

# MORE ON REAL-VALUED MEASURABLE CARDINALS AND FORCING WITH IDEALS

by

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## Abstract.

- (1) It is shown that if  $c$  is real-valued measurable then the Maharam type of  $(c, \mathcal{P}(c), \sigma)$  is  $2^c$ . This answers a question of D. Fremlin [Fr,(P2f)].
- (2) A different construction of a model with a real-valued measurable cardinal is given from that of R. Solovay [So]. This answers a question of D. Fremlin [Fr,(P1)].
- (3) The forcing with a  $\kappa$ -complete ideal over a set  $X$ ,  $|X| \geq \kappa$  cannot be isomorphic to Random $\times$ Cohen or Cohen $\times$ Random. The result for  $X = \kappa$  was proved in [Gi-Sh1] but as was pointed out to us by M. Burke the application of it in [Gi-Sh2] requires dealing with any  $X$ . The application is: if  $A_n$  is a set of reals for  $n < \omega$  then for some pairwise disjoint  $B_n$  (for  $n < \omega$ ) we have  $B_n \subseteq A_n$  but they have the same outer Lebesgue measure.

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In Section 1 we deal with the Maharam types of real-valued measurable cardinals. The result (1) stated in the abstract and its stronger version are proved. The proofs are based on Shelah's strong covering lemmas and his revised power set operation.

In Section 2 a model with a real-valued measurable which is not obtained as the Solovay one by forcing random reals over a model with a measurable.

In Section 3, the result (3) stated in the abstract is proved.

Theorem 1.1 and the construction of Section 2 are due to the first author. Theorem 1.2 is joint and the result of Section 3 is due to the second author.

We are grateful to David Fremlin for bringing the questions on real-valued measurability to our attention. His excellent survey article [Fr] gave the inspiration for the present paper. We wish to thank the Max Burke for pointing out a missing stage in the argument of [Gi-Sh2].

## 1. On Number of Cohen or Random Reals

D. Fremlin asked the following in [Fr,(P2f)]:

If  $c$  is a real-valued measurable with witnessing probability  $\nu$ , does it follow that the Maharam type of  $(c, \mathcal{P}(c), \nu)$  is  $2^c$ ? or in equivalent formulation:

If  $c$  is a real-valued measurable does the forcing with witnessing ideal isomorphic to the forcing for adding  $2^c$  random reals?

The next theorem provides the affirmative answer; see also 1.2.

**Theorem 1.1.** *Suppose that  $I$  is a  $2^{\aleph_0}$ -complete ideal over  $2^{\aleph_0}$  and the forcing with it (i.e.  $\mathcal{P}(2^{\aleph_0})/I$ ) is isomorphic to the adding of  $\lambda$ -Cohen or  $\lambda$ -random reals. Then  $\lambda = 2^{2^{\aleph_0}}$ .*

**Proof:** Suppose otherwise. Denote  $2^{\aleph_0}$  by  $\kappa$ . Let  $j : V \rightarrow N$  be a generic elementary embedding.

**Claim 1.**  $j(\kappa) > (\lambda^+)^V$ .

**Proof:** By a theorem of Prikry [Pr] (see also [Gi-Sh2] for a generalization) for every  $\tau < \kappa$   $2^\tau = 2^{\aleph_0} = \kappa$ . Then, in  $N$ ,  $2^\kappa = j(\kappa)$ . But  $(\mathcal{P}(\kappa))^V \subseteq N$ , so  $j(\kappa) \geq (2^\kappa)^V$ . By

[Gi-Sh2 Lemma 2.2.], then  $(2^\kappa)^V = cov(\lambda, \kappa, \aleph_1, 2)$ . So  $cov(\lambda, \kappa, \aleph_1, 2) \geq \lambda^+$ . Clearly,

$$cov(\lambda, \kappa, \aleph_1, 2) \leq cov(\lambda, \aleph_1, \aleph_1, 2) \leq (cov(\lambda, \aleph_1, \aleph_1, 2))^N.$$

The last inequality holds since  $N$  is obtained by a c.c.c. forcing and so every countable set of ordinals in  $N$  can be covered by a countable set of  $V$ . By Shelah [Sh430,3.2(2)], in  $N$   $cov(\lambda, \aleph_1, \aleph_1, 2) < j(\kappa)$ . Hence  $\lambda^+ \leq cov(\lambda, \kappa, \aleph_1, 2) \leq (cov(\lambda, \aleph_1, \aleph_1, 2))^N < j(\kappa)$ .

□ of the claim.

By Shelah [Sh 430,3.2(1),(2)] for every  $i < 3$  there is  $S_i = \langle s_{i\beta} \mid \beta < \kappa \rangle \subseteq [\kappa]^{\leq \aleph_i}$  unbounded of cardinality  $\kappa$ . We can assume by shrinking  $S_i$  that for every regular  $\delta < \kappa$  if  $\langle s_{i\beta} \mid \beta < \delta \rangle$  is unbounded in  $[\delta]^{\leq \aleph_i}$  then for all  $\gamma, \delta \leq \gamma < \kappa$   $s_{i\gamma} \not\subseteq \delta$ . For  $\alpha < \kappa$  and  $i < 3$  let  $S_i \upharpoonright \alpha = \{P \in S_i \mid P \subseteq \alpha\}$ . Fix a function  $f \in {}^\kappa \kappa$  representing  $\kappa$  in a generic ultrapower and restrict everything to a condition forcing this. Without loss of generality  $f(\alpha) \leq \alpha$  for every  $\alpha < \kappa$ .

**Claim 2.** Let  $i < 3$  then  $\{\alpha < \kappa \mid S_i \upharpoonright f(\alpha) \text{ is unbounded in } [f(\alpha)]^{\leq \aleph_0} \text{ and } |S_i \upharpoonright f(\alpha)| = f(\alpha)\} \in I^*$  where  $I^*$  is the filter dual to  $I$ .

**Proof:** We drop the index  $i$  for a while.  $|S| = \kappa$  in  $V$ , so  $S$  is in a generic ultrapower. Suppose that in a generic ultrapower  $S$  is bounded. There will be some  $t \subseteq \kappa$  countable such that for every  $s \in S$   $s \not\supseteq t$ . Using c.c.c. of the forcing we find a countable subset of  $\kappa$  in  $V$ ,  $t^* \supseteq t$ . Since  $S$  is unbounded in  $V$  some  $s \in S$  contains  $t^*$ . Contradiction. Now,  $j(S) \upharpoonright \kappa = S$ , since  $\kappa$  is regular,  $S = \langle s_\beta \mid \beta < \kappa \rangle$  is unbounded in  $[\kappa]^{\leq \aleph_i}$  and hence no  $s_\gamma \subseteq \kappa$  for every  $\gamma$ ,  $\kappa \leq \gamma < j(\kappa)$ . □ of the claim.

Let  $N$  be a generic ultrapower. By [Gi-Sh1] there are in  $N$  at least  $\kappa$  Cohen (or random) reals over  $V$ .

**Claim 3.** There exists a sequence  $\langle r_\alpha \mid \alpha < \kappa \rangle$  of reals in  $V$  so that

- (1) every real of  $V$  appears in  $\langle r_\alpha \mid \alpha < \kappa \rangle$ .
- (2) for almost all  $\alpha \pmod I$   $\langle r_{\alpha+i} \mid i < f(\alpha) \rangle$  are  $f(\alpha)$ -Cohen (random) generic over  $L[\langle S_\xi \upharpoonright f(\alpha) \mid \xi < 3 \rangle, \langle r_\beta \mid \beta < f(\alpha) \rangle]$ .

**Proof:** Construct  $\langle r_\alpha \mid \alpha < \kappa \rangle$  by induction. On nonlimit stages add reals in order to satisfy (1). For limit  $\alpha$ 's with  $\xi_0 \upharpoonright f(\alpha)$  unbounded in  $[f(\alpha)]^{\leq \aleph_0}$ , for every  $\xi < 3$ , add

$f(\alpha)$ -Cohen (or random) reals. It is possible since there are at least  $\kappa$  candidates in a generic ultrapower by [Gi-Sh1].

□ of the claim.

Now work in  $N$ .  $\text{rng } f \upharpoonright A$  is unbounded in  $\kappa$ , for every  $A \notin I$ . Let  $j(\langle r_\alpha \mid \alpha < \kappa \rangle) = \langle r_\alpha \mid \alpha < j(\kappa) \rangle$  where  $\langle r_\alpha \mid \alpha < \kappa \rangle$  is a sequence given by Claim 3.

Then, using Claim 3 in  $N$  we can find some  $\alpha^* < j(\kappa)$  satisfying (2) of Claim 3 such that  $j(S_\xi) \upharpoonright f(\alpha^*)$  is unbounded in  $[f(\alpha^*)]^{\leq \aleph_0}$ , for every  $\xi < 3$  and  $j(f)(\alpha^*) \geq (\lambda^+)^V$ . It is possible since by Claim 1,  $(\lambda^+)^V < j(\kappa)$  and, in  $V$  the range of  $f$  restricted to a set not in  $I$  is unbounded in  $\kappa$ .

The following will provide the contradiction and complete the proof of the theorem.

**Claim 4.**  $\langle r_{\alpha^*+i} \mid i < j(f)(\alpha^*) \rangle$  is a sequence of Cohen (random) reals over  $V$ .

**Proof:**  $\langle r_{\alpha^*+i} \mid i < j(f)(\alpha^*) \rangle$  is Cohen (random)-generic over  $L[\langle j(S_\xi) \upharpoonright j(f)(\alpha^*) \mid \xi < 3 \rangle, \langle r_\beta \mid \beta < j(f)(\alpha^*) \rangle] =_{df} M(\alpha^*)$ . But  $\langle r_\beta \mid \beta < \kappa \rangle$  is the list of all the reals of  $V$ . So all the reals of  $V$  are in  $M(\alpha^*)$ . Hence it is enough for every  $Q \in ([j(f)(\alpha^*)]^{\leq \aleph_0})^V$  to find  $P \in ([j(f)(\alpha^*)]^{\leq \aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq \aleph_0})^{M(\alpha^*)}$  so that  $P \supseteq Q$ . Then we will have that also  $Q \in ([j(f)(\alpha^*)]^{\leq \aleph_0})^{M(\alpha^*)}$ , which implies that Cohen or Random genericity over  $M(\alpha^*)$  guarantees such genericity over  $V$ .

The following is more than enough.

**Subclaim 5.**  $([j(f)(\alpha^*)]^{\leq \aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq \aleph_0})^{M(\alpha^*)}$  is unbounded in  $[j(f)(\alpha^*)]^{\leq \aleph_0}$  of  $V[G]$ , where  $G$  is the generic subset producing  $j$ .

**Proof:** Denote  $j(f)(\alpha^*)$  by  $\tau$ . We like to apply Shelah's strong covering without squares [Sh580] to both pairs  $(V, V[G])$  and  $(M(\alpha^*), V[G])$  for  $[\tau]^{< \aleph_1}$ . First notice that  $V[G]$  is a c.c.c. extension of  $V$  so it preserves cofinalities and  $([\delta]^{< \mu})^V$  is unbounded (and even stationary) in  $([\delta]^{< \mu})^{V[G]}$  for every regular  $\mu$  and ordinal  $\delta \geq \mu$ . In particular, all the conditions of [Sh580] are satisfied by  $(V, V[G])$ .

Now let us turn to the pair  $(M(\alpha^*), V[G])$ . By the choice of  $\alpha^*$ ,  $j(S_i) \upharpoonright j(f)(\alpha^*)$  is unbounded in  $([j(f)(\alpha^*)]^{< \aleph_i})^N$  for  $i = 1, 2, 3$ . In particular  $\aleph_i^{M(\alpha^*)} = \aleph_i^N$  for  $i = 1, 2, 3$ . However,  $N$  and  $V[G]$  have the same small sequence of ordinals. In particular,  $\aleph_i^N = \aleph_i^{V[G]}$  and  $([j(f)(\alpha^*)]^{< \aleph_i})^N = ([j(f)(\alpha^*)]^{< \aleph_i})^{V[G]}$  for every  $i$ ,  $1 \leq i \leq 3$ .

Then the conditions of 6.7.2 of [Sh580] are satisfied by both pairs  $(V, V[G])$  and  $(M(\alpha^*), V[G])$ . We like to use 3.3 of [Sh580] with cofinal in  $[j(f)(\alpha^*)]^{\leq \aleph_0}$  families  $\mathcal{P}^0 \in V$  and  $\mathcal{P}^1 \in M(\alpha^*)$  so that  $|\mathcal{P}^0|^V \leq j(f)(\alpha^*)$  and  $|\mathcal{P}^1|^{M(\alpha^*)} \leq j(f)(\alpha^*)$ . Take  $\mathcal{P}^1 = j(S_1) \upharpoonright j(f)(\alpha^*)$ . Now, using c.c.c. of the forcing and working with names for elements of  $\mathcal{P}^1$  it is easy to construct  $\mathcal{P}^0 \in V$  such that each element of  $\mathcal{P}^0$  contains an element of  $\mathcal{P}^1$  and  $|\mathcal{P}^0|^V \leq j(f)(\alpha^*)$ . We are now ready to show unboundedness of  $([j(f)(\alpha^*)]^{\leq \aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq \aleph_0})^{M(\alpha^*)}$  in  $[j(f)(\alpha^*)]^{\leq \aleph_0}$  of the largest model  $V[G]$ . Work in  $V[G]$ . Let  $Q \in [j(f)(\alpha^*)]^{\leq \aleph_0}$ . Find  $P_0^0 \in \mathcal{P}^0$ ,  $P_0^0 \supseteq Q$ . Then let  $P_1^0 \in \mathcal{P}^1$   $P_1^0 \supseteq P_0^0$ . Continue by induction and define an increasing sequence  $\langle P_\nu^i \mid \nu < \omega_1, i < 2 \rangle$  so that  $P_0^0 \in \mathcal{P}^0$  and  $P_\nu^0 \subseteq P_\nu^1 \in \mathcal{P}^1$ . Then by 3.3. of [Sh580] sets  $\{\nu < \omega_1 \mid \bigcup_{\nu' < \nu} P_{\nu'}^0 \in V\}$  and  $\{\nu < \omega_1 \mid \bigcup_{\nu' < \nu} P_{\nu'}^1 \in M(\alpha^*)\}$  contain clubs. In particular there will be  $\nu < \omega_1$  in both of them. Then  $P = \bigcup_{\nu' < \nu} P_{\nu'}^0 = \bigcup_{\nu' < \nu} P_{\nu'}^1$  will be in  $V \cap M(\alpha^*)$  and we are done.

□ of subclaim 5.

□ of claim 4

□

Let us now prove a stronger statement which relies on a different property.

**Theorem 1.2.1.** 1) Suppose that  $I$  is a  $\kappa$ -complete ideal over  $\kappa$  and the forcing with it (i.e.  $\mathcal{P}(\kappa)/I$ ) is isomorphic to the forcing for adding  $\lambda$ -Cohen reals. Assume also (\*) some condition forces that “ $j(\kappa) \geq (2^\kappa)^V$ ”, where  $j$  is a generic embedding. Then  $\lambda > 2^{< \kappa}$  implies  $\lambda = 2^\kappa$ .  
2) Similarly for random.

**Proof:** 1) Without loss of generality, let us assume that the weakest condition, i.e.  $\kappa$  forces (\*). Suppose that  $\lambda < 2^\kappa$ . Then  $\lambda < j(\kappa)$  as  $\mathcal{P}(\kappa)^V \subseteq N$ . By [Sh430 3.2(2)] in  $V$  we have  $(\forall \theta < \kappa)(cov(\theta, \aleph_1, \aleph_1, 2) < \kappa)$  hence in a generic ultrapower  $N$   $cov(\lambda, \aleph_1, \aleph_1, 2) < j(\kappa)$ . However, in  $V$   $\lambda^+ \leq cov(\lambda, \kappa, \aleph_1, 2) \leq cov(\lambda, \aleph_1, \aleph_1, 2)^N < j(\kappa)$ . The first inequality holds by [Gi-Sh2]. Hence  $\lambda^+ < j(\kappa)$ .

Now, by [Sh 460, 2.6], there are regular  $\delta < \mu < \kappa$  such that  $cov(\lambda, \mu, \mu, \delta) = \lambda$  (or see [Sh513, §1]).

Let us assume for simplification of the notation that  $\mu = \aleph_2$ ,  $\delta = \aleph_1$ .

Let  $\langle s_\alpha \mid \alpha < \lambda \rangle$  be generic for  $\langle I, \subseteq \rangle$  i.e. a set of  $\lambda$  Cohens representing the generic  $G$ .

**Claim 1.** There is a sequence of reals  $\langle r_\alpha \mid \alpha < \lambda^+ \rangle$  in a generic ultrapower such that for every  $s \subseteq \omega_1$  and even  $s \in {}^{\omega_1}V$  the final segment of  $\langle r_\alpha \mid \alpha < \lambda^+ \rangle$  are Cohen generic over  $L[s]$ .

**Proof:** Let  $N$  be a generic ultrapower. Then  ${}^\kappa N \subseteq N$  where  ${}^\kappa N$  is in the sense of the generic extension. First note that  $\langle s_\alpha \mid \alpha < \kappa \rangle$  is a sequence of  $\kappa$  Cohen reals over  $V$  and it belongs to  $N$ . Clearly every  $s \subseteq \omega_1$  in  $N$  (or the same in  $V[G]$ ) is a name in  $V$  interpreted using  $\aleph_1$  Cohen reals only. Hence for every  $\delta \leq \kappa$  of cofinality  $> \aleph_1$ , some final segment of  $\langle s_\alpha \mid \alpha < \delta \rangle$  will be generic over  $L[s]$ . Then, in  $V$ , for  $I$ -almost every regular  $\delta < \kappa$  there is a sequence  $\bar{t}^\delta = \langle t_\alpha \mid \alpha < \delta \rangle$  such that for every  $s \subseteq \omega_1$  and  $\delta' \leq \delta$  of cofinality  $> \aleph_1$ , some final segment of  $\bar{t}^\delta \upharpoonright \delta'$  is Cohen generic over  $L[s]$ . Back in  $N$ , we use this for some  $\delta \geq \lambda^+$ , for  $\delta' = \lambda^+$  which is still below  $j(\kappa)$ .

□ of the claim.

Let us fix such a sequence  $\langle r_\alpha \mid \alpha < \lambda^+ \rangle$  in  $N$ . We split it into blocks each of the length  $\omega_1$ . Denote such changed sequence by  $\langle r_{\alpha i} \mid \alpha < \lambda^+, i < \omega_1 \rangle$ . Now back in  $V$ , let us use the fact that  $\text{cov}(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$ . We know that for every  $\alpha < \lambda^+$  the block  $\langle r_{\alpha i} \mid i < \omega_1 \rangle$  is added by using only  $\omega_1$  Cohen reals from the  $\lambda$  Cohen reals,  $\langle s_\beta \mid \beta < \lambda \rangle$ . Work in  $V$ . For every  $\alpha < \lambda^+$  and  $i < \omega_1$  pick a condition  $p_{\alpha i}$  in the Cohen forcing for adding  $\lambda$ -Cohen reals which decides the value of  $\mathcal{r}_{\alpha, i}(0)$ . Let  $\rho_{\alpha, i} \in 2$  be such a value, i.e.  $p_{\alpha i} \Vdash \mathcal{r}_{\alpha, i}(0) = \rho_{\alpha, i}$ . W.l. of  $g$ .

$$p_{\alpha 0} \Vdash \left| \{i < \omega_1 \mid p_{\alpha, i} \in \mathcal{G}\} \right| = \aleph_1 .$$

For every  $\alpha < \lambda^+$  and  $i < \omega_1$  let  $\text{dom } p_{\alpha i} = \{\xi_{\alpha, i, \ell} \mid \ell < \ell_{\alpha i} < \omega\} \subseteq \lambda$  and  $p_{\alpha i}(\xi_{\alpha, i, \ell}) = \eta_{\alpha, i, \ell} \in {}^{\omega}2$ . As  $2^{<\kappa} < \lambda$  we can assume that  $\rho_{\alpha, i}, \ell_{\alpha, i}$ 's and  $\eta_{\alpha, i, \ell}$ 's do depend on  $\alpha$  for  $\alpha \in A_0$ ,  $A_0 \subseteq \lambda^+$  unbounded. Thus further we shall drop the index  $\alpha$  in these sets. The number of possibilities for  $\langle \xi_{\alpha, 0, \ell} \mid \ell < \ell_0 \rangle$ 's,  $\alpha \in A_0$  is  $\lambda^{\ell_0} = \lambda$ . So we can assume that for some  $\langle \xi_{0, \ell} \mid \ell < \ell_0 \rangle$  for every  $\alpha \in A_0$   $\langle \xi_{\alpha, 0, \ell} \mid \ell < \ell_0 \rangle = \langle \xi_{0, \ell} \mid \ell < \ell_0 \rangle$ . Hence also  $p_{\alpha, 0}$ 's ( $\alpha \in A_0$ ) will be the same. Drop the index  $\alpha$  and denote  $p_{\alpha, 0}$  by  $p_0$ . Now,

$cov(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$ , so there is a set  $b \subseteq \lambda$  of cardinality  $\aleph_1$  such that the following set is unbounded in  $\lambda^+$ :

$$A = \{\alpha \in A_0 \mid (\exists^{\aleph_1} i)(\forall \ell < l_i)(\xi_{\alpha,i,\ell} \in b) \text{ and } (\forall \ell < l_0)(\xi_{0,\ell} \in b)\}.$$

We actually replace  $\lambda$  in  $cov$  by  $\lambda^{<\omega} \times \omega_1$ . Recall that  $2^{<\kappa} < \lambda$ . We can shrink  $A$  to an unbounded in  $\lambda^+$  set  $A'$  and find uncountable  $a \subseteq \omega_1$  with  $0 \in a$  such that for every  $\alpha, \beta \in A'$ ,  $i \in a$  and  $\ell < l_i$   $\xi_{\alpha,i,\ell} = \xi_{\beta,i,\ell}$ . Again we drop the index  $\alpha$  for  $\alpha$  in  $A'$ ,  $i \in a, \ell < l_i$  and denote  $\xi_{\alpha,i,\ell}$  by  $\xi_{i,\ell}$ . Now,  $p_0$  forces that in  $L[\langle \xi_{i,\ell} \mid i \in a, \ell < l_i \rangle, \langle l_i \mid i \in a \rangle, \langle \eta_{i,\ell} \mid i \in a, \ell < l_i \rangle, \langle \rho_i \mid i \in a \rangle] = \mathcal{M}$  we have information on  $\langle r_{\alpha,i}(0) \mid i \in a \rangle$  for unboundedly many  $\alpha < \lambda^+$ . Thus for  $i \in a$  and  $\alpha \in A'$  the following holds:

if for every  $\ell < l_i$   $s_{\xi_{i,\ell}}$  extends  $\eta_{i,\ell}$  then  $r_{\alpha,i}(0) = \rho_i$ .

Apply the claim 1 to  $M$ . We find some  $\alpha^* < \lambda^+$  such that for every  $\alpha, \alpha^* \leq \alpha < \lambda^+$   $\langle r_{\alpha,i} \mid i < \omega_1 \rangle$  is Cohen generic over  $M$ . Using c.c.c. of the forcing the value of  $\alpha^*$  can be fixed already in  $V$ . Fix  $\alpha^{**} \geq \alpha^*$  an element of  $A'$ . Recall that  $\langle \eta_{i,\ell} \mid i \in a, \ell < l_i \rangle$  is a sequence of elements of  ${}^\omega 2$  which belongs to  $V$  and  $\langle s_{\xi_{i,\ell}} \mid i \in a, \ell < l_i \rangle$  are Cohen reals over  $V$ . So, for some  $a^* \subseteq a$  of cardinality  $\aleph_1$ ,  $a^* \in M$  satisfies the following: for every  $i \in a^*$  and  $\ell < l_i$   $s_{\xi_{i,\ell}}$  extends  $\eta_{i,\ell}$ . Then for every  $i \in a^*$   $r_{\alpha^{**},i}(0) = l_i$ . But  $a^* \in M$ ,  $\langle l_i \mid i \in a^* \rangle \in M$ , so  $\langle r_{\alpha^{**},i} \mid i < \omega_1 \rangle$  cannot be Cohen generic over  $M$ . Contradiction.

2) Let us now deal with the Random reals case. Most of the proof repeats the Cohen reals case, but instead of choosing  $p_{\alpha,i}, \xi_{\alpha,i,\ell}(\ell < l_{\alpha i}), \rho_{\alpha,i}$  we proceed as follows. Find for every  $\alpha < \lambda^+$  and  $i < \omega_1$   $m_{\alpha,i}, n_{\alpha,i} < \omega$ ,  $\xi_{\alpha,i,0} < \xi_{\alpha,i,1} < \dots < \xi_{\alpha,i,m_{\alpha,i}-1}$  and a function  $g_{\alpha,i}$  from  ${}^{m_{\alpha,i}}(n_{\alpha,i} 2)$  to  $\{0, 1\}$  such that

$$\|g_{\alpha,i}(\xi_{\alpha,i,0}, \xi_{\alpha,i,1}, \dots, \xi_{\alpha,i,m_{\alpha,i}-1}) \neq r_{\alpha,i}(0)\| < \frac{1}{4}$$

We view  $g_{\alpha,i}$  as the continuous function on  ${}^{m_{\alpha,i}}(n_{\alpha,i} 2)$  determined by its values on  ${}^{m_{\alpha,i}}(n_{\alpha,i} 2)$ . For every  $\alpha < \lambda^+$  pick an unbounded set  $S_\alpha \subseteq \omega_1$ ,  $m_\alpha, n_\alpha < \omega$  and  $g_\alpha$  so that for every  $i \in S_\alpha$   $m_{\alpha,i} = m_\alpha$ ,  $n_{\alpha,i} = n_\alpha$  and  $g_{\alpha,i} = g_\alpha$ .  $2^{<\kappa} < \lambda$ , so we can assume w.l. of  $g$ . that there are  $S^*, m^*, n^*$  and  $g^*$  such that for some unbounded  $A_0 \subseteq \lambda^+$  for every  $\alpha \in A_0$   $S_\alpha = S^*$ ,  $n_\alpha = n^*$ ,  $m_\alpha = m^*$  and  $g_\alpha = g^*$ . Let  $i_0 < i_1 < \dots < i_n < \dots (n < \omega)$  be any increasing sequence of elements of  $S^*$ . Consider for  $\alpha \in A_0$ ,  $\ell < \omega$

$$\mathfrak{k}_{\alpha,\ell} = |\{\ell' < \ell \mid g^*(\langle \xi_{\alpha,i_{\ell'},m} \mid m < m^* \rangle) \neq r_{\alpha,i_{\ell'}}(0)\}| / \ell.$$

By basic probability some  $q_{\alpha, \langle i_n | n < \omega \rangle}$  forces that

$$\text{“} \lim_{\ell \rightarrow \infty} \inf \{ k_{\alpha, \ell'} \mid \ell' \geq \ell \} < \frac{1}{3}, \text{”}.$$

Now, using  $\text{cov}(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$  for  $[\lambda]^{m^*} \times S^*$  we find in  $V$  a set  $b \subseteq \lambda$ ,  $|b| = \aleph_1$  so that

$$A = \{ \alpha \in A_0 \mid \text{the cardinality of } \{ i \in S^* \mid \forall \ell < m^* \xi_{\alpha, i, \ell} \in b \} \text{ is } \aleph_1 \}$$

is unbounded in  $\lambda^+$ . Again, shrinking  $A$  to an unbounded  $A' \subseteq \lambda^+$ , using  $2^{<\kappa} < \lambda$ , we find  $a \subseteq S^*$  of cardinality  $\aleph_1$  so that for every  $\alpha, \beta \in A'$ ,  $i \in a$  and  $\ell < m^*$   $\xi_{\alpha, i, \ell} = \xi_{\beta, i, \ell} = \xi_{i, \ell}$ .

Consider now in  $V[G]$ ,

$$M = L[\langle s_{\xi_{i, \ell}} \mid i \in a, \ell < m^* \rangle, a, g^*].$$

Back in  $V$ , using Claim 1 for Random and c.c.c. find  $\alpha^* < \lambda^+$  such that the weakest condition forces

“for every  $\alpha, \alpha^* \leq \alpha < \lambda^+$  the sequence  $\langle r_{\alpha, i} \mid i < \omega_1 \rangle$  is Random generic over  $\mathcal{M}$ ”.

Fix some  $\alpha^{**} \geq \alpha^*$  in  $A'$ . Let  $\vec{i} = \langle i_n \mid n < \omega \rangle$  be the sequence of first  $\omega$  elements of  $a$ . Choose a generic set  $G$  with  $q_{\alpha^{**}, \vec{i}} \in G$ . Then,  $\lim_{\ell \rightarrow \infty} \inf \{ k_{\alpha^{**}, \ell'} \mid \ell' \geq \ell \} < \frac{1}{3}$ . But this is impossible since  $\langle r_{\alpha^{**}, i} \mid i < \omega_1 \rangle$  is a sequence random reals over  $M$  and  $\vec{i} \in M$ . Contradiction.  $\square$

## 2. Another Construction of a Model with a Real-Valued Measurable Cardinal

In this section we construct a model with a real-valued measurable cardinal which differs from the Solovay original. This answers negatively a question of D. Fremlin [Fr, (P1)]:

Let  $N$  be a model of ZFC and  $\kappa \in N$  a real-valued measurable cardinal in  $N$ . Does it follow that there are inner models  $M \subseteq N$  such that  $\kappa$  is a measurable in  $M$  and  $M$ -generic filter  $G$  for a random real p.o. set over  $M$  such that  $G \in N$  and  $N \cap \mathcal{P}(\kappa) \subseteq M[G]$ .

Suppose that  $\kappa$  is a measurable and  $GCH$  holds. We define a forcing notion  $P$  as follows:



**Definition 2.1.**  $P$  consists of all triples  $p = \langle p_0, p_1, p_2 \rangle$  so that

- (1)  $p_0 \subseteq \kappa$
- (2)  $p_1$  is a function with domain contained in  $p_0$
- (3)  $p_2$  is a function defined over inaccessibles  $\leq \kappa$
- (4) for every inaccessible  $\delta$   $|p_0 \cap \delta| < \delta$ ,  $|\text{dom } p_1 \cap \delta| < \delta$  and  $|\text{dom } p_2 \cap \delta| < \delta$
- (5) for every  $\alpha \in \text{dom } p_1$   $p_1(\alpha) \subseteq \alpha$
- (6) every element of  $p_0$  is an ordinal of cofinality  $\aleph_0$
- (7) for every limit ordinal  $\beta$  if  $\text{cf } \beta > \aleph_0$ , then  $p_0 \cap \beta$  is not stationary in  $\beta$  and if  $\text{cf } \beta = \aleph_0$  then  $\beta \setminus (p \cap \beta)$  is unbounded in  $\beta$
- (8) for every  $\alpha \in \text{dom } p_2$   $p_2(\alpha)$  is a closed subset of  $\alpha$  disjoint with  $p_0$ .

**Definition 2.2.** Let  $p, q \in P$   $p = \langle p_0, p_1, p_2 \rangle$  and  $q = \langle q_0, q_1, q_2 \rangle$ . Then  $p \geq q$  iff

- (1)  $p_1 \subseteq q_1$
- (2)  $\text{dom } p_2 \supseteq \text{dom } q_2$  and for every  $\alpha \in \text{dom } q_2$   $p_2(\alpha)$  is an end extension of  $q_2(\alpha)$
- (3)  $p_0 \supseteq q_0$
- (4) for every  $\delta < \kappa$ , if  $\delta$  is an inaccessible or a limit of inaccessibles and  $\delta^*$  is the least inaccessible above  $\delta$  then  $p_0 \cap [\delta, \delta^*)$  is an end extension of  $q_0 \cap [\delta, \delta^*)$ .

The forcing  $P$  is intended to add three objects. Thus, the first coordinates of  $P$  are producing a subset  $S$  of  $\kappa$  which is stationary in  $V[S]$  and reflecting only in inaccessibles. The second coordinate is responsible for a kind of diamond sequence over  $S$  and the last coordinate adds clubs preventing reflection of  $S$  at inaccessibles and its stationarity.

The forcing  $P$  destroys the measurability of  $\kappa$  once used over  $V = L[\mu]$ . It is bad for our purpose. We are going to use a certain subforcing of  $P$  which will preserve measurability and contain the projection of  $P$  to the first two coordinates. But first let us study basic properties of  $P$ .

Let  $P_0 = \{p_0 \mid \exists \langle p_1, p_2 \rangle \langle p_0, p_1, p_2 \rangle \in P\}$ ,  $P_{01} = \{\langle p_0, p_1 \rangle \mid \exists p_2 \langle p_0, p_1, p_2 \rangle \in P\}$ . Let  $\alpha$  be an inaccessible. We denote by  $P \upharpoonright \alpha$  the set

$$\{\langle p_0 \cap \alpha, p_1 \upharpoonright \alpha, p_2 \upharpoonright \alpha \rangle \mid \langle p_0, p_1, p_2 \rangle \in P\}$$

and by  $P \setminus \alpha$  the set

$$\{\langle p_0 \setminus \alpha, p_2 \upharpoonright [\alpha, \kappa), p_2 \upharpoonright [\alpha + 1, \kappa) \rangle \mid \langle p_0, p_1, p_2 \rangle \in P\},$$

$P_0 \upharpoonright \alpha, P_{01} \upharpoonright \alpha$  and  $P_0 \setminus \alpha, P_{01} \setminus \alpha$  are defined similarly.

The following is standard.

**Claim 2.3.** Let  $\alpha$  be an inaccessible then the following holds

- (1)  $P = P \upharpoonright \alpha \times P \setminus \alpha$
- (2)  $P_0 = P_0 \upharpoonright \alpha \times P_0 \setminus \alpha$
- (3)  $P_{01} = P_{01} \upharpoonright \alpha \times P_{01} \setminus \alpha$ .

Let  $\alpha < \kappa$  be a limit ordinal and  $Q$  a forcing notion.

Consider the following game  $\text{Game}(Q, \alpha)$ :

$$\begin{array}{ccccccc} \text{I} & q_1 & & q_3 & \cdots & & \cdots \\ & \searrow & & \swarrow & & & \geq \\ \text{II} & & q_2 & & \cdots & & q_\beta \cdots q_\alpha \end{array}$$

where Players I, II are building an increasing sequence of elements of  $Q$ , I at even stages and II at odds. If at some stage  $\beta < \alpha$  II cannot continue i.e. there is no  $q$  above  $\{q'_{\beta'} \mid \beta' < \beta\}$  then I wins. Otherwise II wins.

**Claim 2.4.** The player II has a winning strategy in the game  $\text{Game}(P \setminus \alpha, \alpha^+)$  for every inaccessible  $\alpha$ .

**Proof:** Let  $\alpha$  be an inaccessible. We define a winning strategy  $\sigma$  for Player II in the  $\text{Game}(P \setminus \alpha, \alpha^+)$ .

Let  $\delta > \alpha$  be an inaccessible but not limit one. Denote by  $\delta^-$  the supremum of inaccessibles below  $\delta$ .

Let  $p \in P \setminus \alpha$ . We define  $\bar{p}$  to be the condition obtained from  $p = \langle p_0, p_1, p_2 \rangle$  by adding  $\sup(p_0 \cap [\delta^-, \delta)) + \sup(p_2(\delta))$  to  $p_2(\delta)$  if  $p_0 \cap [\delta^-, \delta) \neq \emptyset$  or  $p_0 \cap [\delta^-, \delta) = \emptyset$  but  $p_0 \cap \delta^-$  is unbounded in  $\delta^-$ , where  $\delta$  runs over inaccessibles above  $\alpha$  which are not limit inaccessibles and  $\sup(p_2(\delta)) = 0$  whenever  $\delta \notin \text{dom } p_2$ .

Now we define  $\sigma$  to be dependent only on the last move of  $I$  at successive stages of the game. Set  $\sigma(p_{\beta+1}) = \bar{p}_{\beta+1}$ . If  $\beta \leq \alpha^+$  is limit and the game up to  $\beta$

$$\begin{array}{ccccccc} p_1 & & p_3 & \cdots & & \cdots & \\ & & p_2 & & \cdots & & p_\gamma \end{array}$$

was played according to  $\sigma$ . Then set  $\sigma(\langle p_\gamma \mid \gamma < \beta \rangle) =$  the closure of  $\bigcup_{\gamma < \beta} p_\gamma$ . More precisely, let  $\sigma(\langle p_\gamma \mid \gamma < \beta \rangle) = \langle p^0, p^1, p^2 \rangle$  where  $p^0 = \bigcup_{\gamma < \beta} p_\gamma^0$ ,  $p^1 = \bigcup_{\gamma < \beta} p_\gamma^1$  and  $\text{dom}(p^2) = \bigcup_{\gamma < \beta} \text{dom}(p_\gamma^2)$ ,  $p^2(\xi) = \bigcup \{p_\gamma^2(\xi) \mid \gamma < \beta, \xi \in \text{dom } p_\gamma^2\} \cup \{\text{sup}(\bigcup \{p_\gamma^2(\xi) \mid \xi \in \text{dom } p_\gamma^2, \gamma < \beta\})\}$ , for  $\xi \in \text{dom } p^2$ .

We need to check that such defined  $p$  is a condition. The only problem is to show that  $p^0$  does not reflect at any  $\tau$ ,  $\aleph_0 < cf\tau < \tau$ . So let  $\tau$  be an ordinal such that  $\aleph_0 < cf\tau < \tau$  and  $p^0 \cap \tau$  is unbounded in  $\tau$ . Pick  $\delta$  to be the first inaccessible above  $\tau$ . Then  $\delta^- \leq \tau$ . If  $\delta^- < \tau$ , then starting with some  $\gamma_0 < \beta$   $p_{\gamma_0}^0 \cap [\delta^-, \delta) \neq \emptyset$ . But then  $p^2(\delta)$  will be a club of  $\tau$  disjoint to  $p^0 \cap \tau$ . Suppose now that  $\delta^- = \tau$ . Then  $C_0 = \{\text{sup}(\bigcup_{\gamma' < \gamma} (p_{\gamma'}^0 \cap \tau)) \mid \gamma < \beta, \gamma \text{ limit}\}$  is a club of  $\tau$ . Also  $C_1 = \{\xi < \tau \mid \xi \text{ is a limit of inaccessibles}\}$  is a club of  $\tau$ . Let  $\xi \in C_0 \cap C_1$ . Then for some inaccessible  $\underline{\delta} < \tau$   $\xi = \underline{\delta}^-$ . Let  $\gamma_\xi < \beta$  be so that  $\xi = \text{sup} \bigcup_{\gamma' < \gamma_\xi} (p_{\gamma'}^0 \cap \tau)$ . Then,  $\xi \in p_{\gamma_\xi}^2(\underline{\delta})$ , by the definition of  $\sigma$  and the procedure. Thus, this provides a club of  $\tau$  disjoint to  $p_\beta^0 \cap \tau$ . □ of the claim.

The following is now trivial.

**Claim 2.5.**  *$P$  preserves cofinalities and does not add new functions from ordinals less than the first inaccessible into  $V$ .*

Let  $U$  be a normal measure over  $\kappa$  and  $j : V \rightarrow N$  the corresponding elementary embedding. Then, in  $N$ ,  $j(P) = j(P) \upharpoonright \kappa \times j(P) \setminus \kappa$ . Clearly,  $(j(P) \upharpoonright \kappa)^N = P$ . Now let us produce inside  $V$  an  $N$ -generic subset of  $j(P) \setminus \kappa$  with the set over the first coordinate nonstationary in  $V$ .

**Claim 2.6.** There exists  $\langle S, F_1, F_2 \rangle$  such that

- (a)  $\langle S, F_1, F_2 \rangle$  is  $j(P) \setminus \kappa^+$  generic over  $N$
- (b)  $S$  is not stationary subset of  $j(\kappa)$
- (c)  $S$  does not reflect.

**Proof:** Let  $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$  be the list of dense open subsets of  $j(P) \setminus \kappa$  of  $N$ . Let  $\sigma \in N$  be a winning strategy for Player II in Game  $(j(P) \setminus \kappa, \kappa^+)$ . It exists by Claim 2.4 applied in  $N$  to  $j(P)$ . Play the game from  $V$  so that I plays at stage  $\beta + 1$  an element  $P_{\beta+1}$  of  $D_\beta$  which is above  $p_\beta$ , where  $\beta < \kappa^+$ . We will finish with a desired  $N$ -generic set.  $\square$

Force with  $P$  over  $V$ . Let  $G$  be a generic subset. We denote  $\bigcup \{p_0 \mid \exists \langle p_1, p_2 \rangle \langle p_0, p_1, p_2 \rangle \in G\}$  by  $S$ . For every  $\alpha \in S$  let  $A_\alpha = \bigcup \{p_1(\alpha) \mid \exists \langle p_0, p_1, p_2 \rangle \in G \text{ and } \alpha \in \text{dom } p_1\}$  and for inaccessible  $\delta \leq \kappa$  let  $C_\delta = \bigcup \{p_2(\delta) \mid \exists \langle p_0, p_1, p_2 \rangle \in G \text{ and } \delta \in \text{dom } p_2\}$ . Then  $S \subseteq \kappa$ , and for every inaccessible  $\delta \leq \kappa$   $C_\delta$  is a club of  $\delta$  disjoint to  $S$ .

**Claim 2.7.**  $S$  is a stationary nonreflecting subset of  $\kappa$  in  $V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle C_\delta \mid \delta \text{ inaccessible and } \delta < \kappa \rangle]$ .

**Proof:**  $\langle C_\delta \mid \delta < \kappa \rangle$  are witnessing the nonreflection. Suppose that  $S$  is nonstationary in  $V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle C_\delta \mid \delta \text{ inaccessible and } \delta < \kappa \rangle]$ . Return back to  $V$  and work with names. Suppose for simplicity that the empty condition forces the nonstationarity of  $S$ . Let  $\tilde{C}$  be a name of witnessing club. Pick an elementary submodel  $N$  of  $V_{2^{2^\kappa}}$  such that  $P, \tilde{C} \in N$ ,  $|N| < \kappa$  and  $N \cap \kappa$  is an ordinal of cofinality  $\aleph_0$ . Let  $\alpha = N \cap \kappa$  and  $\langle \alpha_n \mid n < \omega \rangle$  be a cofinal in  $\alpha$  sequence. Now by induction we construct an increasing sequence  $\langle p_i \mid i < \omega \rangle$  of conditions of  $P \upharpoonright \kappa$  (i.e.  $P$  without the information on a club of  $\kappa$  disjoint to  $S$ ) such that for every  $i < \omega$

- (a)  $p_i \in N$
- (b)  $p_i$  decides the first element of  $\tilde{C}$  above  $\alpha_i$
- (c)  $\text{sup}(p_i)_0 \geq \alpha_i$ .

where  $p_i = \langle (p_i)_0, (p_i)_1, (p_i)_2 \rangle$

Now, in  $V$ , let

$$p = \langle \bigcup_{i < \omega} (p_i)_0 \cup \{\alpha\}, \bigcup_{i < \omega} (p_i)_1, \{ \langle \delta, \cup \{ (p_i)_2(\delta) \mid i < \omega, \delta \in \text{dom}(p_i)_2 \} \rangle \} .$$

Then  $p \Vdash \alpha \in \tilde{C} \cap \tilde{S}$ . Contradiction.  $\square$

**Claim 2.8.**  $\kappa$  is a measurable cardinal in  $V[S, \langle A_\alpha \mid \alpha \in S \rangle]$ .

**Proof:** Just note that in  $N$   $j(P_{01}) = P_{01} \times j(P_{01}) \setminus \kappa^+$ , since nothing is done in the interval  $[\kappa, \kappa^+]$  by this forcing. By Claim 2.6, there is a  $j(P_{01}) \setminus \kappa^+$  generic over  $N$  set in

$V$ . Thus it is easy to extend  $j$  to the embedding of  $V[S, \langle A_\alpha \mid \alpha \in S \rangle]$ . This insures the measurability of  $\kappa$ .  $\square$

Notice, that  $\kappa$  is not measurable in  $V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle C_\delta \mid \delta < \kappa, \delta \text{ inaccessible} \rangle]$  since  $S$  is a stationary nonreflecting subset of  $\kappa$ .

Now, over a model  $V[S, \langle A_\alpha \mid \alpha \in S \rangle]$  we are going to force a Boolean algebra  $B$  such that:

- (a)  $\kappa$  is still measurable in  $V[S, \langle A_\alpha \mid \alpha \in S \rangle, B]$
- (b)  $j^*(B)/G(B)$  is isomorphic to the adding of  $j(\kappa)$ -Random reals, where  $j^*$  the elementary embedding of  $V[S, \langle A_\alpha \mid \alpha \in S \rangle, B]$  into its ultrapower and  $G(B)$  a generic subset of  $B$ .

First let us review some basics of product measure algebras. We refer to D. Fremlin [Fr2] for detailed presentation.

Suppose that  $B$  is a  $\sigma$ -algebra, i.e. a Boolean algebra all of whose countable suprema exist. A *measure on  $B$*  is a function  $\mu : B \rightarrow [0, 1]$  so that: (a)  $\mu(1_B) = 1$ , and (b) whenever  $\{b_n \mid n \in \omega\} \subseteq B$  with  $b_n \wedge b_m = 0$  for  $n \neq m$ , then  $\mu(\bigvee_n b_n) = \sum_n \mu(b_n)$ . If in addition  $\mu$  is *positive* (i.e.  $\mu(b) = 0$  iff  $b = 0$ ), then we say that  $\langle B, \mu \rangle$  is a *measure algebra*. A measure algebra is always a complete Boolean algebra.

Suppose now that  $I$  is a set, and  $\langle B_i, \mu_i \rangle$  for  $i \in I$  are measure algebras. Call  $C \in \prod_{i \in I} B_i$  a *cylinder* iff  $C(i)$  is the unit element of  $B_i$ , except for a finite number of coordinates  $i$ . Let  $B \supseteq \prod_{i \in I} B_i$  be the  $\sigma$ -algebra generated by the cylinders. It is known that there is a unique measure  $\mu$  on  $B$  so that  $\mu(C) = \prod_{i \in I} \mu_i(C(i))$  for any cylinder  $C$ .  $\mu$  may not be positive, but there is a standard strategy: Let  $I = \{b \in B \mid \mu(b) = 0\}$ . Then  $I$  is an ideal, and  $\overline{B} = B/I$  as usual is a  $\sigma$ -algebra consisting of equivalence classes  $[b]$  for  $b \in B$  (where  $[b] = [c]$  iff the symmetric difference  $(b - c) \vee (c - b) \in I$ ). We can define a positive measure  $\overline{\mu}$  on  $\overline{B}$  by:  $\overline{\mu}([b]) = \mu(b)$ . Thus,  $\langle \overline{B}, \overline{\mu} \rangle$  is a measure algebra, called the *product measure algebra* of the  $\langle B_i, \mu_i \rangle$ 's.

Let  $\mathbf{2}$  be the basic measure algebra  $\langle P(2), \mu \rangle$  where  $\mu$  is the measure:  $\mu(\emptyset) = 0$ ,  $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$ , and  $\mu(\{0, 1\}) = 1$ . For any set  $I$ , let  $\mathbf{2}^I$  denote the product measure algebra of  $I$  copies of  $\mathbf{2}$ . We can then force with  $\mathbf{2}^I$  with the natural proviso:  $b$  is a *stronger* condition

than  $c$  iff  $0 < b \leq c$  in  $\mathbf{2}^I$ . This forcing obviously has the  $\omega_1$ -c.c.

For  $I = \omega$ ,  $\mathbf{2}^I$  is just the usual random real forcing and for  $I = \lambda$   $\mathbf{2}^I$  is the  $\lambda$ -random real forcing. Let us denote them by Random and Random( $\lambda$ ) respectively.

We consider the  $\sigma$ -algebras  $B_\alpha \subseteq \prod_{i < \alpha} (\mathcal{P}(2))_i$  generated by the cylinders, where  $\alpha \leq \kappa$  and  $(\mathcal{P}(2))_i$  is just  $i$ -th copy of  $\mathcal{P}(2)$ . The desired algebra  $B$  will be  $B_\kappa/I_\kappa$ , where the ideal  $I_\kappa$  of “null” sets is going to be added generically. More precisely for  $\alpha$ 's of countable cofinality  $I_\alpha$ 's will be added by forcing and then for  $\beta \leq \kappa$  of uncountable cofinality  $I_\beta$  will be the union of  $I_\alpha$ 's, where  $\alpha < \beta$ ,  $cf\alpha = \aleph_0$ . The sequence of ideals  $\langle I_\alpha \mid \alpha \leq \kappa \rangle$  will be in  $V_1 = V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle C_\delta \mid \delta < \kappa, \delta \text{ inaccessible} \rangle]$ .

Let us work in  $V_1$ . We define by induction on  $\alpha < \kappa$  a measure  $\mu_\alpha$  on  $B_\alpha$ . Then  $I_\alpha$  will be the ideal of  $\mu_\alpha$ -measure null sets. Actually there will be a lot of different measures over  $B_\alpha$ 's. We would like to prevent  $B_\kappa$  (and even its subalgebras of power  $\kappa$ ) from carrying a measure. For this purpose, the “diamond” sequence  $\langle A_\alpha \mid \alpha \in S \rangle$  will be used to destroy possible candidates.

If  $\alpha < \min S$ , then let  $\mu_\alpha$  be the usual product measure over  $B_\alpha$ , i.e. one generated by attaching weight  $1/2$  to  $\{0\}$  and  $\{1\}$ ,  $0$  to  $\emptyset$  and  $1$  to  $\{0, 1\}$  in every component  $(\mathcal{P}(2))_i$  ( $i < \alpha$ ) of the product  $\prod_{i < \alpha} (\mathcal{P}(2))_i$ . Set  $I_\alpha = \{X \in B_\alpha \mid \mu_\alpha(X) = 0\}$ .

Suppose now that  $\alpha < \kappa$  and for every  $\beta < \alpha$  the measure  $\mu_\beta$  over  $B_\beta$  was already defined. We need to define  $\mu_\alpha$  over  $B_\alpha$ .

**Case 1.**  $\alpha \notin S$ .

Pick an increasing continuous sequence  $\langle \alpha_\tau \mid \tau < cf\alpha \rangle$  witnessing nonstationarity of  $S \cap \alpha$ . In case,  $cf\alpha = \aleph_0$  just use  $\omega$ -sequence unbounded in  $\alpha$  and disjoint with  $S$ . For every  $\tau < cf\alpha$  let  $\mu(\tau)$  be  $\mu_{\alpha_{\tau+1}} \upharpoonright (B_{\alpha_{\tau+1}} \upharpoonright [\alpha_\tau, \alpha_{\tau+1}))$ , where  $B_{\alpha_{\tau+1}} \upharpoonright [\alpha_\tau, \alpha_{\tau+1})$  is the subalgebra of  $\prod_{\alpha_\tau \leq i < \alpha_{\tau+1}} (\mathcal{P}(2))_i$  generated by the cylinders.

Let  $\mu_\alpha$  be the product measure of  $\langle \langle B_{\alpha_{\tau+1}} \upharpoonright [\alpha_\tau, \alpha_{\tau+1}), \mu(\tau) \rangle \mid \tau < cf\alpha \rangle$ .

Notice that  $\alpha_\tau \notin S$ . Therefore by induction, we can assume that for a limit  $\tau$  the measure  $\mu_{\alpha_\tau}$  over  $B_{\alpha_\tau}$  is the product measure of  $\langle \langle B_{\alpha_\nu} \upharpoonright [\alpha_\nu, \alpha_{\nu+1}), \mu(\nu) \rangle \mid \nu < \tau \rangle$ .

**Case 2.**  $\alpha \in S$ .

Suppose that  $A_\alpha$  codes in some reasonable fashion sequences  $\langle \alpha_n \mid n < \omega \rangle$ ,  $\langle \varphi_n \mid n < \omega \rangle$ ,

$\langle \mu(n) \mid n < \omega \rangle$  and  $\langle a_n \mid n < \omega \rangle$  so that for every  $n < \omega$

- (a)  $\langle \alpha_n \mid n < \omega \rangle$  is a cofinal in  $\alpha$  sequence
- (b)  $a_n$  is a countable subset of  $[\alpha_n, \alpha_{n+1})$
- (c)  $\mu(n)$  is a measure over  $B_{\alpha_{n+1}} \upharpoonright a_n$  respecting the ideal  $I_{\alpha_{n+1}} \upharpoonright a_n$ , i.e. for every  $X \in B_{\alpha_{n+1}} \upharpoonright a_n$   $\mu(n)(X) = 0$  iff  $X \in I_{\alpha_{n+1}}$
- (d)  $\varphi_n: \text{Random} \leftrightarrow B_{\alpha_{n+1}} \upharpoonright a_n$  is a measure algebra isomorphism.

Denote  $B_{\alpha_{n+1}} \upharpoonright a_n$  by  $B(n)$ . Let us define a measure  $\tilde{\mu}(n)$  over  $B(n)$ . Thus for every  $n < \omega$  let us change the value  $\mu(n)(\varphi_n(\{0\}))$  from  $1/2$  to  $1 - \frac{1}{\pi^2 n^2}$  and those of  $\varphi_n(\{1\})$  from  $1/2$  to  $\frac{1}{\pi^2 n^2}$ . Let  $\tilde{\mu}(n)$  be the measure obtained from  $\mu(n)$  in such a fashion. Clearly, such local changes have no effect on the set of measure zero. Namely, for every  $X \in B(n)$   $\mu(n)(X) = 0$  iff  $\tilde{\mu}(n)(X) = 0$ .

Define now the measure  $\mu_\alpha$  over  $B_\alpha$  as the product measure of the measure algebras  $\langle B(n), \tilde{\mu}(n) \rangle$  ( $n < \omega$ ) together with all the rest, i.e.

$$\langle B_{\alpha_{n+1}} \upharpoonright ([\alpha_n, \alpha_{n+1}) \setminus a_n), \mu_{\alpha_{n+1}} \upharpoonright ([\alpha_n, \alpha_{n+1}) \setminus a_n) \rangle .$$

We claim that  $\varphi = \bigcup_{n < \omega} \varphi_n$  cannot be extended to complete embedding into  $\langle B_\alpha, \mu_\alpha \rangle$ . The reason is that under  $\varphi$  the measure of the set  $\bigcap_{n < \omega} \varphi_n(\{0\})$  should be zero, but  $\mu_\alpha(\bigcap_{n < \omega} \varphi_n(\{0\})) = \prod_{n < \omega} \tilde{\mu}(n)(\varphi_n(\{0\})) = \prod_{n < \omega} (1 - \frac{1}{\pi^2 n^2})$  which equals  $\sin(1) \neq 0$  by the Euler formula.

Notice, that the ideal  $I_\alpha$  of  $\mu_\alpha$ -measure zero sets will not be effected if for finitely many  $n$ 's the measures  $\mu(n)$  will be used in the product instead of  $\tilde{\mu}(n)$ 's. Also, if in the previous construction we will do everything above some  $\alpha_{n_0}$  for fixed  $n_0 < \omega$ , i.e. we will define the measure over  $B_\alpha \upharpoonright [\alpha_{n_0}, \alpha)$  instead of all  $B_\alpha$  call it  $\mu_\alpha \upharpoonright [\alpha_{n_0}, \alpha)$  and its ideal  $I_\alpha \upharpoonright [\alpha_{n_0}, \alpha)$ , then, for every  $X \in B_\alpha$   $X \in I_\alpha$  iff  $X \upharpoonright \alpha_{n_0} \in I_{\alpha_{n_0}}$  and  $X \upharpoonright [\alpha_{n_0}, \alpha) \in I_\alpha \upharpoonright [\alpha_{n_0}, \alpha)$ . This means that once having  $I_{\alpha_n}$ 's, initial segments of measures  $\langle \tilde{\mu}(n) \mid n < \omega \rangle$  have no effect on  $I_\alpha$ . This observation will be crucial further for showing measurability of  $\kappa$ .

If  $A_\alpha$  does not guess the sequences as above, then we proceed as in Case 1.

This completes the definition of  $\langle \mu_\alpha \mid \alpha < \kappa \rangle$  and hence also  $\langle I_\alpha \mid \alpha \leq \kappa \rangle$ .

We set  $B = B_\kappa / I_\kappa$ . Let  $V_2 = V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle I_\alpha \mid \alpha < \kappa, \text{cf } \alpha = \aleph_0 \rangle]$ . Then for

every  $\alpha < \kappa$   $I_\alpha \in V_2$ . So  $B \in V_2$ . We will show the following claim which has a proof similar to 2.7.

**Claim 2.9.**  $\text{Random}(\kappa)$  does not embed into  $B$  in  $V_1$  and also in  $V_2$ .

**Proof:** Notice that  $V_2$  and  $V_1$  have the same reals. So if  $\varphi$  is an embedding of  $\text{Random}(\kappa)$  into  $B$  in  $V_2$  then  $\varphi$  will be also such embedding in  $V_1$ . Hence let us prove the claim for  $V_1$ .

Suppose otherwise. Let  $\varphi : \text{Random}(\kappa) \longrightarrow B$  witnessing embedding. Back in  $V$  let us work with names. Let  $\check{\varphi}$  be a name of  $\varphi$  and assume for simplicity that the empty condition forces this.

Pick  $N$  and  $\langle \alpha_n \mid n < \omega \rangle$  to be as in Claim 2.7 with  $\check{\varphi}$  replacing  $\check{C}$ .

We define sequences of conditions of  $P \upharpoonright \kappa$  of  $\{p_n \mid n < \omega\} \subseteq N$  of ordinals  $\langle \beta_n \mid n < \omega \rangle$ , countable sets  $\langle a_n \mid n < \omega \rangle$  and embedding  $\langle \varphi_n \mid n < \omega \rangle$  so that

- (a)  $\sup(p_n)_0 \geq \alpha_n$
- (b)  $\beta_n \geq \alpha_n$
- (c)  $p_n \Vdash'' \check{\varphi} (\{0\}_{\beta_n})$  has nontrivial intersection with  $\check{B} \upharpoonright [\beta_n, \beta_{n+1})''$ , where  $\{0\}_{\beta_n} \in (\mathcal{P}(2))_{\beta_n}$  i.e. the  $\beta_n$ -th copy of  $\mathcal{P}(2)$ .
- (d)  $a_n \subseteq \beta_n$
- (e)  $\varphi_n$  embeds  $(\mathcal{P}(2))_{\beta_n}$  into  $B_\kappa \upharpoonright a_n$
- (f)  $p_n \Vdash'' \check{\varphi}_n$  is equal to  $\check{\varphi} \upharpoonright (\mathcal{P}(2))_{\beta_n} \bmod I_\kappa''$

Since the forcing does not add new countable sequences of elements of  $V$ , there is no problem in carrying out the induction.

Denote by  $\mu(n)$  the measure over  $B_\kappa \upharpoonright a_n$  induced by  $\varphi_n$ .

Now let  $A_\alpha \subseteq \alpha$  be a code for such sequences

$$\langle \beta_n \mid n < \omega \rangle, \langle a_n \mid n < \omega \rangle, \langle \varphi_n \mid n < \omega \rangle$$

and  $\langle \mu(n) \mid n < \omega \rangle$ .

Set  $p = \langle \bigcup_{n < \omega} (p_n)_0 \cup \{\alpha\}, \bigcup_{n < \omega} (p_n)_1 \cup \{\langle \alpha, A_\alpha \rangle\},$

$$\{\langle \delta, \bigcup \{(p_n)_2(\delta) \mid n < \omega, \delta \in \text{dom}(p_n)_2\}\rangle\}$$



Then  $p \Vdash \varphi$  does not embed  $\prod_{n < \omega} (\mathcal{P}(2))_{\beta_n}$  into  $B_\alpha / I_\alpha''$ , by the definition of  $I_\alpha''$ . Hence, also  $p \Vdash \varphi$  does not embed  $\text{Random}(\kappa)$  into  $B''$ . Contradiction.  $\square$

**Claim 2.10.**  $\kappa$  is a measurable cardinal in  $V_1$ .

**Proof:** Let  $j : V \rightarrow N$  be an elementary embedding witnessing the measurability of  $\kappa$ .

We like to extend it to an embedding

$$\begin{aligned} j^* : V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle I_\alpha \mid \alpha < \kappa \text{ and } \text{cf } \alpha = \aleph_0 \rangle] \\ \rightarrow N[S^*, \langle A_\alpha \mid \alpha \in S^* \rangle, \langle I_\alpha \mid \alpha < j(\kappa) \text{ and } \text{cf } \alpha = \aleph_0 \rangle]. \end{aligned}$$

By Claim 2.6,  $j$  extends to

$$j' : V[S, \langle A_\alpha \mid \alpha \in S \rangle] \rightarrow N[S^*, \langle A_\alpha \mid \alpha \in S^* \rangle]$$

where  $\langle S^* \setminus S, \langle A_\alpha \mid \alpha \in S^* \setminus S \rangle \rangle \in V$  is  $j(P_{01}) \setminus \kappa$  generic over  $N$ . We like to produce ideals  $\langle I_\alpha \upharpoonright (B_{j(\kappa)} \upharpoonright [\kappa, j(\kappa)]) \mid \kappa < \alpha < j(\kappa), \text{cf } \alpha = \aleph_0 \rangle$  generic over  $N$  but in  $V$ . In order to define  $\langle I_\alpha \mid \alpha < \kappa, \text{cf } \alpha = \aleph_0 \rangle$  we used clubs witnessing nonreflection of  $S$ , i.e.  $\langle C_\delta \mid \delta < \kappa, \delta \text{ inaccessible} \rangle$ . By Claim 2.6, the only club which is needed in order to extend  $j$  but is missing in  $V$  is  $C_\kappa$ . But, we define generically  $I_\alpha$ 's only for  $\alpha$ 's of cofinality  $\aleph_0$  and moreover initial segments have no influence on such  $I_\alpha$ 's. This means that the definition of  $\langle I_\alpha \upharpoonright B_{j(\kappa)} \upharpoonright [\kappa, j(\kappa)] \mid \kappa < \alpha < j(\kappa), \text{cf } \alpha = \aleph_0 \rangle$  can be carried out completely inside  $N[S^* \setminus S, \langle A_\alpha \mid \alpha \in S^* \setminus S \rangle, \langle C_\delta \mid \kappa < \delta < j(\kappa), \delta \text{ inaccessible} \rangle]$ . All the sets  $S^* \setminus S, \langle A_\alpha \mid \alpha \in S^* \setminus S \rangle$  and  $\langle C_\delta \mid \kappa < \delta < j(\kappa), \delta \text{ inaccessible of } N \rangle$  can be found inside  $V$  by Claim 2.6. Hence we have enough sets to extend  $j$  to  $j^*$ . Thus, the measurability of  $\kappa$  is preserved in  $V_2 = V[S, \langle A_\alpha \mid \alpha \in S \rangle, \langle I_\alpha \mid \alpha < \kappa \text{ and } \text{cf } \alpha = \aleph_0 \rangle]$ .  $\square$

Let  $j^* : V_2 \rightarrow N_2 = N[S^*, \langle A_\alpha \mid \alpha \in S^* \rangle, \langle I_\alpha \mid \alpha < j(\kappa) \text{ and } \text{cf } \alpha = \aleph_0 \rangle]$  be the embedding of Claim 2.10.

**Claim 2.11.**  $j^*(B) \upharpoonright [\kappa, j(\kappa))$  is isomorphic to  $\text{Random}(\kappa^+)$  in  $V_2$ .

**Proof:** By Claim 2.6, there are in  $V$  and hence in  $V_2$  clubs  $\langle C_\delta \mid \kappa < \delta \leq j(\kappa), \delta \text{ is an inaccessible in } N \rangle$  witnessing nonreflection of  $S^* \setminus S$  in every  $N$ -inaccessible  $\delta \leq j(\kappa)$ . Using them we define measures  $\mu_\alpha$  over  $B_\alpha \upharpoonright [\kappa, \alpha)$  agreeing with ideals  $I_\alpha$  for every  $\alpha, \kappa < \alpha \leq$

$j(\kappa)$  as it was done for  $B_\alpha$ 's below  $\kappa$  in  $V_1$ . The final measure  $\mu_{j(\kappa)}$  will turn  $B_{j(\kappa)} \upharpoonright [\kappa, j(\kappa))$  into measure algebra. Since  $|j(\kappa)| = \kappa^+$ , by Maharam theorem, see [Fr2] we obtain the desired result.  $\square$

So we have the following:

**Theorem 2.**  $\kappa$  be the measurable cardinal of  $L[\mu] = V$  (the minimal model with a measurable). Then  $\kappa$  is a real valued measurable in  $V_2^B$  but for every submodel  $V'$  of  $V_2^B$  if  $\kappa$  is a measurable in  $V'$ , then there is no  $G \in V_2^B$  which is  $\text{Random}(\kappa)$  generic over  $V'$ .

**Proof:** Suppose that  $V' \subseteq V_2^B$ ,  $\kappa$  is a measurable in  $V'$ ,  $G \subseteq \text{Random}(\kappa)$  generic over  $V'$  and  $G \in V_2^B$ . But then  $G$  is also  $\text{Random}(\kappa)$  generic over  $V = L[\mu]$ , since  $L[\mu] \subseteq V'$  by its minimality. But  $V$  and  $V_2$  have the some countable sequences of ordinals. So,  $G$  will be  $\text{Random}(\kappa)$ -generic also over  $V_2$ . This means that  $\text{Random}(\kappa)$  embeds  $B$ , which is impossible by Claim 2.9.  $\square$

### 3. The Forcing with Ideal Cannot be Isomorphic to Cohen $\times$ Random or Random $\times$ Cohen

The result for  $\kappa$ -complete ideals over  $\kappa$  was proved in [Gi-Sh1]. Max Bruke pointed out that the application of this in [Gi-Sh2] requires the result also for less than  $\kappa$  complete ideals as well. The purpose of this section is to close this gap.

**Theorem 3.1.** Suppose that  $I$  is a  $\omega_1$ -complete ideal over some  $\kappa$  then the forcing with ideal (i.e.  $\mathcal{P}(\kappa)/I$ ) cannot be isomorphic to Cohen $\times$ Random or Random $\times$ Cohen.

**Proof:** Let us deal with Random $\times$ Cohen case. The Cohen $\times$ Random case is similar.

Suppose otherwise.  $\mathcal{P}(\kappa)/I \simeq \text{Random} \times \text{Cohen}$ . Without loss of generality for some  $\kappa_1 \leq \kappa$  and  $f : \kappa \rightarrow \kappa_1$   $\kappa \Vdash_{\mathcal{P}(\kappa)/I} \kappa_1$  is the critical point of the generic embedding and  $\tilde{f}$  represents  $\kappa_1$  in the ultrapower". Define an ideal  $J$  over  $\kappa_1$  to be the set of all  $A \subseteq \kappa_1$  such that  $f^{-1''}(A) \in I$ . Denote  $Q = \mathcal{P}(\kappa)/I$  and  $Q_1 = \mathcal{P}(\kappa_1)/J$ . Then  $Q_1$  is a complete subordering of  $Q$ . By [Sh480], we can assume that  $Q_1$  is  $\omega^\omega$ -bounding since otherwise it adds a Cohen real which suffices for the argument of [Gi-Sh1]. We define a  $Q_1$ -name  $\tilde{\tau} = \{\eta \in {}^\omega 2 \mid \text{the condition } (\mathbb{1}_{\text{Random}}, \eta) \text{ is compatible with every element of } \tilde{G}(Q_1)\}$ .

For  $\eta, \nu \in {}^\omega 2$  let us write  $\eta \triangleright \nu$  if the sequence  $\eta$  extends the sequence  $\nu$ . The following two claims are obvious.

**Claim 3.2.**  $\tau_{\sim}$  is a  $Q_1$ -name of a nonempty subset of  ${}^\omega 2$  closed under initial segments with no  $\triangleleft$ -maximal element and hence a tree.

**Claim 3.3.**  $\Vdash_{Q_1}''$  the Cohen real is an  $\omega$ -branch of  $\tau_{\sim}$ .

**Claim 3.4.** There is no  $p \in Q_1$  and  $\eta \in {}^\omega 2$  such that  $p \Vdash_{Q_1}$  "for every  $v \in {}^\omega 2$   $v \triangleright \eta$  implies  $v \in \tau_{\sim}$ ".

**Proof:** Suppose otherwise. Let  $p, \eta$  be witnessing this. Then above  $p$  the forcing notion  $Q_1$  is a complete subordering of Random. But it has to add a real. Hence it is isomorphic to Random which is impossible by [Gi-Sh1].  $\square$

Using  ${}^\omega$ -boundness of  $Q_1$ , we can find  $p_0 \in Q_1$  and a function  $h : \omega \rightarrow \omega$  such that  $p_0 \Vdash_{Q_1}$  "for every  $n < \omega$  and  $\eta \in {}^n 2$  there is  $v, \eta \triangleleft v \in {}^{h(n)} 2$  such that  $v \notin \tau_{\sim}$ ". Let us assume for simplicity that this  $p_0$  is just the weakest condition of  $Q_1$ .

Let  $T^* = \{T \mid T \subseteq {}^\omega 2 \text{ is a tree and for every } n < \omega, \eta \in {}^n 2 \text{ there is } v \triangleright \eta \text{ such that } v \in {}^{h(n)} 2 \text{ and } v \notin T\}$ . Consider also  $T_m^* = \{T \cap {}^m 2 \mid T \in T^*\}$  for  $m < \omega$ .  $T^*$  can be viewed as a tree if we identify it with  $\bigcup_{m < \omega} T_m^*$  and define an order by setting  $t_1 \triangleleft t_2$  iff for some  $m < \omega$   $t_1 = t_2 \cap {}^m 2$ . Then, clearly,

$$\Vdash_{Q_1}'' \tau_{\sim} \text{ is an } \omega \text{ - branch of } T^{*\prime} .$$

**Claim 3.5.** Suppose that  $n < \omega$ ,  $q_0 \in \text{Random}$ ,  $\eta \in {}^n 2$ . Then there are  $m < \omega$ ,  $q, v_0, v_1, t_0, t_1$  such that

- (a)  $q \in \text{Random}$  and  $q \geq q_0$ .
- (b)  $\eta \triangleleft v_0, v_1 \in {}^\omega 2$
- (c)  $t_0, t_1 \subseteq {}^{m \geq 2}$  and  $t_0 \neq t_1$ .
- (d)  $(q, v_i) \Vdash_{\sim} \tau \cap {}^{m \geq 2} = t_i$  for  $i < 2$ .

**Proof:** Find first some  $q' \geq q_0$  and  $v_0 \triangleleft \eta$  deciding  $\tau \cap {}^{m \geq 2}$ . Let  $t_0$  be the decided value, i.e.  $(q', v_0) \Vdash_{\sim} \tau \cap {}^{m \geq 2} = t_0$ . By the Claim 3.4 there will be  $m < \omega$  and  $v \triangleleft \eta$ ,

$v \in {}^m 2 \setminus t_0$ . Find some  $(q, v_1) \geq (q', v)$  deciding  $\tau \cap {}^m 2$ . Let  $t_1$  be the forced value, i.e.  $(q, v_1) \Vdash_{\sim} \tau \cap {}^m 2 = t_1$ . Since  $(q, v_1) \Vdash_{\sim} v_1 \in \tau$ , we have  $(q, v_1) \Vdash_{\sim} v = v_1 \upharpoonright m \in \tau$ . But this means  $t_0 \neq t_1$ .  $\square$

**Claim 3.6.** Suppose that  $n, k < \omega$  and  $q_0 \in \text{Random}$ . Then there are  $q \in \text{Random}$ ,  $m < \omega$ ,  $\langle v_{\eta, \ell} \mid \eta \in {}^n 2, \ell < k \rangle$  and  $\langle t_{\eta, \ell} \mid \eta \in {}^n 2, \ell < k \rangle$  such that

- (a)  $q \geq q_0$
- (b)  $m \geq n$
- (c) for every  $\eta_1, \eta_2 \in {}^n 2$ ,  $\ell_1, \ell_2 < k$   $t_{\eta_1, \ell_1} = t_{\eta_2, \ell_2}$  iff  $(\eta_1, \ell_1) = (\eta_2, \ell_2)$
- (d) for every  $\eta \in {}^n 2$ ,  $\ell < k$

$$\eta \triangleleft v_{\eta, \ell} \in {}^{\omega > 2}, t_{\eta, \ell} \in T_m^* \text{ and } (q, v_{\eta, \ell}) \Vdash_{\sim} \tau \cap {}^m 2 = t_{\eta, \ell} .$$

**Proof:** Just use the previous claim enough times. Thus, first, we generate a tree of  $k \cdot (2^n + 1)$  possibilities for one  $\eta \in {}^n 2$  and then we repeat the argument of Claim 3.5 on all  $\eta$ 's.  $\square$

**Claim 3.7.** For every  $n < \omega$ ,  $k < \omega$ ,  $q' \in \text{Random}$  and  $\mathcal{E} > 0$  there are  $m < \omega$ ,  $q \geq q'$ ,  $\{q_\ell \mid \ell < \ell^*\} \subseteq \text{Random}$  pairwise disjoint,  $\langle v_{\eta, \ell, j} \mid \eta \in {}^n 2, \ell < \ell^*, j < k \rangle$  and  $\langle t_{\eta, \ell, j} \mid \eta \in {}^n 2, \ell < \ell^*, j < k \rangle$  such that

- (a)  $Lb(q) \geq 1 - \mathcal{E}$  ( $Lb$  denotes the Lebesgue measure)
- (b)  $q = \bigcup_{\ell < \ell^*} q_\ell$
- (c) if  $\eta \in {}^n 2$ ,  $\ell < \ell^*$  and  $j < k$  then  $v_{\eta, \ell, j} \in {}^{\omega > 2} v_{\eta, \ell, j} \triangleright \eta$  and  $(q_\ell, v_{\eta, \ell, j}) \Vdash_{\sim} \tau \cap {}^m \geq 2 = t_{\eta, \ell, j}$ .
- (d) for every  $\ell < \ell^*$

$$t_{\eta_1, \ell, j_1} = t_{\eta_2, \ell, j_2} \text{ iff } (\eta_1, j_1) = (\eta_2, j_2) .$$

**Proof:** We define by induction  $q_\ell$ 's using Claim 3.6. Thus if  $\langle q_i \mid i \leq \ell \rangle$  is defined then we apply Claim 3.6 to  ${}^{\omega 2} \setminus \bigcup_{i \leq \ell} q_i$ . The process stops after we reach  $\ell^*$  s.t.  $Lb(\bigcup_{\ell < \ell^*} q_\ell) \geq 1 - \mathcal{E}$ .  $\square$

**Claim 3.8.** For every  $n < \omega$  and  $\mathcal{E} > 0$  there are  $m, n \leq m < \omega$  and a function  $H : T_m^* \rightarrow 2$  such that for every  $\eta \in {}^n 2$  and  $i \in 2$  we can find  $q^{i,\eta}, \ell^{i,\eta} < \omega, \langle q_\ell^{i,\eta} \mid \ell < \ell^{i,\eta} \rangle$  and  $\langle v_\ell^{i,\eta} \mid \ell < \ell^{i,\eta} \rangle$  such that for every  $i < 2$  and  $\ell < \ell^{i,\eta}$

- (a)  $\eta \triangleleft v_\ell^{i,\eta} \in {}^\omega 2$
- (b)  $q^{i,\eta}, q_\ell^{i,\eta} \in \text{Random}$  and  $q^{i,\eta} = \bigcup_{\ell < \ell^i} q_\ell^{i,\eta}$
- (c)  $Lb(q^{i,\eta}) \geq 1 - \mathcal{E}$
- (d)  $\langle q_\ell^{i,\eta} \mid \ell < \ell^{i,\eta} \rangle$  are pairwise disjoint
- (e)  $(q_\ell^{i,\eta}, v_\ell^{i,\eta}) \Vdash \text{“} H(\tau \cap^{m \geq 2}) = i \text{”}$ .

**Proof:** For every  $\eta \in {}^n 2$  and  $t \in T_m^*$  let  $I_{\eta,t} = \{q \in \text{Random} \mid \text{there is } v \in {}^\omega 2, v \triangleleft \eta \text{ such that } (q, v) \Vdash \tau \cap^{m \geq 2} = t''\}$ . Let  $\{q_{\eta,t,\ell} \mid \ell < \ell_{\eta,t} \leq \omega\}$  be a maximal antichain subset of  $I_{\eta,t}$ . Let  $q_{\eta,t}^* = \bigcup_{\ell < \ell_{\eta,t}} q_{\eta,t,\ell}$ . Then  $\bigcup_{t \in T_m^*} q_{\eta,t}^* = {}^\omega 2 \text{ mod null set}$ , since  $\{q_{\eta,t,\ell} \mid t \in T_m^*, \ell < \ell_{\eta,t}\}$  is a predense subset of  $\text{Random}$  (but not necessarily antichain). So  $Lb(\bigcup_{t \in T_m^*} q_{\eta,t}^*) = 1$ .

It is enough to prove the following statement:

(\*) There exists  $H : T_m^* \rightarrow 2$  so that for every  $\eta \in {}^n 2$  and  $i < 2$

$$Lb\left(\bigcup\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\}\right) \geq 1 - \frac{\mathcal{E}}{2}.$$

Since then we will be able to find a maximal antichain  $\langle q_\ell^i \mid \ell < \ell^* \leq \omega \rangle$  in  $\text{Random}$  above  $\bigcup\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\}$  together with  $\langle v_\ell^i \mid \ell < \ell^* \rangle$  and  $\langle t_\ell^i \mid \ell < \ell^* \rangle$  so that

$$(q_\ell^i, v_\ell^i) \Vdash \tau \cap^{m \geq 2} = t_\ell^i \text{ and } H(t_\ell) = i''.$$

In order to reduce  $\ell^*$  to a finite  $\ell^i$  we note that the precision here is  $1 - \frac{\mathcal{E}}{2}$  but only  $1 - \mathcal{E}$  is needed.

So let us prove (\*). We consider the set  $\mathcal{H}$  of all functions  $H : T_m^* \rightarrow 2$ . It is finite but more transparent is to look at it as a probability space. All  $H \in \mathcal{H}$  with the same probability. So we choose  $H(t) \in \{0, 1\}$  independently for the  $t \in T_m^*$  with probability  $1/2$ .

We use  $m$  given by Claim 3.7 for our  $n, \mathcal{E}'$  much smaller than  $\mathcal{E}$  and  $k$  large enough. Given  $\eta \in {}^n 2$  and  $i < 2$ . We consider the probability of

$$\left(Lb\left(\bigcup\{q_{\eta,t}^* \mid t \in T_m^*, H(t) = i\}\right)\right) \geq 1 - \frac{\mathcal{E}}{2}$$

in  $\mathcal{H}$ . It is  $\leq 1$ , as the value is always  $\leq 1$  and is  $\geq 1 - \frac{1}{2^k}$ . In order to prove the last inequality, let us use  $\{q_\ell \mid \ell < \ell^*\}$  of Claim 3.7. Thus

$$\begin{aligned} & Lb\left(\bigcup \left\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\right\}\right) \\ &= \sum_{\ell < \ell^*} Lb\left(q_\ell \cap \bigcup \left\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\right\}\right) + Lb\left[\left(\omega 2 \setminus \bigcup_{\ell < \ell^*} q_\ell\right) \cap \right. \\ &\quad \left. \left(\bigcup \left\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\right\}\right)\right] \geq \\ &\geq \sum_{\ell < \ell^*} \left[ Lb(q_\ell) \times \left( Lb\left(q_\ell \cap \bigcup \left\{q_{\eta,t}^* \mid t \in T_m^* \text{ and } H(t) = i\right\}\right) \right) / Lb(q_\ell) - \mathcal{E}' \right] \end{aligned}$$

Since (a) of 3.7,  $Lb\left(\omega 2 \setminus \bigcup_{\ell < \ell^*} q_\ell\right) < \mathcal{E}'$ . Now it suffices to show that for each  $\ell < \ell^*$

$$Lb\left(q_\ell \cap \bigcup \left\{q_{\eta,t}^* \mid t \in T_m^*, H(t) = i\right\}\right) / Lb(q_\ell) \geq 1 - \frac{\mathcal{E}}{4}$$

holds for enough  $H$ 's. But  $v_{\eta,\ell,j}, t_{\eta,\ell,j}$  ( $j < k$ ) of 3.7 are witnessing that the probability in  $\mathcal{H}$  of the failure is  $\leq \frac{1}{2^k}$ . Just in order to fail,  $H$  should take the value  $1 - i$  on  $t_{\eta,\ell,j}$  for every  $j < k$  and the probability of 0,1 are equal. The probability of the failure for some  $\eta \in {}^{n_2}2, i \in 2$  is then  $\leq \frac{2^{n+1}}{2^k}$ . So, picking  $k$  large enough comparatively to  $n$  we will insure that most  $H \in \mathcal{H}$  are fine, whereas we need only one.  $\square$

Now using Claim 3.8, we define by induction on  $j < \omega$   $n_j, m_j, H_j, \langle q_j^{i,\eta} \mid i < 2, \eta \in {}^{n_j}2 \rangle$   $\langle q_{j,\ell}^{i,\eta} \mid \ell < \ell_j, i < 2, \eta \in {}^{n_j}2 \rangle$  and  $\langle v_{j,\ell}^{i,\eta} \mid \ell < \ell_j, i < 2, \eta \in {}^{n_j}2 \rangle$  such that

- (1)  $n_j < m_j < n_{j+1}$
- (2)  $m_j H_j, \langle q_j^{i,\eta} \mid i < 2, \eta \in {}^{n_j}2 \rangle, \langle q_{j,\ell}^{i,\eta} \mid \ell < \ell_j^{n,i}, i < 2, \eta \in {}^{n_j}2 \rangle$  and  $\langle v_{j,\ell}^{i,\eta} \mid \ell < \ell_j^{i,\eta}, i < 2, \eta \in {}^{n_j}2 \rangle$  are given by Claim 3.8 for  $n = n_j$  and  $\mathcal{E} = \frac{1}{2^{2^{n_j}}}$
- (3)  $\text{length}\left(v_{j,\ell}^{i,\eta}\right) < n_{j+1}$  for every  $i < 2, \eta \in {}^{n_j}2, \ell < \ell_j^{i,\eta}$ .

Now define a  $Q_1$ -name  $\sigma \in {}^\omega 2$  by setting

$$\sigma(j) = H_j\left(\tau \cap {}^{m_j}2\right).$$

**Claim 3.9.**  $\Vdash_Q$  “ $\sigma$  is a Cohen real over  $V$ ”.

**Proof:** It is enough to show the following:

for every  $\mathcal{E} > 0$  and  $\eta^* \in {}^\omega 2$  the following holds:

for every  $j < \omega$  large enough and  $\nu \in {}^\omega 2$  of the length  $> j$  there are  $q, \langle q_\ell \mid \ell < \ell^* \rangle$  and  $\langle v_\ell \mid \ell < \ell^* \rangle$  such that

- (a)  $q, \langle q_\ell \mid \ell < \ell^* \rangle$  are in Random
- (b)  $q = \bigcup_{\ell < \ell^*} q_\ell$
- (c)  $Lb(q) \geq 1 - \mathcal{E}$
- (d)  $v_\ell \leq \eta^*$  for every  $\ell < \ell^*$
- (e)  $(q_\ell, v_\ell) \Vdash''_Q \sigma \upharpoonright [j, \text{length } \nu) = \nu \upharpoonright [j, \text{length } \nu)''$ .

**Proof of  $\otimes$ .** Pick  $j < \omega$  such that  $n_j > \text{length } \eta^*$  and  $2^{-j} < \frac{\mathcal{E}}{2}$ . Let  $\nu$  be given. We choose by induction on  $k \in [j, \text{length } \nu)$  a set  $a_k$  and  $\langle q_\eta \mid \eta \in a_k \rangle$  such that

- (a)  $a_k \subseteq {}^{n_k} 2$  is nonempty.
- (b)  $a_j$  is a singleton extending  $\eta^*$
- (c)  $\forall \eta \in a_{k+1} (\eta \upharpoonright n_k \in a_k)$  and  $\forall \eta \in a_k \exists \eta' \in a_{k+1} (\eta \triangleleft \eta')$ .
- (d) for every  $\eta \in a_k$

$$(q_\eta, \eta) \Vdash'' \bigwedge_{\ell=j}^{k-1} H_\ell \left( \tau \cap {}^{m_\ell} 2 \right) = \nu(\ell)''$$

i.e.  $(q_\eta, \eta) \Vdash'' \sigma \upharpoonright [j, k-1) = \nu \upharpoonright [j, k-1)''$

- (e) for  $\eta \in a_j$   $q_\eta = {}^\omega 2$
- (f) for  $\eta \in a_k$   $\langle q_\rho \mid \eta \triangleleft \rho \in a_{k+1} \rangle$  is an antichain of Random above  $q_\eta$  and  $\sum \{Lb(q_\rho) \mid \eta \triangleleft \rho \in a_{k+1}\} / Lb(q_\eta) \geq 1 - \frac{1}{2^k}$ .

There is no problem in carrying on this induction. This completes the proof of  $\otimes$  and hence also the theorem. □

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