i \triangleleft \eta_i$. The set $S = \{s_{\nu_i}\}_{i<\omega} \subseteq A^+_\nu$ is well ordered by the well ordering on A^+_ν and by clause (c) in the definition of $<$, $x_i > x_j \iff \nu_i > \nu_j$ so S is an infinite stricly decreasing subset of A^+_ν – a contradiction.

This finishes the proof that there is a definable well order of T.

♡

Finally we can conclude:

Theorem 5.5. *Let T be a tree. If T is wild or T embeds ${}^{\omega>}2$ then there is no definable choice function on T (by a monadic formula with parameters). If T is tame and does not embed ${}^{\omega>}2$ then there even a definable well ordering of the elements of T by a monadic formula (with parameters) $\varphi(x, y, \bar{P})$.*

♡

As mentioned in the introduction, a tree is tame [wild] [embeds ${}^{\omega>}2$] if and only if it's completion is tame [wild] [embeds ${}^{\omega>}2$]. Moreover looking at the proofs of 4.3, 4.4, 4.5 and 5.1 we note that the counter-examples for the choice function problem are either anti-chains or linearily ordered subsets of T. We conclude:

Conclusion 5.6. *Let T be a tree and T' be it's completion. Then the following are equivalent:*

a) For some $n, l < \omega$, for every anti-chain/branch A of T there is a monadic formula $\varphi_A(x, X, \bar{P}_A)$ with quantifier depth $\leq n$ and $\leq l$ parameters from T, that defines a choice function from non empty subsets of A.

b) There is a monadic formula, with parameters, $\psi(x, y, \bar{P})$ that defines a well ordering of the elements of T.

c) There is a monadic formula, with parameters, $\psi'(x, y, \bar{P}')$ that defines a well ordering of the elements of T'.

♡

REFERENCES

[**BL**] J.R. Büchi and L.H. Landweber, *Solving sequential conditions by finite-state strategies*, ***Transactions of the American Mathematical Society***, vol. 138 (1969), pp. 295–311.

[**Ha**] F. Hausdorff, *Grundzüge einer Theorie der geordnetn Mengen*, ***Math. Ann.***, vol. 65 (1908), pp. 435–505.

[**Gu**] Y. Gurevich, *Monadic Second–order Theories*, ***Model Theoretic Logics***, (J. Barwise and S. Feferman, editors), Springer–Verlag, Berlin 1985, pp. 479–506.

[**GuSh**] Y. Gurevich and S. Shelah, *Rabin's Uniformization Problem*, ***The Journal of Symbolic Logic***, vol. 48 (1983), pp. 1105–1119.

[**Ra**] M.O. Rabin, *Decidability of second-order theories and automata on infinite trees*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[**Sh**] S. Shelah, *The monadic Theory of Order*, ***Annals of Mathematics***, ser. 2, vol. 102 (1975), pp. 379–419.