A POLARIZED PARTITION RELATION AND FAILURE OF GCH AT SINGULAR STRONG LIMIT

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Abstract. The main result is that for \( \lambda \) strong limit singular failing the continuum hypothesis (i.e. \( 2^\lambda > \lambda^+ \)), a polarized partition theorem holds.

\section{0. Introduction}

In the present paper we show a polarized partition theorem for strong limit singular cardinals \( \lambda \) failing the continuum hypothesis. Let us recall the following definition.

\textbf{Definition 0.1.} For ordinal numbers \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and a cardinal \( \theta \), the polarized partition symbol

\[
\left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) \rightarrow \left( \begin{array}{c} \alpha_2 \\ \beta_2 \end{array} \right)^{1,1}_\theta
\]

means:

if \( d \) is a function from \( \alpha_1 \times \beta_1 \) into \( \theta \) then for some \( A \subseteq \alpha_1 \) of order type \( \alpha_2 \) and \( B \subseteq \beta_1 \) of order type \( \beta_2 \), the function \( d \upharpoonright A \times B \) is constant.

We address the following problem of Erdős and Hajnal:

\((*)\) if \( \mu \) is strong limit singular of uncountable cofinality, \( \theta < \text{cf}(\mu) \) does \n
\[
\left( \begin{array}{c} \mu^+ \\ \mu \end{array} \right) \rightarrow \left( \begin{array}{c} \mu \\ \mu \end{array} \right)^{1,1}_\theta
\]

The particular case of this question for \( \mu = \aleph_{\omega_1} \) and \( \theta = 2 \) was posed by Erdős, Hajnal and Rado (under the assumption of GCH) in [EHR65, Problem 11, p.183]). Hajnal said that the assumption of GCH in [EHR65] was not crucial, and he added that the intention was to ask the question “in some, preferably nice, Set Theory”.

Baumgartner and Hajnal have proved that if \( \mu \) is weakly compact then the answer to \((*)\) is “yes” (see [BH95]), also if \( \mu \) is strong limit of cofinality \( \aleph_0 \). But for a weakly compact \( \mu \) we do not know if for every \( \alpha < \mu^+ \):

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\[
\left( \frac{\mu^+}{\mu} \right) \rightarrow \left( \frac{\alpha}{\mu} \right)_\theta^{1,1}.
\]

The first time I heard the problem (around 1990) I noted that (*) holds when \(\mu\) is a singular limit of measurable cardinals. This result is presented in Theorem 1.2. It seemed likely that we can combine this with suitable collapses, to get “small” such \(\mu\) (like \(\aleph_{\omega_1}\)) but there was no success in this direction.

In September 1994, Hajnal reasked me the question putting great stress on it. Here we answer the problem (*) using methods of [Sh:g]. But instead of the assumption of GCH (postulated in [EHR65]) we assume \(2^{\mu} > \mu^+\). The proof seems quite flexible but we did not find out what else it is good for. This is a good example of the major theme of [Sh:g):

**Thesis 0.2.** Whereas CH and GCH are good (helpful, strategic) assumptions having many consequences, and, say, ¬CH is not, the negation of GCH at singular cardinals (i.e. for \(\mu\) strong limit singular \(2^{\mu} > \mu^+\) or, really the strong hypothesis: \(\text{cf}(\mu) < \mu \Rightarrow \text{pp}(\mu) > \mu^+\)) is a good (helpful, strategic) assumption.

Foreman pointed out that the result presented in Theorem 1.1 below is preserved by \(\mu^+\)-closed forcing notions. Therefore, if

\[
\mathcal{V} \models \left( \frac{\lambda^+}{\lambda} \right) \rightarrow \left( \frac{\lambda}{\lambda} \right)_\theta^{1,1}
\]

then

\[
\mathcal{V}^{\text{Levy}(\lambda^+.2^\theta)} \models \left( \frac{\lambda^+}{\lambda} \right) \rightarrow \left( \frac{\lambda}{\lambda} \right)_\theta^{1,1}.
\]

Consequently, the result is consistent with \(2^\lambda = \lambda^+ \& \lambda\) is small. (Note that although our final model may satisfy the Singular Cardinals Hypothesis, the intermediate model still violates SCH at \(\lambda\), hence needs large cardinals, see [Jec03].) For \(\lambda\) not small we can use Theorem 1.2).

Before we move to the main theorem, let us recall an open problem important for our methods:

**Question 0.3.**

1. Let \(\kappa = \text{cf}(\mu) > \aleph_0, \mu > 2^\theta\) and \(\lambda = \text{cf}(\lambda) \in (\mu, \text{pp}^+(\mu))\). Can we find \(\theta < \mu\) and \(\mathcal{G}_\alpha \in [\mu \cap \text{Reg}]^\theta\) such that: \(\lambda \in \text{pcf}(\mathcal{G}_\alpha), \mathcal{G}_\alpha = \bigcup \mathcal{G}_{\alpha_i}, \mathcal{G}_{\alpha_i}\) bounded in \(\mu\) and \(\sigma \in \mathcal{G}_{\alpha_i} \Rightarrow \bigwedge_{\alpha < \sigma} |\alpha|^\theta < \sigma\)?

   For this it is enough to show:

2. If \(\mu = \text{cf}(\mu) > 2^{<\theta}\) but \(\bigvee_{\alpha < \mu} |\alpha|^{<\theta} \geq \mu\) then we can find \(\mathcal{G}_\alpha \in [\mu \cap \text{Reg}]^{<\theta}\) such that \(\lambda \in \text{pcf}(\mathcal{G}_\alpha)\).

As shown in [Sh:g]

**Theorem 0.4.** If \(\mu\) is strong limit singular of cofinality \(\kappa > \aleph_0, 2^\mu > \lambda = \text{cf}(\lambda) > \mu\) then for some strictly increasing sequence \(\langle \lambda_i : i < \kappa \rangle\) of regulars with limit \(\mu\), \(\prod_{i < \kappa} \lambda_i/\mathcal{J}_\kappa^\text{bd}\) has true cofinality \(\lambda\). If \(\kappa = \aleph_0\), it still holds for \(\lambda = \mu^+\).
[More fully, by [Sh:g, Ch.II.§5], we know $pp(\mu) = 2^\mu$ and by [Sh:g, Ch.III.1.6(2)], we know $pp^+(\mu) = pp_{j^\mu}^+(\mu)$. Note that for $\kappa = \aleph_0$ we should replace $J_{\kappa}^{bd}$ by a possibly larger ideal, using [Sh:430, 1.1.6.5] but there is no need here.]

Remark 0.5. Note the problem is $pp = cov$ problem, see more [Sh:430, §1]: so if $\kappa = \aleph_0$, $\lambda < \mu^{+\omega_1}$ the conclusion of 0.4 holds; we allow to increase $J_{\kappa}^{bd}$, even "there are $< \mu^+$ fixed points $< \lambda^{++}$ suffices."
§ 1. Main result

Theorem 1.1. Suppose $\mu$ is strong limit singular satisfying $2^\mu > \mu^+$. Then

(1) \( \left( \frac{\mu^+}{\mu} \right) \rightarrow \left( \frac{\mu + 1}{\mu} \right)_{\theta}^{1,1} \) for any $\theta < \text{cf}(\mu)$,

(2) if $d$ is a function from $\mu^+ \times \mu$ to $\theta$ and $\theta < \text{cf}(\mu)$ then for some sets $A \subseteq \mu^+$ and $B \subseteq \mu$ we have: $\text{otp}(A) = \mu + 1$, $\text{otp}(B) = \mu$ and the restriction $d \upharpoonright A \times B$ does not depend on the first coordinate.

Proof. 1) It follows from part (2), (as if $d(\alpha, \beta) = d'(\beta)$ for $\alpha \in A$, $\beta \in B$, where $d' : B \rightarrow \theta$, and $|B| = \mu$, $\theta < \text{cf}(\mu)$ then there is $B' \subseteq B$, $|B'| = \mu$ such that $d' \upharpoonright B'$ is constant and hence $d \upharpoonright (A \times B')$ is constant as required).

2) Let $d : \mu^+ \times \mu \rightarrow \theta$. Let $\kappa = \text{cf}(\mu)$ and $\bar{\mu} = \{\mu_i : i < \kappa\}$ be a continuous strictly increasing sequence such that $\mu = \sum_{i<\kappa} \mu_i$, $\mu_0 > \kappa$. We can find a sequence $\bar{C} = \{C_\alpha : \alpha < \mu^+\}$ such that:

(A) $C_\alpha \subseteq \alpha$ is closed, $\text{otp}(C_\alpha) < \mu$, 
(B) $\beta \in \text{nacc}(C_\alpha) \Rightarrow C_\beta = C_\alpha \cap \beta$, 
(C) if $C_\alpha$ has no last element then $\alpha = \sup(C_\alpha)$, (so $\alpha$ is a limit ordinal) and any member of nacc($C_\alpha$) is a successor ordinal,
(D) if $\sigma = \text{cf}(\sigma) < \mu$ then the set 
\[ S_\sigma = \{\delta < \mu^+ : \text{cf}(\delta) = \sigma \land \delta = \sup(C_\beta) \land \text{otp}(C_\beta) = \sigma\} \]

is stationary

(possible by \cite{Sh:420, §1}); we could have added

(E) for every $\sigma \in \text{Reg} \cap \mu^+$ and a club $E$ of $\mu^+$, for stationary many $\delta \in S_\sigma$, $E$ separates any two successive members of $C_\delta$.

Let $c$ be a symmetric two place function from $\mu^+$ to $\kappa$ such that for each $i < \kappa$ and $\beta < \mu^+$ the set

\[ \exists_1 (a) \text{ the set } a^\beta_\alpha = \{\alpha < \beta : c(\alpha, \beta) \leq i\} \text{ has cardinality } \leq \mu_i \]

(b) $\alpha < \beta < \gamma \Rightarrow c(\alpha, \gamma) \leq \max(c(\alpha, \beta), c(\beta, \gamma))$ 

(c) $\alpha \in C_\beta$ and $\mu_\beta \geq |C_\beta| \Rightarrow c(\alpha, \beta) \leq i$

(as in \cite{Sh:108}, easily constructed by induction on $\beta$).

Let $\lambda = \{\lambda_i : i < \kappa\}$ be a strictly increasing sequence of regular cardinals with limit $\mu$ such that $\prod_{i<\kappa} \lambda_i / \mathcal{J}^{bd}_\kappa$ has true cofinality $\mu^{++} \leq 2^\mu$. As we can replace $\lambda$ by any subsequence of length $\kappa$, without loss of generality ($\forall i < \kappa)(\lambda_i > 2^{\mu^+})$. Lastly, let $\chi = \mathcal{D}(\mu)^+$ and $<_\chi$ be a well ordering of $\mathcal{H}(\chi) = \{x : \text{the transitive closure of } x \text{ is of cardinality } < \chi\}$.

Now we choose by induction on $\alpha < \mu^+$ sequences $M_\alpha = \langle M_{\alpha, i} : i < \kappa\rangle$ such that:

(i) $M_{\alpha, i} \prec (\mathcal{H}(\chi), \in, <\chi)$,

(ii) $|M_{\alpha, i}| = 2^{\mu^+}$ and $\nu^\chi_i(M_{\alpha, i}) \subseteq M_{\alpha, i}$ and $2^{\mu^+} + 1 \subseteq M_{\alpha, i}$,

(iii) $d, c, \bar{C}, \lambda, \bar{\mu}, \alpha \in M_{\alpha, i}$; $\langle M_{\beta, j} : \beta < \alpha, j < \kappa\rangle$ belongs to $M_{\alpha, i}$.
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There is no problem to carry out the construction. Note that actually the clause (vii) follows from (i)–(vi), as \( a^\alpha \) is defined from \( c, \alpha, i \), see \( \mathbb{B}_1 \).

Our demands imply that

\[ \mathbb{B}_2 \]

(a) \( \beta \in a^\alpha \Rightarrow M^\beta, i < M^\alpha, i \)

(b) \( j < i \Rightarrow M^\alpha, j < M^\alpha, i \)

(c) \( a^\alpha \subseteq M^\alpha, i \), hence \( \alpha \subseteq \bigcup_{i < \kappa} M^\alpha, i \).

For \( \alpha < \mu^+ \) let \( f_\alpha \in \prod_{i < \kappa} \lambda_i \) be defined by \( f_\alpha(i) = \sup(\lambda_i \cap M^\alpha, i) \). Note that \( f_\alpha(i) < \lambda_i \) as \( \lambda_i = \text{cf}(\lambda_i) > 2^{\text{cf}(\lambda_i)} = \|M^\alpha, i\| \). Also, if \( \beta < \alpha \) then for every \( i \in [c(\beta, \alpha), \kappa) \) we have \( \beta \in M^\alpha, i \) and hence \( M^\beta \in M^\alpha, i \). Therefore, as also \( \lambda \in M^\alpha, i \), we have \( f_\beta \in M^\alpha, i \) and \( f_\beta(i) \in M^\alpha, i \cap \lambda_i \).

Consequently

\[ \mathbb{B}_3 \]

(\forall i \in [c(\beta, \alpha), \kappa))(f_\beta(i) < f_\alpha(i)) and thus \( f_\beta < J^\beta f_\alpha \).

Since \( \{f_\alpha : \alpha < \mu^+\} \subseteq \prod_{i < \kappa} \lambda_i \) has cardinality \( \mu^+ \) and \( \prod_{i < \kappa} \lambda_i / J^\beta \) is \( \mu^+ \)-directed, there is \( f^* \in \prod_{i < \kappa} \lambda_i \) such that

\[ \ast_1 (\forall \alpha < \mu^+)(f_\alpha < J^\beta f^*). \]

Let, for \( \alpha < \mu^+ \), \( g_\alpha \in {^\kappa} \theta \) be defined by \( g_\alpha(i) = d(\alpha, f^*(i)) \). Since \( |{^\kappa} \theta| < \mu < \mu^+ = \text{cf}(\mu^+) \), there is a function \( g^* \in {^\kappa} \theta \) such that

\[ \ast_2 \] the set \( A^* = \{\alpha < \mu^+ : g_\alpha = g^*\} \) is unbounded in \( \mu^+ \).

Now choose, by induction on \( \zeta < \mu^+ \), models \( N_\zeta \) such that:

(a) \( N_\zeta \prec (\mathcal{KH}(\omega), \in, <^* \omega) \),

(b) the sequence \( \langle N_\zeta : \zeta < \mu^+ \rangle \) is increasing continuous,

(c) \( \|N_\zeta\| = \mu \) and \( ^{^\kappa} \theta(N_\zeta) \subseteq N_\zeta \) if \( \zeta \) is not a limit ordinal,

(d) \( \langle N_\zeta : \zeta \leq \zeta \rangle \subseteq N_{\zeta + 1} \),

(e) \( \mu + 1 \subseteq N_\zeta \)

(f) \( \bigcup_{\alpha < \zeta} M^\alpha, i \subseteq N_\zeta \)

(g) \( \langle M^\alpha, i : \alpha < \mu^+, i < \kappa \rangle, \{f_\alpha : \alpha < \mu^+\}, g^*, A^*, d \) belong to the first model \( N_\zeta \).

Let \( E =: \{\zeta < \mu^+ : N_\zeta \cap \mu^+ = \zeta\} \). Clearly, \( E \) is a club of \( \mu^+ \), and thus we can find an increasing sequence \( \{\delta_i : i < \kappa\} \) such that

\[ \ast_3 \delta_i \in S^\mu_\zeta \cap \text{acc}(E)(\subseteq \mu^+) \), (see clause \( \text{(D)} \) in the beginning of the proof).
For each $i < \kappa$ choose a successor ordinal $\alpha^*_i \in \text{nacc}(C_\delta) \setminus \bigcup_{i < \kappa} \{ \delta_j + 1 : j < i \}$. Take any $\alpha^* \in A^* \setminus \bigcup_{i < \kappa} \delta_i$.

We choose by induction on $i < \kappa$ an ordinal $j_i$ and sets $A_i$, $B_i$ such that:

\begin{itemize}
  \item[(a)] $j_i < \kappa$ such that $\mu_{j_i} > \lambda_i$ (so $j_i > i$) and $j_i$ strictly increasing in $i$,
  \item[(b)] $f_{\delta_i} \upharpoonright [j_i, \kappa) < f_{\alpha^*_i} \upharpoonright [j_i, \kappa) < f_{\alpha^*} \upharpoonright [j_i, \kappa) < f^* \upharpoonright [\beta_i, \kappa)$,
  \item[(c)] for each $i_0 < i_1$ we have: $c(\delta_{i_0}, \alpha^*_{i_0}) < j_{i_1}$, and $c(\alpha^*_i, \alpha^*_i) < j_{i_1}$, and $c(\alpha^*_i, \alpha^*) < j_{i_1}$,
  \item[(d)] $A_i \subseteq A^* \cap (\alpha^*_i, \delta_i)$,
  \item[(e)] $\text{otp}(A_i) = \mu^+_i$,
  \item[(f)] $A_i \in M_{\delta_i, j_i}$,
  \item[(g)] $B_i \subseteq \lambda_{j_i}$,
  \item[(h)] $\text{otp}(B_i) = \lambda_{j_i}$,
  \item[(i)] $B_{\varepsilon} \in M_{\alpha^*_i, j_i}$ for $\varepsilon < i$ and $B_i \in \bigcup \{ M_{\alpha^*_i, j_i} : j < \kappa \}$
\end{itemize}

(\kappa) for every $\alpha \in \bigcup_{\varepsilon \leq i} A_{\varepsilon} \cup \{ \alpha^* \}$ and $\zeta \leq i$ and $\beta \in B_{\zeta} \cup \{ f^*(j_{i}) \}$ we have
\[ d(\alpha, \beta) = g^*(j_{\zeta}). \]

If we succeed then $A = \bigcup_{\varepsilon < \kappa} A_{\varepsilon} \cup \{ \alpha^* \}$ and $B = \bigcup_{\zeta < \kappa} B_{\zeta}$ are as required. During the induction in stage $i$ concerning (i) we already know $\varepsilon < i \Rightarrow \bigvee_{j < \kappa} B_{\varepsilon} \in M_{\alpha^*_i, j_i}$. So assume that the sequence $\langle (j_\varepsilon, A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ has already been defined.

We can find $j_{i}(0) < \kappa$ satisfying requirements (a), (b), (c), and (i) and such that $\bigwedge_{\varepsilon < i} \lambda_\varepsilon < \mu_{j_{i}(0)}$. Then by "$j_{i}(0)$ satisfies clause (\(\gamma^*\))" for each $\varepsilon < i$ we have $\delta_\varepsilon \in a_{\alpha^*_i}^{\alpha^*_i}(0)$ and hence $M_{\delta_\varepsilon, j_\varepsilon} < M_{\alpha^*_i, j_\varepsilon}(0)$ (for $\varepsilon < i$). But $A_{\varepsilon} \in M_{\delta_\varepsilon, j_\varepsilon}$ (by clause (f)) and $B_{\varepsilon} \in M_{\alpha^*_i, j_\varepsilon}(0)$ (for $\varepsilon < i$), so $\{ A_\varepsilon, B_\varepsilon : \varepsilon < i \} \subseteq M_{\alpha^*_i, j_\varepsilon}(0)$. Since $\bigwedge_{\varepsilon > i} M_{\alpha^*_i, j_\varepsilon}(0) \subseteq M_{\alpha^*_i, j_\varepsilon}(0)$ (see (ii)), the sequence $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ belongs to $M_{\alpha^*_i, j_\varepsilon}(0)$. We know that for $\gamma_1 < \gamma_2$ in $\text{nacc}(C_\delta)$ we have $c(\gamma_1, \gamma_2) \leq i$ (remember clause (B) and the choice of $c$). As $j_{i}(0) > i$ and so $\mu_{j_{i}(0)} \geq \mu^+_i$ the sequence
\[ M^* = : (M_{\alpha^*_i, j_\varepsilon}(0) : \alpha \in \text{nacc}(C_\delta)) \]

is $\prec$-increasing and $M^* \upharpoonright \alpha \in M_{\alpha^*_i, j_\varepsilon}(0)$ for $\alpha \in \text{nacc}(C_\delta)$ and $M_{\alpha^*_i, j_\varepsilon}(0)$ appears in it. Also, as $\delta_i \in \text{acc}(E)$, there is an increasing sequence $\langle \gamma_\xi : \xi < \mu^+_i \rangle$ of members of $\text{nacc}(C_\delta)$ such that $\gamma_0 = \alpha^*_i$ and $\langle \gamma_\xi, \gamma_\xi+1 \rangle \cap E \neq \emptyset$, say $\beta_\xi \in (\gamma_\xi, \gamma_\xi+1) \cap E$. Each element of $\text{nacc}(C_\delta)$ is a successor ordinal, so every $\gamma_\xi$ is a successor ordinal. Each model $M_{\gamma_\xi, j_\varepsilon}(0)$ is closed under sequences of length $\leq \mu^+_i$ by clause (ii), and hence $\langle \gamma_\xi : \xi < \xi \rangle \in M_{\gamma_\xi, j_\varepsilon}(0)$ (by choosing the right $C$ and $\delta_\varepsilon$’s we could have managed to have $\alpha^*_i = \min(C_\delta)$), $\{ \gamma_\xi : \xi < \mu^+_i \} = \text{nacc}(C_\delta)$, without using this amount of closure).

For each $\xi < \mu^+_i$, recalling $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle \in M_{\alpha^*_i, j_\varepsilon}(0)$ we know that
\[ (\mathcal{R}(\chi), \varepsilon, \varphi^*_\varepsilon) = (\{ \exists x \in A^*[x > \gamma_\xi \text{ and } \forall \varepsilon < i)(\forall \gamma \in B_{\varepsilon}) (d(x, y) = g^*(j_{i})) \}) \]

because $x = \alpha^*$ satisfies it. As all the parameters, i.e. $A^*$, $\gamma_\xi$, $d$, $g^*$ and $\langle B_\varepsilon : \varepsilon < i \rangle$, belong to $N_{\beta_\varepsilon}$ (remember clauses (e) and (c)); note that $B_\varepsilon \in M_{\alpha^*_i, j_\varepsilon}(0), \alpha^*_i < \beta_\varepsilon$,
Consequently, and hence $A$ the demands ($\delta$ and it belongs to $M$ and $A$ such that $\mu < \xi < \mu$)

Hence the set $\mu$ has cardinality $j$. Let $\lambda$ has to be unbounded in $n$. We know that the set $A_i$ is a function from $\alpha$.

Clearly $A_i \subseteq A^*$ (as each $\beta_i^* \in A^*$). Now, the sequence $\langle M_{\gamma_i,j_i} : \xi < \mu_i^+ \rangle$ is $\alpha$-increasing, and hence $A_i \subseteq M_{\gamma_i,j_i}$. Since $\mu_i^+ > \mu_i^+ = |A_i|$ we have $A_i \in M_{\gamma_i,j_i}$. Note that at the moment we know that the set $A_i$ satisfies the demands (\delta)–(\zeta). By the choice of $j_i(0)$, as $j_i > j_i(0)$, clearly $M_{\gamma_i,j_i} < M_{\gamma_i,j_i}$, and hence $A_i \in M_{\gamma_i,j_i}$. Similarly, $\langle A_x : x \leq i \rangle \in M_{\gamma_i,j_i}$, $\alpha^* \in M_{\gamma_i,j_i}$, and

$$\sup(M_{\gamma_i,j_i} \cap \lambda_{j_i}) = f_{\gamma_i,j_i} < f^*(j_i).$$

Consequently, $\bigcup_{i < \theta} A_x \cup \{\alpha^* \} \subseteq M_{\gamma_i,j_i}$ (by the induction hypothesis or the above) and it belongs to $M_{\gamma_i,j_i}$. Since $\bigcup_{i < \theta} A_x \cup \{\alpha^* \} \subseteq A^*$, clearly

$$f^*(j_i) = g^*(j_i).$$

Note that

$$\bigcup_{i < \theta} A_x \cup \{\alpha^* \}, g^*(j_i), d, \lambda_{j_i} \in M_{\gamma_i,j_i},$$

and

$$f^*(j_i) \in \lambda_{j_i} \setminus \sup(M_{\gamma_i,j_i} \cap \lambda_{j_i}).$$

Hence the set

$$B_i = \{y < \lambda_{j_i} : (\forall x \in \bigcup_{i < \theta} A_x \cup \{\alpha^* \})(d(x,y) = g^*(j_i))\}$$

has to be unbounded in $\lambda_{j_i}$. It is easy to check that $j_i, A_i, B_i$ satisfy clauses (a)–(b).

Thus we have carried out the induction step, finishing the proof of the theorem.

\[ \square_{1,1} \]

\textit{Theorem 1.2.} Suppose $\mu$ is singular limit of measurable cardinals.

Then

\begin{enumerate}
  \item \( \left( \frac{\mu^+}{\mu} \right) \rightarrow \left( \frac{\mu}{\mu} \right)^{1,1}_\theta \) if \( \theta = 2 \) or at least \( \theta < \text{cf}(\mu) \)
  \item Moreover, if $\alpha^* < \mu^+$ and \( \theta < \text{cf}(\mu) \) then \( \left( \frac{\mu^+}{\mu} \right) \rightarrow \left( \frac{\alpha^*}{\mu} \right)^{1,1}_\theta \)
  \item If \( \theta < \mu, \alpha^* < \mu^+ \) and \( d \) is a function from $\mu^+ \times \mu$ to $\theta$ then for some $A \subseteq \mu^+$, otp($A$) = $\alpha^*$, and $B = \bigcup_{i < \text{cf}(\mu)} B_i \subseteq \mu, |B| = \mu$ we have:

\begin{align*}
  d \upharpoonright A \times B_i & \quad \text{is constant for each } i < \text{cf}(\mu).
\end{align*}
\end{enumerate}
Proof. Easily 3) ⇒ 2) ⇒ 1), so we shall prove part 3).

Let $d : \mu^+ \times \mu \to \theta$. Let $\kappa = \text{cf}(\mu)$. Choose sequences $(\lambda_i : i < \kappa)$ and $(\mu_i : i < \kappa)$ such that $(\mu_i : i < \kappa)$ is increasing continuous, $\mu = \sum_{i < \kappa} \mu_i$, $\mu_0 > \kappa + \theta$, each $\lambda_i$ is measurable and $\mu_i < \lambda_i < \mu_{i+1}$ (for $i < \kappa$). Let $D_i$ be a $\lambda_i$-complete uniform ultrafilter on $\lambda_i$. For $\alpha < \mu^+$ define $g_\alpha \in \theta$ by: $g_\alpha(i) = \gamma$ iff $\{ \beta < \lambda_i : d(\alpha, \beta) = \gamma \} \in D_i$ (as $\theta < \lambda_i$ it exists). The number of such functions is $\theta^\kappa < \mu$ (as $\mu$ is necessarily strong limit), so for some $g^* \in \theta$ the set $A = \{ \alpha < \mu^+ : g_\alpha = g^* \}$ is unbounded in $\mu^+$. For each $i < \kappa$ we define an equivalence relation $e_i$ on $\mu^+$:

$$\alpha e_i \beta \iff (\forall \gamma)(d(\alpha, \gamma) = d(\beta, \gamma)).$$

So the number of $e_i$-equivalence classes is $\le \lambda^i \cdot \theta < \mu$. Hence we can find $\langle \alpha_\zeta : \zeta < \mu^+ \rangle$ an increasing continuous sequence of ordinals $< \mu^+$ such that:

- for each $i < \kappa$ and $e_i$-equivalence class $X$ we have:
  - either $X \cap A \subseteq \alpha_0$
  - or for every $\zeta < \mu^+$, $(\alpha_\zeta, \alpha_{\zeta+1}) \cap X \cap A$ has cardinality $\mu$.

Let $\alpha^* = \bigcup_{i < \kappa} |a_i| = \mu_i$, $(a_i : i < \kappa)$ pairwise disjoint. Now we choose by induction on $i < \kappa$, $A_i$, $B_i$ such that:

- (a) $A_i \subseteq \bigcup_{\zeta < \kappa} \{ \alpha_\zeta, \alpha_{\zeta+1} : \zeta \in a_i \} \cap A$ and each $A_i \cap (\alpha_\zeta, \alpha_{\zeta+1})$ is a singleton,
- (b) $B_i \subseteq D_i$,
- (c) if $\alpha \in A_i, \beta \in B_j, j < i$ then $d(\alpha, \beta) = g^*(j)$.

Now, in stage $i$, $(A_i, B_i) : i < \kappa$ are already chosen. Let us choose $A_i$. For each $\zeta \in a_i$ choose $\beta_\zeta \in (\alpha_\zeta, \alpha_{\zeta+1}) \cap A$ such that if $i > 0$ then for some $\beta' \in A_0, \beta'_e \beta'$, and let $A_i = \{ \beta_\zeta \cap A_i \}$. Now clause (a) is immediate, and the relevant part of clause (c), i.e. $j < i$, is O.K.

Next, as $\bigcup_{j < i} A_j \subseteq A$, the set

$$B_i : = \bigcap_{j < i} \bigcap_{\beta \in A_j} \{ \gamma < \lambda_i : d(\beta, \gamma) = g^*(i) \}$$

is the intersection of $\le \mu_i < \lambda_i$ sets from $D_i$ and hence $B_i \subseteq D_i$. Clearly clauses (b) and the remaining part of clause (c) (i.e. $j = i$) holds. So we can carry the induction and hence finish the proof. □

References


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