On a theorem of Shapiro

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Abstract

We show that a theorem of Leonid B. Shapiro which was proved under MA, is actually independent from ZFC. We also give a direct proof of the Boolean algebra version of the theorem under MA(*Cohen*).

1 Introduction

L.B. Shapiro [8] recently proved the following theorem:

Theorem 1.1 (L.B. Shapiro) (MA(*Cohen*)) *For any compact Hausdorff space* X *of weight* $< 2^{\aleph_0}$ *and* $\aleph_0 \leq \tau < 2^{\aleph_0}$ *the following assertions are equivalent:*

i) *There exists a continuous surjection from* X *onto* ${}^{\tau}\mathbb{I}$;

ii) *There exists a continuous injection from* ${}^{\tau}2$ *into* X;

iii) *There exists a closed subset* $Y \subseteq X$ *such that* $\chi(y, Y) \geq \tau$ *for every* $y \in Y$.

The original proof of Theorem 1.1 by L.B. Shapiro in [8] was formulated under MA. However practically the same proof still works when merely MA(Cohen) is assumed where MA(Cohen) stands for Martin's Axiom restricted to the partial orderings of the form $\mathrm{Fn}(\kappa, 2)$.

A part of the theorem above can be translated into the language of Boolean algebras:

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Corollary 1.2 (Boolean algebra version of Shapiro's theorem) (MA(*Cohen*)) *For any infinite Boolean algebra B of cardinality $< 2^{\aleph_0}$ and any infinite τ, the following are equivalent:*

i') *There exists an injective Boolean mapping from* $\mathrm{Fr}\,\tau$ *into B;*
ii') *There exists a surjective Boolean mapping from B onto* $\mathrm{Fr}\,\tau$.

The implication from ii') to i') as well as the implication from ii) to i) can be proved already in ZFC. For the proof of ii) from i), let $g : {}^{\tau}2 \to X$ be a continuous injection. Note that $g[{}^{\tau}2]$ is a closed subset of X. For any fixed $y_0 \in {}^{\tau}2$ let $f' : X \to {}^{\tau}2$ be defined by

$$f'(x) = \begin{cases} g^{-1}(x) & ; \text{if } x \in g[{}^{\tau}2], \\ y_0 & ; \text{otherwise.} \end{cases}$$

Then f' is a continuous surjection from X onto ${}^{\tau}2$. Let f'' be a continuous surjection from ${}^{\tau}2$ to ${}^{\tau}\mathbb{I}$. E.g. let $h : {}^{\omega}2 \to \mathbb{I}$ be the continuous surjection defined by $u \mapsto$ the real represented by the binary expression $0.u(0)u(1)u(2)\cdots$. $h^{\kappa} : {}^{\kappa}({}^{\omega}2) \to {}^{\kappa}\mathbb{I}$ is then a continuous surjection. Since ${}^{\kappa}({}^{\omega}2)$ is homeomorphic to ${}^{\kappa}2$ we can find a continuous surjection f'' from ${}^{\tau}2$ onto ${}^{\tau}\mathbb{I}$ corresponding to h^{κ}. The mapping $g = f'' \circ f'$ is then as desired. In the next section we shall give a direct proof of $i') \Rightarrow ii')$. For $iii) \Rightarrow i)$ we need some deep results by Shapiro on dyadic compactum (see [8]).

The equivalence of the assertions i') and ii') above is not true in general for Boolean algebras of cardinality $\geq 2^{\aleph_0}$: For any σ-complete Boolean algebra B and any infinite κ, there exits no surjective Boolean mapping $f : B \to \mathrm{Fr}\,\kappa$ (see Lemma 1.3 below). Hence e.g. for Boolean algebra $B = \overline{\mathrm{Fr}\,\omega}$ we have that $|\,B\,| = 2^{\aleph_0}$; $\mathrm{Fr}\,2^{\aleph_0}$ is embeddable into B (by Balcar-Franěk-Theorem, see [1]) but there exists no surjective Boolean mapping from B onto $\mathrm{Fr}\,2^{\aleph_0}$. The non-existence of surjective Boolean mapping from a σ-complete Boolean algebra in the ground model onto $\mathrm{Fr}\,\tau$ is preserved in a generic extension by a partial ordering of cardinality $< \tau$ though B may be no more σ-complete in such a generic extension:

Lemma 1.3 *Let B be a σ-complete Boolean algebra and P a partial ordering. For any $\kappa > |\,P\,|$ we have that*

$$\Vdash_P \text{“}\textit{there exists no surjective Boolean mapping from } B \textit{ onto } \mathrm{Fr}\,\kappa\text{”}.$$

Proof Suppose that there would be a P-name $\dot{f}$ such that

$$\Vdash_P \text{“}\dot{f} : B \to \mathrm{Fr}\,\kappa \text{ is a surjective Boolean mapping”}.$$

For each $p \in P$ let

$$B_p = \{\, b \in B \,:\, p \Vdash_P \text{“}\dot{f}(b) = c \text{ for some } c \in \mathrm{Fr}\,\kappa\text{”} \,\}$$

and

$$C_p = \{\, c \in \mathrm{Fr}\,\kappa \;:\; p \Vdash_P \text{“} \dot{f}(b) = c \text{ for some } b \in B \text{”} \,\}.$$

Then B_p and C_p are subalgebras of B and $\mathrm{Fr}\,\kappa$ respectively. Since $\bigcup_{p \in P} C_p = \mathrm{Fr}\,\kappa$ and $\kappa > |\, P \,|$ there exists some $p \in P$ such that C_p is infinite. Let c_n, $n < \omega$ be pairwise disjoint positive elements of C_p. By the definition of B_p and C_p, there exits pairwise disjoint positive elements b_n, $n < \omega$ of B_p such that $p \Vdash_P$ “$\dot{f}(b_n) = c_n$” holds for every $n < \omega$. Let $X \subseteq \omega$ be such that there exists no $c \in \mathrm{Fr}\,\kappa$ such that $c \cdot c_n = c_n$ holds for all $n \in X$ and $c \cdot c_n = 0$ for all $n < \omega \setminus X$. Let $d = \Sigma^B_{n \in X} b_n$. Then for any $q \leq p$ there can be no $c \in \mathrm{Fr}\,\kappa$ such that $q \Vdash_P$ “$\dot{f}(d) = c$”. This is a contradiction. ❏ (Lemma 1.3)

The lemma above together with Corollary 1.2 yields the following:

Proposition 1.4 *Let B be a complete Boolean algebra with $|\, B \,| = \tau \geq \aleph_0$. Then*

$\Vdash_{\mathrm{Fn}(\kappa,2)}$ “*there exists no surjective Boolean mapping from B onto* $\mathrm{Fr}\,\tau$”

holds if and only if $\kappa < \tau$.

Proof If $\kappa < \tau$ then $|\, \mathrm{Fn}(\kappa, 2) \,| = \kappa < \tau$. Hence by Lemma 1.3,

$\Vdash_{\mathrm{Fn}(\kappa,2)}$ “there exists no surjective Boolean mapping from B onto $\mathrm{Fr}\,\tau$”

holds.

Suppose now that $\kappa \geq \tau$. Then as in the proof of Proposition 2.1, we can show that

$\Vdash_{\mathrm{Fn}(\kappa,2)}$ “there exists a surjective Boolean mapping from B onto $\mathrm{Fr}\,\tau$”

holds. ❏ (Proposition 1.4)

Now, (⫯) (read “stick”, see [2]) is the following principle:

(⫯): There exists a sequence $(x_\alpha)_{\alpha < \omega_1}$ of countable subsets of ω_1 such that for any $y \in [\omega_1]^{\aleph_1}$ there exists $\alpha < \omega_1$ such that $x_\alpha \subseteq y$.

Clearly (⫯) follows from CH. Another combinatorial principle (♣), a strengthning of (⫯), is introduced in Ostaszewski [7]. Let $Lim(\omega_1) = \{\, \gamma < \omega_1 \;:\; \gamma \text{ is a limit} \,\}$.

(♣): There exists a sequence $(x_\gamma)_{\gamma \in Lim(\omega_1)}$ of countable subsets of ω_1 such that for every $\gamma \in Lim(\omega_1)$, x_γ is a cofinal subset of γ, $otp(x_\gamma) = \omega$ and for every $X \in [\omega_1]^{\aleph_1}$ there is $\gamma \in Lim(\omega_1)$ such that $x_\gamma \subseteq X$.

Clearly (⫯) follows from (♣). Unlike (⫯), (♣) does not follow from CH, since (♣) + CH is equivalent with $\diamondsuit$ (K. Devlin, see [7]). For more about the combinatorial principles (⫯) and (♣), and independence results connected with them, see [4].

MA(*countable*) — Martin's axiom restricted to countable partial orderings — and MA(*Cohen*) both add a lot of Cohen reals over any small model of (a sufficiently large finite subset of) ZFC and in many cases where this property is needed, MA(*countable*) is just enough. Hence it seems to be quite natural to ask if these axioms are perhaps equivalent. However they are not. I. Juhász proved in an unpublished note that $\neg$CH + MA(*countable*) + (♣) is consistent (two other constructions of models of $\neg$CH + MA(*countable*) + (♣) are to be found in [5] and [4].). On the other hand, it is easy to see that the negation of $\mathrm{MA}(\mathrm{Fn}(\aleph_1, 2))$ follows from $\neg$CH + (♣): using (⫯) we can obtain a Boolean algebra B of cardinality $\aleph_1$ such that $\mathrm{Fr}\,\omega_1$ is embeddable into B but there is no surjection from B onto $\mathrm{Fr}\,\omega_1$ (see Theorem 4.4). By Proposition 2.1, this shows that $m_{\mathrm{Fn}(\aleph_1,2)} = \aleph_1 < 2^{\aleph_0}$. It follows also that the assertions of Theorem 1.1 and Corollary 1.2 are independent from ZFC and MA(*countable*) is not enough to prove them.

Corollary 1.2 for other variety than Boolean algebras can be simply false. E.g., this is the case in the variety of abelian groups: in [3], an $\aleph_1$-free abelian group G in $\aleph_1$ is constructed (in ZFC) which contains uncountable free subgroup but $Hom(G, Z) = 0$.

2 A proof of the Boolean algebra version of the theorem

In this section we shall prove Corollary 1.2. More precisely we prove the following Proposition 2.1. For any class $\mathcal{C}$ of partial orderings Let

$$m_{\mathcal{C}} = \min\{\,|\,\mathcal{D}\,|\, : \mathcal{D} \text{ is a family of dense subsets of } P \text{ for some } P \in \mathcal{C} \text{ such that there exists no } \mathcal{D}\text{-generic filter over } P\,\}$$

If $\mathcal{C}$ is a singleton $\{\,P\,\}$, we shall write simply m_P in place of $m_{\{\,P\,\}}$. Let us say that two partial orderings P, Q are coabsolute when their completions are isomorphic. It is easy to see that for any class $\mathcal{C}$ of partial orderings $m_{\mathcal{C}} = m_{\tilde{\mathcal{C}}}$ where $\tilde{\mathcal{C}} = \{\,Q : Q \text{ is coabsolute with some } P \in \mathcal{C}\,\}$. If the class $\mathcal{C}$ is introduced by a property $\mathcal{P}$ of Boolean algebras, we also write $m_{\mathcal{P}}$ in place of $m_{\mathcal{D}}$. We also write $m_{countable} = m_{\{\,P\,:\,P \text{ is countable}\,\}}$ and $m_{Cohen} = m_{\{\,P\,:\,P=\mathrm{Fn}(\kappa,2) \text{ for some } \kappa\,\}}$. Hence MA(*Cohen*) (MA(*countable*), MA etc. respectively) holds if and only if $m_{Cohen} = 2^{\aleph_0}$ ($m_{countable} = 2^{\aleph_0}$, $m_{ccc} = 2^{\aleph_0}$ etc. respectively) and we have $m_{ccc} \le m_{Cohen} \le m_{countable}$.

Proposition 2.1 *Let B be a Boolean algebra containing* $\mathrm{Fr}\,\kappa$ *as a subalgebra. If* $|\,B\,| < m_{\mathrm{Fn}(\kappa,2)}$, *then there exists a surjective Boolean mapping from B onto* $\mathrm{Fr}\,\kappa$.

Proof By Sikorski's theorem, there is a Boolean mapping from B to $\overline{\mathrm{Fr}\,\kappa}$ — the completion of $\mathrm{Fr}\,\kappa$, extending the inverse of the canonical embedding of $\mathrm{Fr}\,\kappa$ into B. Hence without loss of generality we may assume that B is a subalgebra of $\overline{\mathrm{Fr}\,\kappa}$. Now let $P = \mathrm{Fn}(\kappa, 3)$. Note that P is coabsolute with $\mathrm{Fn}(\kappa, 2)$. We shall define a family $\mathcal{D}$ of dense subsets of P such that $|\,D\,| < m_{\mathrm{Fn}(\kappa,2)}$ so that among other things (see below), for $\mathcal{D}$-generic set G, $g = \bigcup G$ will be a function from κ to 3 and $X = \{\,\alpha < \kappa \,:\, g(\alpha) = 2\,\}$ will be of cardinality κ. Then we let f be the function on κ defined by:

$$f(\alpha) = \begin{cases} 0_B & ;\text{ if } g(\alpha) = 0,\\ 1_B & ;\text{ if } g(\alpha) = 1,\\ \alpha & ;\text{ otherwise.}\end{cases}$$

Let $\bar{f}$ be the Boolean mapping from $\mathrm{Fr}\,\alpha$ to $\mathrm{Fr}\,X$ generated by f.

Now we are done, if we can show that $\bar{f}$ extends to a Boolean mapping $\tilde{f}$ from B onto $\mathrm{Fr}\,X$. But by the following Lemma 2.2, we can choose $\mathcal{D}$ appropriate for this purpose.

For $p \in P$, let $B_p = \mathrm{Fr}\,\mathrm{dom}(p)$ (hence $B_p \leq B$) and $f_p : B_p \to \mathrm{Fr}\,(p^{-1}[\{2\}])$ be the Boolean mapping generated by the mapping f_p^0 on $\mathrm{dom}(p)$ defined by:

$$f_p^0(\alpha) = \begin{cases} 0_B & ;\text{ if } p(\alpha) = 0,\\ 1_B & ;\text{ if } p(\alpha) = 1,\\ \alpha & ;\text{ otherwise.}\end{cases}$$

Lemma 2.2 *For any $b \in B$ and $p \in P$ there exists $q \leq p$ and b_1, $b_2 \in B_q$ such that $b_1 \leq b$, $b_2 \leq -b$ and $f_q(b_1) + f_q(b_2) = 1$ (i.e, q "forces" $\tilde{f}(b) = f_q(b_1)$).*

For the proof of the Lemma 2.2 we use the following Lemma whose proof is left to the reader:

Lemma 2.3 *Let $b \in \overline{\mathrm{Fr}\,\kappa}$ and let $Y \subseteq \kappa$ be a countable set such that $b \in \overline{\mathrm{Fr}\,Y}$ holds. Let $Y = \{\,\alpha_n \,:\, n < \omega\,\}$. Then there exist an increasing sequence $(l_n)_{n<\omega}$ with $l_n < \omega$ for $n < \omega$ and a sequcence $(i_n)_{n<\omega}$ with $i_n \in {}^{l_n}\{\,-1, 1\,\}$ for $n < \omega$ such that, letting $p_n = \Sigma_{k<l_n} i_n(k) \cdot \alpha_k$ for $n < \omega$,*

i) *either $p_n \leq b$ or $p_n \leq -b$ and*

ii) $\Sigma_{n<\omega} p_n = 1$.

In particular we have $b = \Sigma\{\,p_n \,:\, n < \omega,\, p_n \leq b\,\}$. ❑

Proof of Lemma 2.2 Let $Y = \{\, \alpha_n \,:\, n < \omega \,\}$, $(l_n)_{n<\omega}$, $(i_n)_{n<\omega}$ and p_n, $n < \omega$ be as in Lemma 2.3 for our $b \in B$. Without loss of generality we may assume that $\mathrm{dom}(p) \cap Y = \{\, \alpha_n \,:\, n < k \,\}$ for some $k < \omega$. Let ${}^k\{\, -1, 1 \,\} = \{\, \tau_m \,:\, m < 2^k \,\}$. By induction we can take $n_m < \omega$ for $m < 2^k$ such that

a) i_{n_m} is compatible (as an element of $\mathrm{Fn}(Y, \{\, -1, 1 \,\})$) with τ_m and

b) $\{\, i_{n_m}\restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m < 2^k \,\}$ is pairwise compatible.

Let $\tilde{n} = \max\{\, n_m \,:\, m < 2^k \,\}$, $\tilde{l} = l_{\tilde{n}}$ and $\tilde{i} = \bigcup\{\, i_{n_m}\restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m < 2^k \,\}$. Let $q \leq p$ be such that $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\, \alpha_k, \ldots, \alpha_{\tilde{l}-1} \,\}$, $q\restriction \mathrm{dom}(p) = p$ and

$$q(\alpha_m) = \begin{cases} 1 & ; \text{if } \tilde{i}(\alpha_m) = 1, \\ 0 & ; \text{if } \tilde{i}(\alpha_m) = -1. \end{cases}$$

Then q as above together with $b_1 = \Sigma\{\, p_n \,:\, n < \tilde{n},\, p_n \leq b \,\}$ and $b_2 = \Sigma\{\, p_n \,:\, n < \tilde{n},\, p_n \leq -b \,\}$ is as desired. ❏ (Lemma 2.2)

Now by the lemma above

$$\begin{aligned} \mathcal{D} = & \{\, \{\, p \in P \,:\, \alpha \in \mathrm{dom}(p) \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, p \in P \,:\, \exists \beta > \alpha \; p(\beta) = 2 \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, q \in P \,:\, f_q(b_1) + f_q(b_2) = 1 \text{ for some } b_1 \leq b,\, b_2 \leq -b \,\} \,:\, b \in B \,\} \end{aligned}$$

is a family of dense subsets of P. Clearly the mapping $\bar{f}$ defined as above with respect to this $\mathcal{D}$ can be extended to a Boolean mapping $\tilde{f}$ from B onto $\mathrm{Fr}\, X$.

❏ (Proposition 2.1)

3 Pcf and the theorem of Shapiro

Proposition 3.1 *Assume that*

$\oplus_{\mu,\kappa,\lambda}$ *for any* $\mathcal{F} \subseteq [\lambda]^{\aleph_0}$ *with* $|\, \mathcal{F} \,| < \mu$, *there is* $Y \in [\lambda]^\kappa$ *such that* $a \cap Y$ *is finite for all* $a \in \mathcal{F}$.

Then, for any Boolean algebra B *of cardinality* $< \mu$, *if* $\mathrm{Fr}\, \lambda$ *is embeddable into* B *then there is a surjective Boolean mapping from* B *onto* $\mathrm{Fr}\, \kappa$.

Proof As in the proof of Proposition 2.1, we may assume without loss of generality that $\mathrm{Fr}\, \lambda \leq B \leq \overline{\mathrm{Fr}\, \lambda}$ holds. Let $|\, B \,| = i^*$ $(< \mu)$ and let $(y_i)_{i<i^*}$ be an enumeration of B. Let $y_i = \sum_{n<\omega} \tau_i^n(\alpha(i,n,0), \ldots, \alpha(i,n,m_{i,n}))$ where τ_i^n is a Boolean term with $m_{i,m} + 1$ variables and $\alpha(i,n,0), \ldots, \alpha(i,n,m_{i,n}) < \lambda$ for $i < i^*$ and $n < \omega$. For $i < i^*$, let $w_i = \{\, \alpha(i,n,l) \,:\, n < \omega,\, l \leq m_{i,n} \,\}$. By the assumption, there exists a $Y \in [\lambda]^\kappa$ such that $w_i \cap Y$ is finite for everly $i < i^*$. Let $g : B \to \mathrm{Fr}\, Y$ be defined by

$$g(y_i) = \sum_{n<\omega} \tau_i^n(\alpha^*(i,n,0), \ldots, \alpha^*(i,n,m_{i,n}))$$

where

$$\alpha^*(i,n,l) = \begin{cases} \alpha(i,n,l) & ;\ \text{if } \alpha(i,n,l) \in Y \\ 0_B & ;\ \text{otherwise.} \end{cases}$$

The function g is well-defined since, for each $i < \omega$, $\tau_i^n(\alpha^*(i,n,0), \ldots, \alpha^*(i,n,m_{i,n}))$ is an element of $\mathrm{Fr}\,(w_i \cap Y)$ and $\mathrm{Fr}\,(w_i \cap Y)$ is finite. Clearly this g is as desired.

❏ (Proposition 3.1)

Corollary 3.2 *For any Boolean algebra of cardinality* $< \mathbf{a}$ *(where* $\mathbf{a}$ *is the minimal cardinality of a maximal almost disjoint family in* $[\omega]^{\aleph_0}$*), if* $\mathrm{Fr}\,\omega$ *is embeddable into* B *then there is a surjection from* B *onto* $\mathrm{Fr}\,\omega$*.*

Proof By Proposition 3.1 for $\oplus_{\mathbf{a},\aleph_0,\aleph_0}$. ❏ (Corollary 3.2)

Theorem 3.3 *Assume that*

$(*)_{\mu,\lambda,\kappa}$ *there are* $\underset{\circ}{a}_i \in [Reg \cap (\lambda^+ \setminus \kappa^+)]^{<\aleph_0}$ *for* $i < \kappa$ *such that for every* $a \in [\kappa]^{\aleph_0}$, $\max \mathrm{pcf}(\bigcup_{i\in a} \underset{\circ}{a}_i) \geq \mu$ *holds.*

Then for any Boolean algebra B *of cardinality* $< \mu$*, if* $\mathrm{Fr}\,\kappa$ *is embeddable into* B *then there is a surjective Boolean mapping* g *from* B *onto* $\mathrm{Fr}\,\kappa$*.*

(For more about $(*)_{\mu,\lambda,\kappa}$ see [10]. For pcf theory in general, the reader may consult [11].) The theorem follows from Proposition 3.1 and the following:

Lemma 3.4 *Assume that* $(*)_{\mu,\lambda,\kappa}$ *(as in Theorem 3.3) holds. Then* $\oplus_{\mu,\kappa,\kappa}$ *holds.*

Proof Since max pcf is always regular, we may assume that μ is regular. Let $\underset{\circ}{a} = \bigcup_{i<\kappa} \underset{\circ}{a}_i$. In place of $[\kappa]^{\aleph_0}$, we consider $[Z]^{\aleph_0}$ for $Z = \dot{\bigcup}_{i<\kappa} Z_i$ where $Z_i = \{\, i \,\} \times \prod \underset{\circ}{a}_i$. Hence we assume that $\mathcal{F} \subseteq [Z]^{\aleph_0}$ and $|\, \mathcal{F} \,| < \mu$.

For each $a \in \mathcal{F}$, let $g_a \in \prod \underset{\circ}{a}$ be defined by

$$g_a(\theta) = \sup\{\, \eta(\theta) \,:\, \eta \in a,\, \theta \in \mathrm{dom}(\eta) \,\}$$

for each $\theta \in \underset{\circ}{a}$, where we put $\sup \emptyset = 0$. Since $\prod \underset{\circ}{a} / J_{<\mu}[\underset{\circ}{a}]$ is μ-directed and $|\, \mathcal{F} \,| < \mu$, there is $f^* \in \prod \underset{\circ}{a}$ such that $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$ holds for all $a \in \mathcal{F}$. For $i < \kappa$, let $z_i = \{ (0,i) \} \cup (f^* \restriction \underset{\circ}{a}_i)$. Then $z_i \in Z_i$ for $i < \kappa$. We show that $Y = \{ z_i : i < \kappa \}$ is as required. Suppose not. Then $Y \cap a$ would be infinite for some $a \in \mathcal{F}$. By the assumption, it follows that $\bigcup_{z_i \in Y \cap a} \underset{\circ}{a}_i \notin J_{<\mu}[\underset{\circ}{a}]$. But for $z_i \in Y \cap a$ we have $\{ (0,i) \} \cup (f^* \restriction \underset{\circ}{a}_i) \in a$. It follows that for $\theta \in \underset{\circ}{a}_i$ we have $f^*(\theta) \le g_a(\theta)$. This is a contradiction to $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$. ❑ (Lemma 3.4)

4 Independence of the theorem of Shapiro

The principle (stick) suggests the following cardinal invariant stick:

$$\mathrm{stick} = \min\{ \mid X \mid \; : \; X \subseteq [\omega_1]^{\aleph_0}, \forall y \in [\omega_1]^{\aleph_1} \, \exists x \in X \; x \subseteq y \}.$$

Clearly $\aleph_1 \le \mathrm{stick} \le 2^{\aleph_0}$ and (stick) holds if and only if $\mathrm{stick} = \aleph_1$. We can also consider the following variants of stick:

$$\mathrm{stick}' = \min\{ \kappa : \kappa \ge \aleph_1, \text{ there is an } X \subseteq [\kappa]^{\aleph_0} \text{ such that } \mid X \mid = \kappa \text{ and } \forall y \in [\kappa]^{\aleph_1} \, \exists x \in X \; x \subseteq y \},$$

$$\mathrm{stick}'' = \min\{ \kappa : \kappa \ge \aleph_1, \text{ there is an } X \subseteq [\kappa]^{\aleph_0} \text{ such that } \mid X \mid = \kappa \text{ and } \forall y \in [\kappa]^{\kappa} \, \exists x \in X \; x \subseteq y \}.$$

We have $\aleph_1 \le \mathrm{stick}'' \le \mathrm{stick}' \le 2^{\aleph_0}$ and (stick) holds if and only if $\mathrm{stick} = \mathrm{stick}' = \mathrm{stick}'' = \aleph_1$ holds.

It can be easily shown that $\mathrm{stick} \le \mathrm{stick}'$ holds. Moreover if $\mathrm{stick} < \aleph_{\omega_1}$, then $\mathrm{stick} = \mathrm{stick}'$ holds. The question, if $\mathrm{stick} < \mathrm{stick}'$ is consistent, is connected with some very fundamental unsolved problems on cardinal arithmetics while we can show that $\mathrm{stick}'' < \mathrm{stick}$ is consistent. For more, see [4] and [10].

Proposition 4.1 *There exists a Boolean algebra B such that $\mid B \mid = \mathrm{stick}'$, $\mathrm{Fr}\,\mathrm{stick}'$ is embeddable into B but there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\omega_1$.*

Proof Let $\Phi : \kappa \to \kappa$; $\alpha \mapsto \xi_\alpha$ be the continuously increasing function defined inductively by $\xi_0 = \omega$ and $\xi_{\alpha+1} = \xi_\alpha + \mid \xi_\alpha \mid$. Let $\kappa = \mathrm{stick}'$ and let $X \subseteq [\kappa \times \mathrm{Fr}\,\omega_1]^{\aleph_0}$ be such that $\mid X \mid = \kappa$, $\omega \times \mathrm{Fr}\,\omega \in X$ and $\forall y \in [\kappa \times \mathrm{Fr}\,\omega_1]^{\aleph_1} \, \exists x \in X \; x \subseteq y$ holds. Let $(x_\alpha)_{\alpha<\kappa}$ be an enumeration of X such that $x_\alpha \subseteq \xi_\alpha \times \mathrm{Fr}\,\omega_1$ for all $\alpha < \kappa$.

Now let $(B_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of Boolean algebras such that for all $\alpha < \kappa$

1) the underlying set of B_α is ξ_α;

2) there exits a $b_\alpha \in B_{\alpha+1}$ such that b_α is free over B_α;

3) if x_α generates a Boolean mapping f_α from a subalgebra of B_α onto an infinite subalgebra of $\mathrm{Fr}\,\omega_1$ then $B_{\alpha+1}$ contains an element c_α of the form $\Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ where $Z_\alpha \subseteq \omega$, b^α_n, $n < \omega$ are pairwise disjoint elements in $\mathrm{dom}(f_\alpha)$, $f_\alpha(b^\alpha_n) \neq 0$ for all $n < \omega$ and there is no $d \in \mathrm{Fr}\,\omega_1$ such that $d \cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $d \cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds.

Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. We show that this B is as desired. By *1*) the underlying set of B is κ. By *2*) $\{\, b_\alpha : \alpha < \kappa \,\}$ is an independent subset of B. Hence $\mathrm{Fr}\,\kappa$ is embeddable into B.

Suppose now that there would be a surjective Boolean mapping f from B onto $\mathrm{Fr}\,\omega_1$. Then there is a bijection $g \subseteq f$ from a subset of B onto $\mathrm{Fr}\,\omega_1$. Since g is uncountable there is an $\alpha < \kappa$ such that $x_\alpha \subseteq g$. Since $x_\alpha \subseteq f$, x_α satisfies the condition in *3*). Hence there is a $c_\alpha \in B_{\alpha+1}$ such that $c_\alpha = \Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ for Z_α and b^α_n, $n < \omega$ as un *3*). But then $f(c_\alpha) \cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $f(c_\alpha) \cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds. This is a contradiction to the choice of Z_α. ❑ (Proposition 4.1)

Corollary 4.2 $m_{\mathrm{Fn}(\omega_1,2)} \leq \stick'$.

Proof By Proposition 2.1 and Proposition 4.1. ❑ (Corollary 4.2)

With almost the same proof as in Proposition 4.1 we can also prove the following:

Proposition 4.3 *There exists a Boolean algebra B such that $|\, B \,| = \stick''$,* $\mathrm{Fr}\,\stick''$ *is embeddable into B but there is no surjective Boolean mapping from B onto* $\mathrm{Fr}\,\stick''$. ❑

Since we have $\stick' = \aleph_1$ under ($\stick$), we obtain the following theorem:

Theorem 4.4 *If* ($\stick$) *holds then there exists a Boolean algebra B of cardinality* $\aleph_1$ *such that* $\mathrm{Fr}\,\omega_1$ *is embeddable into B but there is no surjection from B onto* $\mathrm{Fr}\,\omega_1$. ❑

Hence if $\neg$CH and ($\stick$) holds, by Theorem 4.4, there exists a counter-example to the theorem of Shapiro. This shows that we cannot just drop MA(*Cohen*) from Theorem 1.1. Since MA(*countable*) + $\neg$CH + ($\stick$) is consistent (see e.g. [5] or [4]), we see that MA(*countable*) is not enough for Theorem 1.1.

Corollary 4.5 $m_{\mathit{Cohen}} \leq \stick''$.

Proof By Proposition 2.1 and Proposition 4.3. ❏ (Corollary 4.5)

If a Boolean algebra B is atomless then $\mathrm{Fr}\,\omega$ can be embdded into B. By Proposition 2.1, if MA(*countable*) holds and B is of cardinality $< 2^{\aleph_0}$, there exists a surjection from B onto $\mathrm{Fr}\,\omega$. Here again we cannot simply drop the assumption of MA(*countable*):

Proposition 4.6 *It is consistent that there is an atomless Boolean algebra B of cardinality $\aleph_1 < 2^{\aleph_0}$ such that there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\omega$.*

Proof By [9, Theorem 5.12], there is a model of ZFC+ $\neg$CH in which there is an endo-rigid atomless Boolean algebra B of cardinality $\aleph_1$. In particular there is no surjection from B onto $\mathrm{Fr}\,\omega$. ❏ (Proposition 4.6)

Note that, since (†) is consistent with $\neg$CH and MA(*countable*), (†) cannot supply such a Boolean algebra as in the proposition above.

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