COLOURING AND
NON-PRODUCTIVITY OF $\aleph_2$-C.C.

SH572

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Abstract. We prove that colouring of pairs from $\aleph_2$ with strong properties exists. The easiest to state (and quite a well known problem) it solves: there are two topological spaces with cellularity $\aleph_1$ whose product has cellularity $\aleph_2$; equivalently we can speak on cellularity of Boolean algebras or on Boolean algebras satisfying the $\aleph_2$-c.c. whose product fails the $\aleph_2$-c.c. We also deal more with guessing of clubs.

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§1 Retry at $\aleph_2$-c.c. not productive

[We prove $Pr_1(\aleph_1, \aleph_2, \aleph_2, \aleph_0)$ which is a much stronger result].

§2 The implicit properties

[We define a property implicit in §1, note what the proof in §1 gives, and look at related implication for successor of singular non-strong limit and show that $Pr_1$ implies $Pr_6$].

§3 Guessing clubs revisited

[We improve some results mainly from [Sh 413], giving complete proofs. We show that for $\mu$ regular uncountable and $\chi < \mu$ we can find $\langle C_\delta : \delta < \mu^+, cf(\delta) = \mu \rangle$ and functions $h_\delta$, from $C_\delta$ onto $\chi$, such that for every club $E$ of $\mu^+$ for stationarily many $\delta < \mu^+$ we have: $cf(\delta) = \mu$ and for every $\gamma < \chi$ for arbitrarily large $\alpha \in \text{nacc}(C_\delta)$ we have $\alpha \in E$, $h_\delta(\alpha) = \gamma$. Also if $C_\delta = \{\alpha_{\delta, \varepsilon} : \varepsilon < \mu\}$, $(\alpha_{\delta, \varepsilon}$ increasing continuous in $\varepsilon)$ we can demand $\{\varepsilon < \mu : \alpha_{\delta, \varepsilon+1} \in E \text{ (and } \alpha_{\delta, \varepsilon} \in E)\}$ is a stationary subset of $\mu$. In fact for each $\gamma < \mu$ the set $\{\varepsilon < \mu : \alpha_{\delta, \varepsilon+1} \in E, \alpha_{\delta, \varepsilon} \in E \text{ and } f(\alpha_{\delta, \varepsilon+1}) = \gamma\}$ is a stationary subset of $\mu$. We also deal with a parallel to the last one (without $f$) to successor of singulars and to inaccessibles].

§4 More on $Pr_1$

[We prove that $Pr_1(\lambda^+^2, \lambda^+^2, \lambda^+^2, \lambda)$ holds for regular $\lambda$].

On history, references and consequences see [Sh:g, AP1] and [Sh:g, III, §0].
§1 Retry at $\aleph_2$-c.c. not productive

1.1 Theorem. $Pr_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$.

1.2 Remark. 1) Is this hard? A priori it does not look so, but we have worked hard on it several times without success (worse: produce several false proofs). We thank Juhasz and Soukup for pointing out a gap.

2) Remember that

Definition $Pr_1(\lambda, \mu, \theta, \sigma)$ means that there is a symmetric two-place function $d$ from $\lambda$ to $\theta$ such that:

if $\langle u_\alpha : \alpha < \mu \rangle$ satisfies

$$u_\alpha \subseteq \lambda,$$

$$|u_\alpha| < \sigma,$$

$$\alpha < \beta \Rightarrow u_\alpha \cap u_\beta = \emptyset,$$

and $\gamma < \theta$ then for some $\alpha < \beta$ we have

$$\zeta \in u_\alpha \& \xi \in u_\alpha \Rightarrow d(\zeta, \xi) = \gamma.$$

3) If we are content with proving that there is a colouring with $\aleph_1$ colours, then we can simplify somewhat: in stage C we let $c(\beta, \alpha) = d_{sq}(\rho_{h_1}(\beta, \alpha))$ and this shortens stage D.

Proof.

Stage A: First we define a preliminary colouring. There is a function $d_{sq} : ^{\omega^2}(\omega_1) \to \omega_1$ such that:

$$\otimes$$ if $A \subseteq [\omega_1]^{\aleph_1}$ and $\langle (\rho_\alpha, \nu_\alpha) : \alpha \in A \rangle$ is such that $\rho_\alpha \in ^{\omega^2}\omega_1$, $\nu_\alpha \in ^{\omega^2}\omega_1$,

$\alpha \in \text{Rang}(\rho_\alpha) \cap \text{Rang}(\nu_\alpha)$ and $\gamma < \omega_1$ then for some $\zeta < \xi$ from $A$ we have: if $\nu', \rho'$ are subsequences of $\nu_\zeta, \rho_\xi$ respectively and $\zeta \in \text{Rang}(\nu'), \xi \in \text{Rang}(\rho')$ then

$$d_{sq}(\nu'^\zeta, \rho') = \gamma.$$

Proof of $\otimes$. Choose pairwise distinct $\eta_\alpha \in \omega^2$ for $\alpha < \omega_1$. Let $d_0 : [\omega_1]^2 \to \omega_1$ be such that:

$$(*)$$ if $n < \omega$ and $\alpha_\zeta, \ell < \omega_1$ for $\zeta < \omega_1, \ell < n$ are pairwise distinct and $\gamma < \omega_1$ then for some $\zeta < \xi < \omega_1$ we have $\ell < n \Rightarrow \gamma = d_0(\{\alpha_\zeta, \ell, \alpha_\xi, \ell\})$ (exists by [Sh 261, see (2.4), p.176] the $n$ there is 2).
Define $d_{sq}(\nu)$ for $\nu \in \omega^>(\omega_1)$ as follows. If $\ell g(\nu) \leq 1$ or $\nu$ is constant then $d_{sq}(\nu) = 0$. Otherwise let 

$$n(\nu) = \max\{\ell g(\eta_{v(\ell)} \cap \eta_{\nu(k)}) : \ell < k < \ell g(\nu) \text{ and } \nu(\ell) \neq \nu(k)\} < \omega.$$ 

The maximum is on a non-empty set as $\ell g(\nu) \geq 2$ and $\nu$ is not constant, remember $\eta_\alpha \in \omega^2$ were pairwise distinct so $\nu(\ell) \neq \nu(k) \Rightarrow \eta_{\nu(\ell)} \cap \eta_{\nu(k)} \in \omega^2$ (is the largest common initial segment of $\eta_{\nu(\ell)}, \eta_{\nu(k)}$). Let $a(\nu) = \{\ell, k : \ell < k < \ell g(\nu) \text{ and } \ell g(\eta_{v(\ell)} \cap \eta_{\nu(k)}) = n(\nu)\}$ so $a(\nu)$ is non-empty and choose the (lexicographically) minimal pair $(\ell, k, \nu)$ in it. Lastly let 

$$d_{sq}(\nu) = d_0(a(\nu, \nu(k, \nu))).$$ 

So $d_{sq}$ is a function with the right domain and range. Now suppose we are given $A \in [\omega_1]^{\aleph_1}$, $\gamma < \omega_1$ and $\rho_\alpha, \nu_\alpha \in \omega^>(\omega_1)$ for $\alpha \in A$ such that $\alpha \in \text{Rang}(\rho_\alpha) \cap \text{Rang}(\nu_\alpha)$. We should find $\alpha < \beta$ from $A$ such that $d_{sq}(\nu' \cap \rho') = \gamma$ for any subsequences $\nu', \rho'$ subsequences of $\nu_\alpha, \rho_\beta$ respectively such that $\alpha \in \text{Rang}(\nu')$ and $\beta \in \text{Rang}(\rho')$.

For each $\alpha \in A$ we can find $m_\alpha < \omega$ such that:

\[ (*)_0 \text{ if } \ell < k < \ell g(\nu_\alpha \cap \rho_\alpha) \text{ and } (\nu_\alpha \cap \rho_\alpha)(\ell) \neq (\nu_\alpha \cap \rho_\alpha)(k) \text{ then } \eta(\nu_\alpha \cap \rho_\alpha)(\ell) \upharpoonright m_\alpha \neq \eta(\nu_\alpha \cap \rho_\alpha)(k) \upharpoonright m_\alpha. \]

Next we can find $B \subseteq [A]^{\aleph_1}$ such that for all $\alpha \in B$ (the point is that the values do not depend on $\alpha$) we have:

(a) $\ell g(\nu_\alpha) = m_0, \ell g(\rho_\alpha) = m_1$,
(b) $a^* = \{(\ell, k) : \ell < k < m_0 + m_1 \text{ and } (\nu_\alpha \cap \rho_\alpha)(\ell) = (\nu_\alpha \cap \rho_\alpha)(k)\},$
(c) $b^* = \{\ell : m_0 + m_1 - \alpha = (\nu_\alpha \cap \rho_\alpha)(\ell)\},$
(d) $m_\alpha = m_0^2,$
(e) $\eta(\nu_\alpha \cap \rho_\alpha)(\ell) \upharpoonright m_\alpha : \ell < m_0 + m_1 = \tilde{\eta}^*,$
(f) $\text{Rang}(\nu_\alpha \cap \rho_\alpha) : \alpha \in B$ is a $\Delta$-system with heart $w,$
(g) $u^* = \{\ell : (\nu_\alpha \cap \rho_\alpha)(\ell) \in w\} \text{ so } u^* \neq \{\ell : \ell < m_0 + m_1\}$ as $\alpha \in \text{Rang}(\nu_\alpha \cap \rho_\alpha),$
(h) $\alpha_\ell^* = (\nu_\alpha \cap \rho_\alpha)(\ell)$ for $\ell \in u^*,$
(i) if $\alpha < \beta \in B$ then sup $\text{Rang}(\nu_\alpha \cap \rho_\alpha) < \beta.$

For $\zeta \in B$ let $\tilde{\beta}^* = ((\nu_\zeta \cap \rho_\zeta)(\ell) : \ell < m_0 + m_1, \ell \notin u^*)$ and apply $(*)$, i.e. the choice of $d_0$. So for some $\zeta < \xi$ from $B$, we have

$$\ell < m_0 + m_1 \text{ and } \ell \notin u^* \Rightarrow \gamma = d_0((\nu_\zeta \cap \rho_\zeta)(\ell), (\nu_\zeta \cap \rho_\zeta)(\ell)).$$

We shall prove that $\zeta < \xi$ are as required (in $\otimes$). So let $\nu', \rho'$ be subsequences of $\nu_\zeta, \rho_\xi$ (so let $\nu' = \nu_\zeta \upharpoonright v_1$ and $\rho' = \rho_\xi \upharpoonright v_2$) such that $\zeta \in \text{Rang}(\nu')$ and $\xi \in \text{Rang}(\rho')$ and we have to prove $\gamma = d_{sq}(\nu' \cap \rho')$. Let $\tau = \nu' \cap \rho'$, so $\tau = (\nu_\zeta \cap \rho_\xi \upharpoonright (v_1 \cup (m_0 + v_2))$ (in a slight abuse of notation, we look at $\tau$ as a function with domain $v_1 \cup (m_0 + v_2)$ and also as a member of $\omega^>(\omega_1)$ where $m + v =: \{m + \ell : \ell \in v\}$, of course). By the definition of $d_{sq}$ it is enough to prove the following two things:

\[ (*)_1 \text{ if } \ell g(\nu' \cap \rho') \geq m_2 \text{ (see clause } (d) \text{ and } (*)_0 \text{ above),} \]

\[ (*)_2 \text{ for every } \ell_1, \ell_2 \in v_1 \cup (m_0 + v_2) \text{ we have } \ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m_2, \omega) \Rightarrow \gamma = d_0((\tau(\ell_1), \tau(\ell_2))). \]
Proof of \((\ast)_1\). Let \(\ell_1 \in v_1\) and \(\ell_2 \in v_2\) be such that \(\nu_\zeta(\ell_1) = \zeta\) and \(\nu_\xi(\ell_2) = \xi\). So clearly \(\ell_1, m^0 + \ell_2 \in b^*\) (see clause (c)) and 

\[
\eta_{\rho_\xi(\ell_2)} \upharpoonright m^2 = \eta_{\rho_\xi(\ell_2)} \upharpoonright m^2 = \eta_{\nu_\zeta(\ell_1)} \upharpoonright m^2 \quad \text{(first equality as \(\zeta, \xi \in B\)}
\]

and \(m_\zeta = m_\xi = m^2\) (see clause (d) and (e)), second equality as \(\eta_{\rho_\xi(\ell_2)} = \eta_{\nu_\zeta(\ell_1)}\) since \(\ell_1, m^0 + \ell_2 \in b^*\) (see clause (c)). But \(\rho_\xi(\ell_2) = \xi \neq \zeta = \nu_\zeta(\ell_1)\), hence \(\eta_{\rho_\xi(\ell_2)} \neq \eta_{\nu_\zeta(\ell_1)}\), so together with the previous sentence we have

\[
m^2 \leq \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(m^0 + \ell_2)}) < \omega.
\]

Hence \(n(\tau) \geq m^2\) as required in \((\ast)_1\).

Proof of \((\ast)_2\). If \(\ell_1 < \ell_2\) are from \(v_1\), by the choice of \(m^2 = m_\zeta\) it is easy. Namely, if \((\ell_1, \ell_2) \in a(\tau)\) then \((\ell_1, \ell_2) \in a(\nu_\zeta)\) and \(\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) = \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\nu_\zeta(\ell_2)}) < m_\zeta = m^2\). If \(\ell_1, \ell_2 \in m^0 + v^2\), by the choice of \(m^2 = m_\xi\) similarly it is easy to show \(\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) < m^2\). So it is enough to prove

\[
(\ast)_3 \text{ assume } \ell_1 \in v_1, \ell_2 \in v_2 \text{ and } \\
\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) \in [m^2, \omega) \text{ then } \\
g = d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\}).
\]

Now the third assumption in \((\ast)_3\) means \(\eta_{\nu_\zeta(\ell_1)} \upharpoonright m^2 = \eta_{\rho_\xi(\ell_2)} \upharpoonright m^2\). Together we know that \(\eta_{\nu_\zeta(\ell_1)} \upharpoonright m^2 = \eta_{\rho_\xi(\ell_2)} \upharpoonright m^2\), hence by the choice of \(m_\zeta = m^2\) necessarily \(\eta_{\nu_\zeta(\ell_1)} = \eta_{\rho_\xi(\ell_2)}\) so that \(\nu_\zeta(\ell_1) = \rho_\xi(\ell_2)\) and (see clause (b)) also \(\nu_\zeta(\ell_1) = \rho_\xi(\ell_2)\). So

\[
d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\}) = d_0(\{\nu_\zeta(\ell_1), \nu_\zeta(\ell_1)\}).
\]

The latter is the required \(g\) provided that \(\ell_1 \notin u^*\). Equivalently \(\nu_\zeta(\ell_1) \neq \nu_\xi(\ell_1)\) but otherwise also \(\nu_\zeta(\ell_1) = \rho_\xi(\ell_2)\) so \(\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \omega\), contradicting the assumption of \((\ast)_3\) that \(\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m^2, \omega)\) (so it is not equal to \(\omega\)). So we finish\(^1\) proving \((\ast)_2\), hence \(\Diamond\).

Stage B: Like Stage A of [Sh:g, III,4.4,p.164]'s proof. (So for \(\alpha < \beta < \omega_2\), \(\alpha\) does not appear in \(\rho(\beta, \alpha)\)).

Stage C: Defining the colouring:

Remember that \(S^+_\beta = \{\delta < \aleph_\alpha : \text{cf}(\delta) = \aleph_\beta\}\).

For \(\ell = 1, 2\) choose \(h_\ell : \omega_2 \to \omega_\ell\) such that \(S^+_\alpha = S^+_1 \cap h_\ell^{-1}(\{\alpha\})\) is stationary for each \(\alpha < \omega_\ell\). For \(\alpha < \omega_2\), let \(A_\alpha \subseteq \omega_1\) be such that no one is included in the union of finitely many others.

For \(\alpha < \beta < \omega_2\), let \(\ell = \ell_{\beta, \alpha}\) be minimal such that

\[
d_{sq}(\rho_{h_\ell}(\beta, \alpha)) \in A_{\rho(\beta, \alpha)(\ell)}
\]

and lastly let

\(^1\)see alternatively 2.2(1) + 4.1
\[c(\beta, \alpha) = c(\alpha, \beta) =: h_2\left(\rho(\beta, \alpha)(\ell, \alpha)\right).\]

**Stage D:** Proving that the colouring works:

So assume \(n < \omega, (u_\alpha : \alpha < \omega_2)\) is a sequence of pairwise disjoint subsets of \(\omega_2\) of size \(n\) and \(\gamma(*) < \omega_2\) and we should find \(\alpha < \beta\) such that \(c \upharpoonright (u_\alpha \times u_\beta)\) is constantly \(\gamma(*).\)

Without loss of generality \(\alpha < \beta \Rightarrow \max(u_\alpha) < \min(u_\beta)\) and \(\min(u_\alpha) > \alpha\) and let \(E = \{\delta : \delta \text{ a limit ordinal } < \omega_2 \text{ and } ((\forall \alpha)(\alpha < \delta \Rightarrow u_\alpha \subseteq \delta)\}\). Clearly \(E\) is a club of \(\omega_2\). For each \(\delta \in E \cap S^2_1\), there is \(\alpha^*_\delta < \omega_1\) such that \(\alpha \in [\alpha^*_\delta, \delta)\) \& \(\beta \in u_\delta \Rightarrow \rho(\beta, \delta)^{(\delta)} \subseteq \rho(\beta, \alpha)\).

Also for \(\delta \in S^2_1\) let

\[\varepsilon_\delta =: \text{Min}\left\{\varepsilon < \omega_1 : \zeta \in A_\delta \text{ but if } \alpha \in \bigcup_{\beta \in u_\delta} \text{Rang}(\rho(\beta, \delta)) \right\},\]

(so \(\alpha > \delta\) then \(\varepsilon \notin A_\alpha\}).

Note that \(\varepsilon_\delta < \omega_1\) is well defined by the choice of \(A_\alpha\)'s. So, by Fodor’s lemma, for some \(\zeta^* < \omega_1\) and \(\alpha^* < \omega_2\) we have that

\[W =: \{\delta \in S^2_2(\zeta^*): \alpha^*_\delta = \alpha^* \text{ and } \varepsilon_\delta = \varepsilon^*\}\]

is stationary. Let \(h\) be a strictly increasing function from \(\omega_2\) into \(W\) such that \(\alpha^* < h(\delta)\). By the demand on \(\alpha^*\) (and \(W\))

\[\alpha^* < \alpha < \delta \in W \text{ & } \beta \in u_\delta \Rightarrow \rho(\beta, \delta)^{(\delta)} \subseteq \rho(\beta, \alpha)\).

Hence

\[\bigoplus_0 \alpha^* < \alpha < \delta \in W \text{ & } \beta \in u_\delta \Rightarrow \rho(\beta, \delta)^{(\delta)} \subseteq \rho(\beta, \alpha)\).

\[\bigoplus_1 \alpha^* < \alpha < \delta \in S^2_1 \text{ & } \beta \in u_{h(\delta)} \Rightarrow \text{Min}\{\ell : \varepsilon^* \in A_{\rho(\beta, \alpha)(\ell)}\} = \text{Min}\{\ell : \rho(\beta, \delta)(\ell) = h(\delta)\},\]

hence

\[\bigoplus_2 \alpha^* < \alpha < \delta \in S^2_1 \text{ & } \beta \in u_{h(\delta)} \Rightarrow\]

\[h_2\left(\rho(\beta, \delta)\left[\text{Min}\{\ell : \varepsilon^* \in A_{\rho(\beta, \delta)(\ell)}\}\right]\right) = \gamma(*)\).

Let
\[
E_0 =: \left\{ \delta \in \omega_2 : \delta \text{ a limit ordinal}, \delta \in E \text{ and } \alpha < \delta \Rightarrow h(\alpha) < \delta \text{ (hence } \sup(u_{h(\alpha)}) < \delta) \right\}.
\]

For each \( \delta \in S_1^\omega \) there is \( \alpha_{\delta}^{**} < \delta \) such that \( \alpha_{\delta}^{**} > \alpha^* \) and \( \alpha \in [\alpha_{\delta}^{**}, \delta) \& \beta \in u_{h(\delta)} \Rightarrow \rho(\beta, \delta)^-(\delta) \leq \rho(\beta, \alpha). \)

For each \( \gamma < \omega_1, \delta \mapsto \alpha_{\delta}^{**} \) is a regressive function on \( S_1^\omega \), hence for some \( \alpha^{**}(\gamma) < \delta \) the set \( S_1^{\omega} =: \{ \delta \in S_1^\omega \cap E_0 : \alpha_{\delta}^{**} = \alpha^{**}(\gamma) \} \) is stationary.

Let \( \alpha^{**} = \sup\{\alpha^{**}(\gamma) + 1 : \gamma < \omega_1\} \) and note that \( \alpha^{**} < \omega_2 \).

\[
E_1 =: \{ \delta < \omega_2 : \text{for every } \gamma < \omega_1, \delta = \sup(S_1^{\omega} \cap \delta) \text{ and } \delta > \alpha^{**} \},
\]

and note that \( E_1 \) is a club of \( \aleph_2 \) (and as \( S_1^{\omega} \subseteq E_0 \) clearly \( E_1 \subseteq E_0 \)) and choose \( \delta^* \in E_1 \cap S_2^{\gamma(\delta)} \). Then by induction on \( i < \omega_1 \) choose an ordinal \( \zeta_i \) such that \( (\zeta_i : i < \omega_1) \) is strictly increasing with limit \( \delta^* \) and \( \zeta_i \in S_1^{\omega}(\alpha^{**} + 1) \). We know that \( \alpha < \zeta_i \Rightarrow u_\alpha \subseteq \zeta_i \) and \( \alpha < \min(u_\alpha) \), hence for every \( \alpha_i < \zeta_i \) large enough \((\forall \beta \in u_\alpha_i)(\rho(\delta^*, \zeta_i)^-(\zeta_i) \leq \rho(\delta^*, \beta)). \)

Choose such \( \alpha_i \in (\bigcup_{j < i} \zeta_i) \). Lastly for \( i < \omega_1 \) choose \( \beta_i \in E \cap S_1^\omega \) with \( \beta_i > \delta^* \).

Now for each \( i < \omega_1 \) for some \( \xi(i) < \delta^* \),

\[
\alpha \in (\xi(i), \delta^*) \& \beta \in u_{h(\beta_i)} \Rightarrow \rho(\beta, \delta^*)^-(\delta^*) \leq \rho(\beta, \alpha).
\]

As \( \delta^* = \bigcup_{i < \omega_1} \zeta_i \), without loss of generality \( \xi(i) = \zeta_j(i) \), and \( j(i) \) is (strictly) increasing with \( i \) and let \( A =: \{ \varepsilon < \omega_1 : \varepsilon \text{ a limit ordinal and } (\forall i < \varepsilon)(j(i) < \varepsilon) \} \). Clearly \( A \) is a club of \( \omega_1 \). Now putting all of this together we have:

\begin{align*}
(*)_1 & \text{ if } i(0) < i(1) \text{ are in } A, \alpha \in u_{\alpha_{i(1)}}, \beta \in u_{h(\beta_{i(0)})} \text{ then } \\
& \rho(\beta, \alpha) = \rho(\beta, \delta^*)^-(\delta^*) = \rho(\delta^*, \alpha). \\
& \text{[Why? As } j(i(0)) < i(1), \text{ see } \Theta_3].
\end{align*}

\begin{align*}
(*)_2 & \text{ if } i < \omega_1 \text{ then } \beta \in u_{h(\beta_i)} \Rightarrow i \in \text{Rang}(\rho_{h_i}(\beta, \delta^*)) \text{ (witnessed by } \beta_i \text{ which belongs to this set by } \Theta_1). \\
(*)_3 & \text{ if } i < \omega_1 \text{ then } \alpha \in u_{\alpha_i} \Rightarrow i \in \text{Rang}(\rho_{h_i}(\delta^*, \alpha)) \text{ (witnessed by } \zeta_i \text{ which belongs to this set by the choice of } \alpha_i). \\
(*)_4 & \text{ if } i < \omega_1 \text{ and } \beta \in u_{h(\beta_i)} \text{ then } \ell = \text{Min}\{\ell : \zeta^* \in A_{\rho(\beta, \delta^*)(\ell)}\} \text{ is well defined and } h_2(\rho(\beta, \delta^*)(\ell)) = \gamma(\ast). \\
& \text{[Why? By } \Theta_2].
\end{align*}

Now let \( \nu_i \), for \( i < \omega_1 \), be the concatenation of \( \{\rho(\beta, \delta^*) : \beta \in u_{\beta_i}\} \) and \( \rho_i \) be the concatenation of \( \{\rho(\delta^*, \alpha) : \alpha \in u_{\alpha_i}\} \). So we can apply \( \otimes \) of Stage A to \( \langle \nu_i, \rho_i : i < \omega_1 \rangle \) and \( \gamma^* \) (its assumptions hold by \((*)_1 + (*)_2 + (*)_3) \) and get that for
some $i < j < \omega_1$ we have $d_{sq}(\nu' \rho') = \zeta^*$ whenever $\nu'$ is a subsequence of $\nu_i, \rho'$ a subsequence of $\rho_j$ such that $i \in \text{Rang}(\nu'), j \in \text{Rang}(\rho')$. Now for $\beta \in u_{h(\beta_i)}$, $\alpha \in u_{\alpha_j}$ we have

\[ \rho(\beta, \alpha) = \rho(\beta, \delta^*) \cdot \rho(\delta^*, \alpha) \text{ (see } (*)_1) \text{ and } \]

\[ \rho(\beta, \delta^*) \text{ is O.K. as } \nu'. \]

[Why? Because it is a subsequence of $\nu_i$ (see the choice of $\nu_i$) and $i$ belongs to $\text{Rang}(\rho(\beta, \delta^*))$ by $(*)_2$] and

\[ \rho(\delta^*, \alpha) \text{ is O.K. as } \rho'. \]

[Why? Because $\rho(\delta^*, \alpha)$ is a subsequence of $\rho_j$ by the choice of $\rho_j$ and $j$ belongs to $\text{Rang}(\rho(\delta^*, \alpha))$ by $(*)_3$.]

Now by $(*)_4$ the colour $c(\beta, \alpha)$ is $\gamma(\ast)$ as required and get the desired conclusion. \hspace{1cm} \Box_{1,1}

Remark. Can we get $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ for $\lambda$ regulars by the above proof? If $\lambda = \lambda^{<\lambda}$ the same proof works (now $\text{Dom}(d_{sq}) = \omega^>(\lambda^+)$ and $\nu_\alpha, \rho_\alpha \in \lambda^>(\lambda^+)$. See more in §2.
§2 Larger Cardinals: The implicit properties

More generally (than in the remark at the end of §1):

2.1 Definition. 1) $Pr_6(\lambda, \lambda, \theta, \sigma)$ means that there is $d: \omega^+ \rightarrow \theta$ such that:

$$\langle (u_\alpha, v_\alpha) : \alpha < \lambda \rangle$$ satisfies

$$u_\alpha \subseteq \omega^\lambda, v_\alpha \subseteq \omega^\lambda,$$

$$|u_\alpha \cup v_\alpha| < \sigma,$$

$$\nu \in u_\alpha \cup v_\alpha \Rightarrow \alpha \in \text{Rang}(\nu),$$

and $\gamma < \theta$ and $E$ a club of $\lambda$ then for some $\alpha < \beta$ from $E$ we have

$$\nu \in u_\alpha \land \rho \in v_\beta \Rightarrow d(\nu \rho) = \gamma.$$

2) $Pr_6^S(\lambda, \lambda, \theta, \sigma)$ is defined similarly but $\alpha < \beta$ are required to be in $E \cap S$. $Pr_\tau^S(\lambda, \lambda, \theta, \sigma)$ means "for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) \geq \tau\}$ we have $Pr_6^S(\lambda, \lambda, \theta, \sigma)$". If $\tau$ is omitted, we mean $\tau = \sigma$. Lastly $Pr_{nacc}^S(\lambda, \lambda, \theta, \sigma)$ is defined similarly but demanding $\alpha, \beta \in \text{nacc}(E)$ and $Pr_6^S(\lambda, \lambda, \theta, \sigma)$ is defined similarly but $E = \lambda$.

2.2 Lemma. 0) If $Pr_6(\lambda, \lambda, \theta, \sigma)$ and $\theta_1 \leq \theta$ and $\sigma_1 \leq \sigma$ then $Pr_6(\lambda, \lambda, \theta_1, \sigma_1)$ (and similar monotonicity properties for Definition 2.1(2)). Without loss of generality $u_\alpha = v_\alpha$ in Definition 2.1.

1) If $Pr_6(\lambda^+, \lambda^+, \lambda, \lambda)$, then $Pr_1(\lambda^{++}, \lambda^{++}, \lambda^+, \lambda)$.

2) If $Pr_6(\lambda^+, \lambda^+, \theta, \sigma)$, so $\theta \leq \lambda^+$ then $Pr_1(\lambda^{++}, \lambda^{++}, \lambda^{++}, \sigma)$ provided that

$$(\ast) \text{there is a sequence } \bar{A} = \langle A_\alpha : \alpha < \lambda^{++} \rangle \text{ of subsets of } \theta \text{ such that for every } \alpha \in U \subseteq \lambda^{++} \text{ with } u \text{ of cardinality } < \sigma, \text{ we have}$$

$$A_\alpha \uplus \{A_\beta : \beta \in U, \beta \neq \alpha\} \neq \emptyset.$$  

3) If $\lambda$ is regular and $\lambda = \lambda^{<\lambda}$ then $Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$.

4) In [Sh:g, III, 4.7] we can change the assumption accordingly.

Proof. 0) Clear.

1) By part (2) choosing $\theta = \lambda^+, \sigma = \lambda$ as $(\ast)$ holds as $\lambda^+$ is regular (so e.g. choose by induction on $\alpha < \lambda^{++}, A_\alpha \subseteq \lambda^+$ see unbounded non-stationary with $\beta < \alpha \Rightarrow |A_\alpha \cap A_\alpha| \leq \lambda$.

2) Like the proof for $\aleph_2$, only now $\{\delta < \lambda^{++} : \text{cf}(\delta) = \lambda^+\}$ plays the role of $\mathcal{S}_1^2$ and let $h_1 : \lambda^{++} \rightarrow \lambda^+$ and $h_2 : \lambda^{++} \rightarrow \lambda^{++}$ be such that for every $\gamma < \lambda^{+\ell}$ and $\ell \in \{1, 2\}$ the set $S_\gamma^{\ell} = \{\alpha < \lambda^{+\ell} : \text{cf}(\alpha) = \lambda^+ \text{ and } h_\ell(\alpha) = \gamma\}$ is stationary.

Finally, if $dq$ exemplifies $Pr_6(\lambda^+, \lambda^+, \theta, \sigma)$, then in defining $c$ for a given $\alpha < \beta$, let $\ell_{\alpha, \beta}$ be the minimal $\ell$ such that $dq(\rho_{n_1}(\alpha, \beta))$ belongs to $A_{\rho_{n_1}(\alpha, \beta)}(\ell)$ and let $c(\beta, \alpha) = c(\alpha, \beta) = h_2(\rho(\beta, \alpha)(\ell_{\beta, \alpha}))$. Then in stage D without loss of generality
|u_α| = σ < λ for α < λ⁺ and continue as there, but after the definition of E₁ we do not choose ζ₁, α₁ instead we first continue choosing β₁, ξᵢ for i < λ⁺ as there as without loss of generality δ⁺ = \bigcup_{i<λ⁺} ξ(i). Only then we choose by induction on i < λ⁺ the ordinal ζᵢ such that: ζᵢ \in S_i\setminus(α⁺⁺ + 1), ζᵢ > sup\{ξ(j) : j ≤ i\} \cup \{ζ_j : j < i\} and then choose αᵢ < ζᵢ large enough (so no need of the club A of λ⁺).

3) As in the proof of 1.1, Stage A.
4) Combine the proofs here and there (and not used).

This leaves some problems on Pr₁ open; e.g.

2.3 Question. 1) If λ > ℵ₀ is inaccessible, do we have Pr₁(λ⁺, λ⁺, λ⁺, λ) (rather than with σ < λ)?
2) If μ > ℵ₀ is regular (singular) and λ = μ⁺, do we have Pr₁(λ⁺, λ⁺, λ⁺, μ)?

[clearly, yes, for the weaker version: c a symmetric two place function from λ⁺ to λ⁺ such that for every γ < λ⁺ and pairwise disjoint \langle u_α : α < λ⁺ \rangle with u_α ∈ [λ⁺]<λ we have

\[(∃α < β) \forall i \in u_α \forall j \in u_β(γ \in \text{ Rang} \rho_c(j,i))].

See more in §4. Remember that we know Pr₁(λ⁺², λ⁺², λ⁺², σ) for σ < λ.

2.4 Claim. Assume μ is singular, λ = μ⁺, 2κ > μ > κ = κ^θ, θ = cf(θ) ≥ σ and Pr₆(θ, θ, θ, σ). Then Pr₁(μ⁺, μ⁺, θ, σ).

Proof. Let \(\vec{e} = (e_α : α < λ)\) be a club system, \(S \subseteq \{δ < μ⁺ : cf(δ) = θ\}\) stationary such that λ \notin \text{id}^a(\vec{e} | S) and α \in e_δ \Rightarrow cf(α) \neq θ and

δ = sup(δ∩S) & \& \chi < μ \Rightarrow δ = sup\{α \in e_δ : cf(α) > χ + σ⁺, \text{ so } α \in \text{nacc}(e_δ)\}

and α \in e_δ ∩ S \Rightarrow e_α \subseteq e_δ (exists by [Sh 365, 2.10]). Let ċf = \langle f_α : α < θ\rangle, f_α : μ⁺ → κ such that every partial function g from μ⁺ to κ (really σ suffice) of cardinality ≤ θ is included in some f_α (exist by [EK] or see [Sh:g, AP1.7]).

So for some f = f_α(∗) we have

(∗) for every club E of μ⁺ for some δ \in S we have:

(a) \(e_δ \subseteq E\)
(b) if \(χ < μ\) and \(γ < θ\) then
\[δ = \text{ sup}\{α \in \text{nacc}(e_δ) : f(α) = γ \text{ and } cf(α) > χ\}\].

This actually proves \text{id}_{μ⁺}(\vec{e} | S) is not weakly θ⁺-saturated.

The rest is by combining the trick of [Sh:g, III,§4] (using first δ(∗) ∈ S then some suitable α ∈ nacc(e_(δ(∗))) and the proof for ℵ₂. \(\square_{2.4}\)
2.5 Fact. \( Pr_1(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda)) \) implies \( Pr^6(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda)) \).

Remark. This is not totally immediate as in \( Pr_1 \) the sets are required to be pairwise disjoint.

Proof. Let \( \kappa = \text{cf}(\lambda) \) and \( f_\alpha \in {}^*\lambda \) for \( \alpha < \lambda^+ \) be such that \( \alpha < \beta \Rightarrow f_\alpha < {}^*_\kappa f_\beta \).

Let \( d : [\lambda^+]^2 \to \theta \) exemplifies \( Pr_1(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda)) \). Let \( c : \kappa \to \kappa \) be such that for every \( \gamma < \kappa \) for undoubtedly many \( \beta < \kappa \) we have \( c(\beta) = \gamma \). For \( \nu \in \omega^+(\lambda^+) \) we define \( d_{sq}^*(\nu) \) as follows.

If \( \ell g(\nu) \leq 1 \) or \( \nu \) is constant, then \( d_{sq}^*(\nu) = 0 \). So assume \( \ell g(\nu) \geq 2 \) and \( \nu \) is not constant.

For \( \alpha < \beta < \lambda^+ \) let \( s(\beta, \alpha) = s(\alpha, \beta) = \sup\{i + 1 : i < \kappa \text{ and } f_\alpha(i) \geq f_\beta(i)\} \),

\[
s(\alpha, \alpha) = 0,
\]

\[
s(\nu) = \max\{s(\nu(\ell), \nu(\kappa)) : \ell, \kappa < \ell g(\nu) \text{ (so } s \text{ is symmetric)}\},
\]

\[
a(\nu) = \{(\ell, \kappa) : s(\nu(\ell), \nu(\kappa)) = s(\nu) \text{ and } \ell < \kappa < \ell g(\nu)\}.
\]

As \( \ell g(\nu) \geq 2 \) and \( \nu \) is not constant, clearly \( a(\nu) \neq \emptyset \) and \( a(\nu) \) is finite, so let \( (\ell_\nu, k_\nu) \) be the first pair from \( a(\nu) \) in lexicographical ordering.

Lastly \( d_{sq}^*(\nu) = c\left(d(\nu(\ell_\nu), \nu(k_\nu))\right) \).

Now we are given \( \gamma < \theta \), stationary \( S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) \geq \text{cf}(\lambda)\}, \langle u_\alpha : \alpha < \lambda^+ \rangle \) (remember 2.2(0)), \( |u_\alpha| < \text{cf}(\lambda) \), \( u_\alpha \subseteq \omega^+ \lambda \) such that \( \alpha \in \cap\{\text{Rang}(\nu) : \nu \in u_\alpha\} \). Let \( u'_\alpha = \cup\{\text{Rang}(\nu) : \nu \in u_\alpha\} \) and without loss of generality for some stationary \( S' \subseteq S \) and \( \gamma_0, \beta^* \) we have \( \alpha \in S' \Rightarrow \gamma_0 = \min\{\gamma + 1 : \beta_1 < \beta_2 \text{ are in } u'_\alpha \text{ then } f_{\beta_1} \upharpoonright [\gamma, \text{cf}(\lambda)) < f_{\beta_2} \upharpoonright [\gamma, \text{cf}(\lambda))\} < \kappa \) and \( \sup(\cup\{u'_\alpha \cap \alpha : \alpha \in S'\}) < \beta^* < \lambda^+ \).

Now for some \( \gamma_1 \in (\gamma_0, \text{cf}(\lambda)) \) and stationary \( S_0, S_1 \subseteq S' \) and \( \gamma^* < \lambda \) we have

\[
\beta \in u'_\alpha \text{ & } \alpha \in S_0 \Rightarrow f_\beta(\gamma_1) < \gamma^*,
\]

\[
\beta \in u'_\alpha \text{ & } \alpha \in S_1 \Rightarrow f_\beta(\gamma_1) > \gamma^*.
\]

Let \( \{\alpha_\xi : \xi < \lambda\} \) enumerate some unbounded \( S'_\xi \subseteq S_\xi \) in increasing order such that \( \zeta < \xi \Rightarrow \sup(u_{\alpha_\zeta} \cup u_{\alpha_\xi}) < \min(u_{\alpha_\zeta} \cup u_{\alpha_\xi}) \).

Lastly apply the choice of \( d. \)
3.1 Claim. Assume \( \lambda = \mu^+ \), and
\[ S \subseteq \{ \delta < \lambda^+ : cf(\delta) = \lambda \text{ and } \delta \text{ is divisible by } \lambda^2 \} \text{ is stationary.} \]
1) There is a strict club system \( \bar{C} = \langle C_\delta : \delta \in S \rangle \) such that \( \lambda^+ \not\in id^p(\bar{C}) \) and
\[ [\alpha \in nacc(C_\delta) \Rightarrow cf(\alpha) = \lambda] \text{; moreover, there are } h_\delta : C_\delta \rightarrow \mu \text{ such that for every club } E \text{ of } \lambda^+, \text{ for stationarily many } \delta \in S, \]
\[ \bigwedge_{\zeta < \mu} \delta = \sup[h_\delta^{-1}(\{\zeta\}) \cap E \cap nacc(C_\delta)]. \]
2) If \( \bar{C} \) is a strict \( S \)-system, \( \lambda^+ \not\in id^p(\bar{C}, \bar{J}) \), \( J_\delta \) a \( \lambda \)-complete ideal on \( C_\delta \) extending \( J_{bd}^{C_\delta} + \text{acc}(C_\delta) \) (with \( S, \mu \) as above) then the parallel result holds for some
\[ h = \langle h_\delta : \delta \in S \rangle \] where \( h_\delta \) is a function from \( C_\delta \) to \( \mu \), i.e. we have for every club \( E \) of \( \lambda^+ \), for stationarily many \( \delta \in S \cap \text{acc}(E) \) for every \( \gamma < \mu \) the set
\[ \{ \alpha \in C_\delta : h_\delta(\alpha) = \gamma \text{ and } \alpha \in E \} \text{ is } \not= \emptyset \text{ mod } J_\delta. \]

3.2 Remark. 1) This improves [Sh 413, 3.1].
2) Of course, we can strengthen (1) to:
\[ \{ \alpha \in C_\delta : h_\delta(\alpha) = \gamma \text{ and } \alpha \in E \text{ and } \alpha \in nacc(C_\delta) \text{ and } \sup(\alpha \cap C_\delta) \in E \}. \]
E.g. for every thin enough club \( E \) of \( \lambda \), \( \bar{C}E \) will serve where:
\[ C_E^{\delta} = C_\delta \cap E \text{ if } \delta \in \text{acc}(E) \text{ and } C_E^{\delta} = C_\delta, \text{ otherwise.} \]
For 3.1(2) we get slightly less: for some club \( E^* \) : \( \{ \alpha \in C_\delta : \gamma \text{ and } \alpha \in E \text{ and } \alpha \in nacc(C_\delta) \text{ and } \sup(\alpha \cap C_\delta \cap E^*) \in E \}. \]

Proof. 1) Let \( \langle C_\delta : \delta \in S \rangle \) be such that \( \lambda^+ \not\in id^p(\bar{C}) \) and
\[ [\alpha \in nacc(C_\delta) \Rightarrow cf(\delta) = \lambda] \text{ (such a sequence exists by [Sh 365, 2.4(3)]). Let } J_\delta = J_{bd}^{C_\delta} + \text{acc}(C_\delta). \text{ Now apply part (2).} \]
2) For each \( \delta \in S \) let \( \langle A_\delta^{\alpha} : \alpha \in C_\delta \rangle \) be a sequence of distinct non-empty subsets of \( \mu \) to be chosen later. By induction on \( \zeta < \lambda \) we try to define \( E_\zeta, \langle Y_\zeta^{\alpha} : \alpha \in S \rangle, \)
\[ \langle Z_{\delta,\gamma}^{\zeta} : \alpha \in \zeta \text{ and } \gamma < \mu \rangle \text{ such that } \]
\( E_\zeta \) is a club of \( \lambda^+ \), decreasing in \( \zeta \),
for \( \gamma < \mu, \)
\[ Z_{\delta,\gamma}^{\zeta} = \{ \alpha : \alpha \in E_\zeta \cap nacc(C_\delta) \text{ and } \gamma \in A_\delta^{\alpha} \}, \]
\[ Y_\delta^{\zeta} = \{ \gamma < \mu : Z_{\delta,\gamma}^{\zeta} \neq \emptyset \text{ mod } J_\delta \}. \]
\( E_{\zeta+1} \) is such that
\[
\left\{ \delta \in S : Y_\delta^\zeta = Y_\delta^\zeta + 1 \text{ and } \delta \in \text{nacc}(E_{\zeta + 1}) \right\}
\text{ and } E_{\zeta + 1} \cap \text{nacc}(C_\delta) \notin J_\delta \right\}
\text{ is not stationary.}
\]

If we succeed to define \( E_\zeta \), for each \( \zeta < \lambda \), then \( E =: \bigcap_{\zeta < \lambda} E_\zeta \) is a club of \( \lambda^+ \), and since \( \lambda^+ \notin \text{idp}(\bar{C}) \), we can choose \( \delta \in S \) such that \( \delta = \sup(E \cap \text{nacc}(C_\delta)) \) and \( E \cap \text{nacc}(C_\delta) \neq \emptyset \mod J_\delta \). Then as \( \bigcup_{\gamma < \mu} Z_\delta^{\zeta, \gamma} \supseteq E \cap \text{nacc}(C_\delta) \) for each \( \zeta < \lambda \) necessarily (by the requirement on \( J_\delta \)) for some \( \gamma < \mu, Z_\delta^{\zeta, \gamma} \neq \emptyset \mod J_\delta \), hence \( Y_\delta^{\zeta} \neq \emptyset \) so that \( \langle Y_\delta^{\zeta} : \zeta < \lambda \rangle \) is a strictly decreasing sequence of subsets of \( \mu \), and since \( \mu < \text{cf}(\mu^+) = \text{cf}(\lambda) \), we have a contradiction. So necessarily we will be stuck (say) for \( \zeta(*) < \lambda \).

We still have the freedom of choosing \( A_\delta^\alpha \) for \( \alpha \in C_\delta \).

**Case 1:** \( \mu \) regular.

By induction on \( \alpha \in C_\delta \) we can choose sets \( A_\delta^\alpha \) such that

1. \( A_\delta^\alpha \subseteq \mu, |A_\delta^\alpha| = \mu, \langle A_\delta^\alpha : \alpha \in C_\delta, \text{otp}(\alpha \cap C_\delta) < \mu \rangle \) are pairwise disjoint,
2. for \( \beta \in C_\delta \cap \alpha, A_\delta^\alpha \cap A_\beta^\delta \) is bounded in \( \mu \),
3. if \( \mu > \aleph_0 \) then \( A_\delta^\alpha \) is non-stationary (just to clarify their choice).

There is no problem to carry the induction.

We shall prove later that

\((*)\) if \( E \) is a club of \( \lambda^+, \delta \in S \cap \text{acc}(E) \) and \( \delta = \sup(E \cap \text{nacc}(C_\delta)) \) and \( E \cap \text{nacc}(C_\delta) \neq \emptyset \mod J_\delta \) then

\((**)_\delta\) for some \( \alpha_\delta \in E \cap \text{nacc}(C_\delta) \), the following set \( B_\delta \) is unbounded in \( \mu \), where

\[
B_\delta = \left\{ \gamma < \mu : \langle \beta : \beta \in E \cap \text{nacc}(C_\delta) \text{ and } \beta \neq \alpha_\delta \right\}
\text{ and } \gamma = \sup(A_\delta^{\alpha_\delta} \cap A_\delta^\beta) \right\} \neq \emptyset \mod J_\delta \right\}.
\]

Choose the minimal such that \( \alpha_\delta = A_\delta^{E_\delta} \) (for other \( \delta \)'s it does not matter, i.e. for those for which \( \delta > \sup(E \cap \text{nacc}(C_\delta)) \)) or \( E_{\zeta(*)} \cap \text{nacc}(C_\delta) \in J_\delta \).

Clearly if \( E' \supseteq E'' \) and \( A_\delta^{E'}, A_\delta^{E''} \) are defined then \( \alpha_\delta^{E'} \leq \alpha_\delta^{E''} \).

Now for any club \( E^* \subseteq E_{\zeta(*)} \) of \( \lambda^+ \), for \( \delta \in S \cap \text{acc}(E_{\zeta(*)}) \) we define

\( h_{E_\delta}^{E^*} : C_\delta \to \mu \) by letting \( h_{E_\delta}^{E^*}(\beta) = \text{otp}(B_\delta \cap \sup(A_\delta^{\alpha_\delta} \cap A_\delta^\beta)) \) for \( \beta \in C_\delta \setminus \{\alpha_\delta\} \) and \( h_{E_\delta}^{E^*}(\alpha_\delta) = 0. \)
Now for any club $E$ of $\lambda^+$ for stationarily many $\delta \in S \cap \text{acc}(E^* \cap E)$, we have

$$\left\{ \gamma < \mu : \{ \alpha : \alpha \in E^* \cap E \cap E_{\zeta(\ast)} \cap \text{nacc}(C_\delta) \text{ and } \gamma \in A_\delta^\alpha \} \neq \emptyset \mod J_\delta \right\} = Y_\delta^{\zeta(\ast)}$$

(this holds by the choice of $\zeta(\ast)$). Let the set of such $\delta \in S \cap \text{acc}(E^* \cap E)$ be called $Z_E^{E^*}$. Now for each $\delta \in Z_E^{E^*}$, the set

$$B_\delta[E, E^*] = \{ \gamma < \mu : \{ \beta : \beta \in E \cap E^* \cap E_{\zeta(\ast)} \cap \text{nacc}(C_\delta) \text{ and } \beta \neq \alpha_\delta^{E^*} \text{ and } \gamma = \sup(A_\delta^{\alpha_\delta} \cap A_\delta^{\beta}) \} \neq \emptyset \mod J_\delta$$

is necessarily unbounded in $\mu$. So in the same way we have gotten $E_{\zeta(\ast)}$ we can find club $E^*_\ast \subseteq E_{\zeta(\ast)}$ such that for any club $E$ of $\lambda^+$, for stationarily many $\delta \in Z_E^{E^*}$ we have $B_\delta[E, E_{\zeta(\ast)}] = B_\delta[E^*_\ast, E_{\zeta(\ast)}]$ and $\alpha_\delta^{E^*_\ast} = \alpha_\delta^{E^*}$ (note the minimality in the choice of $\alpha_\delta^{E^*}$ so it can change $\leq \lambda + 1$ times; more elaborately if $\langle E_\delta^\ast : \zeta < \lambda \rangle$ is a decreasing sequence of clubs and $\delta \in Z_E^{E^*_\ast}$, where $E^* = \bigcap_{\zeta < \lambda} E_\zeta^\ast$, then $\langle \alpha_\delta^{E_\delta^\ast} : \zeta < \lambda \rangle$ is increasing and bounded in $C_\delta$ (by $\alpha_\delta^{E^*}$), hence is eventually constant). Define $h_\delta : C_\delta \to \mu$ by $h_\delta(\beta) = \text{otp}(B_\delta[E^*_\ast, E_{\zeta(\ast)}] \cap \text{nacc}(A_\delta^{\alpha_\delta} \cap A_\delta^{\beta}))$ if $\beta \neq \alpha_\delta$ and $h_\delta(\beta) = 0$ if $\beta = \alpha_\delta$.

Why does $(\ast)$ hold? If not, let $B = E_{\zeta(\ast)} \cap \text{nacc}(C_\delta)$, so $\text{otp}(B) = \lambda = \mu^+$ and $B \neq \emptyset \mod J_\delta$, so for every $\alpha \in B$ we can find $\varepsilon_\alpha < \mu$ and $Y_\alpha, \varepsilon \in J_\delta$ (for $\varepsilon < \mu$) such that if $\xi \in B \setminus Y_\alpha, \varepsilon \{ \alpha \}$ and $\varepsilon \in [\varepsilon_\alpha, \mu)$ then $\text{sup}(A_\delta^{\alpha} \cap A_\delta^\xi) \neq \varepsilon$. Now let $Y_\alpha = \cup \{ Y_\alpha, \varepsilon : \varepsilon \in [\varepsilon_\alpha, \mu) \} \cup \{ \alpha + 1 \}$ and note that $Y_\alpha \in J_\delta$. So for some $\varepsilon^* < \mu$, $B_1 = \{ \alpha \in B : \varepsilon_\alpha = \varepsilon^* \}$ is $\neq \emptyset \mod J_\delta$. For each $\alpha \in B_1$ choose $\gamma_\alpha \in A_\delta^{\alpha} \setminus (\varepsilon^* + 1)$ (remember $|A_\delta^{\alpha}| = \mu$). So for some $\gamma^* < \mu$ the set $B_2 = \{ \alpha \in B_1 : \gamma_\alpha = \gamma^* \}$ is $\neq \emptyset \mod J_\delta$. Let $\alpha^* = \text{Min}(B_2)$, and for $\gamma \in [\gamma^*, \mu)$ we define

$$B_{\zeta, \gamma} = \{ \alpha \in B_2 : \gamma = \sup(A_\delta^{\alpha} \cap A_\delta^\gamma) \}.$$ So clearly $B_{\zeta, \gamma} = \cup \{ B_{\zeta, \gamma} : \gamma^* \leq \gamma < \mu \}$, hence for some $\gamma^* \in [\gamma^*, \mu)$ we have $B_{\zeta, \gamma^*} \neq \emptyset \mod J_\delta$, hence $\gamma^*$ contradicts the choice of $\varepsilon_\alpha^* = \varepsilon^*$.

Case 2: $\mu$ singular.

Let $\kappa = \text{cf}(\mu)$, so by [Sh:g, II.§1] we can find an increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals $> \kappa$ with limit $\mu$ such that $\lambda = \mu^+ = \text{tcf}(\prod_{i < \kappa} \lambda_i / J^\text{bd}_{\kappa})$, and let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifying this. Without loss of generality $\bigcup_{j < i} \lambda_j < f_\alpha(i) < \lambda_i$. Let

$$g : \kappa \times \mu \times \kappa \times \mu \to \mu$$

be one to one and onto, let $f_\alpha^\delta = f_{\text{otp}(\alpha \cap C_\delta)}$ for $\alpha \in C_\delta$ and let $A_\alpha^\delta = \{ g(i, f_\alpha^\delta(i), j, f_\alpha^\delta(j)) : i, j < \kappa \}$.

\footnote{for the rest of this case $\lambda = \mu^+$ is not used; also $J^\text{bd}_{\kappa}$ can be replaced by any larger ideal}
If \( \delta = \sup(E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta})) \) and \( E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta}) \neq \emptyset \) mod \( J_\delta \) then (as \( J_\delta \) is \( \lambda \)-complete) choose \( Y_\delta \in J_\delta \) such that for each \( i < \kappa, \varepsilon < \lambda_i \) we have

\[
(\ast) \ (\exists \beta)[\beta \in E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta}) \land \beta \notin Y_\delta \land f^\delta_\beta(i) = \varepsilon] \Rightarrow \\
\{ \beta : \beta \in E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta}) \land f^\delta_\beta(i) = \varepsilon \} \neq \emptyset \text{ mod } J_\delta.
\]

Choose \( i(\delta) < \kappa \) such that

\[
B^0_\delta := \{ f^\delta_\beta(i(\delta)) : \beta \in E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta}) \land \beta \notin Y_\delta \}
\]

is unbounded in \( \lambda_i \).

Let \( \xi_\varepsilon = \xi_\varepsilon^\delta \) be the \( \varepsilon \)-th member of \( B^0_\delta \), for \( \varepsilon < \kappa \). For each such \( \varepsilon < \kappa \) for some \( j_\varepsilon = j_\varepsilon^\delta \in (i(\delta) + 1 + \varepsilon, \kappa) \) we have \( B^{1,\delta}_\varepsilon := \{ f^\delta_\beta(j_\varepsilon) : f^\delta_\beta(i(\delta)) = \xi_\varepsilon \} \land \beta \in E_{\zeta(\ast)} \cap \text{nacc}(C_{\delta}) \land \beta \notin Y_\delta \} \) is unbounded in \( \lambda_{j_\varepsilon} \).

Let \( h_{\delta,\varepsilon} \) be a one to one function from \( \bigcup_{j < \varepsilon} \lambda_j, \lambda_\varepsilon \) into \( B^{1,\delta}_\varepsilon \).

Lastly we define \( h_\delta \) as follows:

\[
\text{if } \beta \in C_\delta, \varepsilon < \kappa, f^\delta_\beta(i(\delta)) = \xi_\varepsilon \land h_{\delta,\varepsilon}(\gamma) = f^\delta_\beta(j_\varepsilon) \land \gamma \in \bigcup_{j < \varepsilon} \lambda_j, \lambda_\varepsilon \text{ then } h_\delta(\beta) = \gamma
\]

and \( h_\delta(\beta) = 0 \) otherwise. The rest is similar to the regular case. \( \square_{3.1} \)

**3.3 Claim.** If \( \lambda = \mu^+ \), \( \mu \) regular uncountable and \( S \subseteq \{ \delta < \lambda : cf(\delta) = \mu \} \) is stationary then for some strict \( S \)-club system \( \tilde{C} \) with \( C_\delta = \{ \alpha_{\delta,\zeta} : \zeta < \mu \} \), (where \( \alpha_{\delta,\zeta} \) is strictly increasing continuous in \( \zeta \)) for every club \( E \subseteq \lambda \) for stationarily many \( \delta \in S \),

\[
\{ \zeta < \mu : \alpha_{\delta,\zeta+1} \in E \} \text{ is stationary (as a subset of } \mu \).
\]

**3.4 Remark.** 1) If \( S \in I[\lambda] \) then without loss of generality we can demand (a) or we can demand (b) (but not necessarily both), where

(a) \( X_\alpha = \{ C_\delta \cap \alpha : \delta \in S \} \) is such that \( \alpha \in \text{nacc}(C_\delta) \) has cardinality \( \leq \lambda \),

(b) \( \alpha \in \text{nacc}(C_\delta) \Rightarrow C_\alpha = C_\delta \cap \alpha \) but the conclusion is weakened to:

for every club \( E \subseteq \lambda \) for stationarily many \( \delta \in S \) the set

\[
\{ \zeta < \mu : (\alpha_{\delta,\zeta} : \alpha_{\delta,\zeta+1}) \cap E \neq \emptyset \}
\]

is stationary.

2) In contrast to [Sh 413, 3.4] here we allow \( \mu \) inaccessible.

3) Clearly 3.1(2) can be applied to the results of 3.3 i.e. with

\[
J_\delta = \left\{ A \subseteq C_\delta : \{ \zeta < \lambda : \alpha_{\delta,\zeta+1} \notin A \} \text{ is not stationary} \right\}.
\]

**Proof.** We know that for some strict \( S \)-club system \( \tilde{C}^0 = \{ C^0_\delta : \delta \in S \} \) we have \( \lambda \notin \text{id}_p(\tilde{C}^0) \) (see [Sh 365, 2.3(1)]). Let \( C^0_\delta = \{ \alpha^\delta_\zeta : \zeta < \mu \} \) (increasing continuously
in $\zeta$). We shall prove below that for some sequence of functions $\bar{h} = \langle h_\delta : \delta \in S \rangle$, $h_\delta : \mu \to \mu$ we have

\[
(\ast)_{\bar{h}} \quad \text{for every club } E \text{ of } \mu^+ \text{ for stationarily many } \delta \in S \cap \text{acc}(E),
\]

the following subset of $\mu$ is stationary:

\[
A_{E,\ast}^\delta := \left\{ \zeta < \mu : \alpha_\zeta^\delta \in E \text{ and some ordinal in } \{ \alpha_\zeta^\xi : \zeta < \xi \leq h_\delta(\zeta) \} \right. \left. \text{ belongs to } E \right\}.
\]

The proof now breaks into two parts. **Proving ($\ast$)$_{\bar{h}}$ suffices.**

For each club $E$ of $\lambda$, let $Z_E := \{ \delta \in S : \delta = \sup(E \cap \text{acc}(C_\delta^0)) \}$, and note that this set is a stationary subset of $\lambda$ (by the choice of $C_0^0$). For each such $E$ and $\delta \in Z_E$ let $f_{\delta,E}$ be the partial function from $\mu$ to $\mu$ defined by

\[
f_{\delta,E}(\zeta) = \text{Sup}\{ \xi : \zeta < \xi \leq h_\delta(\zeta) \text{ and } \alpha_\zeta^\delta \in E \}.
\]

So if there is no such $\xi$, then $f_{\alpha,E}(\zeta)$ is not well defined (i.e. if the supremum is on the empty set) but if $\xi = f_{\alpha,E}(\zeta)$ is well defined then $\alpha_\zeta^\delta \in E, \xi \leq h_\delta(\zeta)$ (because $\alpha_\zeta^\delta$ is increasing continuous in $\xi$ and $E$ is a club of $\lambda$). Let $Y_E := \{ \delta \in Z_E : \text{Dom}(f_{\delta,E}) \text{ is a stationary subset of } \mu \}$. So by ($\ast$)$_{\bar{h}}$, we know that

\[
\bigoplus \text{ for every club } E \text{ of } \mu^+ \text{ the set } Y_E \text{ is a stationary subset of } \mu^+.
\]

Also

\[
\bigotimes \text{ if } E_2 \subseteq E_1 \text{ are clubs of } \mu^+ \text{ then } Z_{E_2} \subseteq Z_{E_1} \text{ and } Y_{E_2} \subseteq Y_{E_1} \text{ and for } \delta \in Y_{E_2}, \text{Dom}(f_{\delta,E_2}) \subseteq \text{Dom}(f_{\delta,E_1}) \text{ and } \zeta \in \text{Dom}(f_{\delta,E_2}) \Rightarrow f_{\delta,E_2}(\zeta) \leq f_{\delta,E_1}(\zeta).
\]

We claim that

\[
\bigotimes \text{ for some club } E_0 \text{ of } \mu^+ \text{ for every club } E \subseteq E_0 \text{ of } \mu^+ \text{ for stationarily many } \delta \in S \text{ we have}
\]

(i) $\delta = \sup(E \cap \text{acc } C_\delta)$,

(ii) $\{ \zeta < \mu : \zeta \in \text{Dom}(f_{E,\delta}) \text{ (hence } \zeta \in \text{Dom } f_{E_0,\delta}) \text{ and } f_{E,\delta}(\zeta) = f_{E_0,\delta}(\zeta) \}$ is a stationary subset of $\mu$.

If this fails, then for any club $E_0$ of $\lambda$ there is a club $E(E_0) \subseteq E_0$ of $\lambda$, such that

\[
A_{E_0} = \left\{ \delta : \delta \in S, \delta = \sup(E(E_0) \cap \text{acc}(C_\delta)) \text{ and for some club } e_{E_0,\delta} \text{ of } \mu \text{ we have} \right.
\]

\[
\zeta \in e_{E_0,\delta} \cap \text{Dom}(f_{E(E_0),\delta}) \Rightarrow f_{E(E_0),\delta}(\zeta) = f_{E_0,\delta}(\zeta)
\]

\]
is not a stationary subset of $\lambda = \mu^+$. By obvious monotonicity we can replace $E(E_0)$ by any club of $\mu^+$ which is a subset of it, so without loss of generality $A_{E_0} = \emptyset$.

By induction on $n < \omega$ choose clubs $E_n$ of $\mu^+$ such that $E_0 = \mu^+$ and $E_{n+1} = E(E_n)$. Then $E_\omega := \bigcap_{n<\omega} E_n$ is a club of $\mu^+$ and, by $\bigoplus$ above, $Y_{E_\omega} \subseteq S$ is a stationary subset of $\lambda$, so we can choose a $\delta(*) \in Y_{E_\omega}$. So $f_{E_\omega, \delta(*)}$ has domain a stationary subset of $\mu$ (see the definition of $Y_{E_\omega}$) and by $\bigotimes_1$ we know that $n < \omega \Rightarrow \text{Dom}(f_{E_\omega, \delta(*)}) \subseteq \text{Dom}(f_{E_n, \delta(*)})$. Also there is an $e_{E_n, \delta(*)}$, a club of $\mu$, such that

$$\zeta \in e_{E_n, \delta(*)} \cap \text{Dom}(f_{E_{n+1}, \delta(*)}) \Rightarrow f_{E_{n+1}, \delta(*)}(\zeta) < f_{E_n, \delta(*)}(\zeta)$$

(see the choice of $E_{n+1} = E(E_n)$ i.e. the function $E$). So $e_{\delta(*)} := \bigcap_{n<\omega} e_{E_n, \delta(*)}$ is a club of $\mu$ and, as $\text{Dom}(f_{E_\omega, \delta(*)})$ is a stationary subset of $\mu$, we can find $\zeta(*) \in e_{\delta(*)} \cap \text{Dom}(f_{E_\omega, \delta(*)})$, hence $\zeta(*) \in \bigcap_{n<\omega} \text{Dom}(f_{E_n, \delta(*)}) \cap \bigcap_{n<\omega} e_{E_n, \delta(*)}$, so that $(f_{E_n, \delta(*)}(\zeta(*)) : n < \omega)$ is a well defined strictly increasing $\omega$-sequence of ordinals - a contradiction. So $\bigotimes_2$ cannot fail, and this gives the desired conclusion.

**Proof of $(*)_\tilde{h}$ holds for some $\tilde{h}$.**

So assume that for no $\tilde{h}$ does $(*)_\tilde{h}$ holds, hence (by shrinking $E$) we can assume that for every $\tilde{h} = \langle h_\delta : \delta \in S \rangle$, $h_\delta : \mu \to \mu$, for some club $E$ for every $\delta \in S$, $A_{E_\delta}^\delta$ is not stationary (in $\mu$). By induction on $n < \omega$, we define $E_n$,

$$h^n = \langle h^n_\delta : \delta \in S \rangle, e^n = \langle e^n_\delta : \delta \in S \rangle,$$

with $E_n$ a club of $\lambda$, $e^n_\delta$ a club of $\mu$, $h^n_\delta : \mu \to \mu$ as follows.

Let $E_0 = \lambda$, $h^0_\delta(\zeta) = \zeta + 1$ and $e^0_\delta = \mu$.

If $E_0, \ldots, E_n, h^n_\delta, \ldots, h^n_\delta, e^0_\delta, \ldots, e^n_\delta$ are defined, necessarily $(*)_h^n$ fail, so for some club $E_{n+1}$ of $\lambda$ for every $\delta \in S \cap \text{acc}(E_{n+1})$ there is a club $e^{n+1}_\delta \subseteq \text{acc}(e^n_\delta)$ of $\mu$, such that

$$\zeta \in e^{n+1}_\delta \Rightarrow \{ e^\delta_\xi : \zeta < \xi \leq h_\delta(\zeta) \} \cap E_{n+1} = \emptyset.$$

Choose $h^{n+1}_\delta : \mu \to \mu$ such that $(\forall \zeta < \mu)(h^0_\delta(\zeta) < h^{n+1}_\delta(\zeta))$ and if $\delta = \text{sup}(E_{n+1} \cap \text{acc}(C^0_\delta))$ then $\zeta < \mu \Rightarrow \{ e^\delta_\xi : \zeta < \xi \leq h^{n+1}_\delta(\zeta) \} \cap E_{n+1} \neq \emptyset$.

There is no problem to carry out this inductive definition. By the choice of $C^0$, for some $\delta \in \text{acc}(\bigcap_{n<\omega} E_n)$, we have $\delta = \text{sup}(A')$, where

$$A' := (\text{acc} \bigcap_{n<\omega} E_n) \cap \text{acc}(C^0_\delta).$$

Let $A \subseteq \mu$ be such that $A' = \{ e^\delta_\xi : \zeta \in A \}$ (remember $e^\delta_\xi$ is increasing with $\zeta$) and let $\zeta$ be the second member of $\bigcap_{n<\omega} e^n_\delta$. As $A'$ is unbounded in $\delta$, clearly $A$ is unbounded in $\mu$ and $\bigcap_{n<\omega} e^n_\delta$ is a club of $\mu$ as $\mu = \text{cf}(\mu) > \aleph_0$. Also as $A' \subseteq n\text{acc}(C^0_\delta)$ clearly $A$ is a set of successor ordinals (or zero).
Note that \( \sup(e_n^\delta \cap \zeta) \) is well defined (as \( \Min(e_n^\delta) \leq \Min(\bigcap_{n<\omega} e_n^\eta) < \zeta \)) and \( \sup(e_n^\delta \cap \zeta) < \zeta \) (as \( \zeta \) is a successor ordinal). Now \( \langle \sup(e_n^\delta \cap \zeta) : n < \omega \rangle \) is non-increasing (as \( e_n^\eta \) decreases with \( n \)), hence for some \( n(*) < \omega \) we have \( n(*) \Rightarrow \sup(e_n^\delta \cap \zeta) = \sup(e_n^{\eta(*)} \cap \zeta) \) and call this ordinal \( \xi \) so that \( \xi \in e_n^{\delta_{n(*)+1}} \) and \( h_n^{\eta(*)}(\xi) = h_n^{\delta_{n(*)+1}}(\xi) \), so we get a contradiction for \( n(*) + 1 \).

So \((*)_n\) holds for some \( n \), which suffices, as indicated above. \( \square_{3.3} \)

### 3.5 Discussion

1) We can squeeze a little more, but it is not so clear if with much gain. So assume

\[(*)_0\] \( \mu \) is regular uncountable, \( \lambda = \mu^+ \), \( S \subseteq \{ \delta < \lambda : cf(\delta) = \mu \} \) stationary, \( I \) an ideal on \( S \), \( C = \langle C_\delta : \delta \in S \rangle \) a strict \( S \)-club system, \( J = \langle J_\delta : \delta \in S \rangle \) with \( J_\delta \) an ideal on \( C_\delta \) extending \( J_{C_\delta}^{bd} + (\acc(C_\delta)) \), such that for any club \( E \) of \( \lambda \) we have \( \{ \delta \in S : E \cap C_\delta \neq \emptyset \mod J_\delta \} \neq \emptyset \mod I \).

2) If we immitate the proof of 3.3 we get

\[(*)_1\] if for \( \delta \in S \), \( J_\delta \) is not \( \chi \)-regular (see the definition below) and \( \chi \leq \mu \) then we can find \( \bar{e} = \langle e_\delta : \delta \in S \rangle \) and \( \bar{g} = \langle g_\delta : \delta \in S \rangle \) such that

\[(*)'_1\] \( e_\delta \) is a club of \( \delta \), \( e_\delta \subseteq \acc(C_\delta) \), \( g_\delta : \nacc(C_\delta) \setminus (\min(e_\delta) + 1) \to e_\delta \) is defined by \( g_\delta(\alpha) = \sup(e_\delta \cap \alpha) \) and for every club \( E \) of \( \lambda \)

\[
\left\{ \delta \in S : E \cap \nacc(C_\delta) \neq \emptyset \mod J_\delta \text{ and } \right. \\
\left. Rang(g_\delta | (E \cap \nacc(C_\delta))) \right\} \neq \emptyset \mod I.
\]

3) Definition: An ideal \( J \) on a set \( C \) is \( \chi \)-regular if there is a set \( A \subseteq C \), \( A \neq \emptyset \mod J \) and a function \( f : A \to [\chi]^{<\kappa_0} \) such that

\( \gamma < \chi \Rightarrow \{ x \in A : \gamma \notin f(x) \} = \emptyset \mod J \).

If \( \chi = |C| \), we may omit it.

[How do we prove \((*)'_1\)? Try \( \chi \) times \( E_\xi, \langle e_\xi^\delta : \delta \in S \rangle \) (for \( \zeta < \chi \)).]

4) We can try to get results like 3.1. Now

\[(*)_2\] assume \( \lambda, \mu, S, I, C, J \) are as in \((*)_0\) and \( \bar{e}, \bar{g} \) as in \((*)'_1\) and \( \kappa < \mu \) and for \( \delta \in S \), \( J_0^\delta = \{ a \subseteq e_\delta : \alpha \in \Dom(g_\delta) : g(\alpha) \subseteq a \} \in J_\delta \} \) is weakly normal and \( \mu \) satisfies the condition from \( [\text{Sh} 365, \text{Lemma 2.12}] \). Then we can find \( h_\delta : e_\delta \to \kappa \) such that for every club \( E \) of \( \lambda \),

\( \{ \delta \in S : \text{for each } \gamma < \kappa \text{ the set } \{ \alpha \in \nacc(C_\delta) : h_\delta(g_\delta(\alpha)) = \gamma \} \neq \emptyset \mod J_\delta \} \neq \emptyset \mod I \).

[Why? For each \( \delta \in S, \alpha \in \acc(e_\delta) \) choose a club \( d_{\delta,\alpha} \subseteq e_\delta \cap \alpha \) such that for no club \( d \subseteq e_\delta \) of \( \delta \) do we have \( (\forall \gamma < \delta)(\exists \alpha \in \acc(e_\delta))[d \cap \gamma \subseteq d_{\delta,\alpha}] \). Now for every club \( E \) of \( \lambda \) let

\( S_E = \{ \delta : E \cap \nacc(C_\delta) \neq \emptyset \mod J_\delta, \text{ and } g_\delta''(E \cap \nacc(C_\delta)) \text{ is stationary} \} \) and for \( \delta \in E \) and \( \varepsilon < \mu \), we choose by induction on \( \zeta < \kappa, \xi(\delta, \varepsilon) \) as the first \( \xi \in e_\delta \) such that: \( \xi > \bigcup_{\zeta < \varepsilon} \xi(\delta, \zeta) \) and \( \{ \alpha \in \Dom(g_\delta) : \alpha \in E \text{ and the } \varepsilon \text{-th member of } d_{\delta,g_\delta(\alpha)} \text{ is } \}

\( \zeta < \varepsilon \).]
in the interval $\bigcup_{\zeta < \xi} \zeta \{\xi(\delta, \zeta), \xi(\zeta, \xi)\} \neq \emptyset \bmod J_\delta$.

5) We deal below with successor of singulars and with inaccessibles, we can do parallel things.

3.6 Claim. Suppose $\mu$ is a singular cardinal of cofinality $\kappa, \kappa > \aleph_0$ and $S \subseteq \{\delta < \mu^+ : \text{cf}(\delta) = \kappa\}$ is stationary, and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$-club system satisfying $\mu^+ \not\subseteq \check{\text{id}}(\bar{C}, \check{J}_{\alpha}[\mu])$ where $\check{J}_{\alpha}[\mu] = \langle J_{C_\delta}^{\check{b}[\mu]} : \delta \in S \rangle$ and $J_{C_\delta}^{\check{b}[\mu]} = \{A \subseteq C_\delta : \text{for some } \theta < \mu, \text{ we have } \delta > \sup\{\alpha \in A : \text{cf}(\alpha) > \theta\}\}$. Then we can find a strict $\lambda$-club system $\bar{e}^* = \langle e_{\delta}^* : \delta < \lambda \rangle$ such that

\[\text{(*) for every club } E \text{ of } \mu^+, \text{ for stationarily many } \delta \in S, \text{ for every } \alpha < \delta \text{ and } \theta < \mu \text{ for some } \beta \text{ we have}\]

\[\text{(**) for } E, \beta : \beta \in \text{nacc}(C_\delta) \text{ and } \beta > \alpha \text{ and } \text{cf}(\beta) > \theta \text{ and}\]

\[\{\gamma \in e_{\beta}^* : \gamma \in E \text{ and } \min(e_{\beta}^*(\gamma + 1)) \text{ belongs to } E\}

\text{is a stationary subset of } \beta.

3.7 Remark. 1) We know that for the given $\mu$ and $S$ there is $\bar{C}$ as in the assumption by [Sh 365, §2]. Moreover, if $\kappa > \aleph_0$ then there is such nice strict $\bar{C}$.

2) Remember $J_{\alpha}[\mu] = \{A \subseteq C_\delta : \text{for some } \theta < \mu \text{ and } \alpha < \delta \text{ we have}\}

\{(\forall \beta \in C_\delta)(\beta < \alpha \lor \text{cf}(\beta) < \theta \lor \beta \in \text{nacc}(C_\delta))\}.

Proof. Let $\bar{e} = \langle e_\beta : \beta < \lambda \rangle$ be a strict $\lambda$-club system where $e_\beta = \{\alpha_\zeta^\beta : \zeta < \text{cf}(\beta)\}$ is a (strictly) increasing and continuous enumeration of $e_\beta$ (with limit $\delta$). Now we claim that for some $\bar{h} = \langle h_\beta : \beta < \lambda, \beta \text{ limit}\rangle$ with $h_\beta$ a function from $e_\beta$ to $e_\beta$ and $\bigwedge_{\alpha \in e_\beta} h_\beta(\alpha) > \alpha$, we have

\[\text{(*) for every club } E \text{ of } \mu^+, \text{ for stationarily many } \delta \in S \cap \text{acc}(E), A_\beta^E \not\subseteq J_{C_\delta}^{\check{b}[\mu]}\]

where $A_\beta^E$ is the set of all $\beta \in C_\delta$ such that the following subset of $e_\beta$ is stationary (in $\beta$):

\[\{\gamma \in e_\beta : \gamma \in E \text{ and } \min(e_{\beta}^*(\gamma + 1)) \in E\}.

The rest is like the proof of 3.3 repeating $\kappa^+$ times instead $\omega$ and using "$J_{C_\delta}^{\check{b}[\mu]}$ is $(\leq \kappa)$-based".

\[\square_{3.6}

3.8 Claim. Suppose $\lambda$ is inaccessible, $S \subseteq \lambda$ is a stationary set of inaccessibles, $\bar{C}$ an $S$-club system such that $\lambda \not\subseteq \check{id}(\bar{C})$. Then we can find $\bar{h} = \langle h_\delta : \delta \in S \rangle$ with $h_\delta : C_\delta \rightarrow C_\delta$, such that $\alpha < h(\alpha)$ and

\[\text{(*) for every club } E \text{ of } \lambda, \text{ for stationarily many } \delta \in S \cap \text{acc}(E) \text{ we have that}\]

\[\{\alpha \in C_\delta : \alpha \in E \text{ and } h(\alpha) \in E\} \text{ is a stationary subset of } \delta.\]
So for some $C'_\delta = \{\alpha_{\delta,\zeta} : \zeta < \delta\} \subseteq C_\delta, \alpha_{\delta,\zeta}$ increasing continuous in $\zeta$ we have $h(\alpha_{\delta,\zeta}) = \alpha_{\delta,\zeta+1}$.

Remark. Under quite mild conditions on $\lambda$ and $S$ there is $\bar{C}$ as required - see [Sh 365, 2.12,p.134].

Proof. Like 3.3.

3.9 Claim. Let $\lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \lambda$ stationary, $D$ a normal $\lambda^+$-saturated filter on $\lambda$, $S$ is $D$-positive (i.e. $S \in D^+, \lambda\setminus S \notin D$).

1) Assume $\langle (C_\delta, I_\delta) : \delta \in S \rangle$ is such that

(a) $C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathcal{P}(C_\delta),$

(b) for every club $E$ of $\lambda$,

$$\{\delta \in S : \text{for some } A \in I_\delta \text{ we have } \delta > \sup(A\setminus E)\} \in D^+.$$

Then for some stationary $S_0 \subseteq S, S_0 \in D^+$ we have

$$(b)^+$$ for every club $E$ of $\lambda$

$$\{\delta \in S : \text{for no } A \in I \text{ do we have } \delta > \sup(A\setminus E)\} = \emptyset \mod D.$$

2) Assume $\langle \mathcal{P}_\delta : \delta \in S \rangle$ is such that (here really presaturated is enough)

$$(*)$$ for every $D$-positive $S_0 \subseteq S$ for some $D$-positive $S_1 \subseteq S_0$ and

$\langle (C_\delta, I_\delta) : \delta \in S \rangle$ we have $(C_\delta, I_\delta) \in \mathcal{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathcal{P}(C_\delta)$ and for every club $E$ of $\lambda$

$$\{\delta \in S_1 : \text{for some } A \in I_\delta, \delta > \sup(A\setminus E)\} \neq \emptyset \mod D.$$

Then

$$(**)$$ for some $\langle (C_\delta, A_\delta) : \delta \in S \rangle$ we have $(C_\delta, I_\delta) \in \mathcal{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathcal{P}(C_\delta)$ and for every club $E$ of $\lambda$

$$\{\delta \in S : \text{for no } A \in I_\delta, \delta > \sup(A\setminus E)\} = \emptyset \mod D.$$

Remark. This is a straightforward generalization of [Sh.e, III,§6.2B]. Independently Gitik found related results on generic extensions which were continued in [DjSh 562] and in [GiSh 577].

Proof. The same.
3.10 Lemma. Suppose $\lambda$ is regular uncountable and $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$ is stationary. Then we can find $\langle(C_\delta, h_\delta, \chi_\delta) : \delta \in S\rangle$ and $D$ such that

(A) $D$ is a normal filter on $\lambda^+$,
(B) $C_\delta$ is a club of $\delta$, say $C_\delta = \{\alpha_{\delta, \zeta} : \zeta < \lambda\}$, with $\alpha_{\delta, \zeta}$ increasing continuous in $\zeta$,
(C) $h_\delta$ is a function from $C_\delta$ to $\chi_\delta$, $\chi_\delta \leq \lambda$,
(D) if $A \in D^+$ (i.e. $A \subseteq \lambda^+$ & $\lambda^+ \setminus A \notin D$) and $E$ is a club of $\lambda^+$, then the following set belongs to $D^+$:

$$B_{E, A} = \{\delta : \delta \in A \cap S, \delta \in \text{acc}(E) \text{ and for each } i < \chi_\delta$$

$$\{\zeta < \lambda : \alpha_{\delta, \zeta+1} \in E \text{ and } h_\delta(\alpha_{\delta, \zeta}) = i$$

$$\text{(and } \alpha_{\delta, \zeta} \in E) \} \text{ is a stationary subset of } \lambda\}$$

(hence, for some $\alpha < \lambda^+$ and $\zeta < \lambda$, the set $B_{E, A, \alpha} = \{\delta \in B_{E, A} : \alpha = \alpha_{\delta, \zeta}\}$ is in $D^+$).
(E) If $\gamma < \lambda^+$ and $\chi$ satisfies one of the conditions listed below, then $S_{\gamma, \chi} = \{\delta \in S : \gamma = \text{Min}(C_\delta) \text{ and } \chi_\delta = \chi\} \in D^+$ where

(\alpha) $\lambda = \chi^+$,
(\beta) $\lambda$ is inaccessible not strongly inaccessible, $\chi < \lambda$ and there is $T$ such that

(a) $T$ is a tree with $< \lambda$ nodes and a set $\Gamma$ of branches, $|\Gamma| = \lambda$,
(b) $T'$ if $T' \subseteq T$, $T'$ downward closed and $(\exists \lambda \eta \in \Gamma)$

(\eta a branch of $T'$) then $T'$ has an antichain of cardinality $\geq \chi$,

(\gamma) $\lambda$ is inaccessible not strongly inaccessible and $\chi = \text{Min}\{\chi : \text{for some } \theta \leq \chi \text{ we have } \chi^\theta \geq \lambda\}$,
(\delta) $\lambda$ is strongly inaccessible not ineffable; i.e. $\lambda$ is Mahlo and we can find $\bar{A} = \langle A_\mu : \mu < \lambda \text{ is inaccessible } \rangle$,

$A_\mu \subseteq \mu$ so that for no stationary $\Gamma \subseteq \{\mu < \lambda : \mu \text{ inaccessible}\}$

and $A \subseteq \lambda$ do we have: $\mu \in \Gamma \Rightarrow A_\mu = A \cap \mu$.

3.11 Remark. We can replace $\lambda^+$ in 3.10 and any $\mu = \text{cf}(\mu) > \lambda$, as if $\mu > \lambda^+$ we have even a stronger theorem.
Proof. Let for $\lambda = \text{cf}(\lambda) > \aleph_0$,

$$\Theta = \Theta_\lambda = \left\{ \chi \leq \lambda : \text{if } S' \subseteq \{ \delta < \lambda^+ : \text{cf}(\delta) = \lambda \} \text{ is stationary then} \right.$$  

we can find $\langle (C_\delta, h_\delta) : \delta \in S' \rangle$ such that

(a) $C_\delta$ is a club of $\delta$ of order type $\lambda$,

(b) $h_\delta : C_\delta \to \chi$,

(c) for every club $E$ of $\lambda^+$ for stationarily many

$$\delta \in S' \cap \text{acc}(E) \text{ we have:}$$

$$i < \chi \Rightarrow B_E = \{ \alpha \in C_\delta : \alpha \in E, h(\alpha) = i \text{ and } \min(C_\delta \setminus (\alpha + 1)) \in E \} \text{ is a stationary subset of } \delta \}.$$ 

Now we first show

$\otimes$ for each of the cases from clause (E), the $\chi$ belongs to $\Theta$.

Proof of sufficiency of $\otimes$. We can partition $S$ to $\lambda^+$ stationary sets so we can find a partition $(S_{\chi, \alpha} : \chi \in \Theta \text{ and } \alpha < \lambda^+)$ of $S$ to stationary sets. Without loss of generality, $\alpha \leq \text{Min}(S_{\chi, \alpha})$ and let $\langle (C_\delta^0, h_\delta^0) : \delta \in S_{\chi, \alpha} \rangle$ be as guaranteed by “$\chi \in \Theta$” for the stationary set $S_{\chi, \alpha}$. Now define $C_\delta, h_\delta$ for $\delta \in S$ by:

$C_\delta$ is $C_\delta^0 \cup \{ \alpha \} \text{ if } \delta \in S_{\chi, \alpha}$ and $\alpha < \delta, h_\delta(\beta)$ is $h_\delta(\beta)$ if $\beta \in C_\delta \cap C_\delta^0$ and is zero otherwise. Of course, $\chi_\delta = \chi$ if $\delta \in S_{\chi, \alpha}$.

Lastly, let

$$D = \left\{ A \subseteq \lambda^+ : \text{for some club } E \text{ of } \lambda^+, \text{ for every } \right.$$  

$$\delta \in S \cap \text{acc}(E) \setminus A \text{ for some } i < \chi_\delta,$$

the set $\{ \beta \in C_\delta : \beta \in E, h_\delta(\beta) = i \text{ and } \min(C_\delta \setminus (\beta + 1)) \in E \}$

is not a stationary subset of $\delta \}.$

So $D$ and $\langle (C_\delta, h_\delta, \chi_\delta) : \delta \in S \rangle$ have been defined, and we have to check clauses (A)-(E).

Note that $\Theta \neq \emptyset$ and the proof which appears later does not rely on the intermediate proofs.

Clause (A): Suppose $A_\zeta \in D$ for $\zeta < \lambda$, so for each $\zeta$ there is a club $E_\zeta$ of $\lambda^+$

$\star$ if $\delta \in S_{\chi, \gamma}$ and $\delta \in S \cap \text{acc}(E) \setminus A_\zeta$ then

$$\{ \alpha \in C_\delta : \alpha \in E, \text{Min}(C_\delta \setminus (\alpha + 1)) \in E \text{ and } h_\delta(\alpha) = i_\zeta \} \text{ is not stationary in } \delta.$$
Clearly clubs of $\lambda^+$ belong to $D$.

Clearly $A \supseteq A_\zeta \Rightarrow A \in D$ (by the definition), witnessed by the same $E_\zeta$.

Also $A = A_0 \cap A_1 \in D$ as witnessed by $E = E_0 \cap E_1$.

Lastly, $A = \bigtriangleup A_\zeta = \{ \alpha < \lambda^+ : \alpha \in \bigcap_{\zeta < \lambda^+} A_\zeta \}$ belong to $D$ as witnessed by $E = \{ \alpha < \lambda^+ : \alpha \in \bigcap_{\zeta < \lambda^+} E_\zeta \}$. Note that if $\delta \in S \cap \text{acc}(E) \setminus A$ then for some $\zeta < \delta$

$\delta \in S \cap \text{acc}(E) \setminus A_\zeta \subseteq (S \cap \text{acc}(E_\zeta) \setminus A_\zeta) \cup (1 + \zeta)$

as $E_\zeta \setminus E$ is a bounded subset of $\delta$; included in $1 + \zeta$ so from the conclusion of (*) for $\delta, A_\zeta, E_\zeta$ we get it for $\zeta, A, E$.

Lastly $\emptyset \notin D$; otherwise, let $E$ be a club of $\lambda^+$ witnessing it, i.e. (*) holds in this case. Choose $\chi \in \Theta$ and $\alpha = 0$ and use on it the choice of $\langle C^0_\delta : \delta \in S_{\chi,0} \rangle$ to show that for some $\delta \in S_{\chi,0} \subseteq S$ contradict the implication in (*).

**Clause (B):** Trivial.

**Clause (C):** Trivial.

**Clause (D):** Note that we can ignore the “$\alpha_{\delta,\zeta} \in E$” as $\delta \in \text{acc}(E)$ implies that it holds for a club of $\zeta$’s. Assume $A \in D^+$ (for clause (A)) and $E$ is a club of $\lambda^+$, which contradicts clause (D) so $B_{E,A} \notin D^+$, hence $\lambda^+ \setminus B_{E,A} \notin D$. Also $E$ witnessed that $\lambda^+ \setminus (A \setminus B_{E,A}) \in D$ by the definition of $D$. But by clause (A) we know $D$ is a filter on $\lambda^+$ so $(\lambda^+ \setminus B_{E,A}) \cap (\lambda^+ \setminus (A \setminus B_{E,A})$ belong to $D$, but this is the set $\lambda^+ \setminus (B_{E,A} \setminus (A \setminus B_{E,A})$ which is (as $B_{E,A} \subseteq A$ by its definition) just $\lambda \setminus A$. So $\lambda \setminus A \in D$ hence $A \notin D^+$ - a contradiction.

**Clause (E):** By the proof of $\emptyset \notin D$ above, if $\chi \in \Theta$, also $S_{\chi,\alpha} \in D^+$, and by the definition of $\overline{C}, \overline{C} \upharpoonright S_{\chi,\alpha}$ is as required. So it is enough to show

**3.12 Claim.** If $\chi < \lambda = \text{cf}(\lambda)$ and $\chi$ satisfies one of the clauses of ?, then $\chi \in \Theta$ (from the proof of 3.10).

**Proof.**

Case (α): By 3.1.

Case (β): Like the proof of 3.1, for more details see [Sh 413, §3].

Case (γ): This is a particular case of case (β).

Case (δ): Similar proof (or use 3.13).

More generally (see [Sh 413]):
3.13 Claim. Let $\lambda = \text{cf}(\lambda) > \chi$. A sufficient condition for $\chi \in \Theta_\lambda$ is the existence of some $\zeta < \lambda^+$ such that

\begin{itemize}
  \item in the following game of length $\zeta$, first player has no winning strategy:
  \item in the $\varepsilon$-th move first player chooses a function $f_\varepsilon : \lambda \to \chi$ and second player chooses $\beta_\varepsilon < \chi$. In the end, first player wins the play if
  \item $\{\alpha < \lambda : \text{for every } \varepsilon < \gamma, f_\varepsilon(\alpha) \neq \beta_\varepsilon\}$ is a stationary subset of $\lambda$.
\end{itemize}

(If we weaken the demand in $\Theta_\lambda$ from stationary to unbounded in $\lambda$, we can weaken it here too).
§4 More on $Pr_6$

4.1 Claim. $Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$ for $\lambda$ regular.

Proof. We can find $h : \lambda^+ \to \lambda^+$ such that for every $\gamma < \lambda^+$ the set $S_\gamma = \{ \delta < \lambda^+ : \text{cf}(\delta) = \lambda \text{ and } h(\delta) = \gamma \}$ is stationary, so $\{ S_\gamma : \gamma < \lambda \}$ is a partition of $S = \{ \delta < \lambda^+ : \text{cf}(\delta) = \lambda \}$. We can find $C_\gamma = \{ C_\delta : \delta \in S_\gamma \}$ such that $C_\delta$ is a club of $\delta$ of order type $\lambda$. For any $\nu \in \omega>(\lambda^+)$ we define:

(a) for $\ell < \ell g(\nu)$, if $\nu(\ell) \in S$ then let 
\[ a_\ell = a_{\nu,\ell} = \text{otp}(C_{\nu(\ell)} \cap \nu(k)) : k < \ell g(\nu) \text{ and } \nu(k) < \nu(\ell) \}, \]

(b) $\ell_\nu$ is the $\ell < \ell g(\nu)$ such that 
- (i) $\nu(\ell) \in S$,
- (ii) among those with $\text{sup}(a_{\nu,\ell})$ is maximal, and
- (iii) among those with $\ell$ minimal,

(c) if $\ell_\nu$ is well defined let $d(\nu) = h(\nu(\ell_\nu))$ otherwise let $d(\nu) = 0$.

Now suppose $\langle (u_\alpha, v_\alpha) : \alpha < \lambda^+ \rangle, \gamma < \lambda^+ \text{ and } E \text{ are as in Definition 2.1}$ and we shall prove the conclusion there. Let 
\[ E^* = \{ \delta \in E : \delta \text{ is a limit ordinal and } \alpha < \delta \Rightarrow \delta > \sup[\cup\{ \text{Rang}(\eta) : \eta \in u_\alpha \cup v_\alpha \}] \} \].

Clearly $E^* \subseteq E$ is a club of $\lambda^+$.

For each $\delta \in S_\gamma$ let 
\[ f_0(\delta) = \text{sup}[\text{sup} \cap \bigcup\{ \text{Rang}(\nu) : \nu \in u_\delta \cup v_\delta \}] \].

As $\text{cf}(\delta) = \lambda > |u_\alpha \cup v_\alpha|$ and the sequences are finite clearly $f_0(\delta) < \delta$. Hence by Fodor’s lemma for some $\xi^*, S^1_\gamma = \{ \delta \in S_\gamma : f_0(\delta) = \xi^* \}$ is a stationary subset of $\lambda^+$ (note that $\gamma$ is fixed here). Let $\xi^* = \bigcup_{i<\lambda} a_{2,i}$ where $a_{2,i}$ is increasing with $i$ and $|a_{2,i}| < \lambda$. So for $\delta \in S^1_\gamma$

\[ f_1(\delta) = \text{Min}\left\{ i < \lambda : \delta \cap \bigcup\{ \text{Rang}(\nu) : \nu \in u_\delta \cup v_\delta \} \right. \left. \text{ is a subset of } a_{2,i} \right\} \]

is a well defined ordinal $< \lambda$, hence for some $i^* < \lambda$ the set 
\[ S^2_\gamma = \{ \delta \in S^1_\gamma : f_1(\delta) = i^* \} \]

is a stationary subset of $\lambda^+$. For $\delta \in S^2_\gamma$ let
\[ b_\delta := \begin{cases} 
\text{otp}(C_\beta \cap \alpha) : \alpha < \beta \in S \text{ and both} \\
\text{are in } a_{2,*} \cup \{\delta\} \cup \bigcup \{\text{Rang } \nu : \nu \in u_\delta \cup v_\delta\} \end{cases}. \]

So \( b_\delta \) is a subset of \( \lambda \) of cardinality \( < \lambda \) hence \( \varepsilon_\delta := \sup(b_\delta) < \lambda \), hence for some \( \varepsilon^* \)

\[ S_\gamma^3 := \{ \delta \in S_\gamma^2 : \varepsilon_\delta = \varepsilon^* \} \]

is a stationary subset of \( \lambda^+ \). Choose \( \beta^* \) such that

\[
(*) \quad \beta^* \in S_\gamma^3 \cap E^* \quad \text{and} \quad \beta^* = \sup(\beta^* \cap S_\gamma^3 \cap E^*). 
\]

As \( C_{\beta^*} \) has order type \( \lambda \), (and is a club of \( \beta^* \)) for some \( \alpha^* \in \beta^* \cap S_\gamma^3 \cap E^* \) we have \( \text{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^* \).

We want to show that \( \alpha^*, \beta^* \) are as required. Obviously \( \alpha^* < \beta^*, \alpha^* \in E \) and \( \beta^* \in E \). So assume \( \nu \in u_{\alpha^*}, \rho \in v_{\beta^*} \) and we shall prove that \( d(\nu^* \rho) = \gamma \), which suffices. As \( h(\beta^*) = \gamma \) (as \( \beta^* \in S_\gamma^3 \subseteq S_\gamma \)) it suffices to prove that \( (\nu^* \rho)(\ell_{\nu^* \rho}) = \beta^* \).

Now for some \( \ell_0, \ell_1 \) we have \( \nu(\ell_0) = \alpha^*, \rho(\ell_1) = \beta^* \) (as \( \nu \in u_{\alpha^*}, \rho \in v_{\beta^*} \)) and since \( \text{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^* \), by the definition of \( \ell_{\nu^* \rho} \) it suffices to prove

\[
\otimes \quad \text{if } \ell, k < \ell g(\nu^* \rho), (\nu^* \rho)(\ell) \in S, (\nu^* \rho)(k) < (\nu^* \rho)(\ell) \text{ then } \\
\hspace{1cm} (i) \quad \text{otp}[C_{(\nu^* \rho)(\ell)} \cap (\nu^* \rho)(k)] \leq \varepsilon^* \text{ or} \\
\hspace{1cm} (ii) \quad (\nu^* \rho)(\ell) = \beta^*. 
\]

Assume \( \ell, k \) satisfy the assumption of \( \otimes \) and we shall show its conclusion.

**Case 1:** If \( (\nu^* \rho)(\ell) \) and \( (\nu^* \rho)(k) \) belong to

\[ a_{2,*} \cup \{\beta^*\} \cup \bigcup \{\text{Rang}(\eta) : \eta \in u_{\beta^*} \cup v_{\beta^*}\} \]

then clause (i) holds because

\[
(\alpha) \quad \text{otp}(C_{(\nu^* \rho)(\ell)} \cap (\nu^* \rho)(k)) \in b_{\beta^*} \quad (\text{see the definition of } b_{\beta^*} \text{ and}) \\
(\beta) \quad \sup(b_{\beta^*}) = \varepsilon_{\beta^*} \quad (\text{see the definition of } \varepsilon_{\beta^*} \text{ and}) \\
(\gamma) \quad \varepsilon_{\beta^*} = \varepsilon^* \quad (\text{as } \beta^* \in S_\gamma^3 \text{ and see the choice of } \varepsilon^* \text{ and } S_\gamma^3). 
\]

**Case 2:** If \( (\nu^* \rho)(\ell) \) and \( (\nu^* \rho)(k) \) belong to

\[ a_{2,*} \cup \bigcup \{\text{Rang}(\eta) : \eta \in u_{\alpha^*} \cup v_{\alpha^*}\} \]

then the proof is similar to the proof of the previous case.

**Case 3:** No previous case.
As $\beta^* \in E^*$ and $\beta^* > \alpha^*$ clearly $\sup(\text{Rang}(\nu)) < \beta^*$, but also $(\nu^\ast \rho)(k) < (\nu^\ast \rho)(\ell)$ (see $\otimes$).

Together necessarily $k < \ell g(\nu), \nu(k) \in [\alpha^*, \beta^*), \ell \in [\ell g(\nu), \ell g(\nu) + \ell g(\rho))$ and $\rho(\ell - \ell g(\nu)) \in [\beta^*, \lambda^+).$ If $\rho(\ell) = \beta^*$ then clause $(ii)$ of the conclusion holds. Otherwise necessarily $\nu(\ell) > \beta^*$ hence

$$\text{otp}(C_{(\nu^\ast \rho)(\ell)}) \cap (\nu^\ast \rho)(k) = \text{otp}(C_{(\rho(\ell - \ell g(\nu)) \cap \nu}(k)) \leq \text{otp}(C_{(\rho(\ell - \ell g(\nu)) \cap \beta^*) \leq \sup(a_{\beta^*}) \leq \epsilon^*$$

so clause $(i)$ of $\otimes$ holds. $\Box_{4.1}$

4.2 Conclusion. For $\lambda$ regular, $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ holds.

Proof. By 4.1 and 2.2(1). $\Box_{4.2}$

4.3 Definition. 1) Let $Pr_6(\lambda, \theta, \sigma)$ means that for some $\Xi$, an unbounded subset of $\{\tau : \tau < \sigma, \tau$ is a cardinal (finite or infinite)\}, there is a $d : {\omega^\ast}(\lambda \times \Xi) \to \omega$ such that if $\gamma < \theta$ and $\tau \in \Xi$ are given and $((u_\alpha, v_\alpha) : \alpha < \lambda)$ satisfies

(i) $u_\alpha \subseteq {\omega^\ast}(\lambda \times \Xi) 2\geq (\lambda \times \Xi)$,
(ii) $v_\alpha \subseteq {\omega^\ast}(\lambda \times \Xi) 2\geq (\lambda \times \Xi)$,
(iii) $|u_\alpha| = |v_\alpha| = \tau$,
(iv) $\nu \in u_\beta \Rightarrow \nu(\ell g(\nu) - 1) = (\gamma, \tau)$,
(v) $\rho \in u_\alpha \Rightarrow \rho(0) = (\gamma, \tau)$,
(vi) $\eta \in u_\alpha \cap v_\alpha \Rightarrow (\exists \ell)(\eta(\ell) = (\alpha, \tau))$

then for some $\alpha < \beta$ we have

$$\nu \in u_\beta \& \rho \in v_\alpha \Rightarrow (\nu^\ast \rho)[d(\nu^\ast \rho)] = (\gamma, \tau).$$

2) Let $Pr_6(\lambda, \sigma)$ means $Pr_6(\lambda, \lambda, \sigma)$.

4.4 Fact. $Pr_6(\lambda, \lambda, \theta, \sigma), \theta \geq \sigma$ implies $Pr_6(\lambda, \theta, \sigma)$.

Proof. Let $c$ be a function from ${\omega^\ast}(\lambda \times \Xi)$ to $\theta$ exemplifying $Pr_6(\lambda, \theta, \sigma)$. Let $e$ be a one to one function from $\theta \times \Xi$ onto $\theta$.

Now we define a function $d$ from $({\omega^\ast}(\lambda \times \Xi)$ to $\omega$:

$$d(\nu) = \text{Min}\{\ell : c((e(\nu(m)) : m < \ell g(\nu))) = e(\nu(\ell))\}.$$
4.5 Claim. If $Pr_6(\lambda^+, \sigma), \lambda$ regular and $\sigma \leq \lambda$ then $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$.

Proof. Like the proof of 1.1.

4.6 Remark. As in 4.1, 4.2 we can prove that if $\mu > cf(\mu) + \sigma$ then $Pr_6(\mu^+, \mu^+, \mu^+, \sigma)$, hence $Pr_1(\mu^{+2}, \mu^{+2}, \mu^{+2}, \sigma)$, but this does not give new information.
REFERENCES.


