COLOURING AND NON-PRODUCTIVITY OF $\aleph_2$-C.C.
SH572

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Abstract. We prove that colouring of pairs from $\aleph_2$ with strong properties exists. The easiest to state (and quite a well known problem) it solves: there are two topological spaces with cellularity $\aleph_1$ whose product has cellularity $\aleph_2$; equivalently we can speak on cellularity of Boolean algebras or on Boolean algebras satisfying the $\aleph_2$-c.c. whose product fails the $\aleph_2$-c.c. We also deal more with guessing of clubs.

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Annotated content

§1 Retry at $\aleph_2$-c.c. not productive

[We prove $Pr_1(\aleph_1, \aleph_2, \aleph_2, \aleph_0)$ which is a much stronger result].

§2 The implicit properties

[We define a property implicit in §1, note what the proof in §1 gives, and look at related implication for successor of singular non-strong limit and show that Pr_1 implies Pr_6].

§3 Guessing clubs revisited

[We improve some results mainly from [Sh 413], giving complete proofs. We show that for μ regular uncountable and $\chi < \mu$ we can find $\langle C_\delta : \delta < \mu^+, \mathrm{cf}(\delta) = \mu \rangle$ and functions h_δ, from C_δ onto χ, such that for every club E of μ^+ for stationarily many $\delta < \mu^+$ we have: $\mathrm{cf}(\delta) = \mu$ and for every $\gamma < \chi$ for arbitrarily large $\alpha \in \mathrm{nacc}(C_\delta)$ we have $\alpha \in E, h_\delta(\alpha) = \gamma$. Also if $C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \mu\}$, ($\alpha_{\delta,\varepsilon}$ increasing continuous in ε) we can demand $\{\varepsilon < \mu : \alpha_{\delta,\varepsilon+1} \in E$ (and $\alpha_{\delta,\varepsilon} \in E)\}$ is a stationary subset of μ. In fact for each $\gamma < \mu$ the set $\{\varepsilon < \mu : \alpha_{\delta,\varepsilon+1} \in E, \alpha_{\delta,\varepsilon} \in E$ and $f(\alpha_{\delta,\varepsilon+1}) = \gamma\}$ is a stationary subset of μ. We also deal with a parallel to the last one (without f) to successor of singulars and to inaccessibles.]

§4 More on Pr_1

[We prove that $\mathrm{Pr}_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ holds for regular λ].

On history, references and consequences see [Sh:g, AP1] and [Sh:g, III,§0].

§1 Retry at $\aleph_2$-c.c. not productive

1.1 Theorem. $Pr_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$.

1.2 Remark. 1) Is this hard? Apostriory it does not look so, but we have worked hard on it several times without success (worse: produce several false proofs). We thank Juhasz and Soukup for pointing out a gap.
2) Remember that
Definition $Pr_1(\lambda, \mu, \theta, \sigma)$ means that there is a symmetric two-place function d from λ to θ such that:
if $\langle u_\alpha : \alpha < \mu \rangle$ satisfies

$$u_\alpha \subseteq \lambda,$$

$$|u_\alpha| < \sigma,$$

$$\alpha < \beta \Rightarrow u_\alpha \cap u_\beta = \emptyset,$$

and $\gamma < \theta$ then for some $\alpha < \beta$ we have

$$\zeta \in u_\alpha \ \ \& \ \ \xi \in u_\alpha \Rightarrow d(\zeta, \xi) = \gamma.$$

3) If we are content with proving that there is a colouring with $\aleph_1$ colours, then we can simplify somewhat: in stage C we let $c(\beta, \alpha) = d_{\rm sq}(\rho_{h_1}(\beta, \alpha))$ and this shortens stage D.

Proof.

Stage A: First we define a preliminary colouring.
There is a function $d_{sq} : {}^{\omega>}(\omega_1) \rightarrow \omega_1$ such that:

- $\bigotimes$ if $A \in [\omega_1]^{\aleph_1}$ and $\langle (\rho_\alpha, \nu_\alpha) : \alpha \in A \rangle$ is such that $\rho_\alpha \in {}^{\omega>}\omega_1, \nu_\alpha \in {}^{\omega>}\omega_1$, $\alpha \in$ Rang$(\rho_\alpha) \cap$ Rang(ν_α) and $\gamma < \omega_1$ then for some $\zeta < \xi$ from A we have: if ν', ρ' are subsequences of ν_ζ, ρ_ξ respectively and $\zeta \in$ Rang$(\nu'), \xi \in$ Rang(ρ') then

$$d_{sq}(\nu' \char`\^ \rho') = \gamma.$$

Proof of $\bigotimes$. Choose pairwise distinct $\eta_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$. Let $d_0 : [\omega_1]^2 \rightarrow \omega_1$ be such that:

- $(*)$ if $n < \omega$ and $\alpha_{\zeta,\ell} < \omega_1$ for $\zeta < \omega_1, \ell < n$ are pairwise distinct and $\gamma < \omega_1$ then for some $\zeta < \xi < \omega_1$ we have $\ell < n \Rightarrow \gamma = d_0(\{\alpha_{\zeta,\ell}, \alpha_{\xi,\ell}\})$ (exists by [Sh 261, see (2.4),p.176] the n there is 2).

Define $d_{sq}(\nu)$ for $\nu \in {}^{\omega>}(\omega_1)$ as follows. If $\ell g(\nu) \le 1$ or ν is constant then $d_{sq}(\nu)$ is 0. Otherwise let
$n(\nu) =: \max\{\ell g(\eta_{\nu(\ell)} \cap \eta_{\nu(k)}) : \ell < k < \ell g(\nu) \text{ and } \nu(\ell) \ne \nu(k)\} < \omega$.
The maximum is on a non-empty set as $\ell g(\nu) \ge 2$ and ν is not constant, remember $\eta_\alpha \in {}^\omega 2$ were pairwise distinct so $\nu(\ell) \ne \nu(k) \Rightarrow \eta_{\nu(\ell)} \cap \eta_{\nu(k)} \in {}^{\omega>}2$ (is the largest common initial segment of $\eta_{\nu(\ell)}, \eta_{\nu(k)}$). Let $a(\nu) = \{(\ell, k) : \ell < k < \ell g(\nu)$ and $\ell g(\eta_{\nu(\ell)} \cap \eta_{\nu(k)}) = n(\nu)\}$ so $a(\nu)$ is non-empty and choose the (lexicographically) minimal pair (ℓ_ν, k_ν) in it. Lastly let

$$d_{sq}(\nu) = d_0(\{\nu(\ell_\nu), \nu(k_\nu)\}).$$

So d_{sq} is a function with the right domain and range. Now suppose we are given $A \in [\omega_1]^{\aleph_1}$, $\gamma < \omega_1$ and $\rho_\alpha, \nu_\alpha \in {}^{\omega>}(\omega_1)$ for $\alpha \in A$ such that
$\alpha \in \operatorname{Rang}(\rho_\alpha) \cap \operatorname{Rang}(\nu_\alpha)$. We should find $\alpha < \beta$ from A such that $d_{sq}(\nu' \hat{\ } \rho') = \gamma$ for any subsequences ν', ρ' subsequences of ν_α, ρ_β respectively such that $\alpha \in \operatorname{Rang}(\nu')$ and $\beta \in \operatorname{Rang}(\rho')$.

For each $\alpha \in A$ we can find $m_\alpha < \omega$ such that:

$(*)_0$ if $\ell < k < \ell g(\nu_\alpha \hat{\ } \rho_\alpha)$ and $(\nu_\alpha \hat{\ } \rho_\alpha)(\ell) \ne (\nu_\alpha \hat{\ } \rho_\alpha)(k)$ then $\eta_{(\nu_\alpha \hat{\ } \rho_\alpha)(\ell)} \restriction m_\alpha \ne \eta_{(\nu_\alpha \hat{\ } \rho_\alpha)(k)} \restriction m_\alpha$.

Next we can find $B \in [A]^{\aleph_1}$ such that for all $\alpha \in B$ (the point is that the values do not depend on α) we have:

(a) $\ell g(\nu_\alpha) = m^0, \ell g(\rho_\alpha) = m^1$,
(b) $a^* = \{(\ell, k) : \ell < k < m^0 + m^1 \text{ and } (\nu_\alpha \hat{\ } \rho_\alpha)(\ell) = (\nu_\alpha \hat{\ } \rho_\alpha)(k)\}$,
(c) $b^* = \{\ell < m^0 + m^1 : \alpha = (\nu_\alpha \hat{\ } \rho_\alpha)(\ell)\}$,
(d) $m_\alpha = m^2$,
(e) $\langle \eta_{(\nu_\alpha \hat{\ } \rho_\alpha)(\ell)} \restriction m_\alpha : \ell < m^0 + m^1 \rangle = \bar{\eta}^*$,
(f) $\langle \operatorname{Rang}(\nu_\alpha \hat{\ } \rho_\alpha) : \alpha \in B \rangle$ is a $\triangle$-system with heart w,
(g) $u^* = \{\ell : (\nu_\alpha \hat{\ } \rho_\alpha)(\ell) \in w\}$ (so $u^* \ne \{\ell : \ell < m^0 + m^1\}$ as $\alpha \in \operatorname{Rang}(\nu_\alpha \hat{\ } \rho_\alpha)$),
(h) $\alpha^*_\ell = (\nu_\alpha \hat{\ } \rho_\alpha)(\ell)$ for $\ell \in u^*$,
(i) if $\alpha < \beta \in B$ then $\sup \operatorname{Rang}(\nu_\alpha \hat{\ } \rho_\alpha) < \beta$.

For $\zeta \in B$ let $\bar{\beta}^\zeta =: \langle (\nu_\zeta \hat{\ } \rho_\zeta)(\ell) : \ell < m^0 + m^1, \ell \notin u^* \rangle$ and apply $(*)$, i.e. the choice of d_0. So for some $\zeta < \xi$ from B, we have

$$\ell < m^0 + m^1 \ \& \ \ell \notin u^* \Rightarrow \gamma = d_0\Big(\{(\nu_\zeta \hat{\ } \rho_\zeta)(\ell), (\nu_\xi \hat{\ } \rho_\xi)(\ell)\}\Big).$$

We shall prove that $\zeta < \xi$ are as required (in $\otimes$). So let ν', ρ' be subsequences of ν_ζ, ρ_ξ (so let $\nu' = \nu_\zeta \restriction v_1$ and $\rho' = \rho_\xi \restriction v_2$) such that $\zeta \in \operatorname{Rang}(\nu'), \xi \in \operatorname{Rang}(\rho')$ and we have to prove $\gamma = d_{sq}(\nu' \hat{\ } \rho')$. Let $\tau = \nu' \hat{\ } \rho'$, so $\tau = (\nu_\zeta \hat{\ } \rho_\xi) \restriction (v_1 \cup (m^0 + v_2))$ (in a slight abuse of notation, we look at τ as a function with domain $v_1 \cup (m^0 + v_2)$ and also as a member of ${}^{\omega>}(\omega_1)$ where $m + v =: \{m + \ell : \ell \in v\}$, of course). By the definition of d_{sq} it is enough to prove the following two things:

$(*)_1$ $n(\nu' \hat{\ } \rho') \ge m^2$ (see clause (d) and $(*)_0$ above),
$(*)_2$ for every $\ell_1, \ell_2 \in v_1 \cup (m^0 + v_2)$ we have $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m^2, \omega) \Rightarrow \gamma = d_0(\{\tau(\ell_1), \tau(\ell_2)\})$.

Proof of $(*)_1$. Let $\ell_1 \in v_1$ and $\ell_2 \in v_2$ be such that $\nu_\zeta(\ell_1) = \zeta$ and $\rho_\xi(\ell_2) = \xi$.

So clearly $\ell_1, m^0 + \ell_2 \in b^*$ (see clause (c)) and
$\eta_{\rho_\xi(\ell_2)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2 = \eta_{\nu_\zeta(\ell_1)} \restriction m^2$ (first equality as $\zeta, \xi \in B$ and $m_\zeta = m_\xi = m^2$ (see clause (d) and (e)), second equality as $\eta_{\rho_\zeta(\ell_2)} = \eta_{\nu_\zeta(\ell_1)}$ since $\ell_1, m^0 + \ell_2 \in b^*$ (see clause (c)). But $\rho_\xi(\ell_2) = \xi \neq \zeta = \nu_\zeta(\ell_1)$, hence $\eta_{\rho_\xi(\ell_2)} \neq \eta_{\nu_\zeta(\ell_1)}$, so together with the previous sentence we have

$$m^2 \leq \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(m^0+\ell_2)}) < \omega.$$

Hence $n(\tau) \geq m^2$ as required in $(*)_1$.

Proof of $(*)_2$. If $\ell_1 < \ell_2$ are from v_1, by the choice of $m^2 = m_\zeta$ it is easy. Namely, if $(\ell_1, \ell_2) \in a(\tau)$ then $(\ell_1, \ell_2) \in a(\nu_\zeta)$ and $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) = \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\nu_\zeta(\ell_2)}) < m_\zeta = m^2$. If $\ell_1, \ell_2 \in m^0 + v^2$, by the choice of $m^2 = m_\xi$ similarly it is easy to show $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) < m^2$. So it is enough to prove

$(*)_3$ assume $\ell_1 \in v_1, \ell_2 \in v_2$ and
$\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) \in [m^2, \omega)$ then
$\gamma = d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\})$.

Now the third assumption in $(*)_3$ means $\eta_{\nu_\zeta(\ell_1)} \restriction m^2 = \eta_{\rho_\xi(\ell_2)} \restriction m^2$ and as $\zeta, \xi \in B$ we know that $\eta_{\rho_\xi(\ell_2)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2$. Together we know that $\eta_{\nu_\zeta(\ell_1)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2$, hence by the choice of $m_\zeta = m^2$ necessarily $\eta_{\nu_\zeta(\ell_1)} = \eta_{\rho_\zeta(\ell_2)}$ so that $\nu_\zeta(\ell_1) = \rho_\zeta(\ell_2)$ and (see clause (b)) also $\nu_\xi(\ell_1) = \rho_\xi(\ell_2)$. So

$$d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\}) = d_0(\{\nu_\zeta(\ell_1), \nu_\xi(\ell_1)\}).$$

The latter is the required γ provided that $\ell_1 \notin u^*$. Equivalently $\nu_\zeta(\ell_1) \neq \nu_\xi(\ell_1)$ but otherwise also $\nu_\zeta(\ell_1) = \rho_\xi(\ell_2)$ so $\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \omega$, contradicting the assumption of $(*)_3$ that $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m^2, \omega)$ (so it is not equal to ω).
So we finish[1] proving $(*)_2$, hence $\otimes$.

<u>Stage B</u>: Like Stage A of [Sh:g, III,4.4,p.164]'s proof. (So for $\alpha < \beta < \omega_2$, α does not appear in $\rho(\beta, \alpha)$).

<u>Stage C</u>: Defining the colouring:

Remember that $\mathcal{S}^\alpha_\beta = \{\delta < \aleph_\alpha : \mathrm{cf}(\delta) = \aleph_\beta\}$.

For $\ell = 1, 2$ choose $h_\ell : \omega_2 \to \omega_\ell$ such that $S^\ell_\alpha = \mathcal{S}^2_1 \cap h^{-1}_\ell(\{\alpha\})$ is stationary for each $\alpha < \omega_\ell$. For $\alpha < \omega_2$, let $A_\alpha \subseteq \omega_1$ be such that no one is included in the union of finitely many others.

For $\alpha < \beta < \omega_2$, let $\ell = \ell_{\beta,\alpha}$ be minimal such that

$$d_{sq}\left(\rho_{h_1}(\beta, \alpha)\right) \in A_{\rho(\beta,\alpha)(\ell)}$$

and lastly let

[1] see alternatively 2.2(1) + 4.1

$$c(\beta,\alpha)=c(\alpha,\beta)=: h_2\Big((\rho(\beta,\alpha))(\ell_{\beta,\alpha})\Big).$$

Stage D: Proving that the colouring works:

So assume $n<\omega$, $\langle u_\alpha:\alpha<\omega_2\rangle$ is a sequence of pairwise disjoint subsets of ω_2 of size n and $\gamma(*)<\omega_2$ and we should find $\alpha<\beta$ such that $c\restriction(u_\alpha\times u_\beta)$ is constantly $\gamma(*)$. Without loss of generality $\alpha<\beta\Rightarrow\ \max(u_\alpha)<\ \min(u_\beta)$ and $\min(u_\alpha)>\alpha$ and let $E=\{\delta:\delta$ a limit ordinal $<\omega_2$ and $(\forall\alpha)(\alpha<\delta\Rightarrow u_\alpha\subseteq\delta)\}$. Clearly E is a club of ω_2. For each $\delta\in E\cap\mathcal{S}^2_1$, there is $\alpha^*_\delta<\delta$ such that

$$\alpha\in[\alpha^*_\delta,\delta)\ \ \&\ \ \beta\in u_\delta\Rightarrow\rho(\beta,\delta)\hat{\ }\langle\delta\rangle\trianglelefteq\rho(\beta,\alpha).$$

Also for $\delta\in\mathcal{S}^2_1$ let

$$\begin{aligned}\varepsilon_\delta=:\ \mathrm{Min}\Big\{\varepsilon<\omega_1:&\zeta\in A_\delta\text{ but if }\alpha\in\bigcup_{\beta\in u_\delta}\mathrm{Rang}(\rho(\beta,\delta))\\ &(\text{so }\alpha>\delta)\text{ then }\varepsilon\notin A_\alpha\Big\}.\end{aligned}$$

Note that $\varepsilon_\delta<\omega_1$ is well defined by the choice of A_α's. So, by Fodor's lemma, for some $\zeta^*<\omega_1$ and $\alpha^*<\omega_2$ we have that

$$W=:\{\delta\in S^2_{\gamma(*)}:\alpha^*_\delta=\alpha^*\text{ and }\varepsilon_\delta=\varepsilon^*\}$$

is stationary. Let h be a strictly increasing function from ω_2 into W such that $\alpha^*<h(\delta)$. By the demand on α^* (and W)

$$\bigoplus_0\qquad \alpha^*<\alpha<\delta\in W\ \ \&\ \ \beta\in u_\delta\Rightarrow\rho(\beta,\delta)\hat{\ }\langle\delta\rangle\trianglelefteq\rho(\beta,\alpha).$$

Hence

$$\begin{aligned}\bigoplus_1\qquad \alpha^*<\alpha<\delta\in\mathcal{S}^2_1\ \ \&\ \ \beta\in u_{h(\delta)}\Rightarrow\ &\mathrm{Min}\{\ell:\varepsilon^*\in A_{\rho(\beta,\alpha)(\ell)}\}=\\ &\mathrm{Min}\{\ell:\rho(\beta,\delta)(\ell)=h(\delta)\},\end{aligned}$$

hence

$$\begin{aligned}\bigoplus_2\qquad \alpha^*<\alpha<\delta\in\mathcal{S}^2_1\ \ \&\ \beta\in u_{h(\delta)}\Rightarrow&\\ h_2\Big(\rho(\beta,\delta)\Big[\mathrm{Min}\{\ell:\varepsilon^*\in A_{\rho(\beta,\delta)(\ell)}\}\Big]\Big)&=\gamma(*).\end{aligned}$$

Let

$$E_0 =: \Big\{\delta < \omega_2 : \delta \text{ a limit ordinal, } \delta \in E \text{ and}$$
$$\alpha < \delta \Rightarrow h(\alpha) < \delta \text{ (hence } \sup(u_{h(\alpha)}) < \delta)\Big\}.$$

For each $\delta \in \mathcal{S}^2_1$ there is $\alpha^{**}_\delta < \delta$ such that $\alpha^{**}_\delta > \alpha^*$ and

$$\alpha \in [\alpha^{**}_\delta, \delta) \;\&\; \beta \in u_{h(\delta)} \Rightarrow \rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha).$$

For each $\gamma < \omega_1, \delta \mapsto \alpha^{**}_\delta$ is a regressive function on S^1_γ, hence for some $\alpha^{**}(\gamma) < \delta$ the set $S'_\gamma =: \{\delta \in S^1_\gamma \cap E_0 : \alpha^{**}_\delta = \alpha^{**}(\gamma)\}$ is stationary.

Let $\alpha^{**} = \sup\{\alpha^{**}(\gamma) + 1 : \gamma < \omega_1\}$ and note that $\alpha^{**} < \omega_2$. Let

$$E_1 =: \{\delta < \omega_2 : \text{for every } \gamma < \omega_1, \delta = \sup(S'_\gamma \cap \delta) \text{ and } \delta > \alpha^{**}\},$$

and note that E_1 is a club of $\aleph_2$ (and as $S'_\gamma \subseteq E_0$ clearly $E_1 \subseteq E_0$) and choose $\delta^* \in E_1 \cap S^2_{\gamma(*)}$. Then by induction on $i < \omega_1$ choose an ordinal ζ_i such that $\langle \zeta_i : i < \omega_1\rangle$ is strictly increasing with limit δ^* and $\zeta_i \in S'_i \backslash (\alpha^{**}+1)$. We know that $\alpha < \zeta_i \Rightarrow u_\alpha \subseteq \zeta_i$ and $\alpha < \min(u_\alpha)$, hence for every $\alpha_i < \zeta_i$ large enough $(\forall \beta \in u_{\alpha_i})(\rho(\delta^*,\zeta_i)\hat{\ }(\zeta_i) \trianglelefteq \rho(\delta^*,\beta))$.

Choose such $\alpha_i \in (\bigcup\limits_{j<i} \zeta_j, \zeta_i)$. Lastly for $i < \omega_1$ choose $\beta_i \in E \cap S'_i$ with $\beta_i > \delta^*$. Now for each $i < \omega_1$ for some $\xi(i) < \delta^*$,

$$\bigoplus\nolimits_3 \qquad \alpha \in (\xi(i), \delta^*) \;\&\; \beta \in u_{h(\beta_i)} \Rightarrow \rho(\beta,\delta^*)\hat{\ }\langle\delta^*\rangle \trianglelefteq \rho(\beta,\alpha).$$

As $\delta^* = \bigcup\limits_{i<\omega_1} \zeta_i$, without loss of generality $\xi(i) = \zeta_{j(i)}$, and $j(i)$ is (strictly) increasing with i and let $A =: \{\varepsilon < \omega_1 : \varepsilon \text{ a limit ordinal and } (\forall i < \varepsilon)(j(i) < \varepsilon)\}$. Clearly A is a club of ω_1. Now putting all of this together we have:

- $(*)_1$ if $i(0) < i(1)$ are in $A, \alpha \in u_{\alpha_{i(1)}}, \beta \in u_{h(\beta_{i(0)})}$ then $\rho(\beta,\alpha) = \rho(\beta,\delta^*)\hat{\ }\rho(\delta^*,\alpha)$.
  [Why? As $j(i(0)) < i(1)$, see $\bigoplus_3$].
- $(*)_2$ if $i < \omega_1$ then $\beta \in u_{h(\beta_i)} \Rightarrow i \in \text{Rang}(\rho_{h_1}(\beta,\delta^*))$ (witnessed by β_i which belongs to this set by $\bigoplus_1$).
- $(*)_3$ if $i < \omega_1$ then $\alpha \in u_{\alpha_i} \Rightarrow i \in \text{Rang}(\rho_{h_1}(\delta^*,\alpha))$ (witnessed by ζ_i which belongs to this set by the choice of α_i)
- $(*)_4$ if $i < \omega_1$ and $\beta \in u_{h(\beta_i)}$ then $\ell = \text{Min}\{\ell : \zeta^* \in A_{\rho(\beta,\delta^*)(\ell)}\}$ is well defined and $h_2(\rho(\beta,\delta^*)(\ell)) = \gamma(*)$.
  [Why? By $\bigoplus_2$].

Now let ν_i, for $i < \omega_1$, be the concatanation of $\{\rho(\beta,\delta^*) : \beta \in u_{\beta_i}\}$ and ρ_i be the concatanation of $\{\rho(\delta^*,\alpha) : \alpha \in u_{\alpha_i}\}$. So we can apply $\otimes$ of Stage A to $\langle \nu_i, \rho_i : i < \omega_1\rangle$ and γ^* (its assumptions hold by $(*)_1 + (*)_2 + (*)_3$) and get that for some $i < j < \omega_1$ we have $d_{\rm sq}(\nu'\char`\^\rho') = \zeta^*$ whenever ν' is a subsequence of ν_i, ρ' a subsequence of ρ_j such that $i \in$ Rang$(\nu'), j \in$ Rang(ρ'). Now for $\beta \in u_{h(\beta_i)}$, $\alpha \in u_{\alpha_j}$ we have

$\rho(\beta, \alpha) = \rho(\beta, \delta^*)\char`\^\rho(\delta^*, \alpha)$ (see $(*)_1$) and

$\rho(\beta, \delta^*)$ is O.K. as ν'.

[Why? Because it is a subsequence of ν_i (see the choice of ν_i) and i belongs to Rang$(\rho(\beta, \delta^*))$ by $(*)_2$] and

$\rho(\delta^*, \alpha)$ is O.K. as ρ'

[Why? Because $\rho(\delta^*, \alpha)$ is a subsequence of ρ_j by the choice of ρ_j and j belongs to Rang$(\rho(\delta^*, \alpha))$ by $(*)_3$].

Now by $(*)_4$ the colour $c(\beta, \alpha)$ is $\gamma(*)$ as required and get the desired conclusion. $\square_{1.1}$

Remark. Can we get $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ for λ regulars by the above proof? If $\lambda = \lambda^{<\lambda}$ the same proof works (now Dom$(d_{sq}) = {}^{\omega>}(\lambda^+)$ and $\nu_\alpha, \rho_\alpha \in {}^{\lambda>}(\lambda^+)$). See more in §2.

§2 Larger Cardinals: The implicit properties

More generally (than in the remark at the end of §1):

2.1 Definition. 1) $Pr_6(\lambda,\lambda,\theta,\sigma)$ means that there is $d : {}^{\omega>}\lambda \rightarrow \theta$ such that:
<u>if</u> $\langle (u_\alpha, v_\alpha) : \alpha < \lambda \rangle$ satisfies

$$u_\alpha \subseteq {}^{\omega>}\lambda, v_\alpha \subseteq {}^{\omega>}\lambda,$$

$$|u_\alpha \cup v_\alpha| < \sigma,$$

$$\nu \in u_\alpha \cup v_\alpha \Rightarrow \alpha \in \text{ Rang}(\nu),$$

and $\gamma < \theta$ and E a club of λ <u>then</u> for some $\alpha < \beta$ from E we have

$$\nu \in u_\alpha \ \ \& \ \ \rho \in v_\beta \Rightarrow d(\nu\hat{\ }\rho) = \gamma.$$

2) $Pr^6_S(\lambda,\lambda,\theta,\sigma)$ is defined similarly but $\alpha < \beta$ are required to be in $E \cap S$. $Pr^6_\tau(\lambda,\lambda,\theta,\sigma)$ means "for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) \geq \tau\}$ we have $Pr^6_S(\lambda,\lambda,\theta,\sigma)$". If τ is omitted, we mean $\tau = \sigma$. Lastly $Pr^6_{\text{nacc}}(\lambda,\lambda,\theta,\sigma)$ is defined similarly but demanding $\alpha,\beta \in \text{ nacc}(E)$ and $Pr^-_6(\lambda,\lambda,\theta,\sigma)$ is defined similarly but $E = \lambda$.

2.2 Lemma. *0) If $Pr_6(\lambda,\lambda,\theta,\sigma)$ and $\theta_1 \leq \theta$ and $\sigma_1 \leq \sigma$ <u>then</u> $Pr_6(\lambda,\lambda,\theta_1,\sigma_1)$ (and similar monotonicity properties for Definition 2.1(2)). Without loss of generality $u_\alpha = v_\alpha$ in Definition 2.1.*
1) If $Pr_6(\lambda^+,\lambda^+,\lambda^+,\lambda)$, <u>then</u> $Pr_1(\lambda^{+2},\lambda^{+2},\lambda^{+2},\lambda)$.
2) If $Pr_6(\lambda^+,\lambda^+,\theta,\sigma)$, so $\theta \leq \lambda^+$ <u>then</u> $Pr_1(\lambda^{+2},\lambda^{+2},\lambda^{+2},\sigma)$ provided that

(∗) *there is a sequence $\bar A = \langle A_\alpha : \alpha < \lambda^{++} \rangle$ of subsets of θ such that for every $\alpha \in u \subseteq \lambda^{++}$ with u of cardinality $< \sigma$, we have*

$$A_\alpha \backslash \cup \{A_\beta : \beta \in u, \beta \neq \alpha\} \neq \emptyset.$$

3) If λ is regular and $\lambda = \lambda^{<\lambda}$ <u>then</u> $Pr_6(\lambda^+,\lambda^+,\lambda^+,\lambda)$.
4) In [Sh:g, III,4.7] we can change the assumption accordingly.

Proof. 0) Clear.
1) By part (2) choosing $\theta = \lambda^+, \sigma = \lambda$ as (∗) holds as λ^+ is regular (so e.g. choose by induction on $\alpha < \lambda^{++}, A_\alpha \subseteq \lambda^+$ see unbounded non-stationary with $\beta < \alpha \Rightarrow |A_\alpha \cap A_\alpha| \leq \lambda$.
2) Like the proof for $\aleph_2$, only now $\{\delta < \lambda^{++} : \text{cf}(\delta) = \lambda^+\}$ plays the role of $\mathcal{S}^2_1$ and let $h_1 : \lambda^{++} \rightarrow \lambda^+$ and $h_2 : \lambda^{++} \rightarrow \lambda^{++}$ be such that for every $\gamma < \lambda^{+\ell}$ and $\ell \in \{1,2\}$ the set $S^\ell_\gamma = \{\alpha < \lambda^{+2} : \text{cf}(\alpha) = \lambda^+ \text{ and } h_\ell(\alpha) = \gamma\}$ is stationary. Finally, if dq exemplifies $Pr_6(\lambda^+,\lambda^+,\theta,\sigma)$, then in defining c for a given $\alpha < \beta$, let $\ell_{\alpha,\beta}$ be the minimal ℓ such that $dq(\rho_{h_1}(\alpha,\beta))$ belongs to $A_{\rho_{h_1}(\alpha,\beta)(\ell)}$ and let $c(\beta,\alpha) = c(\alpha,\beta) = h_2\left(\rho(\beta,\alpha)(\ell_{\beta,\alpha})\right)$. Then in stage D without loss of generality

$|u_\alpha| = \sigma < \lambda$ for $\alpha < \lambda^+$ and continue as there, but after the definition of E_1 we do not choose ζ_i, α_i instead we first continue choosing β_i, ξ_i for $i < \lambda^+$ as there as without loss of generality $\delta^* = \bigcup\limits_{i<\lambda^+} \xi(i)$. Only then we choose by induction on $i < \lambda^+$ the ordinal ζ_i such that: $\zeta_i \in S'_i \backslash (\alpha^{**}+1), \zeta_i > \sup[\{\xi(j) : j \le i\} \cup \{\zeta_j : j < i\}]$ and then choose $\alpha_i < \zeta_i$ large enough (so no need of the club A of λ^+).
3) As in the proof of 1.1, Stage A.
4) Combine the proofs here and there (and not used). $\square_{2.2}$

This leaves some problems on Pr_1 open; e.g.

2.3 Question. 1) If $\lambda > \aleph_0$ is inaccessible, do we have $Pr_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ (rather than with $\sigma < \lambda$)?
2) If $\mu > \aleph_0$ is regular (singular) and $\lambda = \mu^+$, do we have $Pr_1(\lambda^+, \lambda^+, \lambda^+, \mu)$?
[clearly, yes, for the weaker version: c a symmetric two place function from λ^+ to λ^+ such that for every $\gamma < \lambda^+$ and pairwise disjoint $\langle u_\alpha : \alpha < \lambda^+\rangle$ with $u_\alpha \in [\lambda^+]^{<\lambda}$ we have

$$(\exists \alpha < \beta) \forall i \in u_\alpha\, \forall j \in u_\beta \Big(\gamma \in \text{ Rang } \rho_c(j,i)\Big)].$$

See more in §4. Remember that we know $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$ for $\sigma < \lambda$.

2.4 Claim. *Assume μ is singular, $\lambda = \mu^+$, $2^\kappa > \mu > \kappa = \kappa^\theta, \theta = cf(\theta) \ge \sigma$ and $Pr_6(\theta, \theta, \theta, \sigma)$. <u>Then</u> $Pr_1(\mu^+, \mu^+, \theta, \sigma)$.*

Proof. Let $\bar e = \langle e_\alpha : \alpha < \lambda\rangle$ be a club system, $S \subseteq \{\delta < \mu^+ : cf(\delta) = \theta\}$ stationary such that $\lambda \notin \text{ id}^a(\bar e \restriction S)$ and $\alpha \in e_\delta \Rightarrow \text{ cf}(\alpha) \ne \theta$ and

$$\delta = \sup(\delta \cap S) \;\&\; \chi < \mu \Rightarrow \delta = \sup\big(\{\alpha \in e_\delta : cf(\alpha) > \chi + \sigma^+, \text{ so } \alpha \in \text{ nacc}(e_\delta)\}\big)$$

and $\alpha \in e_\beta \cap S \Rightarrow e_\alpha \subseteq e_\beta$ (exists by [Sh 365, 2.10]). Let $\bar f = \langle f_\alpha : \alpha < \theta\rangle, f_\alpha : \mu^+ \to \kappa$ such that every partial function g from μ^+ to κ (really σ suffice) of cardinality $\le \theta$ is included in some f_α (exist by [EK] or see [Sh:g, AP1.7]).

So for some $f = f_{\alpha(*)}$ we have

(∗) for every club E of μ^+ for some $\delta \in S$ we have:
  (a) $e_\delta \subseteq E$
  (b) if $\chi < \mu$ and $\gamma < \theta$ then
  $\delta = \text{ sup}(\{\alpha \in \text{ nacc}(e_\delta) : f(\alpha) = \gamma \text{ and cf}(\alpha) > \chi\})$.

This actually proves $\text{id}_p(\bar e \restriction S)$ is not weakly θ^+-saturated.
The rest is by combining the trick of [Sh:g, III,§4] (using first $\delta(*) \in S$ then some suitable $\alpha \in \text{ nacc}(e_{\delta(*)})$) and the proof for $\aleph_2$. $\square_{2.4}$

2.5 Fact. $Pr_1(\lambda^+,\lambda^+,\theta,\mathrm{cf}(\lambda))$ implies $Pr^6(\lambda^+,\lambda^+,\theta,\mathrm{cf}(\lambda))$.

Remark. This is not totally immediate as in Pr_1 the sets are required to be pairwise disjoint.

Proof. Let $\kappa=\mathrm{cf}(\lambda)$ and $f_\alpha\in{}^\kappa\lambda$ for $\alpha<\lambda^+$ be such that $\alpha<\beta\Rightarrow f_\alpha<^*_{J^{bd}_\kappa}f_\beta$. Let $d:[\lambda^+]^2\to\theta$ exemplifies $Pr_1(\lambda^+,\lambda^+,\theta,\mathrm{cf}(\lambda))$. Let $c:\kappa\to\kappa$ be such that for every $\gamma<\kappa$ for undoubtedly many $\beta<\kappa$ we have $c(\beta)=\gamma$. For $\nu\in{}^{\omega>}(\lambda^+)$ we define $d^*_{sq}(\nu)$ as follows.

If $\ell g(\nu)\le 1$ or ν is constant, then $d^*_{\mathrm{sq}}(\nu)=0$. So assume $\ell g(\nu)\ge 2$ and ν is not constant.

$$\text{For } \alpha<\beta<\lambda^+ \text{ let } \mathbf{s}(\beta,\alpha)=\mathbf{s}(\alpha,\beta)=\sup\{i+1:i<\kappa \text{ and } f_\alpha(i)\ge f_\beta(i)\},$$

$$\mathbf{s}(\alpha,\alpha)=0,$$

$$\mathbf{s}(\nu)=\max\{\mathbf{s}(\nu(\ell),\nu(k)):\ell,k<\ell g(\nu) \text{ (so } \mathbf{s} \text{ is symmetric)}\},$$

$$a(\nu)=\{(\ell,k):\mathbf{s}(\nu(\ell),\nu(k))=\mathbf{s}(\nu) \text{ and } \ell<k<\ell g(\nu)\}.$$

As $\ell g(\nu)\ge 2$ and ν is not constant, clearly $a(\nu)\ne\emptyset$ and $a(\nu)$ is finite, so let (ℓ_ν,k_ν) be the first pair from $a(\nu)$ in lexicographical ordering.

Lastly $d^*_{\mathrm{sq}}(\nu)=c\Big(d(\{\nu(\ell_\nu),\nu(k_\nu)\})\Big)$.

Now we are given $\gamma<\theta$, stationary $S\subseteq\{\delta<\lambda^+:\mathrm{cf}(\delta)\ge\mathrm{cf}(\lambda)\}$, $\langle u_\alpha:\alpha<\lambda^+\rangle$ (remember 2.2(0)), $|u_\alpha|<\mathrm{cf}(\lambda), u_\alpha\subseteq{}^{\omega>}\lambda$ such that $\alpha\in\cap\{\mathrm{Rang}(\nu):\nu\in u_\alpha\}$. Let $u'_\alpha=\cup\{\mathrm{Rang}(\nu):\nu\in u_\alpha\}$ and without loss of generality for some stationary $S'\subseteq S$ and γ_0,β^* we have $\alpha\in S'\Rightarrow\gamma_0=\min\{\gamma+1:$ if $\beta_1<\beta_2$ are in u'_α then $f_{\beta_1}\restriction[\gamma,\mathrm{cf}(\lambda))<f_{\beta_2}\restriction[\gamma,\mathrm{cf}(\lambda))\}<\kappa$ and $\sup(\cup\{u'_\alpha\cap\alpha:\alpha\in S'\})<\beta^*<\lambda^+$. Now for some $\gamma_1\in(\gamma_0,\mathrm{cf}(\lambda))$ and stationary $S_0,S_1\subseteq S'$ and $\gamma^*<\lambda$ we have

$$\beta\in u'_\alpha \;\&\; \alpha\in S_0\Rightarrow f_\beta(\gamma_1)<\gamma^*,$$

$$\beta\in u'_\alpha \;\&\; \alpha\in S_1\Rightarrow f_\beta(\gamma_1)>\gamma^*.$$

Let $\{\alpha^\ell_\zeta:\zeta<\lambda\}$ enumerate some unbounded $S'_\ell\subseteq S_\ell$ in increasing order such that $\zeta<\xi\Rightarrow\sup(u_{\alpha^0_\zeta}\cup u_{\alpha^1_\zeta})<\min(u_{\alpha^0_\xi}\cup u_{\alpha^1_\xi})$.

Lastly apply the choice of d. $\square_{2.5}$

§3 Guessing Clubs Revisited

3.1 Claim. *Assume $\lambda = \mu^+$, and*
$S \subseteq \{\delta < \lambda^+ : cf(\delta) = \lambda$ and δ is divisible by $\lambda^2\}$ is stationary.
1) There is a strict club system $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that $\lambda^+ \notin id^p(\bar{C})$ and $[\alpha \in nacc(C_\delta) \Rightarrow cf(\alpha) = \lambda]$; moreover, there are $h_\delta : C_\delta \to \mu$ such that for every club E of λ^+, for stationarily many $\delta \in S$,

$$\bigwedge_{\zeta<\mu} \delta = \sup[h_\delta^{-1}(\{\zeta\}) \cap E \cap nacc(C_\delta)].$$

2) If $\bar{C}$ is a strict S-system, $\lambda^+ \notin id^p(\bar{C}, \bar{J})$, J_δ a λ-complete ideal on C_δ extending $J^{bd}_{C_\delta} + acc(C_\delta)$ (with S, μ as above) <u>then</u> the parallel result holds for some $\bar{h} = \langle h_\delta : \delta \in S\rangle$ where h_δ is a function from C_δ to μ, i.e. we have for every club E of λ^+, for stationarily many $\delta \in S \cap acc(E)$ for every $\gamma < \mu$ the set $\{\alpha \in C_\delta : h_\delta(\alpha) = \gamma$ and $\alpha \in E\}$ is $\neq \emptyset$ mod J_δ.

3.2 Remark. 1) This improves [Sh 413, 3.1].
2) Of course, we can strengthen (1) to:

$$\{\alpha \in C_\delta : h_\delta(\alpha) = \gamma \text{ and } \alpha \in E \text{ and } \alpha \in \text{ nacc}(C_\delta) \text{ and } \sup(\alpha \cap C_\delta) \in E\}.$$

E.g. for every thin enough club E of $\lambda, \bar{C}^E$ will serve where: $C^E_\delta = C_\delta \cap E$ if $\delta \in$ acc(E) and $C^E_\delta = C_\delta$, otherwise.
For 3.1(2) we get slightly less: for some club $E^* : \{\alpha \in C_\delta : h_\delta(\alpha) = \gamma$ and $\alpha \in E$ and $\alpha \in$ nacc(C_δ) and $\sup(\alpha \cap C_\delta \cap E^*) \in E\}$.

Proof. 1) Let $\langle C_\delta : \delta \in S\rangle$ be such that $\lambda^+ \notin$ id$^p(\bar{C})$ and $[\alpha \in$ nacc$(C_\delta) \Rightarrow cf(\delta) = \lambda]$ (such a sequence exists by [Sh 365, 2.4(3)]). Let $J_\delta = J^{bd}_{C_\delta} +$ acc(C_δ). Now apply part (2).
2) For each $\delta \in S$ let $\langle A^\alpha_\delta : \alpha \in C_\delta\rangle$ be a sequence of distinct non-empty subsets of μ to be chosen later. By induction on $\zeta < \lambda$ we try to define $E_\zeta, \langle Y^\zeta_\alpha : \alpha \in S\rangle$, $\langle Z^\zeta_{\alpha,\gamma} : \alpha \in \zeta$ and $\gamma < \mu\rangle$ such that

$$E_\zeta \text{ is a club of } \lambda^+, \text{ decreasing in } \zeta,$$

for $\gamma < \mu$,

$$Z^\zeta_{\delta,\gamma} = \{\alpha : \alpha \in E_\zeta \cap \text{ nacc}(C_\delta) \text{ and } \gamma \in A^\alpha_\delta\},$$

$$Y^\zeta_\delta = \{\gamma < \mu : Z^\zeta_{\delta,\gamma} \neq \emptyset \text{ mod } J_\delta\}.$$

$E_{\zeta+1}$ is such that

$$\Big\{\delta \in S : Y^\zeta_\delta = Y^{\zeta+1}_\delta \text{ and } \delta \in \operatorname{nacc}(E_{\zeta+1}) \text{ and } E_{\zeta+1} \cap \operatorname{nacc}(C_\delta) \notin J_\delta\Big\} \text{ is not stationary.}$$

If we succeed to define E_ζ, for each $\zeta < \lambda$, then $E =: \bigcap_{\zeta<\lambda} E_\zeta$ is a club of λ^+, and since $\lambda^+ \notin \operatorname{id}^p(\bar{C})$, we can choose $\delta \in S$ such that $\delta = \sup(E \cap \operatorname{nacc} C_\delta)$ and $E \cap \operatorname{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$. Then as $\bigcup_{\gamma<\mu} Z^\zeta_{\delta,\gamma} \supseteq E \cap \operatorname{nacc}(C_\delta)$ for each $\zeta < \lambda$ necessarily (by the requirement on J_δ) for some $\gamma < \mu, Z^\zeta_{\delta,\gamma} \neq \emptyset \bmod J_\delta$, hence $Y^\zeta_\delta \neq \emptyset$ so that $\langle Y^\zeta_\delta : \zeta < \lambda\rangle$ is a strictly decreasing sequence of subsets of μ, and since $\mu < \operatorname{cf}(\mu^+) = \operatorname{cf}(\lambda)$, we have a contradiction. So necessarily we will be stuck (say) for $\zeta(*) < \lambda$.
We still have the freedom of choosing A^α_δ for $\alpha \in C_\delta$.

Case 1: μ regular.
By induction on $\alpha \in C_\delta$ we can choose sets A^α_δ such that

(i) $A^\alpha_\delta \subseteq \mu, |A^\alpha_\delta| = \mu, \langle A^\alpha_\delta : \alpha \in C_\delta, \operatorname{otp}(\alpha \cap C_\delta) < \mu\rangle$ are pairwise disjoint,
(ii) for $\beta \in C_\delta \cap \alpha$, $A^\alpha_\delta \cap A^\beta_\delta$ is bounded in μ,
(iii) if $\mu > \aleph_0$ then A^α_δ is non-stationary (just to clarify their choice).

There is no problem to carry the induction.
We shall prove later that

$(*)$ if E is a club of $\lambda^+, \delta \in S \cap \operatorname{acc}(E)$ and $\delta = \sup(E \cap \operatorname{nacc} C_\delta)$ and $E \cap \operatorname{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$ then

$(**)_\delta$ for some $\alpha_\delta \in E \cap \operatorname{nacc}(C_\delta)$, the following set B_δ is unbounded in μ, where

$$B_\delta = \Big\{\gamma < \mu : \{\beta : \beta \in E \cap \operatorname{nacc}(C_\delta) \text{ and } \beta \neq \alpha_\delta \text{ and } \gamma = \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\} \neq \emptyset \bmod J_\delta\Big\}.$$

Choose the minimal such that $\alpha_\delta = \alpha^E_\delta$ (for other δ's it does not matter, i.e. for those for which $\delta > \sup(E \cap \operatorname{nacc}(C_\delta))$ or $E_{\zeta(*)} \cap \operatorname{nacc}(C_\delta) \in J_\delta$).
Clearly if $E' \supseteq E''$ and $\alpha^{E'}_\delta, \alpha^{E''}_\delta$ are defined then $\alpha^{E'}_\delta \leq \alpha^{E''}_\delta$.
Now for any club $E^* \subseteq E_{\zeta(*)}$ of λ^+, for $\delta \in S \cap \operatorname{acc}(E_{\zeta(*)})$ we define $h^{E^*}_\delta : C_\delta \to \mu$ by letting $h^{E^*}_\delta(\beta) = \operatorname{otp}(B_\delta \cap \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta))$ for $\beta \in C_\delta \setminus \{\alpha_\delta\}$ and $h^{E^*}_\delta(\alpha_\delta) = 0$.

Now for any club E of λ^+ for stationarily many $\delta \in S \cap \operatorname{acc}(E^* \cap E)$, we have

$$\left\{\gamma < \mu : \{\alpha : \alpha \in E^* \cap E \cap E_{\zeta(*)} \cap \operatorname{nacc}(C_\delta) \text{ and } \gamma \in A^\alpha_\delta\} \neq \emptyset \bmod J_\delta\right\} = Y^{\zeta(*)}_\delta$$

(this holds by the choice of $\zeta(*)$). Let the set of such $\delta \in S \cap \operatorname{acc}(E^* \cap E)$ be called $Z^{E^*}_E$. Now for each $\delta \in Z^{E^*}_E$, the set

$$B_\delta[E, E^*] =: \left\{\gamma < \mu : \{\beta : \beta \in E \cap E^* \cap E_{\zeta(*)} \cap \operatorname{nacc}(C_\delta) \text{ and } \beta \neq \alpha^{E^*}_\delta \text{ and } \gamma = \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\} \neq \emptyset \bmod J_\delta\right\}$$

is necessarily unbounded in μ. So in the same way we have gotten $E_{\zeta(*)}$ we can find club $E^* \subseteq E_{\zeta(*)}$ such that for any club E of λ^+, for stationarily many $\delta \in Z^{E^*}_E$ we have $B_\delta[E, E_{\zeta(*)}] = B_\delta[E^*, E_{\zeta(*)}]$ and $\alpha^E_\delta = \alpha^{E^*}_\delta$ (note the minimality in the choice of α^E_δ so it can change $\leq \lambda+1$ times; more elaborately if $\langle E^*_\zeta : \zeta < \lambda\rangle$ is a decreasing sequence of clubs and $\delta \in Z^{E^*}_{E^*}$, where $E^* = \bigcap\limits_{\zeta<\lambda} E^*_\zeta$, then $\langle \alpha^{E^*_\zeta}_\delta : \zeta < \lambda\rangle$ is increasing and bounded in C_δ (by $\alpha^{E^*}_\delta$), hence is eventually constant). Define $h_\delta : C_\delta \to \mu$ by $h_\delta(\beta) = \operatorname{otp}\left(B_\delta[E^*, E_{\zeta(*)}] \cap \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\right)$ if $\beta \neq \alpha_\delta$ and $h_\delta(\beta) = 0$ if $\beta = \alpha_\delta$.

Why does $(*)$ hold?
If not, let $B = E_{\zeta(*)} \cap \operatorname{nacc}(C_\delta)$, so $\operatorname{otp}(B) = \lambda = \mu^+$ and $B \neq \emptyset \bmod J_\delta$, so for every $\alpha \in B$ we can find $\varepsilon_\alpha < \mu$ and $Y_{\alpha,\varepsilon} \in J_\delta$ (for $\varepsilon < \mu$) such that if $\xi \in B \backslash Y_{\alpha,\varepsilon} \backslash \{\alpha\}$ and $\varepsilon \in [\varepsilon_\alpha, \mu)$ then $\sup(A^\alpha_\delta \cap A^\xi_\delta) \neq \varepsilon$. Now let $Y_\alpha =: \cup\{Y_{\alpha,\varepsilon} : \varepsilon \in [\varepsilon_\alpha, \mu)\} \cup \{\alpha+1\}$ and note that $Y_\alpha \in J_\delta$. So for some $\varepsilon^* < \mu$, $B_1 =: \{\alpha \in B : \varepsilon_\alpha = \varepsilon^*\}$ is $\neq \emptyset \bmod J_\delta$. For each $\alpha \in B_1$ choose $\gamma_\alpha \in A^\delta_\alpha \backslash (\varepsilon^*+1)$ (remember $|A^\delta_\alpha| = \mu$). So for some $\gamma^* < \mu$ the set $B_2 =: \{\alpha \in B_1 : \gamma_\alpha = \gamma^*\}$ is $\neq \emptyset \bmod J_\delta$. Let $\alpha^* = \operatorname{Min}(B_2)$, and for $\gamma \in [\gamma^*, \mu)$ we define
$B_{\zeta,\gamma} = \{\alpha \in B_2 : \gamma = \sup(A^{\alpha^*}_\delta \cap A^\alpha_\delta\}$. So clearly $B_2 = \cup\{B_{\zeta,\gamma} : \gamma^* \leq \gamma < \mu\}$, hence for some $\gamma^{**} \in [\gamma^*, \mu)$ we have $B_{\zeta,\gamma^{**}} \neq \emptyset \bmod J_\delta$, hence γ^{**} contradicts the choice of $\varepsilon_{\alpha^*} = \varepsilon^*$.

Case 2: μ singular.
Let $\kappa = cf(\mu)$, so by [Sh:g, II,§1] we can find an increasing sequence $\langle \lambda_i : i < \kappa\rangle$ of regular cardinals $> \kappa$ with limit μ such that $\lambda = \mu^+ = \operatorname{tcf}(\prod_{i<\kappa} \lambda_i / J^{bd}_\kappa)$, and[2] let $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifying this. Without loss of generality $\bigcup\limits_{j<i} \lambda_j < f_\alpha(i) < \lambda_i$. Let $g : \kappa \times \mu \times \kappa \times \mu \to \mu$ be one to one and onto, let $f^\delta_\alpha = f_{\operatorname{otp}(\alpha \cap C_\delta)}$ for $\alpha \in C_\delta$ and let $A^\delta_\alpha = \{g(i, f^\delta_\alpha(i), j, f^\delta_\alpha(j)) : i, j < \kappa\}$.

[2]for the rest of this case "$\lambda = \mu^+$" is not used; also J^{bd}_κ can be replaced by any larger ideal

If $\delta = \sup(E_{\zeta(*)} \cap \text{ nacc}(C_\delta))$ and $E_{\zeta(*)} \cap \text{ nacc}(C_\delta) \neq \emptyset \mod J_\delta$ then (as J_δ is λ-complete) choose $Y_\delta \in J_\delta$ such that for each $i < \kappa, \varepsilon < \lambda_i$ we have

$$(*)\ (\exists\beta)[\beta \in E_{\zeta(*)} \cap \text{ nacc}(C_\delta)\ \&\ \beta \notin Y_\delta\ \&\ f^\delta_\beta(i) = \varepsilon] \Rightarrow$$
$$\{\beta : \beta \in E_{\zeta(*)} \cap \text{ nacc}(C_\delta)\ \&\ f^\delta_\beta(i) = \varepsilon\} \neq \emptyset \mod J_\delta.$$

Choose $i(\delta) < \kappa$ such that

$$B^0_\delta =: \{f^\delta_\beta(i(\delta)) : \beta \in E_{\zeta(*)} \cap \text{ nacc}(C_\delta) \text{ and } \beta \notin Y_\delta\}$$

is unbounded in λ_i.

Let $\xi_\varepsilon = \xi^\delta_\varepsilon$ be the ε-th member of B^0_δ, for $\varepsilon < \kappa$. For each such $\varepsilon < \kappa$ for some $j_\varepsilon = j^\delta_\varepsilon \in (i(\delta) + 1 + \varepsilon, \kappa)$ we have $B^{1,\delta}_\varepsilon =: \{f^\delta_\beta(j_\varepsilon) : f^\delta_\beta(i(\delta)) = \xi^\delta_\varepsilon$ and $\beta \in E_{\zeta(*)} \cap \text{ nacc}(C_\delta)$ and $\beta \notin Y_\delta\}$ is unbounded in $\lambda_{j^\delta_\varepsilon}$.

Let $h_{\delta,\varepsilon}$ be a one to one function from $[\bigcup\limits_{j<\varepsilon} \lambda_j, \lambda_\varepsilon)$ into $B^{1,\delta}_\varepsilon$.

Lastly we define h_δ as follows:

$$\underline{\text{if}}\ \beta \in C_\delta, \varepsilon < \kappa, f^\delta_\beta(i(\delta)) = \xi^\delta_\varepsilon \text{ and } h_{\delta,\varepsilon}(\gamma) = f^\delta_\beta(j^\delta_\varepsilon)$$
$$(\text{so } \gamma \in [\bigcup_{j<\varepsilon} \lambda_j, \lambda_\varepsilon))\ \underline{\text{then}}\ h_\delta(\beta) = \gamma$$

and $h_\delta(\beta) = 0$ otherwise. The rest is similar to the regular case. $\square_{3.1}$

3.3 Claim. *If $\lambda = \mu^+, \mu$ regular uncountable and $S \subseteq \{\delta < \lambda : cf(\delta) = \mu\}$ is stationary* *then* *for some strict S-club system $\bar C$ with $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \mu\}$, (where $\alpha_{\delta,\zeta}$ is strictly increasing continuous in ζ) for every club $E \subseteq \lambda$ for stationarily many $\delta \in S$,*

$$\{\zeta < \mu : \alpha_{\delta,\zeta+1} \in E\} \text{ is stationary (as a subset of } \mu).$$

3.4 Remark. 1) If $S \in I[\lambda]$ then without loss of generality we can demand (a) or we can demand (b) (but not necessarily both), where

(a) $X_\alpha = \{C_\delta \cap \alpha : \delta \in S,$ is such that $\alpha \in \text{ nacc}(C_\delta)\}$ has cardinality $\leq \lambda$,
(b) $\alpha \in \text{ nacc}(C_\delta) \Rightarrow C_\alpha = C_\delta \cap \alpha$ but the conclusion is weakened to: for every club E of λ for stationarily many $\delta \in S$ the set $\{\zeta < \mu : (\alpha_{\delta,\zeta}, \alpha_{\delta,\zeta+1}) \cap E \neq \emptyset\}$ is stationary.

2) In contrast to [Sh 413, 3.4] here we allow μ inaccessible.
3) Clearly 3.1(2) can be applied to the results of 3.3 i.e. with

$$J_\delta = \left\{A \subseteq C_\delta : \{\zeta < \lambda : \alpha_{\delta,\zeta+1} \notin A\} \text{ is not stationary}\right\}.$$

Proof. We know that for some strict S-club system $\bar C^0 = \langle C^0_\delta : \delta \in S\rangle$ we have $\lambda \notin \text{ id}_p(\bar C^0)$ (see [Sh 365, 2.3(1)]). Let $C^0_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\}$ (increasing continuously in ζ). We shall prove below that for some sequence of functions $\bar{h} = \langle h_\delta : \delta \in S \rangle$, $h_\delta : \mu \to \mu$ we have

$(*)_{\bar{h}}$ for every club E of μ^+ for stationarily many $\delta \in S \cap \operatorname{acc}(E)$, the following subset of μ is stationary:

$$A^{\delta,*}_E =: \Big\{ \zeta < \mu : \alpha^\delta_\zeta \in E \text{ and some ordinal in } \{\alpha^\delta_\xi : \zeta < \xi \le h_\delta(\zeta)\} \text{ belongs to } E \Big\}.$$

The proof now breaks into two parts.
<u>Proving</u> $(*)_{\bar{h}}$ <u>suffices</u>.

For each club E of λ, let $Z_E =: \{\delta \in S : \delta = \sup(E \cap \operatorname{nacc}(C^0_\delta))\}$, and note that this set is a stationary subset of λ (by the choice of $\bar{C}^0$). For each such E and $\delta \in Z_E$ let $f_{\delta,E}$ be the partial function from μ to μ defined by

$$f_{\delta,E}(\zeta) = \operatorname{Sup}\{\xi : \zeta < \xi \le h_\delta(\zeta) \text{ and } \alpha^\delta_\xi \in E\}.$$

So if there is no such ξ, then $f_{\alpha,E}(\zeta)$ is not well defined (i.e. if the supremum is on the empty set) but if $\xi = f_{\alpha,E}(\zeta)$ is well defined then $\alpha^\delta_\xi \in E, \xi \le h_\delta(\zeta)$ (because α^δ_ξ is increasing continuous in ξ and E is a club of λ). Let
$Y_E =: \{\delta \in Z_E : \operatorname{Dom}(f_{\delta,E}) \text{ is a stationary subset of } \mu\}$. So by $(*)_{\bar{h}}$, we know that

$\bigoplus$ for every club E of μ^+ the set Y_E is a stationary subset of μ^+.

Also

$\bigotimes_1$ if $E_2 \subseteq E_1$ are clubs of μ^+ then $Z_{E_2} \subseteq Z_{E_1}$ and $Y_{E_2} \subseteq Y_{E_1}$ and for $\delta \in Y_{E_2}, \operatorname{Dom}(f_{\delta,E_2}) \subseteq \operatorname{Dom}(f_{\delta,E_1})$ and $\zeta \in \operatorname{Dom}(f_{\delta,E_2}) \Rightarrow f_{\delta,E_2}(\zeta) \le f_{\delta,E_1}(\zeta)$.

We claim that

$\bigotimes_2$ for some club E_0 of μ^+ for every club $E \subseteq E_0$ of μ^+ for stationarily many $\delta \in S$ we have
  (i) $\delta = \sup(E \cap \operatorname{nacc} C_\delta)$,
  (ii) $\{\zeta < \mu : \zeta \in \operatorname{Dom}(f_{E,\delta})$ (hence $\zeta \in \operatorname{Dom} f_{E_0,\delta}$) and $f_{E,\delta}(\zeta) = f_{E_0,\delta}(\zeta)\}$ is a stationary subset of μ.

If this fails, then for any club E_0 of λ there is a club $E(E_0) \subseteq E_0$ of λ, such that

$$A_{E_0} = \Big\{ \delta : \delta \in S, \delta = \sup(E(E_0) \cap \operatorname{nacc}(C_\delta)) \text{ and for some club } e_{E_0,\delta} \text{ of } \mu \text{ we have } \zeta \in e_{E_0,\delta} \cap \operatorname{Dom}(f_{E(E_0),\delta}) \Rightarrow f_{E(E_0),\delta}(\zeta) = f_{E_0,\delta}(\zeta) \Big\}$$

is not a stationary subset of $\lambda = \mu^+$. By obvious monotonicity we can replace $E(E_0)$ by any club of μ^+ which is a subset of it, so without loss of generality $A_{E_0} = \emptyset$.

By induction on $n < \omega$ choose clubs E_n of μ^+ such that $E_0 = \mu^+$ and $E_{n+1} = E(E_n)$.
Then $E_\omega =: \bigcap\limits_{n<\omega} E_n$ is a club of μ^+ and, by $\bigoplus$ above, $Y_{E_\omega} \subseteq S$ is a stationary subset of λ, so we can choose a $\delta(*) \in Y_{E_\omega}$. So $f_{E_\omega,\delta(*)}$ has domain a stationary subset of μ (see the definition of Y_{E_ω}) and by $\bigotimes_1$ we know that $n < \omega \Rightarrow \text{ Dom}(f_{E_\omega,\delta(*)}) \subseteq \text{ Dom}(f_{E_n,\delta(*)})$. Also there is an $e_{E_n,\delta(*)}$, a club of μ, such that

$$\zeta \in e_{E_n,\delta(*)} \cap \text{ Dom}(f_{E_{n+1},\delta(*)}) \Rightarrow f_{E_{n+1},\delta(*)}(\zeta) < f_{E_n,\delta(*)}(\zeta)$$

(see the choice of $E_{n+1} = E(E_n)$ i.e. the function E). So $e_{\delta(*)} =: \bigcap\limits_{n<\omega} e_{E_n,\delta(*)}$ is a club of μ and, as $\text{Dom}(f_{E_\omega,\delta(*)})$ is a stationary subset of μ, we can find $\zeta(*) \in e_{\delta(*)} \cap \text{ Dom}(f_{E_\omega,\delta(*)})$, hence $\zeta(*) \in \bigcap\limits_{n<\omega} \text{ Dom}(f_{E_n,\delta(*)}) \cap \bigcap\limits_{n<\omega} e_{E_n,\delta(*)}$, so that $\langle f_{E_n,\delta(*)}(\zeta(*)) : n < \omega\rangle$ is a well defined strictly increasing ω-sequence of ordinals - a contradiction. So $\bigotimes_2$ cannot fail, and this gives the desired conclusion.

Proof of $(*)_{\bar h}$ *holds for some* $\bar h$.

So assume that for no $\bar h$ does $(*)_{\bar h}$ holds, hence (by shrinking E) we can assume that for every $\bar h = \langle h_\delta : \delta \in S\rangle, h_\delta : \mu \to \mu$, for some club E for every $\delta \in S, A^{\delta,*}_E$ is not stationary (in μ). By induction on $n < \omega$, we define E_n, $\bar h^n = \langle h^n_\delta : \delta \in S\rangle, \bar e^n = \langle e^n_\delta : \delta \in S\rangle$, with E_n a club of λ, e^n_δ club of $\mu, h^n_\delta : \mu \to \mu$ as follows.
Let $E_0 = \lambda$, $h^0_\delta(\zeta) = \zeta + 1$ and $e^n_\delta = \mu$.
If $E_0, ..., E_n, \bar h^0, ..., \bar h^n, \bar e^0, ..., \bar e^n$ are defined, necessarily $(*)_{\bar h^n}$ fail, so for some club E_{n+1} of λ for every $\delta \in S \cap \text{ acc}(E_{n+1})$ there is a club $e^{n+1}_\delta \subseteq \text{ acc}(e^n_\delta)$ of μ, such that

$$\zeta \in e^{n+1}_\delta \Rightarrow \{\alpha^\delta_\xi : \zeta < \xi \leq h_\delta(\zeta)\} \cap E_{n+1} = \emptyset.$$

Choose $h^{n+1}_\delta : \mu \to \mu$ such that $(\forall \zeta < \mu)(h^n_\delta(\zeta) < h^{n+1}_\delta(\zeta))$ and if $\delta = \sup(E_{n+1} \cap \text{ nacc}(C_\delta))$ then $\zeta < \mu \Rightarrow \{\alpha^\delta_\xi : \zeta < \xi \leq h^{n+1}_\delta(\zeta)\} \cap E_{n+1} \neq \emptyset$.

There is no problem to carry out this inductive definition. By the choice of $\bar C^0$, for some $\delta \in \text{ acc}(\bigcap\limits_{n<\omega} E_n)$, we have $\delta = \sup(A')$, where $A' =: (\text{acc} \bigcap\limits_{n<\omega} E_n) \cap \text{ nacc}(C^0_\delta)$. Let $A \subseteq \mu$ be such that $A' = \{\alpha^\delta_\zeta : \zeta \in A\}$ (remember α^δ_ζ is increasing with ζ) and let ζ be the second member of $\bigcap\limits_{n<\omega} e^n_\delta$. As A' is unbounded in δ, clearly A is unbounded in μ and $\bigcap\limits_{n<\omega} e^n_\delta$ is a club of μ as $\mu = \text{cf}(\mu) > \aleph_0$. Also as $A' \subseteq \text{ nacc}(C^0_\delta)$ clearly A is a set of successor ordinals (or zero).

Note that $\sup(e_n^\delta \cap \zeta)$ is well defined (as $\mathrm{Min}(e_n^\delta) \le \mathrm{Min}(\bigcap\limits_{n<\omega} e_\delta^n) < \zeta$) and $\sup(e_n^\delta \cap \zeta) < \zeta$ (as ζ is a successor ordinal). Now $\langle \sup(e_n^\delta \cap \zeta) : n < \omega \rangle$ is non-increasing (as e_δ^n decreases with n), hence for some $n(*) < \omega$ we have $n > n(*) \Rightarrow \sup(e_\delta^n \cap \zeta) = \sup(e_\delta^{n(*)} \cap \zeta)$ and call this ordinal ξ so that $\xi \in e^\delta_{n(*)+1}$ and $h_\delta^{n(*)}(\xi) = h_\delta^{n(*)+1}(\xi)$, so we get a contradiction for $n(*)+1$.
So $(*)_{\bar h}$ holds for some $\bar h$, which suffices, as indicated above. $\square_{3.3}$

3.5 Discussion. 1) We can squeeze a little more, but it is not so clear if with much gain. So assume

$(*)_0$ μ is regular uncountable, $\lambda = \mu^+, S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ stationary, I an ideal on S, $\bar C = \langle C_\delta : \delta \in S \rangle$ a strict S-club system, $\bar J = \langle J_\delta : \delta \in S \rangle$ with J_δ an ideal on C_δ extending $J^{bd}_{C_\delta} + (\mathrm{acc}(C_\delta))$, such that for any club E of λ we have $\{\delta \in S : E \cap C_\delta \ne \emptyset \bmod J_\delta\} \ne \emptyset \bmod I$.

2) If we immitate the proof of 3.3 we get

$(*)_1$ if for $\delta \in S, J_\delta$ is not χ-regular (see the definition below) and $\chi \le \mu$ then we can find $\bar e = \langle e_\delta : \delta \in S \rangle$ and $\bar g = \langle g_\delta : \delta \in S \rangle$ such that

$(*)_1'$ e_δ is a club of $\delta, e_\delta \subseteq \mathrm{acc}(C_\delta), g_\delta : \mathrm{nacc}(C_\delta) \backslash (\min(e_\delta)+1) \to e_\delta$ is defined by $g_\delta(\alpha) = \sup(e_\delta \cap \alpha)$ and for every club E of λ

$$\Big\{\delta \in S : E \cap \mathrm{nacc}(C_\delta) \ne \emptyset \bmod J_\delta \text{ and } \mathrm{Rang}(g_\delta \restriction (E \cap \mathrm{nacc}(C_\delta))) \text{ is a stationary subset of } \delta\Big\} \ne \emptyset \bmod I.$$

3) <u>Definition</u>: An ideal J on a set C is χ-regular if there is a set $A \subseteq C$, $A \ne \emptyset \bmod J$ and a function $f : A \to [\chi]^{<\aleph_0}$ such that
$\gamma < \chi \Rightarrow \{x \in A : \gamma \notin f(x)\} = \emptyset \bmod J$.
If $\chi = |C|$, we may omit it.
[How do we prove $(*)_1'$? Try χ times $E_\zeta, \langle e_\delta^\zeta : \delta \in S \rangle$ (for $\zeta < \chi$)].
4) We can try to get results like 3.1. Now

$(*)_2$ assume $\lambda, \mu, S, I, \bar C, \bar J$ are as in $(*)_0$ and $\bar e, \bar g$ as in $(*)_1'$ and $\kappa < \mu$ and for $\delta \in S, J_\delta^0 =: \{a \subseteq e_\delta : \{\alpha \in \mathrm{Dom}(g_\delta) : g(\alpha) \in a\} \in J_\delta\}$ is weakly normal and μ satisfies the condition from [Sh 365, Lemma 2.12]. <u>Then</u> we can find $h_\delta : e_\delta \to \kappa$ such that for every club E of λ,
$\{\delta \in S :$ for each $\gamma < \kappa$ the set $\{\alpha \in \mathrm{nacc}(C_\delta) : h_\delta(g_\delta(\alpha)) = \gamma\}$ is $\ne \emptyset \bmod J_\delta\} \ne \emptyset \bmod I$.

[Why? For each $\delta \in S, \alpha \in \mathrm{acc}(e_\delta)$ choose a club
$d_{\delta,\alpha} \subseteq e_\delta \cap \alpha$ such that for no club $d \subseteq e_\delta$ of δ do we
have $(\forall \gamma < \delta)(\exists \alpha \in \mathrm{acc}(e_\delta))[d \cap \gamma \subseteq d_{\delta,\alpha}]$. Now for every club E of λ let
$S_E = \{\delta : E \cap \mathrm{nacc}(C_\delta) \ne \emptyset \bmod J_\delta$, and $g_\delta''(E \cap \mathrm{nacc}(C_\delta))$ is stationary$\}$ and for $\delta \in E$ and $\varepsilon < \mu$, we choose by induction on $\zeta < \kappa, \xi(\delta, \varepsilon)$ as the first $\xi \in e_\delta$ such that: $\xi > \bigcup\limits_{\zeta<\varepsilon} \xi(\delta, \zeta)$ and $\{\alpha \in \mathrm{Dom}(g_\delta) : \alpha \in E$ and the ε-th member of $d_{\delta, g_\delta(\alpha)}$ is

in the interval $[\bigcup_{\zeta<\varepsilon} \xi(\delta,\zeta),\xi)]\} \neq \emptyset \bmod J_\delta$.

5) We deal below with successor of singulars and with inaccessibles, we can do parallel things.

3.6 Claim. *Suppose μ is a singular cardinal of cofinality $\kappa, \kappa > \aleph_0$ and $S \subseteq \{\delta < \mu^+ : cf(\delta) = \kappa\}$ is stationary, and $\bar C = \langle C_\delta : \delta \in S\rangle$ is an S-club system satisfying $\mu^+ \notin id^p(\bar C, \bar J^{b[\mu]})$ where $\bar J^{b[\mu]} = \langle J^{b[\mu]}_{C_\delta} : \delta \in S\rangle$ and $J^{b[\mu]}_{C_\delta} =: \{A \subseteq C_\delta : \text{for some } \theta < \mu, \text{ we have } \delta > \sup\{\alpha \in A : cf(\alpha) > \theta\}\}$. Then we can find a strict λ-club system $\bar e^* = \langle e^*_\delta : \delta < \lambda\rangle$ such that*

- $(*)$ *for every club E of μ^+, for stationarily many $\delta \in S$, for every $\alpha < \delta$ and $\theta < \mu$ for some β we have*
- $(**)_{E,\beta}$ *$\beta \in nacc(C_\delta)$ and $\beta > \alpha$ and $cf(\beta) > \theta$ and $\{\gamma \in e^*_\beta : \gamma \in E \text{ and } min(e^*_\beta \backslash (\gamma+1)) \text{ belongs to } E\}$ is a stationary subset of β.*

3.7 Remark. 1) We know that for the given μ and S there is $\bar C$ as in the assumption by [Sh 365, §2]. Moreover, if $\kappa > \aleph_0$ then there is such nice strict $\bar C$.
2) Remember $J^{b[\mu]}_\delta = \{A \subseteq C_\delta : \text{for some } \theta < \mu \text{ and } \alpha < \delta \text{ we have}$
$$(\forall \beta \in C_\delta)(\beta < \alpha \vee \text{ cf}(\beta) < \theta \vee \beta \in \text{ nacc}(C_\delta))\}.$$

Proof. Let $\bar e = \langle e_\beta : \beta < \lambda\rangle$ be a strict λ-club system where $e_\beta = \{\alpha^\beta_\zeta : \zeta < \text{ cf}(\beta)\}$ is a (strictly) increasing and continuous enumeration of e_β (with limit δ). Now we claim that for some $\bar h = \langle \bar h_\beta : \beta < \lambda, \beta \text{ limit}\rangle$ with h_β a function from e_β to e_β and $\bigwedge_{\alpha \in e_\beta} h_\beta(\alpha) > \alpha$, we have

$(*)_{\bar h}$ for every club E of μ^+, for stationarily many $\delta \in S \cap \text{ acc}(E)$, $A^\delta_E \notin J^{b[\mu]}_{C_\delta}$ where A^δ_E is the set of all $\beta \in C_\delta$ such that the following subset of e_β is stationary (in β):
$$\{\gamma \in e_\beta : \gamma \in E \text{ and } \min(e_\beta \backslash (\gamma+1)) \in E\}.$$

The rest is like the proof of 3.3 repeating κ^+ times instead ω and using "$J^{b[\mu]}_{C_\delta}$ is $(\leq \kappa)$-based". $\square_{3.6}$

3.8 Claim. *Suppose λ is inaccessible, $S \subseteq \lambda$ is a stationary set of inaccessibles, $\bar C$ an S-club system such that $\lambda \notin id^p(\bar C)$. Then we can find $\bar h = \langle h_\delta : \delta \in S\rangle$ with $h_\delta : C_\delta \to C_\delta$, such that $\alpha < h(\alpha)$ and*

- $(*)$ *for every club E of λ, for stationarily many $\delta \in S \cap acc(E)$ we have that*
$$\{\alpha \in C_\delta : \alpha \in E \text{ and } h(\alpha) \in E\} \text{ is a stationary subset of } \delta.$$

So for some $C'_\delta = \{\alpha_{\delta,\zeta} : \zeta < \delta\} \subseteq C_\delta, \alpha_{\delta,\zeta}$ *increasing continuous in* ζ *we have* $h(\alpha_{\delta,\zeta}) = \alpha_{\delta,\zeta+1}$.

Remark. Under quite mild conditions on λ and S there is $\bar{C}$ as required - see [Sh 365, 2.12,p.134].

Proof. Like 3.3.

3.9 Claim. *Let* $\lambda = cf(\lambda) > \aleph_0, S \subseteq \lambda$ *stationary,* D *a normal* λ^+*-saturated filter on* λ, S *is* D*-positive (i.e.* $S \in D^+$, $\lambda \backslash S \notin D$*).*
1) Assume $\langle (C_\delta, I_\delta) : \delta \in S \rangle$ *is such that*

(a) $C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathcal{P}(C_\delta)$,
(b) *for every club* E *of* λ,

$$\{\delta \in S : \textit{for some } A \in I_\delta \textit{ we have } \delta > \sup(A \backslash E)\} \in D^+.$$

Then for some stationary $S_0 \subseteq S, S_0 \in D^+$ *we have*
$(b)^+$ *for every club* E *of* λ

$$\{\delta \in S : \textit{for no } A \in I \textit{ do we have } \delta > \sup(A \backslash E)\} = \emptyset \textit{ mod } D.$$

2) Assume $\langle \mathcal{P}_\delta : \delta \in S \rangle$ *is such that (here really presaturated is enough)*

$(*)$ *for every* D*-positive* $S_0 \subseteq S$ *for some* D*-positive* $S_1 \subseteq S_0$ *and* $\langle (C_\delta, I_\delta) : \delta \in S \rangle$ *we have* $(C_\delta, I_\delta) \in \mathcal{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathcal{P}(C_\delta)$ *and for every club* E *of* λ
$\{\delta \in S_1 : \textit{for some } A \in I_\delta, \delta > \sup(A \backslash E)\} \neq \emptyset \textit{ mod } D$.

Then

$(**)$ *for some* $\langle (C_\delta, A_\delta) : \delta \in S \rangle$ *we have* $(C_\delta, I_\delta) \in \mathcal{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta)$, $I_\delta \subseteq \mathcal{P}(C_\delta)$ *and for every club* E *of* λ

$$\{\delta \in S : \textit{ for no } A \in I_\delta, \delta > \sup(A \backslash E)\} = \emptyset \textit{ mod } D.$$

Remark. This is a straightforward generalization of [Sh:e, III,§6.2B]. Independently Gitik found related results on generic extensions which were continued in [DjSh 562] and in [GiSh 577].

Proof. The same.

3.10 Lemma. *Suppose λ is regular uncountable and $S \subseteq \{\delta < \lambda^+ : cf(\delta) = \lambda\}$ is stationary. Then we can find $\langle (C_\delta, h_\delta, \chi_\delta) : \delta \in S\rangle$ and D such that*

(A) *D is a normal filter on λ^+,*
(B) *C_δ is a club of δ, say $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \lambda\}$, with $\alpha_{\delta,\zeta}$ increasing continuous in ζ,*
(C) *h_δ is a function from C_δ to $\chi_\delta, \chi_\delta \leq \lambda$,*
(D) *if $A \in D^+$ (i.e. $A \subseteq \lambda^+$ & $\lambda^+ \backslash A \notin D$) and E is a club of λ^+, then the following set belongs to D^+:*

$$B_{E,A} =: \Big\{\delta : \delta \in A \cap S, \delta \in \ acc(E) \text{ and for each } i < \chi_\delta$$
$$\{\zeta < \lambda : \alpha_{\delta,\zeta+1} \in E \text{ and } h_\delta(\alpha_{\delta,\zeta}) = i$$
$$\text{(and } \alpha_{\delta,\zeta} \in E)\} \text{ is a stationary subset of } \lambda\Big\}$$

(hence, for some $\alpha < \lambda^+$ and $\zeta < \lambda$, the set $B_{E,A,\alpha} =: \{\delta \in B_{E,A} : \alpha = \alpha_{\delta,\zeta}\}$ is in D^+).
(E) *If $\gamma < \lambda^+$ and χ satisfies one of the conditions listed below, then $S_{\gamma,\chi} = \{\delta \in S : \gamma = \ Min(C_\delta) \text{ and } \chi_\delta = \chi\} \in D^+$ where*

(α) $\lambda = \chi^+$,

(β) *λ is inaccessible not strongly inaccessible, $\chi < \lambda$ and there is T such that*

(a) *T is a tree with $< \lambda$ nodes and a set Γ of branches, $|\Gamma| = \lambda$,*

(b)$'$ *if $T' \subseteq T, T'$ downward closed and $(\exists^\lambda \eta \in \Gamma)$*

(η a branch of T') then T' has an antichain of cardinality $\geq \chi$,

(γ) *λ is inaccessible not strongly inaccessible and $\chi = \ Min\{\chi : \text{for some } \theta \leq \chi \text{ we have } \chi^\theta \geq \lambda\}$,*

(δ) *λ is strongly inaccessible not ineffable; i.e. λ is Mahlo and we can find $\bar{A} = \langle A_\mu : \mu < \lambda \text{ is inaccessible } \rangle$, $A_\mu \subseteq \mu$ so that for no stationary $\Gamma \subseteq \{\mu < \lambda : \mu \text{ inaccessible}\}$ and $A \subseteq \lambda$ do we have: $\mu \in \Gamma \Rightarrow A_\mu = A \cap \mu$.*

3.11 Remark. We can replace λ^+ in 3.10 and any $\mu = \ \mathrm{cf}(\mu) > \lambda$, as if $\mu > \lambda^+$ we have even a stronger theorem.

Proof. Let for $\lambda = \text{cf}(\lambda) > \aleph_0$,

$$\Theta = \Theta_\lambda = \Big\{\chi \le \lambda : \text{if } S' \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\} \text{ is stationary}$$

$$\underline{\text{then}} \text{ we can find } \langle (C_\delta, h_\delta) : \delta \in S'\rangle \text{ such that}$$
$$(a) \quad C_\delta \text{ is a club of } \delta \text{ of order type } \lambda,$$
$$(b) \quad h_\delta : C_\delta \to \chi,$$
$$(c) \quad \text{for every club } E \text{ of } \lambda^+ \text{ for stationarily many}$$
$$\delta \in S' \cap \text{ acc}(E) \text{ we have:}$$
$$i < \chi \Rightarrow B_E = \{\alpha \in C_\delta : \alpha \in E, h(\alpha) = i \text{ and}$$
$$\min(C_\delta \backslash (\alpha + 1)) \in E\}$$
$$\text{is a stationary subset of } \delta\Big\}.$$

Now we first show

$\bigotimes$ for each of the cases from clause (E), the χ belongs to Θ.

Proof of sufficiency of $\bigotimes$. We can partition S to λ^+ stationary sets so we can find a partition $\langle S_{\chi,\alpha} : \chi \in \Theta$ and $\alpha < \lambda^+\rangle$ of S to stationary sets. Without loss of generality, $\alpha \le \text{ Min}(S_{\chi,\alpha})$ and let $\langle (C^0_\delta, h^0_\delta) : \delta \in S_{\chi,\alpha}\rangle$ be as guaranteed by "$\chi \in \Theta$" for the stationary set $S_{\chi,\alpha}$. Now define C_δ, h_δ for $\delta \in S$ by:

C_δ is $C^0_\delta \cup \{\alpha\} \backslash \alpha$ <u>if</u> $\delta \in S_{\chi,\alpha}$ and $\alpha < \delta, h_\delta(\beta)$ is $h_\delta(\beta)$ if $\beta \in C_\delta \cap C^0_\delta$ and is zero otherwise. Of course, $\chi_\delta = \chi$ if $\delta \in S_{\chi,\alpha}$.

Lastly, let

$$D = \Big\{A \subseteq \lambda^+ : \text{for some club } E \text{ of } \lambda^+, \text{ for every}$$
$$\delta \in S \cap \text{ acc}(E) \backslash A \text{ for some } i < \chi_\delta,$$
$$\text{the set } \{\beta \in C_\delta : \beta \in E, h_\delta(\beta) = i \text{ and } \min(C_\delta \backslash (\beta + 1) \in E\}$$
$$\text{is not a stationary subset of } \delta\Big\}.$$

So D and $\langle (C_\delta, h_\delta, \chi_\delta) : \delta \in S\rangle$ have been defined, and we have to check clauses (A)-(E).

Note that $\Theta \ne \emptyset$ and the proof which appears later does not rely on the intermediate proofs.

<u>Clause (*A*)</u>: Suppose $A_\zeta \in D$ for $\zeta < \lambda$, so for each ζ there is a club E_ζ of λ^+

(∗) if $\delta \in S_{\chi,\gamma}$ and $\delta \in S \cap \text{ acc}(E) \backslash A_\zeta$ then $\{\alpha \in C_\delta : \alpha \in E, \text{Min}(C_\delta \backslash (\alpha + 1)) \in E$ and $h_\delta(\alpha) = i_\zeta\}$ is not stationary in δ.

Clearly clubs of λ^+ belong to D.
Clearly $A \supseteq A_\zeta \Rightarrow A \in D$ (by the definition), witnessed by the same E_ζ.
Also $A = A_0 \cap A_1 \in D$ as witnessed by $E = E_0 \cap E_1$.
Lastly, $A = \underset{\zeta<\lambda}{\triangle} A_\zeta = \{\alpha < \lambda^+ : \alpha \in \bigcap\limits_{\zeta<1+\alpha} A_\zeta\}$ belong to D as witnessed by $E = \{\alpha < \lambda^+ : \alpha \in \bigcap\limits_{\zeta<1+\alpha} E_\zeta\}$. Note that if $\delta \in S \cap \text{ acc}(E) \backslash A$ then for some $\zeta < \delta$

$$\delta \in S \cap \text{ acc}(E) \backslash A_\zeta \subseteq (S \cap \text{ acc}(E_\zeta) \backslash A_\zeta) \cup (1+\zeta)$$

as $E_\zeta \backslash E$ is a bounded subset of δ; included in $1+\zeta$ so from the conclusion of $(*)$ for δ, A_ζ, E_ζ we get it for ζ, A, E.

Lastly $\emptyset \notin D$; otherwise, let E be a club of λ^+ witnessing it, i.e. $(*)$ holds in this case. Choose $\chi \in \Theta$ and $\alpha = 0$ and use on it the choice of $\langle C^0_\delta : \delta \in S_{\chi,0}\rangle$ to show that for some $\delta \in S_{\chi,0} \subseteq S$ contradict the implication in $(*)$.

Clause (B): Trivial.

Clause (C): Trivial.

Clause (D): Note that we can ignore the "$\alpha_{\delta,\zeta} \in E$" as $\delta \in \text{ acc}(E)$ implies that it holds for a club of ζ's. Assume $A \in D^+$ (for clause (A)) and E is a club of λ^+, which contradicts clause (D) so $B_{E,A} \notin D^+$, hence $\lambda^+ \backslash B_{E,A} \in D$. Also E witnessed that $\lambda^+ \backslash (A \backslash B_{E,A}) \in D$ by the definition of D. But by clause (A) we know D is a filter on λ^+ so $(\lambda^+ \backslash B_{E,A}) \cap (\lambda^+ \backslash (A \backslash B_{E,A})$ belong to D, but this is the set $\lambda^+ \backslash B_{E,A} \backslash (A \backslash B_{E,A})$ which is (as $B_{E,A} \subseteq A$ by its definition) just $\lambda \backslash A$. So $\lambda \backslash A \in D$ hence $A \notin D^+$ - a contradiction.

Clause (E): By the proof of $\emptyset \notin D$ above, if $\chi \in \Theta$, also $S_{\chi,\alpha} \in D^+$, and by the definition of $\bar C, \bar C \restriction S_{\chi,\alpha}$ is as required. So it is enough to show

3.12 Claim. *If $\chi < \lambda =$ cf(λ) and χ satisfies one of the clauses of ?, then $\chi \in \Theta$ (from the proof of 3.10).*

Proof.
Case (α): By 3.1.

Case (β): Like the proof of 3.1, for more details see [Sh 413, §3].

Case (γ): This is a particular case of case (β).

Case (δ): Similar proof (or use 3.13). $\square_{3.12}\square_{3.10}$

More generally (see [Sh 413]):

3.13 Claim. *Let $\lambda = \mathrm{cf}(\lambda) > \chi$. A sufficient condition for $\chi \in \Theta_\lambda$ is the existence of some $\zeta < \lambda^+$ such that*

$\otimes$ *in the following game of length ζ, first player has no winning strategy: in the ε-th move first player chooses a function $f_\varepsilon : \lambda \to \chi$ and second player chooses $\beta_\varepsilon < \chi$. In the end, first player wins the play if $\{\alpha < \lambda : \text{for every } \varepsilon < \gamma, f_\varepsilon(\alpha) \neq \beta_\varepsilon\}$ is a stationary subset of λ.*

(If we weaken the demand in Θ_λ from stationary to unbounded in λ, we can weaken it here too).

§4 More on Pr_6

4.1 Claim. *$Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$ for λ regular.*

Proof. We can find $h : \lambda^+ \to \lambda^+$ such that for every $\gamma < \lambda^+$ the set $S_\gamma =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda \text{ and } h(\delta) = \gamma\}$ is stationary, so $\langle S_\gamma : \gamma < \lambda \rangle$ is a partition of $S =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$. We can find $\bar{C}^\gamma = \langle C_\delta : \delta \in S_\gamma \rangle$ such that C_δ is a club of δ of order type λ. For any $\nu \in {}^{\omega >}(\lambda^+)$ we define:

(a) for $\ell < \ell g(\nu)$, if $\nu(\ell) \in S$ then let
$a_\ell = a_{\nu,\ell} = \{\mathrm{otp}(C_{\nu(\ell)} \cap \nu(k)) : k < \ell g(\nu) \text{ and } \nu(k) < \nu(\ell)\}$,

(b) ℓ_ν is the $\ell < \ell g(\nu)$ such that
  (i) $\nu(\ell) \in S$,
  (ii) among those with $\sup(a_{\nu,\ell})$ is maximal, and
  (iii) among those with ℓ minimal,

(c) if ℓ_ν is well defined let $d(\nu) = h(\nu(\ell_\nu))$ otherwise let $d(\nu) = 0$.

Now suppose $\langle (u_\alpha, v_\alpha) : \alpha < \lambda^+ \rangle, \gamma < \lambda^+$ and E are as in Definition 2.1 and we shall prove the conclusion there. Let
$E^* = \{\delta \in E : \delta \text{ is a limit ordinal and } \alpha < \delta \Rightarrow \delta > \sup[\cup\{\mathrm{Rang}(\eta) : \eta \in u_\alpha \cup v_\alpha\}]\}$.
Clearly $E^* \subseteq E$ is a club of λ^+.

For each $\delta \in S_\gamma$ let

$$f_0(\delta) =: \sup[\delta \cap \bigcup\{\mathrm{Rang}(\nu) : \nu \in u_\delta \cup v_\delta\}].$$

As $\mathrm{cf}(\delta) = \lambda > |u_\alpha \cup v_\alpha|$ and the sequences are finite clearly $f_0(\delta) < \delta$. Hence by Fodor's lemma for some $\xi^*, S^1_\gamma =: \{\delta \in S_\gamma : f_0(\delta) = \xi^*\}$ is a stationary subset of λ^+ (note that γ is fixed here). Let $\xi^* = \bigcup\limits_{i<\lambda} a_{2,i}$ where $a_{2,i}$ is increasing with i and $|a_{2,i}| < \lambda$. So for $\delta \in S^1_\gamma$

$$f_1(\delta) = \mathrm{Min}\Big\{i < \lambda : \delta \cap \bigcup\{\mathrm{Rang}(\nu) : \nu \in u_\delta \cup v_\delta\} \text{ is a subset of } a_{2,i}\Big\}$$

is a well defined ordinal $< \lambda$, hence for some $i^* < \lambda$ the set

$$S^2_\gamma =: \{\delta \in S^1_\gamma : f_1(\delta) = i^*\}$$

is a stationary subset of λ^+. For $\delta \in S^2_\gamma$ let

$$b_\delta =: \Big\{ \mathrm{otp}(C_\beta \cap \alpha) : \alpha < \beta \in S \text{ and both}$$
$$\text{are in } a_{2,i^*} \cup \{\delta\} \cup \bigcup \{\mathrm{Rang}\, \nu : \nu \in u_\delta \cup v_\delta\} \Big\}.$$

So b_δ is a subset of λ of cardinality $< \lambda$ hence $\varepsilon_\delta =: \sup(b_\delta) < \lambda$, hence for some ε^*

$$S^3_\gamma =: \{\delta \in S^2_\gamma : \varepsilon_\delta = \varepsilon^*\}$$

is a stationary subset of λ^+. Choose β^* such that

(∗) $\beta^* \in S^3_\gamma \cap E^*$ and $\beta^* = \sup(\beta^* \cap S^3_\gamma \cap E^*)$.

As C_{β^*} has order type λ, (and is a club of β^*) for some $\alpha^* \in \beta^* \cap S^3_\gamma \cap E^*$ we have $\mathrm{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^*$.
We want to show that α^*, β^* are as required. Obviously $\alpha^* < \beta^*, \alpha^* \in E$ and $\beta^* \in E$. So assume $\nu \in u_{\alpha^*}, \rho \in v_{\beta^*}$ and we shall prove that $d(\nu\char`^\rho) = \gamma$, which suffices. As $h(\beta^*) = \gamma$ (as $\beta^* \in S^3_\gamma \subseteq S_\gamma$) it suffices to prove that $(\nu\char`^\rho)(\ell_{\nu\char`^\rho}) = \beta^*$. Now for some $\ell_0, \ell_1$ we have $\nu(\ell_0) = \alpha^*, \rho(\ell_1) = \beta^*$ (as $\nu \in u_{\alpha^*}, \rho \in v_{\beta^*}$) and since $\mathrm{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^*$, by the definition of $\ell_{\nu\char`^\rho}$ it suffices to prove

⊗ <u>if</u> $\ell, k < \ell g(\nu\char`^\rho), (\nu\char`^\rho)(\ell) \in S, (\nu\char`^\rho)(k) < (\nu\char`^\rho)(\ell)$ <u>then</u>
  (i) $\mathrm{otp}[C_{(\nu\char`^\rho)(\ell)} \cap (\nu\char`^\rho)(k)] \le \varepsilon^*$ or
  (ii) $(\nu\char`^\rho)(\ell) = \beta^*$.

Assume ℓ, k satisfy the assumption of ⊗ and we shall show its conclusion.

<u>Case 1</u>: If $(\nu\char`^\rho)(\ell)$ and $(\nu\char`^\rho)(k)$ belong to

$$a_{2,i^*} \cup \{\beta^*\} \cup \bigcup \{\mathrm{Rang}(\eta) : \eta \in u_{\beta^*} \cup v_{\beta^*}\}$$

then clause (i) holds because

(α) $\mathrm{otp}(C_{(\nu\char`^\rho)(\ell)} \cap (\nu\char`^\rho)(k)) \in b_{\beta^*}$ (see the definition of b_{β^*}) and
(β) $\sup(b_{\beta^*}) = \varepsilon_{\beta^*}$ (see the definition of ε_{β^*}) and
(γ) $\varepsilon_{\beta^*} = \varepsilon^*$ (as $\beta^* \in S^3_\gamma$ and see the choice of ε^* and S^3_γ).

<u>Case 2</u>: If $(\nu\char`^\rho)(\ell)$ and $(\nu\char`^\rho)(k)$ belong to

$$a_{2,i^*} \cup \bigcup \{\mathrm{Rang}(\eta) : \eta \in u_{\alpha^*} \cup v_{\alpha^*}\}$$

then the proof is similar to the proof of the previous case.

<u>Case 3</u>: No previous case.

So $(\nu\hat{\ }\rho)(\ell)$ and $(\nu\hat{\ }\rho)(k)$ are not in a_{2,i^*}, hence (as $\{\nu,\rho\} \subseteq (u_{\alpha^*} \cup v_{\beta^*})$, and $\{\alpha^*,\beta^*\} \subseteq S^2_\gamma \subseteq S^1_\gamma$)

$$m \in \{\ell,k\} \ \& \ m < \ell g(\nu) \Rightarrow (\nu\hat{\ }\rho)(m) = \nu(m) \geq \alpha^*,$$

$$m \in \{\ell,k\} \ \& \ m \geq \ell g(\nu) \Rightarrow (\nu\hat{\ }\rho)(m) = \rho(m - \ell g(\nu)) \geq \beta^*.$$

As $\beta^* \in E^*$ and $\beta^* > \alpha^*$ clearly $\sup(\mathrm{Rang}(\nu)) < \beta^*$, but also $(\nu\hat{\ }\rho)(k) < (\nu\hat{\ }\rho)(\ell)$ (see $\bigotimes$).

Together necessarily $k < \ell g(\nu), \nu(k) \in [\alpha^*,\beta^*)$, $\ell \in [\ell g(\nu), \ell g(\nu) + \ell g(\rho))$ and $\rho(\ell - \ell g(\nu)) \in [\beta^*,\lambda^+)$. If $\rho(\ell) = \beta^*$ then clause (ii) of the conclusion holds. Otherwise necessarily $\nu(\ell) > \beta^*$ hence

$$\begin{aligned} \mathrm{otp}(C_{(\nu\hat{\ }\rho)(\ell)}) \cap (\nu\hat{\ }\rho)(k)) &= \mathrm{otp}(C_{\rho(\ell-\ell g(\nu))} \cap \nu(k)) \\ &\leq \ \mathrm{otp}(C_{\rho(\ell-\ell g(\nu))} \cap \beta^*) \leq \sup(a_{\beta^*}) \leq \varepsilon^* \end{aligned}$$

so clause (i) of $\otimes$ holds. $\square_{4.1}$

4.2 Conclusion. For λ regular, $Pr_1(\lambda^{+2},\lambda^{+2},\lambda^{+2},\lambda)$ holds.

Proof. By 4.1 and 2.2(1). $\square_{4.2}$

4.3 Definition. 1) Let $Pr_6(\lambda,\theta,\sigma)$ means that for some Ξ, an unbounded subset of $\{\tau : \tau < \sigma, \tau$ is a cardinal (finite or infinite)$\}$, there is a $d : {}^{\omega>}(\lambda \times \Xi) \to \omega$ such that if $\gamma < \theta$ and $\tau \in \Xi$ are given and $\langle (u_\alpha, v_\alpha) : \alpha < \lambda\rangle$ satisfies

(i) $u_\alpha \subseteq {}^{\omega>}(\lambda \times \Xi) \backslash {}^{2\geq}(\lambda \times \Xi)$,
(ii) $v_\alpha \subseteq {}^{\omega>}(\lambda \times \Xi) \backslash {}^{2\geq}(\lambda \times \Xi)$,
(iii) $|u_\alpha| = |v_\alpha| = \tau$,
(iv) $\nu \in u_\beta \Rightarrow \nu(\ell g(\nu) - 1) = \langle \gamma, \tau\rangle$,
(v) $\rho \in u_\alpha \Rightarrow \rho(0) = \langle \gamma,\tau\rangle$,
(vi) $\eta \in u_\alpha \cup v_\alpha \Rightarrow (\exists \ell)(\eta(\ell) = \langle \alpha,\tau\rangle)$

<u>then</u> for some $\alpha < \beta$ we have

$$\nu \in u_\beta \ \& \ \rho \in v_\alpha \Rightarrow (\nu\hat{\ }\rho)[d(\nu\hat{\ }\rho)] = \langle \gamma,\tau\rangle.$$

2) Let $Pr_6(\lambda,\sigma)$ means $Pr_6(\lambda,\lambda,\sigma)$.

4.4 Fact. $Pr_6(\lambda,\lambda,\theta,\sigma), \theta \geq \sigma$ implies $Pr_6(\lambda,\theta,\sigma)$.

Proof. Let c be a function from ${}^{\omega>}\lambda$ to θ exemplifying $Pr_6(\lambda,\lambda,\theta,\sigma)$. Let e be a one to one function from $\theta \times \Xi$ onto θ.

Now we define a function d from ${}^{\omega>}(\lambda \times \Xi)$ to ω:

$$d(\nu) = \ \mathrm{Min}\{\ell : c(\langle e(\nu(m)) : m < \ell g(\nu)\rangle) = e(\nu(\ell))\}.$$

$\square_{4.4}$

4.5 Claim. *If* $Pr_6(\lambda^+, \sigma), \lambda$ *regular and* $\sigma \leq \lambda$ *then* $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$.

Proof. Like the proof of 1.1.

4.6 Remark. As in 4.1, 4.2 we can prove that if $\mu > \operatorname{cf}(\mu) + \sigma$ then $Pr^6(\mu^+, \mu^+, \mu^+, \sigma)$, hence $Pr_1(\mu^{+2}, \mu^{+2}, \mu^{+2}, \sigma)$, but this does not give new information.

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