In the random graph $G(n,p), p = n^{-a}$: if ψ has probability $O(n^{-\varepsilon})$ for every $\varepsilon > 0$ then it has probability $O(e^{-n^{\varepsilon}})$ for some $\varepsilon > 0$

Sh551

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§0 Introduction

Shelah Spencer [ShSp 304] proved the $0-1$ law for the random graphs $G(n, p_n)$, $p_n = n^{-\alpha}$, $\alpha \in (0,1)$ irrational (set of nodes in $[n] = \{1, \dots, n\}$, the edges are drawn independently, probability of edge is p_n). One may wonder what can we say on sentences ψ for which $\mathrm{Prob}(G(n,p_n) \models \psi)$ converge to zero, Lynch [L] asked the question and did the analysis, getting (for every ψ):

(α) $\mathrm{Prob}[G(n,p_n) \models \psi] = cn^{-\beta} + O(n^{-\beta-\varepsilon})$ for some ε such that $\beta > \varepsilon > 0$

or

(β) $\mathrm{Prob}(G(n,p_n) \models \psi) = O(n^{-\varepsilon})$ for every $\varepsilon > 0$.

Lynch conjectured that in case (β) we have

(β^+) $\mathrm{Prob}(G(n,p_n) \models \psi) = O(e^{-n^{\varepsilon}})$ for some $\varepsilon > 0$.

We prove it here.

Notation Let ℓ, m, n, k be natural numbers.
Let $\varepsilon, \zeta, \alpha, \beta, \gamma$ be positive reals.
$[n] = \{1, \dots, n\}$.
$\mathbb{R}$ is the set of reals.
$\mathbb{R}^+$ is the set of reals > 0.

§1

1.1 Theorem. *1) For any first order sentence ψ in the language of graphs and irrational $\alpha \in (0,1)_{\mathbb{R}}$ we have (where $p_n = n^{-\alpha}$ and $Prob(G_{n,p_n} \models \psi) \to 0$):*

$$\underline{\text{either}}\ Prob(G_{n,p_n} \models \psi) \text{ is } cn^{-\beta} + O(n^{\beta-\varepsilon}) \text{ for some reals } \beta > \varepsilon > 0 \text{ and } c > 0$$

$$\underline{\text{or}}\ Prob(G_{n,p_n} \models \psi) \text{ is } O(e^{-n^{\varepsilon}}) \text{ for some real } \varepsilon > 0.$$

2) However, this is not recursive.

Proof. We change the context generalizing it.

1.2 Definition of the Probability Context.

(a) $Q_n \subseteq \{1,\dots,n\}, G^*_{Q_n}$ a graph on Q_n.
(b) We consider first order sentences or formulas with vocabulary $\subseteq \tau = \{=, Q, R\}$, ($=$ is equality, Q is a monadic predicate, R is a symmetric irreflexive binary relation (will be "being an edge").)
(c) $G = G_{n,p_n}[G^*_Q]$ a graph on $[n]$, $G \restriction Q = G^*_Q$, and except this, G is random with edge probability p_n (i.e. for every edge not included in Q we flip a coin with probability p_n and do it independently for the set of edges). We consider G a τ-model with $Q^M = Q, R$ the edge relation.

Remark. The point is that $|Q|$ will be required to be just $< n^{\varepsilon}$ not say $< \log(n)$.

Proof. We consider only graphs H in $\{H : H$ a graph whose set of nodes include Q, moreover $H \restriction Q = G^*_Q\}$. First, we repeat the proof in Shelah Spencer [ShSp 304], section 4, starting in p.105. In our context we define "$[H_0, H_1)$ has type (v,e)", it holds if $v = |H_1 \backslash H_0 \backslash Q|$, and

$$e = \left|\left\{\{x,y\} \in E(G_{n,p}) : \{x,y\} \subseteq H_1 \cup Q, \{x,y\} \nsubseteq H_0 \cup Q\right\}\right|,$$

(where for a graph G, $E(G)$ is the set of edges of G).
Then define dense, sparse, safe, rigid, hinged as there adding "over Q and/or inside G" for definiteness. We also define $cl_\ell(H_0; H_1)$ as in p.107, line 7. Later we write $cl_\ell(H_0; Q)$. All claims hold, but arriving to Theorem 1.3 (bottom of p.107) we should be careful. We consider only embeddings which are the identity on Q.

1.3 Lemma. *1) Let $\ell^* \in \mathbb{N}$. For every small enough $\varepsilon > 0$, for some $\xi > 0$, for every n large enough, if $|Q| \le n^\xi, Q \subseteq [n]$ we have: if (H_0, H_1) is safe of type (v, e) and f embeds H_0 into G (and f is the identity on Q) and $|H_1 \backslash Q| \le \ell^*$ then:*

$$Prob\Big(\neg[n^{v-\alpha\, e-\varepsilon} < N(f, H_0, H_1) < n^{v-\alpha\, e+\varepsilon}]\Big) < e^{-n^\xi}$$

(where $N(f, H_0, H_1)$ is the number of extensions $g : H_1 \to G$ satisfying:
$x \in H_0 \Rightarrow g(x) = f(n)$ and $\{x, b\} \in E(H_1), b \notin H_0 \Rightarrow \{g(x), g(y)\} \in E(G)$).
2) Let $\varepsilon \in \mathbb{R}^+$ and $\ell^* \in \mathbb{N}$ be given, then for some $\xi > 0$ for every n large enough and any $Q \subseteq [n], |Q| \le n^\varepsilon$ and graph G^*_Q on Q we consider only embeddings which are the identity on Q. Then

(∗) *if H_1 is a graph with $|H_1 \backslash Q| \le \ell^*$, $H_0 \subseteq H_1$, we assume f embeds H_0 into Q, f is the identity on H_0 and (H_0, H_1) is rigid then:*

$$Prob\Big(N(f, H_0, H_1, G_{n,p_n}) > 0\Big) < n^{-\varepsilon}.$$

Proof. 1) As in [ShSp 304, Theorem 3,p.107] + extra computation by the central limit theorem or see [Sh 550, §5] for more.
2) As in [ShSp 304].

4 Lemma. *For any $k, m \in \mathbb{N}$ there are ℓ^* and $\varepsilon^* > 0$ depending on k only such that the following holds:*

(∗) *For any formula $\psi = \psi(x_1, \dots, x_m)$ of quantifier depth $\le k$ in the vocabulary $\{=, Q, R\}$ there is a formula $\theta_\psi = \theta_\psi(x_1, \dots, x_m)$ in the vocabulary $\{=, Q, R\}$ such that:*

(∗∗) *For every n large enough, $Q \subseteq \{1, \dots, n\}, |Q| \le n^{\varepsilon^*}$, and graph G^*_Q on Q and $G = G_{n,p_n}[G^*_Q]$ such that the small probability cases from Lemma ?,? (for (H_1, H_2) of type $(v, e), v \le 2\ell^*$), or just $\otimes^1_{\ell^*} + \otimes^2_{\ell^*}$ below do not occur, we have:*

(∗∗∗) *for every $a_1, \dots, a_m \in \{1, \dots, m\}$ we have*

$$(\{1, \dots, n\}, Q, R) \models \psi[a_1, \dots, a_m] \ \underline{\text{iff}}$$
$$\Big(Q \cup \{a_1, \dots, a_m\}, Q, R \restriction (Q \cup \{a_1, \dots, a_m\})\Big) \models \theta_\psi[a_1, \dots, a_m].$$

where

$\bigotimes^1_{\ell^*}$ *if (H_0, H_1) is safe (so $Q \subseteq H_0$) $|H_1 \backslash Q| \le \ell^*, H_0 \subseteq G_{n,p_n}[G^*_Q]$ then we can extend id_{H_0} to an embedding g of H_1 into $G_{n,p_n}[G^*_Q]$ such that $c\ell_{\ell^*}\big(g(H_1), G_{n,p}[G^*_q]\big) = g(H_1) \cup c\ell_{\ell^*}\big(f(H_0, G_{n,p_n}[G^*_Q]\big)$*

$\bigotimes^2_{\ell^*}$ *if (H_0, H_1) is rigid, $|H_1 \backslash Q| \le \ell^*, H_0 = G^*_Q$ then there is no extension of f of id_{H_0} to an embedding of H_1 into $G_{n,p_n}[G^*_Q]$.*

Proof. Similar to the proof in [ShSp 304], (and is a particular case of [Sh 467, §2] (see related)).

Proof of Theorem 1.1. Part (1) Let θ_ψ be from the analysis (i.e. Lemma ? for the ψ from Theorem 1.1) for the original sentence ψ.

Case A. For some finite graph G^* on say $\{1, \dots, m^*\}$ we have $G^* \models \theta_\psi$.

In this case the probability that G^* can be embedded into G_{n,p_n} is $\geq O(n^{-\beta})$ for some $\beta \in (0, \infty)$ if $n \geq m^*$ of course; so this means that one of the $\leq n^{m^*}$ possible mapping is an embedding, but more convenient is to consider the event $G \restriction [m^*] = G^*$ which also has probability $\geq n^{-\beta}$ for some β. Now modulo this event the probability that the conclusion of Lemma ? fails is (for n large enough) much smaller than n^{-m^*}. So we can assume that for $G \restriction [m^*] \cong G^*$ and that the conclusion of Lemma ? holds for this. Now check and if we succeed by Lemma ?, we are done, i.e. the probability that $G_{n,p_n} \models \psi$ is quite high.

Case B. For no finite graph $G^*, G^* \models \theta_\psi$.

Choose $\ell^* \in \mathbb{N}$ large enough as needed for our sentence ψ in Lemma 4.

Let $\zeta \in \mathbb{R}^+$ be such that:
$v \in \{0, \dots, 2\ell^*\}, e \in \mathbb{N} \Rightarrow |v - \alpha e| \geq \zeta$ and it satisfies the requirements on ζ in Lemma ? (for $2\ell^*$ (readily follows).)
(The $2\ell^*$ rather than ℓ^* is for the bound on $\text{Prob}(\mathcal{E}_2)$.) Clearly ζ exists and if (H_0, H_1) is hinged and $|H_1 \backslash H_0| \leq \ell^*$ and (H_0, H_1) is of type (v, e) then $v - \alpha e < -\zeta$.

Let $\varepsilon(\ell^*), \xi$ be such that:

(*a*) $\varepsilon(\ell^*) \in \mathbb{R}^+$ and $\varepsilon(\ell^*) < \zeta/2, \xi < \zeta/2$
(*b*) in Lemma ? $\varepsilon(\ell^*), \xi$ satisfies the requirements of ε, ξ respectively.

We shall prove that for n large enough $\text{Prob}(G_{n,p_n} \models \psi)$ is $\leq e^{-(n^\xi)}$, this is enough.

For any $G = G_{n,p_n}$, we define by induction on $j \leq n$, a subset $P_j = P_j[G]$ of $\{1, \dots, n\}$ as follows:

$$P_0 = \emptyset$$

$$P_{j+1} = P_j \cup \{H : P_j \subseteq H \subseteq G, |H \backslash P_j| \leq \ell^*, H \neq P_j \text{ and } (P_j, H) \text{ is rigid in } G\}.$$

For some $j(*) < n$ we have $P_{j(*)} = P_{j(*)+1}$ (hence $P_{j(*)+1} = P_{j(*)+2}$, etc).
If $|P_{j(*)}| \leq n^{\varepsilon(\ell^*)}$ and $\otimes^1_{\ell^*}$ holds then, (as $P_{j(*)} = P_{j(*)+1}$) this implies $\otimes^2_{\ell^*}$ and then by Lemma ? we are done ($P_{j(*)}$ is Q). So it is enough to give an upper bound of the form e^{-n^ε} to the probability $\text{Prob}(\mathcal{E}_1) + \text{Prob}(\mathcal{E}_2)$ were $\mathcal{E}_1$ is the event $|P_{j(*)}| > n^{\varepsilon(\ell^*)}$ and $\mathcal{E}_2$ is the event $|P_{j(*)}| \leq n^{\varepsilon(\ell^*)}$ & [$\otimes^1_{\ell^*}$ fails].

<u>On Prob($\mathcal{E}_1$)</u>. If $|P_{j(*)}| \geq n^{\varepsilon(\ell^*)}$ then we can find $a_{j,\ell}$ for $j < [n^{\varepsilon(\ell^*)}/\ell^*]$ and $\ell < \ell_j \leq \ell^*$ such that

$\Big(H_i \cap \{a_{i,\ell} : \ell < \ell_i\}, \{a_{i,\ell} : \ell < \ell_i\}\Big)$ (in G) is rigid of type (v_i, e_i) where $H_i =: \{a_{j,\ell} : j < i \text{ and } \ell < \ell_j\}$ (so we may have not used all $P_{j(*)}$). Clearly there is a real $\zeta > 0$ depending on ℓ^*, α only such that $v_i - e_i\alpha \leq -\zeta$, (simply, there are only finitely many possible pairs (v, e)).

Let I be a sequence describing this situation, i.e. it contains

$$\langle \ell_i : i < [n^{\varepsilon(\ell^*)}/\ell^*] \rangle$$

$$\{((i_1, m_1), (i_2, m_2)) : a_{\ell_1, m_1} = a_{i_2, m_2}\}$$

$$\{(i, m_1, m_2) : a_{i,m_1} R^G a_{i,m_2}\}.$$

There are $\prod\limits_{i<[n^{\varepsilon(\ell^*)}/\ell^*]} (\ell^* \times (\ell^* \times i)^{\ell^*} \times 2^{2\ell^*})$ possible such sequences I (an overkill). [Why? The ith term in the product is an upper bound on the number of choices in stage i, there ℓ^* is the number of possible ℓ_i, $\ell^* \times i$ is an upper bound on the number $|\{a_{j,\ell} : j < i, \ell < \ell_j\}|$, $(\ell^* \times i)^{\ell^*}$ is an upper bound to the number of choices of $\langle a_{i,\ell} : \ell < \ell^*, a_{i,\ell} \in \{a_{j,s} : j < i, s < \ell_j\}\rangle$, and $2^{2\ell^*}$ is an upper bound to the number of possible $G \restriction \{a_{i,\ell} : \ell < \ell_i\}$].

Now for some constants c_0, c_1 depending only on ℓ^* (i.e. ψ) this number is $\leq c_0^{n^{\varepsilon(\ell^*)}/\ell^*} \times [(n^{\varepsilon(k^*)}/\ell^*)!]^{\ell^*} \leq n^{\varepsilon(\ell^*)n^{\varepsilon(\ell^*)}}$. For each I the number of possibilities for the $a_{i,\ell}$ is $\leq \prod\limits_i n^{v_i}$, and the probability it holds in G is $\prod\limits_i n^{-\alpha e_i}$, hence the expected value is

$$\prod_i n^{(v_i - \alpha e_i)} \leq \prod_i n^{-\zeta} = n^{-\zeta(n^{\varepsilon(\ell^*)}/\ell^*)}.$$

So the expected number of number of such $\langle a_{i,\ell} : i < n^{\varepsilon(\ell^*)}/\ell^* \text{ and } \ell < \ell_i\rangle$ for some I is $\leq n^{(\varepsilon(\ell^*)-\zeta)n^{\varepsilon(\ell^*)}}$ and as we have $\varepsilon(\ell^*) < \zeta$ the conclusion should be clear.

Probability of $\mathcal{E}_2$. Should be clear by Lemma ?; i.e. except suitably small probability the number of extensions of f to embedding of H_1 is much larger than the number of such extensions failing the requirement in $\otimes^1_{\ell^*}$.

Proof of Theorem 1.1-part (2). In non-trivial cases for some ℓ and pair (H_0, H_1) we have $H_1 \neq H_0$ and $H_1 \subseteq cl_\ell(H_0)$.
Now for n large enough (if $|cl_\ell(H_0)| \ll \log n$),
on $cl_\ell(H_0)$ in G_{n,p_n} we can interpret arithmetic on $cl_\ell(H_0)$ (with parameters) and all subsets and all second place relations. Fix H_0, ℓ.

For a sentence ψ speaking on $\mathbb{N} \restriction k$, (or 2^k) we can compute ψ^* in the vocabulary of graphs saying

(∗) there is a copy H_0' of H_0 such that

$$\mathbb{N} \restriction |cl_\ell(H_0')| \models \psi^*.$$

So for every function $h : \mathbb{N} \to \mathbb{N}$ converging to infinity

$$\text{Lim inf}_n \left(\text{Prob}(G_{n,p_n} \models \psi^*)/n^{-h(n)} \right) \geq 1 \underline{\text{ iff }} \bigvee_k [\mathbb{N} \restriction k \models \psi].$$

But the set $\{\psi : (\exists k)[|N \restriction k| \models \psi]\}$ is like the set of sentences having a finite model (i.e. same Turing degree) so is not recursive.

Concluding Remarks. 1) In fact, we have to consider P_j (in case B during the proof of Theorem 1) only for $j \leq 2^r$, where r is the quantifier depth of the sentence ψ (for which we are proving Theorem 1.1).

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