

# The Consistency of $\text{ZFC} + 2^{\aleph_0} > \aleph_\omega + \mathcal{I}(\aleph_2) = \mathcal{I}(\aleph_\omega)$

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## 1 Introduction

The <sup>1</sup> basic notion that will be studied in this work is that of an *identity*. It arises naturally in a Ramsey theory setting when considering the coloring patterns on finite sets that occur when coloring infinite complete graphs with infinitely many colors. We first give some definitions and establish some notation.

An  $\omega$ -coloring is a pair  $\langle f, B \rangle$  where  $f : [B]^2 \rightarrow \omega$ . The set  $B$  is the *field* of  $f$  and denoted  $\text{Fld}(f)$ .

**Definition 1** Let  $f, g$  be  $\omega$ -colorings. We say that  $f$  realizes the coloring  $g$  if there is a one-one function  $k : \text{fld}(g) \rightarrow \text{fld}(f)$  such that for all  $\{x, y\}, \{u, v\} \in \text{dom}(g)$

$$f(\{k(x), k(y)\}) \neq f(\{k(u), k(v)\}) \Rightarrow g(\{x, y\}) \neq g(\{u, v\}).$$

We write  $f \simeq g$  if  $f$  realizes  $g$  and  $g$  realizes  $f$ . It should be clear that  $\simeq$  induces an equivalence relation on the class of  $\omega$ -colorings. We call the  $\simeq$ -classes of  $\omega$ -colorings with finite fields *identities*.

If  $f, g, h, k$  are  $\omega$ -colorings, with  $f \simeq g$  and  $h \simeq k$ , then  $f$  realizes  $h$  if and only if  $g$  realizes  $k$ . Thus without risk of confusion we may speak of identities

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realizing colorings and of identities realizing other identities. We say that an identity  $I$  is of *size*  $r$  if  $|\text{fld}(f)| = r$  for some (all)  $f \in I$ .

Let  $\kappa$  be a cardinal and  $f : [\kappa]^2 \rightarrow \omega$ . We define  $\mathcal{I}(f)$  to be the collection of identities realized by  $f$  and  $\mathcal{I}(\kappa)$  to be  $\bigcap \{\mathcal{I}(f) \mid f : [\kappa]^2 \rightarrow \omega\}$ . We now define a specific collection of identities. Let  $h : {}^{<\omega}2 \rightarrow \omega$  be 1-1. Define  $f : [2^\omega]^2 \rightarrow \omega$  by  $f(\{\alpha, \beta\}) = h(\alpha \cap \beta)$ . We define  $\mathcal{J} = \mathcal{I}(f)$ . Note that  $\mathcal{J}$  is independent of the choice of  $h$ . In [7], the second author proved that  $2^{\aleph_0} > \aleph_\omega$  implies  $\mathcal{I}(\aleph_\omega) = \mathcal{J}$ .

In [2], was shown consistency of  $ZFC + \mathcal{I}(\aleph_2) \neq \mathcal{I}(\aleph_\omega)$ . Here we will show

**Main Theorem.** If  $ZFC$  is consistent then  $ZFC + 2^{\aleph_0} > \aleph_\omega + \mathcal{I}(\aleph_2) = \mathcal{I}(\aleph_\omega)$  is consistent.

This is accomplished by adding  $\nu > \aleph_\omega$  random reals to a model of  $GCH$ . As  $2^{\aleph_0} > \aleph_\omega$  holds in the resulting model we need only show that  $\mathcal{I}(\aleph_2) \supseteq \mathcal{J}$  is true.

## 2 The Partial Order

We establish the notation necessary to add many random reals to a model of ZFC. For a more detailed explanation see [3]. Let  $\nu > \aleph_\omega$  be a cardinal. Let  $\Omega = {}^\nu\{0, 1\}$ . Let  $T$  be the set of functions  $t$  from a finite subset of  $\nu$  into  $\{0, 1\}$ . For each  $t \in T$ , let  $S_t = \{f \in \Omega : t \subset f\}$  and let  $\mathcal{S}$  be the  $\sigma$ -algebra generated by  $\{S_t : t \in T\}$ . The product measure  $m$  on  $\mathcal{S}$  is the unique measure so that  $m(S_t) = \frac{1}{2^{|t|}}$ . We define  $\mathcal{B}_1$  to be the boolean algebra  $\mathcal{S}/J$  where  $J$  is the ideal of all  $X \in \mathcal{S}$  of measure 0. We define a partial order  $\langle \mathbb{P}, < \rangle$  by letting  $\mathbb{P} = \mathcal{B}_1 \setminus J$  and the order be inclusion modulo  $J$ . The following two theorems can be found in [3].

**Theorem 1**  $\mathbb{P}$  is c.c.c.

**Theorem 2** Let  $M$  be a model of set theory and  $G$  be  $\mathbb{P}$ -generic. Then  $M[G]$  satisfies  $2^{\aleph_0} \geq \nu$ .

Let  $Y = \{y_\alpha : \alpha < \nu\}$ . Let  $\Gamma$  denote the collection of all  $\tau(\bar{y})$  where  $\bar{y}$  is a tuple from  $Y$  and  $\tau(\bar{x})$  is a boolean term with free variables  $\bar{x}$ . For  $\alpha < \nu$  denote by  $t_\alpha \in T$  the function whose domain is  $\{\alpha\}$  such that  $t_\alpha(\alpha) = 0$ .

There is an obvious embedding of  $\Gamma$  into  $\mathcal{S}$  which extends the map  $y_\alpha \mapsto S_{t_\alpha}$  and respects the boolean operations. We denote by  $\mathcal{B}_0$  the image of  $\Gamma$  in  $\mathcal{S}$ . It should be clear that  $\mathcal{B}_0$  is a boolean algebra. We call the elements of  $Y$  *generators*. Elements of  $\mathcal{B}_0$  are denoted by their preimage in  $\Gamma$ . The following theorem should be clear.

**Theorem 3** *For  $p \in \mathcal{S}$  and  $\epsilon > 0$  there exists a finite  $u \subset Y$  and a boolean formula  $\tau(\bar{x})$  such that  $\mu(\tau(\bar{u})\Delta p) < \epsilon$ , where  $\Delta$  denotes the symmetric difference.*

### 3 A Combinatorial Statement

Here we formulate a combinatorial statement  $[I, \kappa, \lambda, g, f]$  which will play a crucial role in the proof of the main result. We require some preliminary definitions. Let  $Y, \mathcal{S}, \mathcal{B}_0, \mathcal{B}_1, \mu$  and  $\mathbb{P}$  be as in the previous section. Let  $g, f : \omega \rightarrow \omega$ . For each  $L < \omega$  let  $\mathcal{T}_L$  be a finite set of boolean terms  $\tau(\bar{x})$  where  $\bar{x} = (x_1, \dots, x_{f(L)})$  which is complete in the sense that for any boolean term  $\sigma(\bar{x})$  there is some  $\tau(\bar{x}) \in \mathcal{T}_L$  such that  $\sigma(\bar{x}) = \tau(\bar{x})$  is a valid formula of the theory of boolean algebras. Let  $\mathcal{T} = \bigcup\{\mathcal{T}_L : L < \omega\}$ . In the following we work only with boolean formulas in  $\mathcal{T}$ . List  $\mathcal{T}_L$  as  $\{\tau_i^L : i \leq h(L)\}$ . For  $L < \omega$  define  $\mathbb{T}_L = (\mathcal{T}_L)^{g(L)}$ . For  $w \in [\kappa]^2$  and  $L < \omega$  define

$$\mathbb{T}_{w,L} = \{\langle \tau_1(\bar{x}_L^{w,t}), \dots, \tau_{g(L)}(\bar{x}_L^{w,t}) \rangle : t = \langle \tau_1, \dots, \tau_{g(L)} \rangle \in \mathbb{T}_L\}$$

where  $\bar{x}_L^{w,t} = \langle x_{L,1}^{w,t}, \dots, x_{L,f(L)}^{w,t} \rangle$  is a sequence of distinct variables for each triple  $(w, t, L)$ , and where

$$\bar{x}_L^{w,t} \cap \bar{x}_M^{v,u} \neq \emptyset \Rightarrow (t = u \wedge w = v \wedge L = M).$$

Let  $X$  denote

$$\bigcup\{\bar{x}_L^{w,t} : t \in \mathbb{T}_L, L < \omega, w \in [\kappa]^2\}.$$

Let  $\mathcal{C}(P, L)$  denote

$$\{c : c \text{ is a mapping of } [P]^2 \text{ into } \{1, \dots, g(L)\}\}.$$

**Definition 2** *Let  $k, m < \omega$  and  $\langle \tau_n(\bar{x}) : n \leq k \rangle$  be a sequence of  $m$ -ary boolean formulas. Let  $\bar{u}$  be an  $m$ -tuple from  $Y$ . Then  $\langle \tau_n(\bar{u}) : n \leq k \rangle$  is called a partition sequence if  $\mu(\tau_m(\bar{u}) \cap \tau_n(\bar{u})) = 0$  for all  $m, n$  with  $m \neq n$ , and  $\mu(\bigcup\{\tau_n(\bar{u}) : n \leq k\}) = 1$ .*

The combinatorial statement will now be defined.

**Definition 3** Let  $I$  be an  $r$ -identity,  $\lambda \leq \omega$  and  $\kappa$  a cardinal. We say that  $[I, \kappa, \lambda, g, f]$  holds if the following is true: there exist  $\bar{u}_{w,L}, \tau_{L,m}^w (w \in [\kappa]^2, L < \lambda, 1 \leq m \leq g(L))$  such that for all  $w \in [\kappa]^2, L < \lambda$  and  $P \in [\kappa]^r$

- C1.  $\bar{u}_{w,L}$  is a tuple in  $Y$  of length  $f(L)$
- C2.  $\tau_{L,m}^w \in \mathcal{T}_L, \langle \tau_{L,1}^w, \dots, \tau_{L,g(L)}^w \rangle \in \mathbb{T}_L$
- C3.  $\langle \tau_{L,m}^w(\bar{u}_{w,L}) : 1 \leq m \leq g(L) \rangle$  is a partition sequence
- C4. for  $N \leq L, \mu(\bigcup \{ \tau_{N,m}^w(\bar{u}_{w,N}) \cap \tau_{L,m}^w(\bar{u}_{w,L}) : m \leq g(N) \}) \geq 1 - 1/2^N$
- C5. the measure of

$$\bigcup \left\{ \bigcap \{ \tau_{L,c(z)}^z(\bar{u}_{z,L}) : z \in [P]^2 \} : c \in \mathcal{C}(P, L) \wedge c \text{ realizes } I \right\}$$

is less than  $1/L$ .

## 4 Proof of the Main Theorem

The theorem follows from the following three lemmas which will be proved later.

**Lemma 1** Let  $I \in \mathcal{J}$ . For no  $g, f : \omega \rightarrow \omega$  and  $\kappa > \aleph_\omega$  do we have  $[I, \kappa, \omega, g, f]$ .

**Lemma 2** Let  $I \in \mathcal{J}, \kappa \geq \aleph_0$  and  $g, f : \omega \rightarrow \omega$  be such that  $[I, \kappa, \omega, g, f]$  fails. Then there exists  $m < \omega$  such that  $[I, m, m, g, f]$  fails.

**Lemma 3** Let  $I \in \mathcal{J}$  and  $M$  be a model of set theory satisfying  $GCH$ . Let  $G$  be  $\mathbb{P}$ -generic over  $M$ . If it is true in  $M[G]$  that  $I \notin \mathcal{I}(\aleph_2)$ , then in  $M$  there exists  $g, f : \omega \rightarrow \omega$  such that  $[I, m, m, g, f]$  holds for all  $m < \omega$ .

We suppose that these lemmas are true and prove the main result. Let  $M$  be a model of  $ZFC + GCH$ . Let  $I \in \mathcal{J}$  and towards a contradiction suppose that  $I \notin \mathcal{I}(\aleph_2)$  in  $M[G]$  where  $G$  is  $\mathbb{P}$ -generic over  $M$ . By lemma 3 in  $M$  there exist  $g, f : \omega \rightarrow \omega$  such that  $[I, m, m, g, f]$  holds for all  $m < \omega$ . But from lemma 1,  $[I, (\aleph_\omega)^+, \omega, g, f]$  fails, and so by lemma 2 there exists  $m < \omega$  such that  $[I, m, m, g, f]$  fails, contradiction.

## 4.1 Proof of the first lemma

Assume that the conclusion of lemma fails. Let  $\kappa > \aleph_\omega$ . Let  $g, f : \omega \rightarrow \omega$  be such that  $[I, \kappa, \omega, g, f]$  holds. We force with the partial order  $\mathbb{P}$ , where  $\mathbb{P}$  is defined with  $\nu = \kappa$ . Let  $G \subseteq \mathbb{P}$  be a generic set. For  $L < \omega$  we define  $c_L : [\kappa]^2 \rightarrow \omega$  by  $c_L(w) = m$  if  $\tau_{L,m}^w(\bar{u}_{w,L})/J \in G$ .

**Proposition 1** *For all  $w \in [\kappa]^2$  there exists  $N < \omega, m < \omega$  such that  $c_L(w) = m$  for all  $L > N$ .*

**Proof:** For  $w \in [\kappa]^2$  define

$$D_w = \{p \in \mathbb{P} : p \Vdash \exists N \exists m (c_L(w) = m \text{ for all } L > N)\}.$$

We claim that  $D_w$  is dense in  $\mathbb{P}$ . To this end choose  $p^* \in \mathbb{P}$  and let  $p \in \mathcal{S}$  be such that  $p/J = p^*$ . Let  $\mu(p) = \delta$ . As  $\delta > 0$  we can choose  $N$  such that  $\sum_{L > N} 1/2^L < \delta/3$ . By C4 of the definition of  $[I, \kappa, \omega, g, f]$ ,

$$\mu(\bigcup \{ \bigcap \{ \tau_{L,m}^w(\bar{u}_{w,L}) : L > N \} : m \leq g(N) \}) > 1 - (\delta/3).$$

Thus

$$\mu(\bigcup \{ \bigcap \{ \tau_{L,m}^w(\bar{u}_{w,L}) : L > N \} : m \leq g(N) \} \cap p) > \delta/3.$$

There is thus an  $m \leq g(N)$  such that  $\mu(q) > 0$ , where

$$q = \bigcap \{ \tau_{L,m}^w(\bar{u}_{w,L}) : L > N \} \cap p.$$

Clearly  $q/J \Vdash c_L(w) = m$  for all  $L > N$ . Thus the proposition is proved.  $\square$

We now continue with the proof of the lemma. Define  $c : [\kappa]^2 \rightarrow \omega$  in  $M[G]$  by  $c(w) = \lim_{L \rightarrow \omega} c_L(w)$ . Fix  $P \in [\kappa]^r$ . By property C5 of  $[I, \kappa, \omega, g, f]$ ,

$$\sup \{ \mu(p) : p/J \Vdash \text{“}c_L \text{ realizes } I \text{ on } P\text{”} \} < 1/L.$$

Thus

$$\sup \{ \mu(p) : p/J \Vdash \text{“}c \text{ realizes } I \text{ on } P\text{”} \} < 1/L$$

for all sufficiently large  $L < \omega$ . Hence this set has measure 0 and so it is true that  $c$  does not induce  $I$  in any generic extension. A contradiction occurs as  $\kappa > \aleph_\omega$  and by [7] every coloring  $c : [\kappa]^2 \rightarrow \omega$  must realize  $I$ . Thus the lemma is proved.

## 4.2 Proof of the second lemma

The proof of lemma 2 is accomplished by showing that it is possible to represent the statement  $[I, \kappa, \omega, g, f]$  by a theory in a language of propositional constants when the propositional constants are assigned suitable meanings. The compactness theorem is then used to show that the failure of  $[I, \kappa, \omega, g, f]$  implies the failure of  $[I, m, m, g, f]$  for all sufficiently large  $m$  in  $\omega$ .

Throughout this section fix  $g, f : \omega \rightarrow \omega$ . Let  $\mathcal{B}_0$  and  $\mu$  be as previously defined. Let  $I$  be an  $r$ -identity for some  $r < \omega$ . Consider  $X$ , the collection of free variables previously defined. Define  $\mathcal{L} = \{p_w : w \in [X]^2\}$  to be a collection of propositional constants. For each partition  $\mathcal{P}$  of  $X$  let  $\sim_{\mathcal{P}}$  denote the associated equivalence relation. Let

$$\mathcal{A} : [\kappa]^2 \times \{(L, m) : L < \omega \wedge 1 \leq m \leq g(L)\} \rightarrow \mathcal{T}$$

be such that  $\mathcal{A}(w, L, m) \in \mathcal{T}_L$  for all  $w \in [\kappa]^2$  and  $1 \leq m \leq g(L)$ . Let

$$\mathcal{Q} = \{q_{L,m,i}^w : w \in [\kappa]^2, L < \omega, 1 \leq m \leq g(L), i \leq h(L)\}$$

be a collection of propositional constants. Denote  $\mathcal{R} = \mathcal{L} \cup \mathcal{Q}$ . For each  $\mathcal{P}$  a partition of  $X$  and function  $\mathcal{A}$  define a truth valuation  $V_{\mathcal{P},\mathcal{A}} : \mathcal{R} \rightarrow \{\mathbf{T}, \mathbf{F}\}$  by  $V_{\mathcal{P},\mathcal{A}}(p_w) = \mathbf{T}$  iff  $w = \{i, j\} \wedge i \sim_{\mathcal{P}} j$  and  $V_{\mathcal{P},\mathcal{A}}(q_{L,m,i}^w) = \mathbf{T}$  iff  $\mathcal{A}(w, L, m) = \tau_i^L$ . There is a propositional theory  $T_0$  such that a truth valuation  $V$  models  $T_0$  if and only if  $V = V_{\mathcal{P},\mathcal{A}}$  for some function  $\mathcal{A}$  and partition  $\mathcal{P}$ .

Let  $V$  be a truth valuation that models the theory  $T_0$ . Denote by  $\mathcal{P}_V$  the partition of  $X$  defined by  $x_1 \sim_{\mathcal{P}_V} x_2 \Leftrightarrow V(p_{\{x_1, x_2\}}) = \mathbf{T}$ . Fix a mapping  $v_V : X \rightarrow Y$  such that  $v_V(x) = v_V(y) \Leftrightarrow x \sim_{\mathcal{P}_V} y$ . For  $L < \omega, 1 \leq m \leq g(L)$  and  $w \in [\kappa]^2$  define  $\tau_{L,m}^{V,w}$  to be  $\tau_i^L$  if  $V(q_{L,m,i}^w) = \mathbf{T}$ . Let  $t = t_L^{V,w}$  denote  $\langle \tau_{L,1}^{V,w}, \dots, \tau_{L,g(L)}^{V,w} \rangle \in \mathbb{T}_L$ . For each such sequence let  $\bar{x}_L^{V,w,t}$  denote  $\bar{x}_L^{w,t}$  and write  $\tau_{L,m}^{V,w}(\bar{u}_L^{V,w})$  for the  $\mathcal{B}_0$ -term obtained from  $\tau_{L,m}^{V,w}(\bar{x}_L^{V,w,t})$  by substituting the variables  $\bar{x}_L^{V,w,t}$  by their image under  $v_V$ . Note that since  $\mathbb{T}_L$  is finite, for each  $L < \omega$  and  $w \in [\kappa]^2$ ,

$$X_L^w =_{\text{def}} \bigcup \{\bar{x}_L^{V,w,t} : t = t_L^{V,w} \in \mathbb{T}_L \wedge V \text{ models } T_0\}$$

is finite.

**Lemma 4** *Let  $k < \omega$  and  $\sigma(x_1, \dots, x_k)$  be a boolean term. For  $1 \leq i \leq k$  let  $L_i < \omega$ ,  $1 \leq m_i \leq g(L_i)$  and  $w_i \in [\kappa]^2$ . Let  $\theta(y)$  be a statement of one of the forms  $\mu(y) < 1/n$ ,  $\mu(y) > 1/n$  or  $\mu(y) = 0$ , where  $y$  runs through  $\mathcal{B}_0$ . There exists a propositional formula  $\chi$  such that for all valuations  $V$  modelling  $T_0$ ,  $V$  models  $\chi$  if and only if  $\theta(\sigma(\tau_{L_1, m_1}^{V, w_1}(\bar{u}_{L_1}^{V, w_1}), \dots, \tau_{L_k, m_k}^{V, w_k}(\bar{u}_{L_k}^{V, w_k})))$ .*

**Proof:** Let  $W = \bigcup\{X_{L_i}^{w_i} : 1 \leq i \leq k\}$ . Define  $\mathcal{V} = \{V : V \text{ is a truth valuation modelling } T_0\}$ . Since  $\mathcal{T}_{L_i}$  is finite for all  $1 \leq i \leq k$  the collection  $S = \{\langle \tau_{L_i, m_i}^{V, w_i} : 1 \leq i \leq k \rangle : V \in \mathcal{V}\}$  is a finite set. For each  $s \in S$  define  $\mathcal{V}_s = \{V \in \mathcal{V} : \langle \tau_{L_i}^{V, w_i} : 1 \leq i \leq k \rangle = s\}$

For the moment fix  $s \in S$ . Each  $V \in \mathcal{V}_s$  induces a partition,  $\mathcal{P}_V$  of  $X$  and thus of  $W$ . Since every permutation of  $Y$  induces an automorphism of  $\mathcal{B}_0$  which preserves the measure, for  $V_1, V_2 \in \mathcal{V}_s$ ,  $\mathcal{P}_{V_1} \upharpoonright W = \mathcal{P}_{V_2} \upharpoonright W$  implies

$$\begin{aligned} & \mu(\sigma(\tau_{L_1, m_1}^{V_1, w_1}(\bar{u}_{L_1}^{V_1, w_1}), \dots, \tau_{L_k, m_k}^{V_1, w_k}(\bar{u}_{L_k}^{V_1, w_k}))) \\ &= \mu(\sigma(\tau_{L_1, m_1}^{V_2, w_1}(\bar{u}_{L_1}^{V_2, w_1}), \dots, \tau_{L_k, m_k}^{V_2, w_k}(\bar{u}_{L_k}^{V_2, w_k}))). \end{aligned}$$

As there are only finitely many partitions of  $W$  there is a formula  $\chi_s$  that chooses those partitions in  $\{\mathcal{P}_V : V \in \mathcal{V}_s\}$  that produce the desired measure. We define  $\chi = \bigvee_{s \in S} (\eta_s \Rightarrow \chi_s)$ , where  $\eta_s$  is a formula such that  $V \in \mathcal{V}$  implies  $s = \langle \tau_{L_i, m_i}^{V, w_i} : 1 \leq i \leq k \rangle$  if and only if  $V(\eta_s) = \mathbf{T}$ .  $\square$

**Lemma 5** *There is a propositional theory  $T$  such that  $T$  is consistent if and only if  $[I, \kappa, \omega, g, f]$  holds.*

**Proof:** By the previous lemma, for each triple  $(w, L, P)$  where  $w \in [\kappa]^2$ ,  $L < \omega$  and  $P \in [\kappa]^r$  there exists a formula  $\chi_{w, L, P}$  such that a truth valuation  $V$  models  $T_0 \cup \{\chi_{w, L, P}\}$  implies C1-C5 hold for  $w, L, P$  and the sequences of boolean terms and generators defined by the valuation. We define  $T$  to be  $T_0 \cup \{\chi_{w, L, P} : w \in [\kappa]^2, L < \omega \text{ and } P \in [\kappa]^r\}$ . It is easily seen that the consistency of  $T$  implies that  $[I, \kappa, \omega, g, f]$  holds. In this regard one should observe that  $Y$  is large enough to realize any desired partition.

Now suppose that  $[I, \kappa, \omega, g, f]$  holds. The existence of the sequences of terms  $t_L^w = \langle \tau_{L, 1}^w, \dots, \tau_{L, g(L)}^w \rangle$  and generators  $\bar{u}_{w, L} = \langle u_{w, L, 1}, \dots, u_{w, L, f(L)} \rangle$  defines a function  $\mathcal{A}$  and partition  $\mathcal{P}$  in the following manner. Let  $\mathcal{A}(w, L, m) =$

$\tau_i^L$  if  $\tau_{L,m}^w = \tau_i^L$ . A partition  $\mathcal{P}'$  of  $\bigcup\{\bar{x}_L^{w,t} : t = t_L^w, w \in [\kappa]^2, L < \omega\}$  is first defined by setting  $x_{L,i}^{w,t} \sim_{\mathcal{P}'} x_{M,j}^{v,u}$  if  $u_{w,L,i} = u_{v,M,j}$  where  $t = t_L^w$  and  $s = t_M^v$ . We choose a partition of  $X$  which is an extension of  $\mathcal{P}'$  and denote it by  $\mathcal{P}$ . The truth valuation  $V_{\mathcal{P},\mathcal{A}}$  models the theory  $T$ . This completes the proof of lemma 5.  $\square$

Lemma 2 follows from the compactness theorem for propositional logic.

### 4.3 Proof of the third lemma

Towards a contradiction let  $I$  be an identity on  $r < \omega$  elements,  $d$  a  $\mathbb{P}$ -name for a function and  $p \in \mathbb{P}$  such that

$$p \Vdash "d : [\aleph_2]^2 \longrightarrow \omega \wedge d \text{ does not realize } I".$$

Without loss of generality we assume that  $p = 1_{\mathbb{P}}$ . For each  $w \in [\aleph_2]^2$  choose a sequence  $\langle b_n^w : n < \omega \rangle$  and a sequence  $\langle p_n^w : n < \omega \rangle \in [\mathcal{S}]^\omega$  such that  $\langle p_n^w/J : n < \omega \rangle$  is a maximal antichain in  $\mathbb{P}$  and  $p_n^w/J \Vdash d(w) = b_n^w$ . Let  $b : [\aleph_2]^2 \times \omega \longrightarrow \omega$  be defined by  $b(w, n) = b_n^w$ .

For  $w \in [\aleph_2]^2, L < \omega$  choose  $g(w, L)$  so that  $\sum_{n > g(w, L)} \mu(p_n^w) < 1/(2^{L+5}L)$ . The next lemma follows from theorem 3.

**Lemma 6** *There exists a function  $f : [\aleph_2]^2 \times \omega \longrightarrow \omega$  sequences of boolean terms  $\langle \sigma_{L,m}^w : m \leq g(w, L) \rangle$  and generators  $\bar{v}_{w,L}(w \in [\aleph_2]^2, L < \omega)$  such that:*

1.  $\bar{v}_{w,L} = \{y_{w,L,k} : k \leq h(w, L)\}$
2. For  $m \leq g(w, L)$  we have

$$\mu(p_m^w \Delta \sigma_{L,m}^w(\bar{v}_{w,L})) < \frac{1}{(L2^{L+5}[g(w, L)]^{r^2+1})}.$$

**Lemma 7** *There exists a function  $f : [\aleph_2]^2 \times \omega \longrightarrow \omega$  sequences of boolean terms  $\langle \rho_{L,m}^w : m \leq g(w, L) \rangle$  and generators  $\bar{v}_{w,L}(w \in [\aleph_2]^2, L < \omega)$  such that:*

1.  $\bar{v}_{w,L} = \{y_{w,L,k} : k \leq f(w, L)\}$
2.  $\langle \rho_{L,m}^w(\bar{v}_{w,L}) : m \leq g(w, L) \rangle$  is a partition sequence



3. For  $m < g(w, L)$  we have

$$\mu(p_m^w \Delta \rho_{L,m}^w(\bar{v}_{w,L})) < \frac{1}{2^{L+3} L [g(w, L)]^{r^2}}$$

4.  $\mu(p_{g(w,L)}^w \Delta \rho_{L,g(w,L)}^w(\bar{v}_{w,L})) < \frac{1}{L 2^{L+3}}$ .

**Proof:** Let  $f, \sigma_{L,m}^w$ , and  $\bar{v}_{w,L}$  satisfy the conclusion of the last lemma. For  $m < g(w, L)$  define  $\rho_{L,m}^w(\bar{v}_{w,L}) = \sigma_{L,m}^w(\bar{v}_{w,L}) \setminus \bigcup \{\sigma_{L,i}^w(\bar{v}_{w,L}) : i < m\}$ . Define  $\rho_{L,g(w,L)}^w(\bar{v}_{w,L}) = 1 \setminus \bigcup \{\sigma_{L,i}^w(\bar{v}_{w,L}) : i < g(w, L)\}$ .

Part 1 and 2 of the conclusion clearly hold. For  $m < g(w, L)$ ,

$$\mu(p_m^w \Delta \rho_{L,m}^w(\bar{v}_{w,L})) \leq \sum_{i \leq m} \mu(p_i^w \Delta \sigma_{L,i}^w(\bar{v}_{w,L}))$$

$$\leq g(w, L) / 2^{L+5} L [g(w, L)]^{r^2+1} = 1 / 2^{L+5} L [g(w, L)]^{r^2}.$$

For  $m = g(w, L)$

$$\begin{aligned} & \mu(p_{g(w,L)}^w \Delta \rho_{L,g(w,L)}^w(\bar{v}_{w,L})) \\ & \leq \sum_{i \leq g(w,L)} \mu(p_i^w \Delta \sigma_{L,i}^w(\bar{v}_{w,L})) + \mu(\bigcup \{p_i^w : i > g(w, L)\}) \\ & \leq g(w, L) / (L 2^{L+5} [g(w, L)]) + 1 / L 2^{L+5}. \end{aligned}$$

This concludes the proof of lemma 7.  $\square$

**Lemma 8 (GCH)** Let  $s < \omega$  and for  $1 \leq i \leq s$  let  $h_i : [\aleph_2]^2 \times \omega \longrightarrow \omega$ . There exists  $A = \langle \alpha_i : i < \omega \rangle \in [\aleph_2]^\omega$  and for  $1 \leq i \leq s$  there exist functions  $\hat{h}_i : \omega \longrightarrow \omega$  such that

$$\forall n < \omega \forall m \leq n \forall w \in [\{\alpha_i : n < i < \omega\}]^2 (h_i(w, m) = \hat{h}_i(m)).$$

**Proof:** A standard ramification argument will show that there exists  $Z_0 \subseteq \aleph_2$  of order type  $\aleph_1$  such that for  $\alpha < \beta < \gamma$  in  $Z_0, L < \omega$ , and  $1 \leq i \leq s (h_i(\{\alpha, \beta\}, L) = h_i(\{\alpha, \gamma\}, L))$ . See [4, 1] for details. For  $\alpha \in Z_0, L < \omega$  and  $1 \leq i \leq s$  define  $h_{i,\alpha}(L) = h_i(\{\alpha, \beta\}, L)$  where  $\beta > \alpha$  is chosen in  $Z_0$ .

By cardinality considerations there exists a sequence  $\langle Z_i : 1 \leq i < \omega \rangle$  of subsets of  $Z_0$  such that for all  $k < \omega$ , we have  $Z_{k+1} \subseteq Z_k$ ,  $|Z_k| = \aleph_1$  and for all  $\alpha, \beta \in Z_{k+1}$ ,  $h_{i,\alpha} \upharpoonright (k+1) = h_{i,\beta} \upharpoonright (k+1)$ . We define  $A = \{\alpha_i : i < \omega\}$  in the following manner. Let  $\alpha_0$  be minimal in  $Z_1$  and inductively define  $\alpha_i$  to be minimal in  $Z_{i+1} \setminus \{\alpha_0, \dots, \alpha_{i-1}\}$ . We then define the functions  $\hat{h}_i$  by  $\hat{h}_i(k) = h_{i,\alpha_k}(k)$ .

To verify the lemma let  $n < \omega$  and  $m \leq n$ . Choose  $w = \{\alpha_t, \alpha_v\} \in [\{\alpha_k : n < k < \omega\}]^2$ . Then for  $1 \leq i \leq s$  ( $h_i(w, m) = h_i(\{\alpha_t, \alpha_v\}, m) = h_{i,\alpha_t}(m) = h_{i,\alpha_m}(m) = \hat{h}_i(m)$ ). Thus the lemma is proved.  $\square$

Let  $b, g : [\aleph_2]^2 \times \omega \rightarrow \omega$  be the functions chosen above and  $f, \rho_{L,m}^w, \bar{v}_L^w$  satisfy the conclusion of lemma 7. Let  $A = \langle \alpha_i : i < \omega \rangle \in [\aleph_2]^\omega, \hat{b}, \hat{g}, \hat{f}; \omega \rightarrow \omega$  be the set an functions obtained when the lemma 8 is applied with  $s = 3$  and  $(h_1, h_2, h_3) = (b, g, f)$ . We now verify that  $[I, n, n, \hat{g}, \hat{f}]$  holds for all  $n < \omega$ . To this end fix  $n < \omega$ . Define  $t < \omega$  to be  $n + \max\{g(m) : m \leq n\} + 1$ . For  $w = \{i, j\} \in [n]^2$  define  $w^*$  to be  $\{\alpha_{t+i}, \alpha_{t+j}\}$ . Then for  $w \in [n]^2, L < n, 1 \leq m \leq \hat{g}(L)$  define  $\tau_{L,m}^w$  to be  $\rho_{L,m}^{w^*}$  and  $\bar{u}_{w,L}$  to be  $\bar{v}_{w^*,L}$ .

We will now verify that C1-C5 hold for these sequences of boolean terms and generators. C1-C3 will follow from lemma 9, C4 from lemma 10 and C5 from lemma 11.

**Lemma 9** *Let  $\hat{g}, \hat{f} : \omega \rightarrow \omega, A \subset \aleph_2$  and  $\tau_{L,m}^w, \bar{u}_{w,L}, (w \in [n]^2, L < n, 1 \leq m \leq \hat{g}(L))$  be as defined above. Then*

1.  $\bar{u}_{w,L} = \{y_{w,L,k} : k \leq \hat{f}(L)\}$
2.  $\langle \tau_{L,m}^w(\bar{u}_{w,L}) : m \leq \hat{g}(L) \rangle$  is a partition sequence
3. For  $m < \hat{g}(L)$  we have

$$\mu(p_m^{w^*} \Delta \tau_{L,m}^w(\bar{u}_{w,L})) < \frac{1}{2^{L+3} L [\hat{g}(L)]^{r^2}}$$

$$4. \mu(p_{\hat{g}(L)}^{w^*} \Delta \tau_{L,\hat{g}(L)}^w(\bar{u}_{w,L})) < \frac{1}{L^{2L+3}}.$$

**Proof:** For  $w \in [n]^2, L < n$  ( $g(w^*, L) = \hat{g}(L)$  and  $f(w^*, L) = \hat{f}(L)$ ).  $\square$

**Lemma 10** *Let  $w \in [n]^2$  and  $N < L < n$ . For the sequences of boolean terms defined above*

$$(\mu(\bigcup\{\tau_{N,m}^w(\bar{u}_{w,N}) \cap \tau_{L,m}^w(\bar{u}_{w,L}) : m \leq \hat{g}(N)\}) > 1 - 1/(2^N).$$

**Proof:**

$$\begin{aligned} & \mu(\bigcup\{\tau_{N,m}^w(\bar{u}_{w,N}) \cap \tau_{L,m}^w(\bar{u}_{w,L}) : m \leq \hat{g}(N)\}) \\ \geq & \mu(\bigcup\{\tau_{N,m}^w(\bar{u}_{w,N}) \cap \tau_{L,m}^w(\bar{u}_{w,L}) \cap p_m^{w*} : m \leq \hat{g}(N)\}) \\ = & 1 - [\mu((\bigcup\{\tau_{N,m}^w(\bar{u}_{w,N}) \cap \tau_{L,m}^w(\bar{u}_{w,L}) \cap p_m^{w*} : m \leq \hat{g}(N)\})^c)] \\ \geq & 1 - (\sum_{n < \hat{g}(N)} \mu(p_n^{w*} \Delta \tau_{N,n}^w(\bar{u}_{w,N})) + \sum_{n < \hat{g}(L)} \mu(p_n^{w*} \Delta \tau_{L,n}^w(\bar{u}_{w,L})) \\ & + \mu(p_{\hat{g}(N)}^{w*} \Delta \tau_{L,\hat{g}(N)}^w(\bar{u}_{w,N})) + \mu(p_{\hat{g}(L)}^{w*} \Delta \tau_{L,\hat{g}(L)}^w(\bar{u}_{w,L})) + \mu(\bigcup\{p_m^{w*} : m > \hat{g}(N)\})) \\ \geq & 1 - (3/2^{N+2}) \\ \geq & 1 - 1/2^N. \end{aligned}$$

This concludes the proof of lemma 10.  $\square$

**Lemma 11** *Let  $L < n$  and  $P \in [n]^r$ . The measure of*

$$\bigcup\{\bigcap\{\tau_{L,c(z)}^z(\bar{u}_{z,L}) : z \in [P]^2\} : c \in \mathcal{C}(P, L) \wedge c \text{ realizes } I\}$$

*is less than  $1/L$ .*

**Proof:** First note that for  $z \in [P]^2$  and  $1 \leq m \leq \hat{g}(L)$ ,

$$p_m^{z*}/J \Vdash d(z^*) = b(z^*, m).$$

Now  $z^* \in [\{\alpha_s : s \geq t\}]^2$  and  $m < t$  so  $b(z^*, m) = \hat{b}(m)$ . Thus, for  $c \in \mathcal{C}(P, L)$ ,

$$q =_{\text{def}} \bigcap\{p_{c(z)}^{z*} : z \in [P]^2\}/J \Vdash (\forall z \in [P]^2(d(z^*) = b(c(z))))$$

if  $q \neq J$ . Thus if  $c$  realizes  $I$  on  $P$  and  $q \neq J$  then in some generic extension,  $d$  realizes  $I$  on  $P^* = \{\alpha_{t+i} : i \in P\}$ . Since we assume that  $d$  does not realize  $I$  we can conclude that  $q = J$  and  $\mu(\bigcap\{p_{c(z)}^{z*} : z \in [P]^2\}) = 0$ . Secondly note that  $|\mathcal{C}(P, L)| < g(L)^{r^2}$ .

We first examine those colorings that induce  $I$  and involve at least one color other than  $g(L)$ . For each such  $c$ ,

$$\mu\left(\bigcap\{\tau_{L,c(z)}^z(\bar{u}_{z,L}) : z \in [P]^2\}\right) \leq \min\{\mu(\tau_{L,c(z)}^z(\bar{u}_{z,L}) \Delta p_{c(z)}^{z*}) : z \in [P]^2\}.$$

By lemma 9 this measure is at most  $1/(2L[g(L)]^{r^2})$ . Thus the probability of any of the colorings under consideration inducing  $I$  is less than  $1/2L$ . In the case that the coloring induces  $I$  and uses only the color  $g(L)$  (implying that there is only one such coloring),

$$\mu\left(\bigcap\{\tau_{L,g(L)}^z(\bar{u}_{z,L}) : z \in [P]^2\}\right) \leq \min\{\mu(\tau_{L,g(L)}^z(\bar{u}_{z,L}) \Delta p_{g(L)}^{z*}) : z \in [P]^2\}.$$

By lemma 9 this value is less than  $1/2L$ . Thus lemma 11 is proved.  $\square$

This finishes the proof of lemma 3 and concludes the proof of the main theorem.

For the work in this paper,  $\omega$ -colorings were defined as mappings from pairs of ordinals into  $\omega$ . Clearly this can be generalized so that they are mappings from  $r$ -tuples of ordinals into  $\omega$ . The concept of an  $r$ -identity can then be defined as can the collection of  $r$ -identities realized by an  $\omega$ -coloring, and the collection (denoted  $\mathcal{I}^r(\kappa)$ ) of  $r$ -identities realized by all  $\omega$ -colorings,  $f : [\kappa]^2 \rightarrow \omega$ . We believe that the results of this paper can be extended to show that  $\mathcal{I}^r(\aleph_r) = \mathcal{I}(\aleph_\omega)$ . We also believe that these results can be demonstrated by adding many Cohen reals.

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