A COMPLETE BOOLEAN ALGEBRA THAT HAS NO PROPER ATOMLESS COMPLETE SUBLAGEBRA

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Abstract. There exists a complete atomless Boolean algebra that has no proper atomless complete subalgebra.

An atomless complete Boolean algebra B is called *simple* [5] if it has no atomless complete subalgebra A such that $A \neq B$. We prove below that such an algebra exists.

The question whether a simple algebra exists was first raised in [8] where it was proved that B has no proper atomless complete subalgebra if and only if B is *rigid* and *minimal*. For more on this problem, see [4], [5] and [1, p. 664].

Properties of complete Boolean algebras correspond to properties of generic models obtained by forcing with these algebras. (See [6], pp. 266–270; we also follow [6] for notation and terminology of forcing and generic models.) When in [7] McAloon constructed a generic model with all sets ordinally definable he noted that the corresponding complete Boolean algebra is *rigid*, i.e. admitting no nontrivial automorphisms. In [9] Sacks gave a forcing construction of a real number of minimal degree of constructibility. A complete Boolean algebra B that adjoins a minimal

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set (over the ground model) is *minimal* in the following sense:

(1) If A is a complete atomless subalgebra of B then there exists a partition W of 1 such that for every $w \in W$, $A_w = B_w$, where $A_w = \{a \cdot w : a \in A\}$.

In [3], Jensen constructed, by forcing over L, a definable real number of minimal degree. Jensen's construction thus proves that in L there exists rigid minimal complete Boolean algebra. This has been noted in [8] and observed that B is rigid and minimal if and only if it has no proper atomless complete subalgebra. McAloon then asked whether such an algebra can be constructed without the assumption that $V = L$. In [5] simple complete algebras are studied systematically, giving examples (in L) for all possible cardinalities.

In [10] Shelah introduced the (f, g)-bounding property of forcing and in [2] developed a method that modifies Sacks' perfect tree forcing so that while one adjoins a minimal real, there remains enough freedom to control the (f, g)-bounding property. It is this method we use below to prove the following Theorem:

Theorem. *There is a forcing notion $\mathcal{P}$ that adjoins a real number g minimal over V and such that $B(\mathcal{P})$ is rigid.*

Corollary. *There exists a countably generated simple complete Boolean algebra.*

The forcing notion $\mathcal{P}$ consists of finitely branching perfect trees of height ω. In order to control the growth of trees $T \in \mathcal{P}$, we introduce a *master tree* $\mathcal{T}$ such that every $T \in \mathcal{P}$ will be a subtree of $\mathcal{T}$. To define $\mathcal{T}$, we use the following fast growing sequences of integers $(P_k)_{k=0}^{\infty}$ and $(N_k)_{k=0}^{\infty}$:

(2) $$P_0 = N_0 = 1, \quad P_k = N_0 \cdot \ldots \cdot N_{k-1}, \quad N_k = 2^{P_k}$$

(Hence $N_k = 1, 2, 4, 256, 2^{2^{11}}, \ldots$).

Definition. The *master tree* $\mathcal{T}$ and the *index function* ind:

(3)(i) $\mathcal{T} \subset [\omega]^{<\omega}$,

(ii) ind is a one-to-one function of $\mathcal{T}$ onto ω,

(iii) ind $(< >) = 0$,

(iv) if $s, t \in \mathcal{T}$ and length(s) < length(t) then $\text{ind}(s) < \text{ind}(t)$,

(v) if $s, t \in \mathcal{T}$, length(s) = length(t) and $s <_{lex} t$ then $\text{ind}(s) < \text{ind}(t)$,

(vi) if $s \in \mathcal{T}$ and $\text{ind}(s) = k$ then s has exactly N_k successors in $\mathcal{T}$, namely all $s^\frown i$, $i = 0, \ldots, N_k - 1$.

The forcing notion $\mathcal{P}$ is defined as follows:

Definition. $\mathcal{P}$ is the set of all subtrees T of $\mathcal{T}$ that satisfy the following:

(4) for every $s \in T$ and every m there exists some $t \in T$, $t \supset s$, such that t has at least ${P_{\text{ind}(t)}}^m$ successors in T.

(We remark that $\mathcal{T} \in \mathcal{P}$ because for every m there is a K such that for all $k \geq K$, ${P_k}^m \leq 2^{P_k} = N_k$.)

When we need to verify that some T is in $\mathcal{P}$ we find it convenient to replace (4) by an equivalent property:

Lemma. *A tree $T \subseteq \mathcal{T}$ satisfies* (4) *if and only if*

(5)(i) *every $s \in T$ has at least one successor in T,*

(ii) *for every n, if ind$(s) = n$ and $s \in T$ then there exists a k such that if ind$(t) = k$ then $t \in T$, $t \supset s$ and t has at least ${P_k}^n$ successors in T.*

Proof. To see that (5) is sufficient, let $s \in T$ and let m be arbitrary. Find some $\bar{s} \in T$ such that $\bar{s} \supset s$ and $\text{ind}(\bar{s}) \geq m$, and apply (5ii). □

The forcing notion $\mathcal{P}$ is partially ordered by inclusion. A standard forcing argument shows that if G is a generic subset of $\mathcal{P}$ then $V[G] = V[g]$ where g is the *generic branch*, i.e. the unique function $g : \omega \to \omega$ whose initial segments belong to all $T \in G$. We shall prove that the generic branch is minimal over V, and that the complete Boolean algebra $B(\mathcal{P})$ admits no nontrivial automorphisms.

First we introduce some notation needed in the proof:

(6) For every k, s_k is the unique $s \in \mathcal{T}$ such that $\text{ind}(s) = k$.

(7) If T is a tree then $s \in \mathrm{trunk}(T)$ if for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$.

(8) If T is a tree and $a \in T$ then $(T)_a = \{s \in T : s \subseteq a \text{ or } a \subseteq s\}$.

Note that if $T \in \mathcal{P}$ and $a \in T$ then $(T)_a \in \mathcal{P}$. We shall use repeatedly the following technique:

Lemma. *Let $T \in \mathcal{P}$ and, let l be an integer and let $U = T \cap \omega^l$ (the l^{th} level of T). Let $\dot{x}$ be a name for some set in V. For each $a \in U$ let $T_a \subseteq (T)_a$ and x_a be such that $T_a \in \mathcal{P}$ and $T_a \Vdash \dot{x} = x_a$.*

Then $T' = \bigcup\{T_a : a \in U\}$ is in $\mathcal{P}$, $T' \subseteq T$, $T' \cap \omega^l = T \cap \omega^l = U$, and $T' \Vdash \dot{x} \in \{x_a : a \in U\}$. □

We shall combine this with *fusion*, in the form stated below:

Lemma. *Let $(T_n)_{n=0}^\infty$ and $(l_n)_{n=0}^\infty$ be such that each T_n is in $\mathcal{P}$, $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$, $l_0 < l_1 < \cdots < l_n < \ldots$, $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n}$, and such that*

(9) *for every n, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that t has at least ${P_{\mathrm{ind}(t)}}^n$ successors in T_{n+1}.*

Then $T = \bigcap_{n=0}^\infty T_n \in \mathcal{P}$.

Proof. To see that T satisfies (5), note that if $s_n \in T$ then $s_n \in T_n$, and the node t found by (9) belongs to T. □

We shall now prove that the generic branch is minimal over V:

Lemma. *If $X \in V[G]$ is a set of ordinals, then either $X \in V$ or $g \in V[X]$.*

Proof. The proof is very much like the proof for Sacks' forcing. Let $\dot{X}$ be a name for X and let $T_0 \in \mathcal{P}$ force that $\dot{X}$ is not in the ground model. Hence for every $T \leq T_0$ there exist T', $T'' \leq T$ and an ordinal α such that $T' \Vdash \alpha \in \dot{X}$ and $T'' \Vdash \alpha \notin \dot{X}$. Consequently, for any $T_1 \leq T$ and $T_2 \leq T$ there exist $T_1' \leq T_1$ and $T_2' \leq T_2$ and an α such that both T_1' and T_2' decide "$\alpha \in \dot{X}$" and $T_1' \Vdash \alpha \in \dot{X}$ if and only if $T_2' \Vdash \alpha \notin \dot{X}$.

Inductively, we construct $(T_n)_{n=0}^{\infty}$, $(l_n)_{n=0}^{\infty}$, $U_n = T_n \cap \omega^{l_n}$, and ordinals $\alpha(a,b)$ for all $a, b \in U_n$, $a \neq b$, such that

(10)(i) $T_n \in \mathcal{P}$ and $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$,

(ii) $l_0 < l_1 < \cdots < l_n < \cdots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) for every n, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that t has at least ${P_{\mathrm{ind}(t)}}^n$ successors in T_{n+1},

(v) for every n, for all $a, b \in U_n$, if $a \neq b$ then both $(T_n)_a$ and $(T_n)_b$ decide "$\alpha(a,b) \in \dot{X}$" and $(T_n)_a \Vdash \alpha(a,b) \in \dot{X}$ if and only if $(T_n)_b \Vdash \alpha(a,b) \in \dot{X}$.

When such a sequence has been constructed, we let $T = \bigcap_{n=0}^{\infty} T_n$. As (9) is satisfied, we have $T \in \mathcal{P}$ and $T \leq T_0$. If G is a generic such that $T \in G$ and if X is the G-interpretation of $\dot{X}$ then the generic branch g is in $V[X]$: for every n, $g \restriction l_n$ is the unique $a \in U_n$ with the property that for every $b \in U_n$, $b \neq a$, $\alpha(a,b) \in X$ if and only if $(T)_a \Vdash \alpha(a,b) \in \dot{X}$.

To construct $(T_n)_{n=0}^{\infty}$, $(l_n)_{n=0}^{\infty}$ and $\alpha(a,b)$, we let $l_0 = 0$ (hence $U_0 = \{s_0\}$) and proceed by induction. Having constructed T_n and l_n, we first find $l_{n+1} > l_n$ as follows: If $s_n \in T_n$, we find $t \in T_n$, $t \supset s_n$, such that t has at least ${P_{\mathrm{ind}(t)}}^n$ successors in T_n. Let $l_{n+1} = \text{length}\ (t) + 1$. (If $s_n \notin T_n$, let $l_{n+1} = l_n + 1$.) Let $U_{n+1} = T_n \cap \omega^{l_{n+1}}$.

Next we consider, in succession, all pairs $\{a, b\}$ of district elements of U_{n+1}, eventually constructing conditions T_a, $a \in U_{n+1}$, and ordinals $\alpha(a,b)$, $a, b \in U_{n+1}$, such that for all a, $T_a \leq (T_n)_a$ and if $a \neq b$ then either $T_a \Vdash \alpha(a,b) \in \dot{X}$ and $T_b \Vdash \alpha(a,b) \notin \dot{X}$, or $T_a \Vdash \alpha(a,b) \notin \dot{X}$ and $T_b \Vdash \alpha(a,b) \in \dot{X}$. Finally, we let $T_{n+1} = \bigcup\{T_a : a \in U_{n+1}\}$.

It follows that $(T_n)_{n=0}^{\infty}$, $(l_n)_{n=0}^{\infty}$ and $\alpha(a,b)$ satisfy (10). □

Let B be the complete Boolean algebra $B(\mathcal{P})$. We shall prove that B is rigid. Toward a contradiction, assume that there exists an automorphism π of B that is not the identity. First, there is some $u \in B$ such that $\pi(u) \cdot u = 0$. Let $p \in \mathcal{P}$ be such that $p \leq u$ and let $q \in \mathcal{P}$ be such that $q \leq \pi(p)$. Since $q \not\leq p$, there is some

$s \in q$ such that $s \notin p$. Let $T_0 = (q)_s$.

Note that for all $t \in T_0$, if $t \supseteq s$ then $t \notin p$. Let

$$A = \{\mathrm{ind}(t) : t \in p\},$$

and consider the following property $\varphi(x)$ (with parameters in V):

(11)
$\varphi(x) \leftrightarrow$ if x is a function from A into ω

such that $x(k) < N_k$ for all k, then there exists

a function u on A in the ground model V such that the values of u are finite sets of

integers and for every $k \in A$, $u(k) \subseteq \{0, ..., N_k - 1\}$ and $|u(k)| \leq P_k$,

and $x(k) \in u(k)$.

We will show that

$$p \Vdash \exists x \neg\varphi(x), \tag{12}$$

and

$$\text{there exists a } T \leq T_0 \text{ such that } T \Vdash \forall x \varphi(x). \tag{13}$$

This will yield a contradiction: the Boolean value of the sentence $\exists x \neg\varphi(x)$ is preserved by π, and so

$$T_0 \leq q \leq \pi(p) \leq \pi(\|\exists x \neg\varphi(x)\|) = \|\exists x \neg\varphi(x)\|,$$

contradicting (13).

In order to prove (12), consider the following (name for a) function $\dot{x} : A \to \omega$. For every $k \in A$, let

$$\dot{x}(k) = \dot{g}(\text{length}\,(s_k) + 1) \text{ if } s_k \subset \dot{g}, \text{ and } \dot{x}(k) = 0 \quad \text{otherwise.}$$

Now if $p_1 < p$ and $u \in V$ is a function on A such that $u(k) \subseteq \{0, ..., N_k - 1\}$ and $|u(k)| \leq P_k$ then there exist a $p_2 < p_1$ and some $k \in A$ such that $s_k \in p_2$ has at least P_k^2 successors, and there exist in turn a $p_3 < p_2$ and some $i \notin u(k)$ such that $s_k \frown i \in \text{trunk}(p_3)$. Clearly, $p_3 \Vdash \dot{x}(k) \notin u(k)$.

Property (13) will follow from this lemma:

Lemma. *Let $T_1 \le T_0$ and $\dot{x}$ be such that T_1 forces that $\dot{x}$ is function from A into ω such that $x(k) < N_k$ for all $k \in A$. There exist sequences $(T_n)_{n=1}^{\infty}$, $(l_n)_{n=1}^{\infty}$, $(j_n)_{n=1}^{\infty}$, $(U_n)_{n=1}^{\infty}$ and sets z_a, $a \in U_n$, such that*

(14)(i) $T_n \in \mathcal{P}$ *and* $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n \supseteq \ldots$,

(ii) $l_1 < l_2 < \cdots < l_n < \ldots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) *for every n, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that t has at least ${P_{\mathrm{ind}(t)}}^n$ successors in T_{n+1},*

(v) $j_1 < j_2 < \cdots < j_n < \ldots$,

(vi) *for every $a \in U_n$, $(T_n)_a \Vdash \langle \dot{x}(k) : k \in A \cap j_n \rangle = z_a$,*

(vii) *for every $k \in A$, if $k \ge j_n$ then $|U_n| < P_k$,*

(viii) *for every $k \in A$, if $k < j_n$ then $|\{z_a(k) : a \in U_n\}| \le P_k$.*

Granted this lemma, (13) will follow: If we let $T = \bigcap_{n=1}^{\infty} T_n$, then $T \in \mathcal{P}$ and $T \le T_1$ and for every $k \in A$, $T \Vdash \dot{x}(k) \in u(k)$ where $u(k) = \{z_a(k) : a \in U_n\}$ (for any and all $n > k$).

Proof of Lemma. We let $l_1 = j_1 = \mathrm{length}(s)$, $U_1 = \{s\}$ and strengthen T_1 if necessary so that T_1 decides $\langle \dot{x}(k) : k \in A \cap j_1 \rangle$, and let z_s be the decided value. We also assume that length$(s) \ge 2$ so that $|U_1| = 1 < P_k$ for every $k \in A$, $k \ge j_1$. Then we proceed by induction.

Having constructed T_n, l_n, j_n etc., we first find $l_{n+1} > l_n$ and $j_{n+1} > j_n$ as follows: If $s_n \notin T_n$ (Case I), we let $l_{n+1} = l_n + 1$ and $j_{n+1} = j_n + 1$. Thus assume that $s_n \in T_n$ (Case II).

Since length$(s_n) \le n \le l_n$, we choose some $v_n \in U_n$ such that $s_n \subseteq v_n$. By (4) there exists some $t \in T_n$, $t \supset v_n$, so that if ind$(t) = m$ then t has at least ${P_m}^{n+1}$ successors in T_n. Moreover we choose t so that $m = \mathrm{ind}(t)$ is big enough so that there is at least one $k \in A$ such that $j_n \le k < m$. We let $l_{n+1} = \mathrm{length}(t) + 1$ and $j_{n+1} = m = \mathrm{ind}(t)$.

Next we construct $U_{n+1}, \{z_a : a \in U_{n+1}\}$ and T_{n+1}. In Case I, we choose for each $u \in U_n$ some successor $a(u)$ of u and let $U_{n+1} = \{a(u) : u \in U_n\}$. For every

$a \in U_{n+1}$ we find some $T_a \subseteq (T_n)_a$ and z_a so that $T_a \Vdash \langle \dot{x}(k) : k \in A \cap j_{n+1} \rangle = z_a$, and let $T_{n+1} = \bigcup \{T_a : a \in U_{n+1}\}$. In this case $|U_{n+1}| = |U_n|$ and so (vii) holds for $n+1$ as well, while (viii) for $n+1$ follow either from (viii) or from (vii) for n (the latter if $j_n \in A$).

Thus consider Case II. For each $u \in U_n$ other than v_n we choose some $a(u) \in T_n$ of length l_{n+1} such that $a(u) \supset u$, and find some $T_{a(u)} \subseteq (T_n)_{a(u)}$ and $z_{a(u)}$ so that $T_{a(u)} \Vdash \langle \dot{x}(k) : k \in A \cap m \rangle = z_{a(u)}$.

Let S be the set of all successors of t (which has been chosen so that $|S| \geq {P_m}^{n+1}$ where $m = \operatorname{ind}(t)$); every $a \in S$ has length l_{n+1}. For each $a \in S$ we choose $T_a \subseteq (T_n)_a$ and z_a, so that $T_a \Vdash \langle \dot{x}(k) : k \in A \cap m \rangle = z_a$. If we denote $K = \max(A \cap m)$ then we have

$$|\{z_a : a \in S\}| \leq \prod_{i \in A \cap m} N_i \leq \prod_{i=0}^{K} N_i = P_{K+1} \leq P_m,$$

while $|S| \geq {P_m}^{n+1}$. Therefore there exists a set $U \subset S$ of size ${P_m}^n$ such that for every $a \in U$ the set z_a is the same. Therefore if we let

$$U_{n+1} = U \cup \{a(u) : u \in U_n - \{v_n\}\},$$

and $T_{n+1} = \bigcup \{T_a : a \in U_{n+1}\}$, T_{n+1} satisfies property (iv). It remains to verify that (vii) and (viii) hold.

To verify (vii), let $k \in A$ be such that $k \geq j_{n+1} = m$. Since $m = \operatorname{ind}(t)$, we have $m \notin A$ and so $k > m$. Let $K \in A$ be such that $j_n \leq K < m$. Since $|U_n| < P_K$, we have

$$|U_{n+1}| < |U_n| + |U| < P_K + N_m < P_m \cdot N_m = P_{m+1} \leq P_k.$$

To verify (viii), it suffices to consider only those $k \in A$ such that $j_n \leq k < m$. But then $|U_n| < P_k$ and we have

$$|\{z_a(k) : a \in U_{n+1}\}| \leq |\{z_a : a \in U_{n+1}\}| \leq |U_n| + 1 \leq P_k.$$

□

References

[1] M. Bekkali and R. Bonnet, *Rigid Boolean algebras*, in "Handbook of Boolean algebras" (J. D. Monk, ed.), vol. 2, pp. 637–678.

[2] M. Goldstern and S. Shelah, *Many simple cardinal invariants*, Arch. Math. Logic **32** (1993), 203–221, (Publ. no. 448).

[3] R. B. Jensen, *Definable sets of minimal degree*, in "Mathematical Logic and Foundations of Set Theory" (Y.Bar-Hillel, ed.), pp. 122–128, North-Holland Publ. Co., 1970.

[4] T. Jech, *A propos d'algèbres de Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **274** (1972), 371–372.

[5] T. Jech, *Simple complete Boolean algebras*, Israel J. Math. **18** (1974), 1–10.

[6] T. Jech, *Set Theory*, Academic Press, New York 1978.

[7] K. McAloon, *Consistency results about ordinal definability*, Ann. Math. Logic **2** (1971), 449–467.

[8] K. McAloon, *Les algèbres de Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **272** (1971), 89–91.

[9] G. Sacks, *Forcing with perfect closed sets*, in "Axiomatic Set Theory" (D. Scott, ed.), Proc. Symp. Pure Math. 13, I, pp. 331–355, American Math. Society 1971.

[10] S. Shelah, *Vive la différence I: nonisomorphism of ultrapowers of countable models*, in "Set Theory of the Continuum" (H. Judah et al., eds.) MSRI vol. 26, pp.357–405, Springer-Verlag 1992.

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