

Evasion and prediction II

JÖRG BRENDLE¹

Mathematisches Institut der Universität, Auf der Morgenstelle 10, 72076 Tübingen,
Germany; email: jobr@michelangelo.mathematik.uni-tuebingen.de

SAHARON SHELAH^{2,3}

Department of Mathematics, Hebrew University, Givat Ram, 91904 Jerusalem, Israel
and
Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

ABSTRACT

A subgroup $G \leq \mathbb{Z}^\omega$ exhibits the Specker phenomenon if every homomorphism $G \rightarrow \mathbb{Z}$ maps almost all unit vectors to 0. We give several combinatorial characterizations of the cardinal \mathfrak{se} , the size of the smallest $G \leq \mathbb{Z}^\omega$ exhibiting the Specker phenomenon. We also prove the consistency of $\mathfrak{b} < \mathfrak{e}$, where \mathfrak{b} is the unbounding number and \mathfrak{e} the evasion number. Our results answer several questions addressed by Blass.

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Introduction

Specker [Sp] proved that given a homomorphism h from \mathbb{Z}^ω to the infinite cyclic group \mathbb{Z} , where \mathbb{Z}^ω denotes the direct product of countably many copies of \mathbb{Z} , we have $h(e_n) = 0$ for all but finitely many unit vectors $e_n \in \mathbb{Z}^\omega$ (in other words, the n -th component of e_n is 1, and its other components are 0). Blass [Bl] studied the *Specker–Eda number* \mathfrak{se} , the size of the smallest subgroup $G \leq \mathbb{Z}^\omega$ containing all unit vectors which still has the property that every homomorphism $h : G \rightarrow \mathbb{Z}$ annihilates almost all unit vectors. We will give various (mostly less algebraic) characterizations of \mathfrak{se} (some of which already play a prominent role in Blass’ work); we will also study some related cardinal invariants of the continuum.

To be more explicit, let \leq^* denote the *eventual domination order* on the Baire space ω^ω ; i.e. $f \leq^* g$ iff $f(n) \leq g(n)$ for all but finitely many n . We shall usually abbreviate the statement in italics by $\forall^\infty n$; similarly we will write $\exists^\infty n$ for *there are infinitely many n* . The *unbounding number* \mathfrak{b} is the smallest size of a \leq^* -unbounded family \mathcal{F} of functions in ω^ω (i.e., given any $g \in \omega^\omega$, there is $f \in \mathcal{F}$ with $\exists^\infty n (f(n) > g(n))$). Given a σ -ideal \mathcal{I} on ω^ω , the *additivity* $\text{add}(\mathcal{I})$ is the least cardinality of a family \mathcal{F} of members of \mathcal{I} whose union is not in \mathcal{I} . We shall use this cardinal only in the cases $\mathcal{I} = \mathcal{M}$, the ideal of meager sets, and $\mathcal{I} = \mathcal{L}$, the ideal of Lebesgue null sets. — While the preceding invariants have been studied by a number of people in the last two decades, the following concept was introduced only recently by Blass [Bl]. Given an at most countable set S , an *S -valued predictor* is a pair $\pi = (D_\pi, \langle \pi_n; n \in D_\pi \rangle)$ where $D_\pi \subseteq \omega$ is infinite and for each $n \in D_\pi$, π_n is a function from S^n to S . π *predicts* $f \in S^\omega$ iff for all but finitely many $n \in D_\pi$, we have $f(n) = \pi_n(f \upharpoonright n)$; otherwise f *evades* π . The *evasion number* \mathfrak{e} is the smallest size of a family \mathcal{F} of functions in ω^ω such that no ω -valued predictor predicts all $f \in \mathcal{F}$. A \mathbb{Z} -valued predictor is *linear* iff all $\pi_n : \mathbb{Z}^n \rightarrow \mathbb{Q}$ are \mathbb{Q} -linear maps. The corresponding *linear evasion number* shall be denoted by \mathfrak{e}_ℓ (i.e., $\mathfrak{e}_\ell = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \mathbb{Z}^\omega \text{ and no linear } \mathbb{Z}\text{-valued predictor predicts all } f \in \mathcal{F}\}$). (Blass’ definition of linear evading [Bl, section 4] is slightly different; however, it gives rise to the same cardinal; we use the present definition because we shall work with functions in \mathbb{Z}^ω in 2.2.)

These notions enable us to phrase our main results.

Theorem A. *It is consistent with ZFC to assume $\mathfrak{b} < \mathfrak{e}$.*

Theorem B. $\mathfrak{se} = \mathfrak{e}_\ell = \min\{\mathfrak{e}, \mathfrak{b}\}$.

They will be proved in sections 1 and 2 of our work. Section 2 also contains a further purely combinatorial characterization of the cardinal \mathfrak{se} (subsections 2.4 and 2.5). To put our results into a somewhat larger context, we point out the following consequences which involve some earlier results, due mostly to Blass [Bl].

Corollary. (a) $\text{add}(\mathcal{L}) \leq \mathfrak{se} \leq \text{add}(\mathcal{M}) \leq \mathfrak{b}$;

(b) *any of the inequalities in (a) can be consistently strict;*

(c) *it is consistent with ZFC to assume $\mathfrak{e}_\ell < \mathfrak{e}$.*

Theorems A and B together with the Corollary give a complete solution to Questions (1) through (3) in [Bl, section 5]. Note in particular that the cardinals (2) through (5) in Corollary 8 in [Bl, section 3] are indeed equal.

PROOF OF COROLLARY. (a) This follows from Theorem B and Blass' results [Bl, Theorems 12 and 13]. The well-known inequality $\text{add}(\mathcal{M}) \leq \mathfrak{b}$ is due to Miller [Mi].

(b) The consistency of $\text{add}(\mathcal{M}) < \mathfrak{b}$ is well-known (it holds e.g. in the Mathias or Laver real models); for the consistency of $\text{add}(\mathcal{L}) < \mathfrak{se}$ see [Bl] (in particular [Bl, Theorem 9]); the consistency of $\mathfrak{se} < \text{add}(\mathcal{M})$ follows from Theorem B and [Br, Theorem A].

(c) This is immediate from Theorems A and B. \square

A set of reals predicted by a single predictor is small in various senses; it belongs, in particular, both to \mathcal{M} and \mathcal{L} . This motivates us to introduce the σ -ideal \mathcal{J} on ω^ω generated by such sets of reals (see [Br, section 4] for more on this). Clearly, the uniformity of \mathcal{J} (i.e., the size of the smallest set of reals not in \mathcal{J}) is closely related to the evasion number. In fact, $\mathfrak{e} \leq \mathfrak{e}(\omega)$ where $\mathfrak{e}(\omega)$ denotes the former cardinal. We shall show in section 3 that these two cardinals are equal under some additional assumption, thus giving a partial answer to [Br, section 6, question (4)].

The results of this work are due to the second author. It was the first author's task to work them out and to write up the paper.

Notational remarks. A p.o. \mathbb{P} is σ -centered iff there are $\mathbb{P}_n \subseteq \mathbb{P}$ ($n \in \omega$) so that $\mathbb{P} = \bigcup_n \mathbb{P}_n$ and given $n \in \omega$, $F \subseteq \mathbb{P}_n$ finite, there is $p \in \mathbb{P}$ extending all $q \in F$. \mathbb{P} -names are denoted by symbols like $\dot{h}, \dot{\pi}, \dot{D}, \dots$ | stands for *divides*; \nmid means *does not divide*.

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§ 1. Proof of Theorem A

1.1. We shall use a finite support iteration of *ccc* p.o.'s of length κ (where $\kappa \geq \omega_2$ is a regular cardinal) over a model V for CH to prove the consistency of $\mathfrak{e} > \mathfrak{b}$. In fact, in the resulting model, $\mathfrak{b} = \omega_1$ and $\mathfrak{e} = \kappa$. We start with defining the p.o. \mathbb{P} we want to iterate. Notice that it is quite similar to the one used in [Br, 4.3.] for predicting below a given function.

$$\begin{aligned} \langle d, \pi, F \rangle \in \mathbb{P} &\iff d \in 2^{<\omega} \text{ is a finite partial function,} \\ \pi &= \langle \pi_n; n \in d^{-1}(\{1\}) \rangle \text{ and } \pi_n : \omega^n \rightarrow \omega \text{ is a finite partial function,} \\ F &\subseteq \omega^\omega \text{ is finite and } (f \neq g \in F \longrightarrow \max\{n; f \upharpoonright n = g \upharpoonright n\} < |d|). \end{aligned}$$

The order is given by:

$$\begin{aligned} \langle d', \pi', F' \rangle \leq \langle d, \pi, F \rangle &\iff d' \supseteq d, \pi' \supseteq \pi, F' \supseteq F \text{ and} \\ &(f \in F, n \in (d')^{-1}(\{1\}) \setminus d^{-1}(\{1\}) \longrightarrow \pi'_n(f \upharpoonright n) = f(n)) \\ &\text{(in particular } \pi'_n(f \upharpoonright n) \text{ is defined).} \end{aligned}$$

Notice that we use the convention that stronger conditions are smaller in the p.o. — The first two coordinates of a condition are intended as a finite approximation to a generic predictor; the third coordinate then guarantees that functions are predicted from some point on. Thus it is straightforward that \mathbb{P} adjoins a predictor which predicts all ground-model functions. Hence iterating \mathbb{P} increases \mathfrak{e} .

Furthermore \mathbb{P} is σ -centered (and thus in particular *ccc*). To see this simply notice that conditions with the same initial segment in the first two coordinates are compatible.

So it remains to show that $\mathfrak{b} = \omega_1$ after iterating \mathbb{P} . For this it suffices to show the following:

- (*) whenever $G \in W$ is an unbounded family of functions from ω to ω , and $\mathbb{P} \in W$ is the p.o. defined above, then

$$\Vdash_{\mathbb{P}} \text{“}G \text{ is unbounded”}.$$

Using (*) we can show that $\omega^\omega \cap V$ is still unbounded in the final model: (*) guarantees that it stays unbounded in successor steps of the iteration; and one of the usual preservation results for finite support iterations (see, e.g., [JS, Theorem 2.2]) shows that it does so in limit steps of the iteration as well. Now, $V \models CH$; hence $\omega^\omega \cap V$ is an unbounded family of size ω_1 in the final model.

To start with the proof of (*), let \dot{h} be a \mathbb{P} -name for a function in ω^ω . For each $d \in 2^{<\omega}$, $\pi = \langle \pi_n; n \in d^{-1}(\{1\}) \rangle$ an initial segment of a predictor (as in the definition of \mathbb{P}), $k \in \omega$ and $\bar{f}^* = \langle f_\ell^* \in \omega^{|\dot{d}|}; \ell < k \rangle$ we define $h = h_{d,\pi,\bar{f}^*} \in (\omega + 1)^\omega$ by

$$h(n) := \min\{m \leq \omega; \text{for no } p \in \mathbb{P} \text{ with } p = \langle d, \pi, F \rangle, F = \{f_\ell; \ell < k\}, f_\ell \upharpoonright |d| = f_\ell^*, \\ \text{do we have } p \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) > m\text{”}\}.$$

1.2. Main Claim. $h \in \omega^\omega$.

1.3. PROOF OF (*) FROM THE MAIN CLAIM. Let $h^* \in \omega^\omega$ such that for all d, π, \bar{f}^* as above we have $h_{d,\pi,\bar{f}^*} \leq^* h^*$. As G is unbounded we can find $f \in G$ such that there are infinitely many n with $f(n) > h^*(n)$. We claim that $\Vdash_{\mathbb{P}} \text{“}\exists^\infty n (f(n) > \dot{h}(n))\text{”}$. This will show (*).

Assume $m \in \omega$ and $p \in \mathbb{P}$ are such that

$$p \Vdash_{\mathbb{P}} \text{“}\forall n \geq m (f(n) \leq \dot{h}(n))\text{”}.$$

Find d, π, \bar{f}^* such that $p = \langle d, \pi, F \rangle$ where $F = \{f_\ell; \ell < k\}$ and $f_\ell \upharpoonright |d| = f_\ell^*$. Find $n \geq m$ such that $f(n) > h^*(n)$ and $h^*(n) \geq h_{d,\pi,\bar{f}^*}(n)$. Then

$$p \Vdash_{\mathbb{P}} \text{“}h_{d,\pi,\bar{f}^*}(n) < f(n) \leq \dot{h}(n)\text{”},$$

contradicting the definition of h_{d,π,\bar{f}^*} . \square

1.4. PROOF OF THE MAIN CLAIM (1.2.). Let $d, \pi, k, \bar{f}^* = \langle f_\ell^*; \ell < k \rangle$ as above and $n \in \omega$ be fixed. Now assume that we have $p_i = \langle d, \pi, \{f_\ell^i; \ell < k\} \rangle$ with $f_\ell^i \upharpoonright |d| = f_\ell^*$ and

$$p_i \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) > i\text{”}.$$

We shall reach a contradiction. As we can replace $\langle p_i; i \in \omega \rangle$ by a subsequence, if necessary, we may assume that for all $\ell < k$:

- either (a) $_\ell$ for some $g_\ell \in \omega^\omega$ for all i ($f_\ell^i \upharpoonright i = g_\ell \upharpoonright i$)
- or (b) $_\ell$ for some $i_\ell \in \omega$ and $\hat{g}_\ell \in \omega^{i_\ell}$ ($f_\ell^i \upharpoonright i_\ell = \hat{g}_\ell \wedge f_\ell^i(i_\ell) > i$).

Notice that $i_\ell \geq |d|$ in the latter case. — Let $d^* := d \cup 0_{[|d|, \max(i_\ell; (b)_\ell \text{ holds})+1]}$; i.e. the function d^* takes value 0 between $|d|$ and the maximum of the i_ℓ . Put $F^* := \{g_\ell; (a)_\ell \text{ holds}\}$. Then clearly $p^* = \langle d^*, \pi, F^* \rangle \in \mathbb{P}$. Now choose ℓ^* and $q \leq p^*$ such that

$$q \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) = \ell^*\text{”}.$$

We shall find $i > \ell^*$ so that q and p_i are compatible; this is a contradiction because q and p_i force contradictory statements.

Assume $q = \langle d^q, \pi^q, F^q \rangle$. Choose $i \geq \ell^*$ large enough such that:

(A) $i \geq |d^q|$;

(B) $i \geq \max\{\max\{\sigma(j); \sigma \in \text{dom}(\pi_m^q) \wedge j \in m\}; m \in (d^q)^{-1}(\{1\})\}$.

Notice that (A) implies that $f_\ell^i \upharpoonright |d^q| = g_\ell \upharpoonright |d^q|$ whenever $(a)_\ell$ holds, while $f_\ell^i(i_\ell) > \max\{\max\{\sigma(j); \sigma \in \text{dom}(\pi_m^q) \wedge j \in m\}; m \in (d^q)^{-1}(\{1\})\}$ by (B) in case $(b)_\ell$ holds. For such i let $q^i = \langle d^i, \pi^i, F^i \rangle$ where

- $d^i = d^q \cup 0_{[|d^q|, a]}$, where a is large enough such that all functions in F^i disagree before a ;
- $\pi^i \supseteq \pi^q$ such that for all $m \in (d^q)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$ and all f_ℓ^i so that $(b)_\ell$ holds, we have

$$f_\ell^i(m) = \pi_m^i(f_\ell^i \upharpoonright m). \quad (\star)$$

(This can be done because, by (B), π_m^q was not defined yet for sequences of the form $f_\ell^i \upharpoonright m$.)

- $F^i = F^q \cup \{f_\ell^i; \ell < k\}$.

Now we clearly have $q^i \in \mathbb{P}$ and $q^i \leq q$. So we are left with checking $q^i \leq p_i$. The inclusion relations are all satisfied. Hence it suffices to see that for all $\ell < k$ and $m \in (d^i)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$, we have

$$f_\ell^i(m) = \pi_m^i(f_\ell^i \upharpoonright m). \quad (+)$$

In case $(b)_\ell$ holds this is true by (\star) . In case $(a)_\ell$ holds we have $f_\ell^i \upharpoonright (m+1) = g_\ell \upharpoonright (m+1)$ for all such m . As $q \leq p^*$ we have $\pi_m^i(g_\ell \upharpoonright m) = \pi_m^q(g_\ell \upharpoonright m) = g_\ell(m)$ for such m , and $(+)$ holds again. This completes the proof of the Main Claim. \square

§ 2. Proof of Theorem B

2.1. Theorem. $\mathfrak{se} \leq \mathfrak{e}$.

PROOF. Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \mathfrak{se}$. By Blass' result $\mathfrak{se} \leq \mathfrak{b}$ [Bl, Theorem 2], there is $g \in \omega^\omega$ such that for all $f \in \mathcal{F} \forall^\infty n (f(n) < g(n))$. Without loss g is strictly increasing. We let $\langle p_n; n \in \omega \rangle$ be a sequence of distinct primes such that $p_n \gg g(n)$ and $p_n \gg \prod_{\ell < n} p_\ell$. For $f \in \mathcal{F}$, let $a_f \in \omega^\omega$ be defined by

$$a_f(n) := f(n) \cdot \prod_{\ell \leq n} p_\ell.$$

Let $G \leq \mathbb{Z}^\omega$ be the pure closure of the subgroup generated by the unit vectors e_n , $n \in \omega$, and the a_f , $f \in \mathcal{F}$. Clearly $|G| < \mathfrak{se}$. Hence there is $h : G \rightarrow \mathbb{Z}$ a homomorphism such that $W := \{n; h(e_n) \neq 0\}$ is infinite.

Let us define

$$W^* := \{n \in \omega; \exists i > n (p_i | h(e_m) \text{ whenever } m \in \{n+1, \dots, i-1\} \text{ but } p_i \nmid h(e_n))\}.$$

We claim that W^* is an infinite subset of W . To see this, first note that trivially $W^* \subseteq W$, by the clause $p_i \nmid h(e_n)$. Next, given $n_0 \in W$, find $i > n_0$ so that $p_i \nmid h(e_{n_0})$. Then clearly there is $n \geq n_0$ so that $n \in W$ and $p_i \nmid h(e_n)$ and for all $m \in \{n+1, \dots, i-1\}$, $p_i | h(e_m)$. Thus $n \in W^*$. This shows that W^* is infinite.

We introduce a predictor $\pi = (W^*, \langle \pi_n; n \in W^* \rangle)$ as follows. Given $n \in W^*$ and $s \in \omega^n$ so that $\max \text{rng}(s) < g(n-1)$, if there is $f \in \mathcal{F}$ with $s \subseteq f$ and $f(n) < g(n)$ and $|h(a_f)| < p_{n-1}$, then let $\pi_n(s) = f(n)$ for some f with the above property. Otherwise $\pi_n(s)$ is arbitrary.

We claim that π predicts all $f \in \mathcal{F}$. This clearly finishes the proof. Assume this were false, i.e. there is $f \in \mathcal{F}$ which evades π . Let $n \in W^*$ be large enough, such that $\max \text{rng}(f \upharpoonright n) < g(n-1)$, $f(n) < g(n)$, $|h(a_f)| < p_{n-1}$ and $\pi_n(f \upharpoonright n) \neq f(n)$. Then, by the definition of π , there must be $f' \in \mathcal{F}$ with $f' \upharpoonright n = f \upharpoonright n$, $f'(n) < g(n)$, $|h(a_{f'})| < p_{n-1}$ and $\pi_n(f' \upharpoonright n) = f'(n) \neq f(n)$. Now, for $k \in \{f, f'\}$, we let

$$\begin{aligned} a_k^0 &= (a_k(0), \dots, a_k(n-1), 0, \dots) \\ a_k^1 &= (0, \dots, 0, a_k(n), 0, \dots) \\ a_k^2 &= (0, \dots, 0, a_k(n+1), \dots, a_k(i-1), 0, \dots) \\ a_k^3 &= (0, \dots, 0, a_k(i), a_k(i+1), \dots) \end{aligned}$$

where i witnesses that $n \in W^*$. So we have $a_k = a_k^0 + a_k^1 + a_k^2 + a_k^3$. Thus

$$h(a_{f'} - a_f) = h(a_{f'}^0 - a_f^0) + h(a_{f'}^1 - a_f^1) + h(a_{f'}^2 - a_f^2) + h(a_{f'}^3 - a_f^3). \quad (\star)$$

Clearly $h(a_{f'}^0 - a_f^0) = h(0) = 0$. Next, $p_i \cdot \prod_{\ell \leq n} p_\ell$ divides $h(a_{f'}^3 - a_f^3)$ by definition of the a_k ; it also divides $h(a_{f'}^2 - a_f^2)$ by definition of the a_k and because $p_i | h(e_m)$ for $m \in \{n+1, \dots, i-1\}$ as i witnesses $n \in W^*$. Thus (\star) yields the equation

$$h(a_{f'} - a_f) = h(a_{f'}^1 - a_f^1) \quad \text{in} \quad \mathbb{Z}/(p_i \cdot \prod_{\ell \leq n} p_\ell)\mathbb{Z}. \quad (\star\star)$$

The right-hand side in $(\star\star)$ must be non-zero, because $p_i \nmid h(e_n)$ (as i witnesses $n \in W^*$) and $p_i \nmid (a_{f'}(n) - a_f(n)) = \prod_{\ell \leq n} p_\ell \cdot (f'(n) - f(n))$ (as $f'(n), f(n) < g(n) \ll p_n \ll p_i$). However, it certainly is divisible by $\prod_{\ell \leq n} p_n$, whereas the left-hand side in $(\star\star)$ is not unless it is zero (as $|h(a_f)|, |h(a_{f'})| < p_{n-1} \ll p_n$). This shows that the equation $(\star\star)$ cannot hold, the final contradiction. \square

Note that this result improves [Br, Theorem 3.2].

2.2. Lemma. $\epsilon_\ell \geq \min\{\mathfrak{e}, \mathfrak{b}\}$.

PROOF. Let $\mathcal{F} \subseteq \mathbb{Z}^\omega$, $|\mathcal{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find $g \in \omega^\omega$ strictly increasing so that for all $f \in \mathcal{F}$, we have $|f| <^* g$, where $|f|(n) = |f(n)|$. We partition ω into intervals I_n , $n \in \omega$, so that $\max(I_n) + 1 = \min(I_{n+1})$, as follows. $I_0 = \{0\}$. Assume I_n is defined; choose I_{n+1} so that $|I_{n+1}| > [2 \cdot g(\max(I_n))]^{\sum_{i \leq n} |I_i|}$. For $f \in \mathcal{F}$, define \bar{f} by $\bar{f}(n) := f \upharpoonright I_n$, and let $\bar{\mathcal{F}} = \{\bar{f}; f \in \mathcal{F}\}$. Use $|\bar{\mathcal{F}}| < \mathfrak{e}$ to get a single predictor $\bar{\pi} = (\bar{D}, \langle \bar{\pi}_n; n \in \bar{D} \rangle)$ predicting all the $\bar{f} \in \bar{\mathcal{F}}$. For $n \in \bar{D}$, let $\Gamma_n := \text{rng}(\bar{\pi}_n \upharpoonright (-g(\max(I_{n-1})), g(\max(I_{n-1}))) \cup_{i < n} I_i) \cap \mathbb{Z}^{I_n}$. So $|\Gamma_n| < |I_n|$; hence for some $i_n \in I_n$, the vector $\bar{x}_{i_n} = \langle t(i_n); t \in \Gamma_n \rangle$ depends on the vectors $\{\bar{x}_i = \langle t(i); t \in \Gamma_n \rangle; \min(I_n) \leq i < i_n\}$. Say $\bar{x}_{i_n} = \sum_{\min(I_n) \leq i < i_n} q_i^n \bar{x}_i$, where $q_i^n \in \mathbb{Q}$. In particular, for fixed $t \in \Gamma_n$, we have $t(i_n) = \sum_{\min(I_n) \leq i < i_n} q_i^n t(i)$. This allows us to define a linear predictor $\pi = (D, \langle \pi_n; n \in D \rangle)$ with $D = \{i_n; n \in \omega\}$ and $\pi_{i_n}(s) = \sum_{\min(I_n) \leq i < i_n} q_i^n s(i)$. Note that if $n \in \omega$ is such that $\max \text{rng}(|f| \upharpoonright \cup_{i < n} I_i) < g(\max(I_{n-1}))$ and $\bar{\pi}_n(\bar{f} \upharpoonright n) = \bar{f}(n)$, then $\pi_{i_n}(f \upharpoonright i_n) = f(i_n)$. Hence, as $\bar{\pi}$ predicts all $\bar{f} \in \bar{\mathcal{F}}$, π predicts all $f \in \mathcal{F}$. \square

2.3. Clearly, Theorem B follows from 2.1., 2.2. and Blass' results $\epsilon_\ell \leq \mathfrak{se} \leq \mathfrak{b}$ [Bl, Theorem 2, Corollary 8 and Theorem 10]. \square

2.4. Definition. Given $D \subseteq \omega$ infinite and $\bar{a} = \langle a_n \in [\omega]^{\leq n}; n \in D \rangle$, the *slalom* $S_{\bar{a}}^D$ is the set of all functions f in ω^ω with $f(n) \in a_n$ for almost all $n \in D$.

Using this notion we can give a combinatorial characterization of the cardinal $\mathfrak{e}_\ell = \mathfrak{se}$.

2.5. Lemma. $\min\{\mathfrak{e}, \mathfrak{b}\} = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \omega^\omega \text{ and for all } D \subseteq \omega \text{ and } \bar{a} = \langle a_n \in [\omega]^{\leq n}; n \in D \rangle \text{ there is } f \in \mathcal{F} \setminus S_{\bar{a}}^D\}$.

NOTE. It is immediate that the cardinal on the right-hand side is bigger than or equal to the additivity of Lebesgue measure $\text{add}(\mathcal{L})$, by Bartoszyński's characterization of that cardinal ([Ba 1], [Ba 2]). We also note that the original proof of $\text{add}(\mathcal{L}) \leq \text{add}(\mathcal{M})$ [Ba 1] shows in fact that this cardinal is $\leq \text{add}(\mathcal{M})$ as well. This gives an alternative proof of Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \text{add}(\mathcal{M})$ [Bl, Theorem 13].

PROOF. " \geq ". By Theorem B, it suffices to show that \mathfrak{e}_ℓ is bigger than or equal to the cardinal on the right-hand side. However, this is exactly like Blass' original proof of $\text{add}(\mathcal{L}) \leq \mathfrak{e}_\ell$ [Bl, Theorem 12], and we therefore leave details to the reader.

" \leq ". This argument is very similar to the one in Lemma 2.2. So we just stress the differences.

Take $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find g strictly increasing and eventually dominating all functions from \mathcal{F} . As before, partition ω into intervals I_n , $n \in \omega$; this time we require that $i_{n+1} := g(\max(I_n)) \sum_{i \leq n} |I_i| \in I_{n+1}$. \bar{f} , $\bar{\mathcal{F}}$ and $\bar{\pi}$, \bar{D} are defined as before.

We put $D := \{i_n; n \in \bar{D}\}$ and $a_{i_n} = \{\bar{\pi}_n(s)(i_n); s \in g(\max(I_{n-1})) \bigcup_{i < n} I_i\} \in [\omega]^{\leq i_n}$, and leave it to the reader to check that $\mathcal{F} \subseteq S_{\bar{a}}^D$. \square

2.6. The notion of linear predicting can be generalized as follows (see [Br, section 4] for details). Let \mathbb{K} be an at most countable field. A \mathbb{K} -valued predictor $\pi = (D_\pi, \langle \pi_n; n \in D_\pi \rangle)$ is *linear* iff all $\pi_n : \mathbb{K}^n \rightarrow \mathbb{K}$ are linear. $\mathfrak{e}_\mathbb{K}$ is the corresponding *linear evasion number*. We easily see $\mathfrak{e}_\mathbb{Q} = \mathfrak{e}_\ell$. Rewriting the proof of 2.2. in this more general context gives $\mathfrak{e}_\mathbb{K} \geq \min\{\mathfrak{e}, \mathfrak{b}\}$ for arbitrary \mathbb{K} and $\mathfrak{e}_\mathbb{K} \geq \mathfrak{e}$ in case \mathbb{K} is finite. As $\mathfrak{e}_\mathbb{K} \leq \mathfrak{b}$ for infinite \mathbb{K} [Br, 5.4.], we get $\mathfrak{e}_\mathbb{K} = \min\{\mathfrak{e}, \mathfrak{b}\}$ for such fields — in particular all $\mathfrak{e}_\mathbb{K}$ for \mathbb{K} a countable field are equal. We do not know whether this is true for finite \mathbb{K} . Note that $\mathfrak{e}_\mathbb{K} > \mathfrak{e}, \mathfrak{b}$ is consistent for such fields [Br, section 4].

§ 3. Some results on evasion ideals

3.1. Definition. We say a predictor $\pi = (D, \langle \pi_n; n \in D \rangle)$ predicts a function $f \in \omega^\omega$ everywhere if $\pi_n(f \upharpoonright n) = f(n)$ holds for all $n \in D$. We put $\epsilon(\omega) := \min\{|\mathcal{F}|; \mathcal{F} \subseteq \omega^\omega \wedge \text{for all countable families of predictors } \Pi \text{ there is } f \in \mathcal{F} \text{ evading all } \pi \in \Pi\}$, the *uniformity of the evasion ideal* \mathcal{J} . — As usual, $\text{cov}(\mathcal{M})$ denotes the *covering number* of the ideal \mathcal{M} , i.e. the smallest size of a family $\mathcal{F} \subseteq \mathcal{M}$ so that $\bigcup \mathcal{F} = \omega^\omega$.

3.2. Observation. Assume $\langle D^n; n \in \omega \rangle$ is a decreasing sequence of infinite subsets of ω , and $\langle \pi^n = (D^n, \langle \pi_k^n; k \in D^n \rangle); n \in \omega \rangle$ is a sequence of predictors. Then there are a set $D \subseteq \omega$, almost included in all D^n , and a predictor $\pi = (D, \langle \pi_k; k \in D \rangle)$ predicting all functions which are predicted by one of the π^n .

PROOF. We can assume that each function which is predicted by some π^n is predicted everywhere by some π^m — otherwise go over to sequences $\langle E^n; n \in \omega \rangle$ and $\langle \bar{\pi}^n = (E^n, \langle \bar{\pi}_k^n; k \in E^n \rangle); n \in \omega \rangle$ such that (i) for all $n \in \omega$ there is $m \in \omega$ so that $E^m \subseteq D^n$ and $\bar{\pi}_k^m = \pi_k^n$ for $k \in E^m$ and (ii) for all $n, m \in \omega$ there is $\ell \in \omega$ so that $E^\ell \subseteq E^n \setminus m$ and $\bar{\pi}_k^\ell = \bar{\pi}_k^n$ for $k \in E^\ell$.

Choose $d^n \in D^n$ minimal with $d^n > d^{n-1}$, and put $D = \{d^n; n \in \omega\}$. Fix $n \in \omega$ and $s \in \omega^{d^n}$. To define $\pi_{d^n}(s)$, choose, if possible, $i \leq n$ minimal so that for all $k \in D^i \cap d^n$, we have $\pi_k^i(s \upharpoonright k) = s(k)$, and let $\pi_{d^n}(s) = \pi_{d^n}^i(s)$. If this is impossible, let $\pi_{d^n}(s)$ be arbitrary.

To see that this works, take $f \in \omega^\omega$ and $i \in \omega$ minimal so that π^i predicts f everywhere. As the set of functions predicted everywhere by a single predictor is closed, there are $n \geq i$ and $s \in \omega^{d^n}$ so that $s \subseteq f$ and s is not predicted everywhere by any of the π^j where $j < i$. Then $\pi_{d^m}(f \upharpoonright d^m) = \pi_{d^m}^i(f \upharpoonright d^m)$ for all $m \geq n$, as required. \square

3.3. Theorem. $\epsilon \geq \min\{\epsilon(\omega), \text{cov}(\mathcal{M})\}$; thus either $\epsilon < \text{cov}(\mathcal{M})$ or $\epsilon(\omega) \leq \text{cov}(\mathcal{M})$ imply $\epsilon = \epsilon(\omega)$.

REMARK. The statement is very similar to a recent result of Kamburelis who proved $\mathfrak{s} \geq \min\{\mathfrak{s}(\omega), \text{cov}(\mathcal{M})\}$, where \mathfrak{s} is the splitting number and $\mathfrak{s}(\omega)$ the \aleph_0 -splitting number.

PROOF. The second statement easily follows from the first. To prove the latter, let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \min\{\epsilon(\omega), \text{cov}(\mathcal{M})\}$. We shall show $|\mathcal{F}| < \epsilon$. For $\sigma \in \omega^{<\omega} \setminus \{\langle \rangle\}$, we construct recursively sets $D^\sigma \subseteq \omega$ and predictors $\pi^\sigma = (D^\sigma, \langle \pi_n^\sigma; n \in D^\sigma \rangle)$ such that:

- (i) $D^{\sigma \upharpoonright i} \supseteq D^\sigma$ for $i \in |\sigma|$;
- (ii) for all $f \in \mathcal{F}$ and all $\sigma \in \omega^{<\omega}$ there is $i \in \omega$ so that f is predicted by $\pi^{\sigma \wedge i}$.

First construct $\pi^{(i)} = (D^{(i)}, \langle \pi_n^{(i)}; n \in D^{(i)} \rangle)$ satisfying (ii) by applying $|\mathcal{F}| < \mathfrak{e}(\omega)$.

To do the recursion, assume $\pi^\sigma = (D^\sigma, \langle \pi_n^\sigma; n \in D^\sigma \rangle)$ is constructed for some fixed $\sigma \in \omega^{<\omega}$. Given $f \in \omega^\omega$, define f^σ by:

$$f^\sigma(i) := f(k_i^\sigma),$$

where $\{k_i^\sigma; i \in \omega\}$ is the increasing enumeration of the set D^σ . Let $\mathcal{F}^\sigma = \{f^\sigma; f \in \mathcal{F}\}$.

Again we get ω many predictors $\bar{\pi}^{\sigma \wedge i} = (\bar{D}^{\sigma \wedge i}, \langle \bar{\pi}_n^{\sigma \wedge i}; n \in \bar{D}^{\sigma \wedge i} \rangle)$, $i \in \omega$, so that every $f^\sigma \in \mathcal{F}^\sigma$ is predicted by some $\bar{\pi}^{\sigma \wedge i}$. Let $D^{\sigma \wedge i} = \{k_j^\sigma; j \in \bar{D}^{\sigma \wedge i}\}$. Fix $j \in \bar{D}^{\sigma \wedge i}$ and $s \in \omega^{k_j^\sigma}$. Let $\bar{s} \in \omega^j$ be defined by $\bar{s}(\ell) = s(k_\ell^\sigma)$. Put $\pi_{k_j^\sigma}^{\sigma \wedge i}(s) := \bar{\pi}_j^{\sigma \wedge i}(\bar{s})$. Now it is easy to see that $\pi^{\sigma \wedge i}$ predicts f whenever $\bar{\pi}^{\sigma \wedge i}$ predicts f^σ . Thus (i) and (ii) hold. This completes the recursive construction.

Given $f \in \omega^\omega$, let $T_f = \{\sigma \in \omega^{<\omega}; \text{ for all } i \leq |\sigma| \text{ } (\pi^{\sigma \upharpoonright i} \text{ does not predict } f \text{ everywhere})\}$. By the above construction, the sets $[T_f]$ are nowhere dense for $f \in \mathcal{F}$. As $|\mathcal{F}| < \text{cov}(\mathcal{M})$, there must be $g \in \omega^\omega \setminus \bigcup_{f \in \mathcal{F}} [T_f]$. Now use the Observation (3.2.) to construct a new predictor from the $\langle \pi^{g \upharpoonright n}; n \in \omega \rangle$ which will predict all $f \in \mathcal{F}$. \square

3.4. It is unclear whether $\mathfrak{e} = \mathfrak{e}(\omega)$ can be proved in *ZFC*. In view of Theorem 3.3 it seems reasonable to ask first

QUESTION. *Is $\mathfrak{e} > \text{cov}(\mathcal{M})$ consistent?*

Of course, we may also consider the cardinal $\mathfrak{e}_\ell(\omega)$, the smallest size of a family \mathcal{F} of functions from ω to ω such that no countable family of linear predictors predicts all $f \in \mathcal{F}$. However, it is now easy to see that $\mathfrak{e}_\ell(\omega) = \mathfrak{e}_\ell$. This is so because $\mathfrak{e}_\ell(\omega) \leq \min\{\mathfrak{e}(\omega), \mathfrak{b}\} \leq \min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathfrak{e}_\ell$. To see the first inequality, note that the argument for $\mathfrak{e}_\ell \leq \mathfrak{b}$ gives $\mathfrak{e}_\ell(\omega) \leq \mathfrak{b}$ as well (see [Br, section 5.4] for a stronger result); for the second inequality, $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} \leq \text{cov}(\mathcal{M})$ by rewriting Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \text{cov}(\mathcal{M})$ [Bl, Theorem 13] and thus $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} = \min\{\mathfrak{e}(\omega), \text{cov}(\mathcal{M}), \mathfrak{b}\} \leq \min\{\mathfrak{e}, \mathfrak{b}\}$ by Theorem 3.3; the third inequality is Lemma 2.2.

3.5. **Duality.** Most of the cardinal invariants of the continuum come in pairs and results about them usually can be dualized (see [Br, section 4.5] for details). In our situation, the dual cardinals are: the *dominating number* \mathfrak{d} (dual to \mathfrak{b}), the smallest size of a family

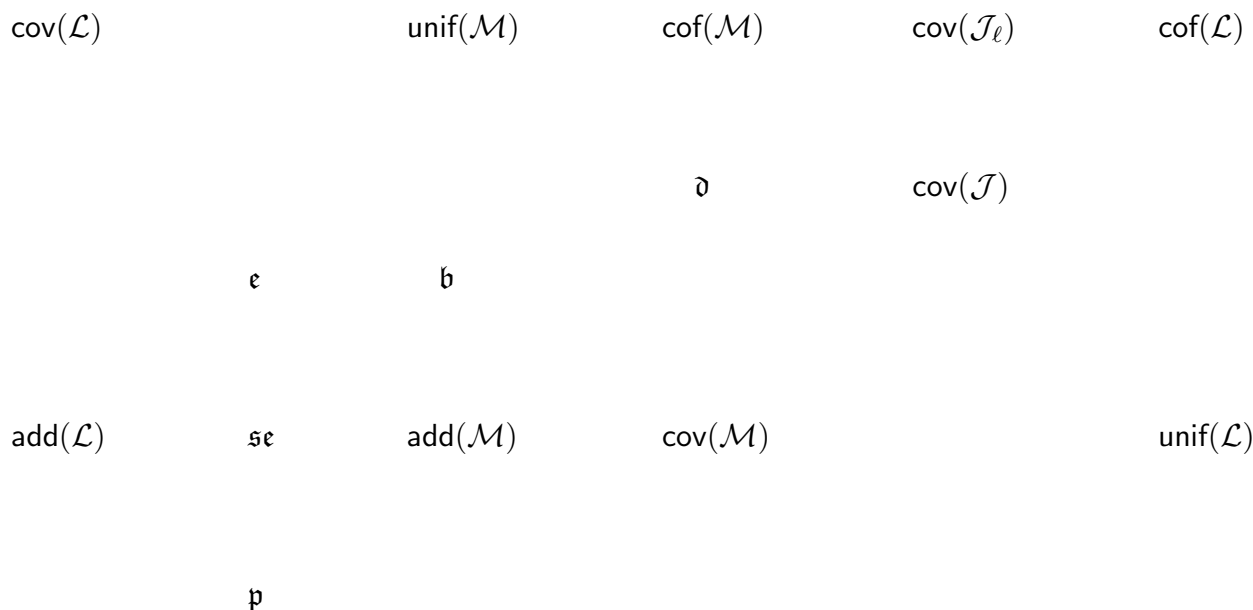
$\mathcal{F} \subseteq \omega^\omega$ such that given any $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $g \leq^* f$; the (linear) covering number $\text{cov}(\mathcal{J})$ ($\text{cov}(\mathcal{J}_\ell)$) of the ideal \mathcal{J} (\mathcal{J}_ℓ) (the first being dual to both \mathfrak{e} and $\mathfrak{e}(\omega)$, the second being dual to \mathfrak{e}_ℓ), the least cardinality of a family of (linear) predictors Π such that every function $f \in \omega^\omega$ (\mathbb{Z}^ω) is predicted by some $\pi \in \Pi$. Then we get:

THEOREM. (a) *It is consistent with ZFC to assume $\mathfrak{d} > \text{cov}(\mathcal{J})$.*

(b) $\text{cov}(\mathcal{J}_\ell) = \max\{\text{cov}(\mathcal{J}), \mathfrak{d}\} = \min\{|\mathcal{S}|; \mathcal{S} \text{ consists of slaloms } S_{\bar{a}}^{\bar{a}} \text{ where } \bar{a} = \langle a_n \in [\omega]^{\leq n}; n \in D \rangle \text{ and } D \subseteq \omega \text{ is infinite and } \forall f \in \omega^\omega \exists S_{\bar{a}}^{\bar{a}} \in \mathcal{S} \forall^\infty n \in D (f(n) \in a_n)\}$.

PROOF. These dualizations are standard, and we therefore refrain from giving detailed proofs. The model for (a) is gotten by iterating the p.o. \mathbb{P} from § 1 ω_1 times with finite support over a model for $MA + \neg CH$. (b) is the dual version of Theorem B and Lemma 2.5. \square

We close our work with a diagram showing the relations between the cardinal invariants considered in this work (in particular, the Specker–Eda number \mathfrak{se} and the evasion number \mathfrak{e}) and some other cardinal invariants of the continuum (in particular, those of Cichoń’s diagram). We refer the reader to [Bl], [Br] or [Fr] for the cardinals not defined here. A similar diagram was drawn in [Br, section 4].



In the diagram, cardinals increase as one moves up and to the right. To enhance readability, we omitted the relations $\mathfrak{e} \leq \text{unif}(\mathcal{L})$, and its dual $\text{cov}(\mathcal{L}) \leq \text{cov}(\mathcal{J})$. The dotted lines give the relations $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$, $\mathfrak{se} = \min\{\mathfrak{e}, \mathfrak{b}\}$, and their dual versions.

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