NOT COLLAPSING CARDINALS $\leq \kappa$ IN $(<\kappa)$–SUPPORT ITERATIONS

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Abstract. We deal with the problem of preserving various versions of completeness in $(<\kappa)$–support iterations of forcing notions, generalizing the case "S–complete proper is preserved by CS iterations for a stationary co-stationary $S \subseteq \omega_1$". We give applications to Uniformization and the Whitehead problem. In particular, for a strongly inaccessible cardinal κ and a stationary set $S \subseteq \kappa$ with fat complement we can have uniformization for $\langle A_\delta : \delta \in S' \rangle$, $A_\delta \subseteq \delta = \sup A_\delta$, $\mathrm{cf}(\delta) = \mathrm{otp}(A_\delta)$ and a stationary non-reflecting set $S' \subseteq S$.

Annotated Content

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CASE B

Here we deal with κ strongly inaccessible, $S \subseteq \kappa$ usually a stationary "thin" set of singular cardinals. There is no point to ask even for $\aleph_1$–completeness , so the completeness demands are only on sequences of models.

Section B.5: More on complete forcing notions We define completeness of forcing notions with respect to a suitable family $\hat{\mathcal{E}}$ of increasing sequences $\bar{N}$ of models, say, such that $\bigcup_{j<\delta} N_j \cap \kappa \notin S$ for limit $\delta \leq \ell g(\bar{N})$. S is the non-reflecting stationary set where "something is done". The suitable preservation theorem for $(< \kappa)$–support iterations is proved in B.5.6. So this $\hat{\mathcal{E}}$ plays a role of $\mathcal{S}_0$ of Case A, and the presrvation will play the role of preservation of strong $\mathcal{S}_0$–completeness. We end defining the version of completeness (which later we prove is preserved; it is parallel to $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$–completeness of Case A).

Section B.6: Examples for an inaccessible cardinal κ We present a forcing notion taking care of Pr_S, at least for cases which are locally OK, say, $S \subseteq \kappa$ is stationary non-reflecting. We show that it satisfies the right properties (for iterating) for the naturally defined $\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1$. Then we turn to the related problem of Whitehead group.

Section B.7: The iteration theorem for inaccessible κ We show that completeness for $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$is preserved in $(< \kappa)$–support iterations (this covers the uniformization). Then we prove the κ^+–cc for the simplest cases.

Section B.8: The Axiom and its applications We phrase the axiom and prove its consistency. The main case is for a stationary set $S \subseteq \kappa$ whose complement is fat, but checking that forcing notions fit is clear for forcing notion related to non-reflecting subsets $S' \subseteq S$, So S can have stationary intersection with S^κ_σ for any regular $\sigma < \kappa$. The instance of $S \cap$ inaccessible is not in our mind, but it is easier – similar to the successor case. Next we show the consistency of "GCH + there are almost free Abelian groups in κ, and all of them are Whitehead". We start with an enough indestructible weakly compact cardinal and a stationary non-reflecting set $S \subseteq \kappa$, for simplicity $S \subseteq S^\kappa_{\aleph_0}$, and then we force the axiom. Enough weak compactness remains, so that we have: every stationary set $S' \subseteq \kappa \setminus S$ reflects in inaccessibles, hence "G almost free in κ" implies $\Gamma(G) \subseteq S \mod D_\lambda$", but the axiom makes all of them Whitehead.

0. Introduction

In the present paper we deal with the following question from the Theory of Forcing:

Problem we address 0.1. Iterate with $(<\kappa)$–support forcing notions not collapsing cardinals $\leq \kappa$ preserving this property, generalizing "S–complete proper is preserved by CS iterations for a stationary co-stationary $S \subseteq \omega_1$".
We concentrate on the ZFC case (i.e., we prefer to avoid the use of large cardinals, or deal with cardinals which may exists in $\mathbf{L}$) and we demand that no bounded subsets of κ are added.

We use as our test problems instances of uniformization (see 0.2 below) and Whitehead groups (see 0.3 below), but the need for 0.1 comes from various questions of Set Theory. The case of CS iteration and $\kappa = \aleph_1$ has gotten special attention (so we generalize *no new real* case by S–completeness, see [16, Ch V]) and is a very well understood case, but still with consequences in CS iterations of S–complete forcing notions. This will be our starting point.

One of the questions which caused us to look again in this direction was:

> is it consistent with ZFC + GCH that for some regular κ there is an almost free Abelian group of cardinality κ, but every such Abelian group is a Whitehead one?

By Göbel and Shelah [3], we have strong counterexamples for $\kappa = \aleph_n$: an almost free Abelian group G on κ with $\mathrm{HOM}(G, \mathbb{Z}) = \{0\}$. Here, the idea is that we have an axiom for G with $\Gamma(G) \subseteq S$ (to ensure being Whitehead) and some reflection principle gives

$$\Gamma(G) \setminus S \text{ is stationary} \quad \Rightarrow \quad G \text{ is not almost free in } \kappa,$$

(see B.8). This stream of investigations has a long history already, one of the starting points was [14] (see earlier references there too), and later Mekler and Shelah [8], [7].

Definition 0.2. Let $\kappa \geq \lambda$ be cardinals.

(1) We let $S^\kappa_\lambda \stackrel{\text{def}}{=} \{\delta < \kappa : \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$.

(2) *A (κ, λ)–ladder system* is a sequence $\bar{\mathcal{A}} = \langle A_\delta : \delta \in S \rangle$ such that the set $\mathrm{dom}(\bar{\mathcal{A}}) = S$ is a stationary subset of S^κ_λ and
$$(\forall \delta \in S)(A_\delta \subseteq \delta = \sup(A_\delta) \quad \& \quad \mathrm{otp}(A_\delta) = \mathrm{cf}(\lambda)).$$
When we say that $\bar{\mathcal{A}}$ is a (κ, λ)–ladder system on S, then we mean that $\mathrm{dom}(\bar{\mathcal{A}}) = S$.

(3) Let $\bar{\mathcal{A}}$ be a (κ, λ)–ladder system. We say that $\bar{\mathcal{A}}$ has *the h^*–**U**niformization **P**roperty* (and then we may say that it has h^*–**UP**) if $h^* : \kappa \longrightarrow \kappa$ and

for every sequence $\bar{h} = \langle h_\delta : \delta \in S \rangle$, $S = \mathrm{dom}(\bar{\mathcal{A}})$, such that
$$(\forall \delta \in S)(h_\delta : A_\delta \longrightarrow \kappa \ \ \& \ \ (\forall \alpha \in A_\delta)(h_\delta(\alpha) < h^*(\alpha)))$$
there is a function $h : \kappa \longrightarrow \kappa$ with
$$(\forall \delta \in S)(\sup\{\alpha \in A_\delta : h_\delta(\alpha) \neq h(\alpha)\} < \delta\}).$$
If h^* is constantly μ, then we may write μ–**UP**; if $\mu = \lambda$, then we may omit it.

(4) For a stationary set $S \subseteq S^\kappa_\lambda$, let $\mathrm{Pr}_{S,\mu}$ be the following statement
$\mathrm{Pr}_{S,\mu} \equiv$ each (κ,λ)–ladder system $\bar{A}$ on S has the μ–Uniformization Property.
We may replace μ by h^*; if $\mu = \lambda$ we may omit it.

There are several works on the **UP**, for example the author proved that it is consistent with GCH that *there is* a (λ^+,λ)–ladder system on $S^{\lambda^+}_\lambda$ with the Uniformization Property (see Steinhorn and King [18], for more general cases see [14]), but necessarily not every such system has it (see [16, AP, §3]). In the present paper we are interested in a stronger statement: we want to have the **UP** for *all* systems on S (i.e., Pr_S).

We work mostly without large cardinals. First we concentrate on the case when $\kappa = \lambda^+$, λ a regular cardinal, and then we deal with the related problem for inaccessible κ. The following five cases should be treated somewhat separately.

Case A: $\kappa = \lambda^+$, $\lambda = \lambda^{<\lambda}$, $S \subseteq S^\kappa_\lambda$, and the set $S^\kappa_\lambda \setminus S$ is stationary;
Case B: κ is (strongly) inaccessible (e.g. the first one), S is a "thin" set of singulars;
Case C: λ is singular, $S \subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is a non–reflecting stationary set;
Case D: κ is strongly inaccessible, the set

$$\{\delta < \kappa : \delta \in S \text{ and } \delta \text{ is not strongly inaccessible }\}$$

is not stationary;
Case E: $S = S^\kappa_\lambda$, $\kappa = \lambda^+$, $\lambda = \lambda^{<\lambda}$.

We may also consider

Case F: $\kappa = \kappa^{<\kappa}$, $\theta^+ < \kappa = 2^\theta$.
Case G: $S = S^\kappa_\lambda$, $\kappa = \lambda^+$, $\lambda = \lambda^{<\lambda}$ and we make $2^\lambda > \kappa$.

In the present paper we will deal with the first two (i.e., **A** and **B**) cases. The other cases will be considered in subsequent papers, see [9], [17].

Note that $\diamondsuit_S$ excludes the Uniformization Property for systems on S. Consequently we have some immediate limitations and restrictions. Because of a theorem of Jensen, in case **B** we have to consider $S \subseteq \kappa$ which is not too large (e.g. not reflecting). In the context of case **C**, one should remember that by Gregory [4] when λ is regular, and by [13] generally: if $\lambda^{<\lambda} = \lambda$ or λ is strong limit singular, $2^\lambda = \lambda^+$ and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$ is stationary, then $\diamondsuit_S$ holds true.

By [14, §3], if λ is a strong limit singular cardinal, $2^\lambda = \lambda^+$, $\square_\lambda$ and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ reflects on a stationary set then $\diamondsuit_S$ holds; more results in this direction can be found in Džamonja and Shelah [1].

In the cases **A**, **E**, **G** we are assuming that $\lambda^{<\lambda} = \lambda$. We will start with the first (i.e., **A**) case which seems to be easier. The forcing notions which we will use will be quite complete, mainly "outside" S (see A.1.1, A.1.7, A.1.16 below). Having this amount of completeness we will be able to put weaker requirements on the forcing notion for S.

Finally note that we cannot expect here a full parallel of properness for $\lambda = \aleph_0$, as even for λ^+–cc the parallel of *FS iteration preserves ccc* fails.

We deal here with cases **A** and **B**, other will appear in Part II, [17], [9]. For iterating $(< \lambda)$–complete forcing notions possibly adding subsets to λ, $\kappa = \lambda^+$, see [9]. In [17] we show a weaker κ^+–cc (parallel to pic, eec in [16, Ch VII, VIII]) suffices. We also show that for a strong limit singular λ cardinal and a stationary set $S\subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$, Pr_S (the uniformization for S) fails, but it may hold for many S–ladder systems (so we have consequences for the Whitehead groups).

This paper is based on my lectures in Madison, Wisconsin, in February and March 1996, and was written up by Andrzej Rosłanowski to whom I am greatly indebted.

0.1. **Background: Abelian groups.** We try to be self-contained, but for further references see Eklof and Mekler [2].

Definition 0.3. (1) An Abelian group G is *a Whitehead group* if for every homomorphism $h:H\stackrel{\text{onto}}{\longrightarrow}G$ from an Abelian group H onto G such that $\mathrm{Ker}(h)\cong\mathbb{Z}$ there is a lifting g (i.e., a homomorphism $g:G\longrightarrow H$ such that $h\circ g=\mathrm{id}_G$).

(2) Let $h:H\longrightarrow G$ be as above, G_1 be a subgroup of G. A homomorphism $g:G_1\longrightarrow H$ is a lifting for G_1 (and h) if $h\circ g_1=\mathrm{id}_{G_1}$.

(3) We say that an Abelian group G is a *direct sum* of its subgroups $\langle G_i:i\in J\rangle$ (and then we write $G=\bigoplus\limits_{i\in J}G_i$) if

(a) $G=\langle\bigcup\limits_{i\in J}G_i\rangle_G$ (where for a set $A\subseteq G$, $\langle A\rangle_G$ is the subgroup of G generated by A; $\langle A\rangle_G=\{\sum\limits_{\ell<k}a_\ell x_\ell : k<\omega,\ a_\ell\in\mathbb{Z},\ x_\ell\in A\}$), and

(b) $G_i\cap\langle\bigcup\limits_{i\neq j}G_j\rangle_G=\{0_G\}$ for every $i\in J$.

Remark 0.4. Concerning the definition of a Whitehead group, note that if $h:H\stackrel{\text{onto}}{\longrightarrow}G$ is a homomorphism such $\mathrm{Ker}(h)=\mathbb{Z}$ and $H=\mathbb{Z}\oplus H_1$, then $h\restriction H_1$ is a homomorphism from H_1 into G with kernel $\{0\}$ (and so it is one-to-one, and "onto"). Thus $h\restriction H_1$ is an isomorphism and $g\stackrel{\text{def}}{=}(h\restriction H_1)^{-1}$ is a required lifting.

Also conversely, if $g:G\longrightarrow H$ is a homomorphism such that $h\circ g=\mathrm{id}_G$ then $H=\mathbb{Z}\oplus g[G]$.

The reader familiar with the Abelian group theory should notice that a group G is Whitehead if and only if $\mathrm{Ext}(G,\mathbb{Z})=\{0\}$.

Proposition 0.5. (1) *If $h:H\stackrel{\text{onto}}{\longrightarrow}G$ is a homomorphism, $G_1\oplus G_2\subseteq G$ and g_ℓ is a lifting for G_ℓ (for $\ell=1,2$), then there is a unique lifting g for $G_1\oplus G_2$ (called $\langle g_1,g_2\rangle$) extending both g_1 and g_2; clearly $g(x_1+x_2)=g_1(x_1)+g_2(x_2)$ whenever $x_1\in G_1$, $x_2\in G_2$.*

(2) *Similarly for $\bigoplus\limits_{i\in J}G_i$, g_i a lifting for G_i.*

(3) *If $h:H\stackrel{\text{onto}}{\longrightarrow}G$, $\mathrm{Ker}(h)\cong\mathbb{Z}$ and $G_1\subseteq G$ is isomorphic to $\mathbb{Z}$, then there is a lifting for G_1.*

Definition 0.6. Let λ be an uncountable cardinal, and let G be an Abelian group.

(a) G is *free* if and only if $G=\bigoplus\limits_{i\in J}G_i$ where each G_i is isomorphic to $\mathbb{Z}$.

(b) G is *λ–free* if its every subgroup of size $<\lambda$ is free.

(c) G is *strongly λ–free* if for every $G'\subseteq G$ of size $<\lambda$ there is G'' such that

(α) $G'\subseteq G''\subseteq G$ and $|G''|<\lambda$,

(β) G'' is free,

(γ) G/G'' is λ–free.

(d) G is *almost free in* λ if it is strongly λ–free of cardinality λ but it is not free.

Remark 0.7. Note that the *strongly* in 0.6(d) does not have much influence. In particular, for κ inaccessible, "strongly κ–free" is equivalent to "κ–free".

Proposition 0.8. *Assume G/G'' is λ–free. Then for every $K \subseteq G$, $|K| < \lambda$ there is a free Abelian group $L \subseteq G$ such that $K \subseteq G'' \oplus L \subseteq G$.*

Definition 0.9. Assume that κ is a regular cardinal. Suppose that G is an almost free in κ Abelian group (so by 0.6(d) it is of size κ). Let $\bar{G} = \langle G_i : i < \kappa \rangle$ be a filtration of G, i.e., $\langle G_i : i < \kappa \rangle$ is an increasing continuous sequence of subgroups of G, each of size less than κ. We define

$$\gamma(\bar{G}) = \{i < \kappa : G/G_i \text{ is not } \kappa\text{–free}\},$$

and we let $\Gamma[G] = \gamma(\bar{G})/D_\kappa$ for any filtration $\bar{G}$, where D_κ is the club filter on κ (see [2]).

Proposition 0.10. *Suppose that G, κ and $\langle G_i : i < \kappa \rangle$ are as above.*

(1) *G is free if and only if $\gamma(G)$ is not stationary.*
(2) *$\gamma[G]$ cannot reflect in inaccessibles.*

The problem which was the *raison d'etre* of the paper is the following question of Göbel.

Göbel's question 0.11. Is it consistent with *GCH* that for some regular cardinal κ we have:

(a) every almost free in κ Abelian group is Whitehead, and
(b) there are almost free in κ Abelian groups ?

Remark 0.12. The point in 0.11(b) is that without it we have a too easy solution: any weakly compact cardinal will do the job. This demand is supposed to be a complement of Göbel Shelah [3] which proves that, say for $\kappa = \aleph_n$, there are (under GCH) almost free in κ groups H with $\mathrm{HOM}(H, \mathbb{Z}) = \{0\}$.

Now, our conclusion B.8.4 gives that

(a)' every almost free in κ Abelian group G with $\Gamma[G] \subseteq S/D_\kappa$ is Whitehead,
(b)' there are almost free in κ Abelian groups H with $\Gamma[H] \subseteq S/D_\kappa$.

It can be argued that this answers the question if we understand it as whether from an almost free in κ Abelian group we can build a non–Whitehead one, so the further restriction of the invariant to be $\subseteq S$ does not influence the answer.

However we can do better, starting with a weakly compact cardinal κ we can manage that in addition to (a)', (b)' we have

$(b)^+$ (i) every stationary subset of $\kappa \setminus S$ reflects in inaccessibles,
(ii) for every almost free in κ Abelian group H, $\Gamma[H] \subseteq S/D_\kappa$.

(In fact, for an uncountable inaccessible κ, (i) implies (ii)). So we get a consistency proof for the original problem. This will be done here.

We may ask, can we do it for small cardinals? Successor of singular? Successor of regular? For many cardinals simultaneously? We may get consistency and ZFC+GCH information, but the consistency strength is never small. That is, we need a regular cardinal κ and a stationary set $S \subseteq \kappa$ such that we have enough uniformization on S. Now, for a Whitehead group G: if $\bar{G} = \langle G_i : i < \kappa \rangle$ is a filtration of G, $S=\gamma(\bar{G})$, $\lambda_i=|G_{i+1}/G_i|$ for $i\in S$, for simplicity $\lambda_i=\lambda$, then we need a version of $\mathrm{Pr}_{S,\lambda}$ (see Definition 0.2(4)). We would like to have a suitable reflection (see Magidor and Shelah [6]); for a stationary $S'\subseteq\kappa\setminus S$ this will imply $0^{\#}$.

0.2. **Background: forcing.** Let us review some basic facts concerning iterated forcing and establish our notation. First remember that in forcing considerations we keep the convention that

a stronger condition (i.e., carrying more information) is the larger one.

For more background than presented here we refer the reader to either [16] or Jech [5, Ch 4].

Definition 0.13. Let κ be a cardinal number. We say that $\bar{\mathbb{Q}}$ is *a* $(<\kappa)$*–support iteration of length* γ *(of forcing notions* $\underset{\sim}{\mathbb{Q}}_\alpha$*)* if $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\beta:\alpha\leq\gamma,\ \beta<\gamma\rangle$ and for every $\alpha\leq\gamma$, $\beta<\gamma$:

(a) $\mathbb{P}_\alpha$ is a forcing notion,
(b) $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$–name for a forcing notion with the minimal element $\mathbf{0}_{\underset{\sim}{\mathbb{Q}}_\beta}$
[for simplicity we will assume that $\underset{\sim}{\mathbb{Q}}_\beta$ is a partial order on an ordinal; remember that each partial order is isomorphic to one of this form],
(c) a condition f in $\mathbb{P}_\alpha$ is a partial function such that $\mathrm{dom}(f)\subseteq\alpha$, $\|\mathrm{dom}(f)\|<\kappa$ and

$$(\forall\xi\in\mathrm{dom}(f))(f(\xi)\text{ is a }\mathbb{P}_\xi\text{–name and }\Vdash_{\mathbb{P}_\xi}f(\xi)\in\underset{\sim}{\mathbb{Q}}_\xi)$$

[we will keep a convention that if $f\in\mathbb{P}_\alpha$, $\xi\in\alpha\setminus\mathrm{dom}(f)$ then $f(\xi)=\mathbf{0}_{\underset{\sim}{\mathbb{Q}}_\xi}$; moreover we will assume that each $f(\xi)$ is a canonical name for an ordinal, i.e., $f(\xi)=\{\langle q_i,\gamma_i\rangle:i<i^*\}$ where $\{q_i:i<i^*\}\subseteq\mathbb{P}_\xi$ is a maximal antichain of $\mathbb{P}_\xi$ and for every $i<i^*$: γ_i is an ordinal and $q_i\Vdash_{\mathbb{P}_\xi}$ "$f(\xi)=\gamma_i$"],
(d) the order of $\mathbb{P}_\alpha$ is given by
$$f_1\leq_{\mathbb{P}_\alpha}f_2\quad\text{if and only if}\quad(\forall\xi\in\alpha)(f_2{\restriction}\xi\Vdash_{\mathbb{P}_\xi}f_1(\xi)\leq_{\underset{\sim}{\mathbb{Q}}_\xi}f_2(\xi)).$$

Note that the above definition is actually an inductive one (see below too).

Remark 0.14. The forcing notions which we will consider will satisfy *no new sequences of ordinals of length* $<\kappa$ *are added*, or maybe at least *any new set of ordinals of cardinality* $<\kappa$ *is included in an old one*. Therefore there will be no need to consider the revised support iterations.

Let us recall that:

Fact 0.15. *Suppose* $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\beta:\alpha\leq\gamma,\ \beta<\gamma\rangle$ *is a* $(<\kappa)$*–support iteration,* $\beta<\alpha\leq\gamma$*. Then*

(a) $p\in\mathbb{P}_\alpha$ *implies* $p{\restriction}\beta\in\mathbb{P}_\beta$,
(b) $\mathbb{P}_\beta\subseteq\mathbb{P}_\alpha$,
(c) $\leq_{\mathbb{P}_\beta}=\leq_{\mathbb{P}_\alpha}{\restriction}\mathbb{P}_\beta$,
(d) *if* $p\in\mathbb{P}_\alpha$, $p{\restriction}\beta\leq_{\mathbb{P}_\beta}q\in\mathbb{P}_\beta$ *then the conditions* p,q *are compatible in* $\mathbb{P}_\alpha$*; in fact* $q\cup p{\restriction}[\beta,\alpha)$ *is the least upper bound of* p,q *in* $\mathbb{P}_\alpha$*,*
consequently
(e) $\mathbb{P}_\beta\lessdot\mathbb{P}_\alpha$ *(i.e., complete suborder).*

Fact 0.16. (1) *If $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma,\ \beta < \gamma \rangle$ is a $(<\kappa)$–support iteration of length γ, $\underset{\sim}{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$–name for a forcing notion (on an ordinal), then there is a unique $\mathbb{P}_{\gamma+1}$ such that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma+1,\ \beta < \gamma+1 \rangle$ is a $(<\kappa)$–support iteration.*

(2) *If $\langle \gamma_i : i < \delta \rangle$ is a strictly increasing continuous sequence of ordinals with limit γ_δ, δ is a limit ordinal, and for each $i < \delta$ the sequence $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma_i,\ \beta < \gamma_i \rangle$ is a $(<\kappa)$–support iteration, then there is a unique $\mathbb{P}_{\gamma_\delta}$ such that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma_\delta,\ \beta < \gamma_\delta \rangle$ is a $(<\kappa)$–support iteration.*

Because of Fact 0.16(2), we may write $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ when considering iterations (with $(<\kappa)$–support), as $\mathbb{P}_\gamma$ is determined by it (for $\gamma = \beta+1$ essentially $\mathbb{P}_\gamma = \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$). For $\gamma' < \gamma$ and an iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ we let

$$\bar{\mathbb{Q}} \restriction \gamma' = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma' \rangle.$$

Fact 0.17. *For every function $\mathbf{F}$ (even a class) and an ordinal γ there is a unique $(<\kappa)$–support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma' \rangle$, $\gamma' \leq \gamma$ such that $\underset{\sim}{\mathbb{Q}}_\alpha = \mathbf{F}(\bar{\mathbb{Q}} \restriction \alpha)$ for every $\alpha < \gamma'$ and*

$$\text{either} \quad \gamma' = \gamma \quad \text{or} \quad \mathbf{F}(\bar{\mathbb{Q}}) \ \text{ is not of the right form.}$$

For a forcing notion $\mathbb{Q}$, the completion of $\mathbb{Q}$ to a complete forcing will be denoted by $\hat{\mathbb{Q}}$ (see [16, Ch XIV]). Thus $\mathbb{Q}$ is a dense suborder of $\hat{\mathbb{Q}}$ and in $\hat{\mathbb{Q}}$ any increasing sequence of conditions which has an upper bound has a least upper bound. In this context note that we may define and prove by induction on α^* the following fact.

Fact 0.18. *Assume $\langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\alpha : \alpha < \alpha^* \rangle$ is a $(<\kappa)$–support iteration. Let $\mathbb{P}_\alpha$, $\underset{\sim}{\mathbb{Q}}_\alpha$ be such that for $\alpha < \alpha^*$*

(1) $\mathbb{P}_\alpha = \{f \in \mathbb{P}'_\alpha : (\forall \xi < \alpha)(f(\xi)$ *is a $\mathbb{P}_\xi$–name for an element of* $\underset{\sim}{\mathbb{Q}}_\xi)\}$,

(2) $\underset{\sim}{\mathbb{Q}}_\alpha$ *is a $\mathbb{P}_\alpha$–name for a dense suborder of $\underset{\sim}{\mathbb{Q}}'_\alpha$.*

Then for each $\alpha \leq \alpha^*$, $\mathbb{P}_\alpha$ is a dense suborder of $\mathbb{P}'_\alpha$ and $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \alpha^* \rangle$ is a $(<\kappa)$–support iteration.

We finish our overview of basic facts with the following observation, which will be used several times later (perhaps even without explicit reference).

Fact 0.19. *Let $\mathbb{Q}$ be a forcing notion which does not add new $(<\theta)$–sequences of elements of λ (i.e., $\Vdash_{\mathbb{Q}}$ "$\lambda^{<\theta} = \lambda^{<\theta} \cap \mathbf{V}$"). Suppose that N is an elementary submodel of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\|N\| = \lambda$, $\mathbb{Q} \in N$, and $N^{<\theta} \subseteq N$. Let $G \subseteq \mathbb{Q}$ be a generic filter over $\mathbf{V}$. Then*

$$\mathbf{V}[G] \models N[G]^{<\theta} \subseteq N[G].$$

Proof. Suppose that $\bar{x} = \langle x_i : i < i^* \rangle \in N[G]^{<\theta}$, $i^* < \theta$. By the definition of $N[G]$, for each $i < i^*$ there is a $\mathbb{Q}$–name $\underset{\sim}{\tau}_i \in N$ such that $x_i = \underset{\sim}{\tau}_i^G$. Look at the sequence $\langle \underset{\sim}{\tau}_i : i < i^* \rangle \in \mathbf{V}[G]$. By the assumptions on $\mathbb{Q}$ we know that $\langle \underset{\sim}{\tau}_i : i < i^* \rangle \in \mathbf{V}$ (remember $i^* < \theta$, $\|N\| = \lambda$) and therefore, as each $\underset{\sim}{\tau}_i$ is in N and $N^{<\theta} \subseteq N$, we have $\langle \underset{\sim}{\tau}_i : i < i^* \rangle \in N$. This implies that $\bar{x} \in N[G]$. □

0.3. **Notation.** We will define several properties of forcing notions using the structure $(\mathcal{H}(\chi), \in, <^*_\chi)$ (where $\mathcal{H}(\chi)$ is the family of sets hereditarily of size less than χ, and $<^*_\chi$ is a fixed well ordering of $\mathcal{H}(\chi)$). In all these definitions any "large enough" regular cardinal χ works.

Definition 0.20. *For most $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ with PROPERTY we have...* will mean:

> there is $x\in\mathcal{H}(\chi)$ such that
> if $x\in N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ and N has the *PROPERTY*, then ....

Similarly, *for most sequences $\bar{N}=\langle N_i:i<\alpha\rangle$ of elementary submodels of* $(\mathcal{H}(\chi),\in,<^*_\chi)$ *with PROPERTY we have...* will mean:

> there is $x\in\mathcal{H}(\chi)$ such that
> if $x\in N_0$, $\bar{N}=\langle N_i:i<\alpha\rangle$, $N_i\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ and $\bar{N}$ has the *PROPERTY*, then ....

In these situations we call the element $x\in\mathcal{H}(\chi)$ *a witness.*

Notation 0.21. We will keep the following rules for our notation:

(1) $\alpha,\beta,\gamma,\delta,\xi,\zeta,i,j\dots$ will denote ordinals,
(2) $\kappa,\lambda,\mu,\mu^*\dots$ will stand for cardinal numbers,
(3) a bar above a name indicates that the object is a sequence, usually $\bar{X}$ will be $\langle X_i:i<\ell g(\bar{X})\rangle$, where $\ell g(\bar{X})$ denotes the length of $\bar{X}$,
(4) a tilde indicates that we are dealing with a name for an object in a forcing extension (like $\underaccent{\tilde}{x}$),
(5) S, S_i, S^j_i, E, E_i, $E^j_i\dots$ will be used to denote sets of ordinals,
(6) $\mathcal{S}$, $\mathcal{S}_i$, $\mathcal{S}^j_i$, $\mathcal{E}$, $\mathcal{E}_i$, $\mathcal{E}^j_i\dots$ will stand for families of sets of ordinals of size $<\kappa$, and finally
(7) $\hat{\mathcal{S}}$, $\hat{\mathcal{S}}_i$, $\hat{\mathcal{S}}^j_i$, $\hat{\mathcal{E}}$, $\hat{\mathcal{E}}_i$, $\hat{\mathcal{E}}^j_i$ will stand for families of sequences of sets of ordinals of size $<\kappa$.
(8) The word group will mean here Abelian group. In groups we will use the additive convention (so in particular 0_G will stand for the neutral element of the group G). G,H,K,L will denote (always Abelian) groups.

Case: A

In this part of the paper we are dealing with the **Case A** (see the introduction), so naturally we assume the following.

Our Assumptions 1. λ, κ, μ^* are uncountable cardinal numbers such that

$$\lambda^{<\lambda}=\lambda<\lambda^+=2^\lambda=\kappa\leq\mu^*.$$

We will keep these assumptions for some time (unless stated otherwise) and we may forget to remind the reader of them.

A.1. Complete forcing notions

In this section we introduce several notions of completeness of forcing notions and prove basic results about them. We define when a forcing notion $\mathbb{Q}$ is: (θ, S)–strategically complete, $(<\lambda)$-strategically complete, strongly $\mathcal{S}$–complete, $(\mathcal{S}_0, \mathcal{S}_1)$–complete basically $(\mathcal{S}_0, \hat{\mathcal{S}}_1)$–complete and really $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$–complete. The notions which we will use are strong $\mathcal{S}_0$–completeness and real $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$–completeness, however the other definitions seem to be interesting too. They are, in some sense, successive approximations to real completeness (which is as weak as the iteration theorem allows) and they might be of some interest in other contexts. But a reader not interested in a general theory may concentrate on definitions A.1.1(3), A.1.5, A.1.7(3) and A.1.16 only.

Definition A.1.1. Let $\mathbb{Q}$ be a forcing notion, and let θ be an ordinal and $S \subseteq \theta$.

(1) For a condition $r \in \mathbb{Q}$, let $\mathcal{G}^\theta_S(\mathbb{Q}, r)$ be the following game of two players, COM (for *complete*) and INC (for *incomplete*):
   the game lasts θ moves and during a play the players construct a sequence $\langle (p_i, q_i) : i < \theta\rangle$ of conditions from $\mathbb{Q}$ in such a way that $(\forall j < i < \theta)(r \le p_j \le q_j \le p_i)$ and at the stage $i < \theta$ of the game:
   if $i \in S$, then COM chooses p_i and INC chooses q_i, and
   if $i \notin S$, then INC chooses p_i and COM chooses q_i.
   The player COM wins if and only if for every $i < \theta$ there are legal moves for both players.
(2) We say that the forcing notion $\mathbb{Q}$ is (θ, S)*–strategically complete* if the player COM has a winning strategy in the game $\mathcal{G}^\theta_S(\mathbb{Q}, r)$ for each condition $r \in \mathbb{Q}$. We say that $\mathbb{Q}$ is *strategically* $(<\theta)$*–complete* if it is $(\theta, \emptyset)$–strategically complete.
(3) We say that the forcing notion $\mathbb{Q}$ is $(<\theta)$*–complete* if every increasing sequence $\langle q_i : i < \delta\rangle \subseteq \mathbb{Q}$ of length $\delta < \theta$ has an upper bound in $\mathbb{Q}$.

Proposition A.1.2. *Let $\mathbb{Q}$ be a forcing notion. Suppose that θ is an ordinal and $S \subseteq \theta$.*

(1) *If $\mathbb{Q}$ is $(<\theta)$–complete, then it is (θ, S)–strategically complete.*
(2) *If $S' \subseteq S$ and $\mathbb{Q}$ is (θ, S')–strategically complete, then it is (θ, S)–strategically complete.*
(3) *If $\mathbb{Q}$ is (θ, S)–strategically complete, then the forcing with $\mathbb{Q}$ does not add new sequences of ordinals of length $< \theta$.*

Proof. 1) and 3) should be clear.

2) Note that if all members of S are limit ordinals, or at least $\alpha \in S \quad \Rightarrow \quad \alpha+1 \notin S$, then one may easily translate a winning strategy for COM in $\mathcal{G}^\theta_{S'}(\mathbb{Q}, r)$ to the one in $\mathcal{G}^\theta_S(\mathbb{Q}, r)$. In the general case, however, we have to be slightly more careful. First note that we may assume that θ is a limit ordinal (if θ is not limit consider the game $\mathcal{G}^{\theta+\omega}_S(\mathbb{Q}, r)$). Now, for a set $S \subseteq \theta$ and a condition $r \in \mathbb{Q}$ we define a game ${}^*\mathcal{G}^\theta_S(\mathbb{Q}, r)$:

the game lasts θ moves and during a play the players construct a sequence $\langle p_i : i < \theta\rangle$ of conditions from $\mathbb{Q}$ such that $r \le p_i \le p_j$ for each $i < j < \theta$ and

if $i \in S$, then p_i is chosen by COM,
if $i \notin S$, then p_i is determined by INC.
The player COM wins if and only if there are legal moves for each $i < \theta$.

Note that clearly, if $S' \subseteq S \subseteq \theta$ and Player COM has a winning strategy in ${}^*\mathcal{G}^\theta_{S'}(\mathbb{Q}, r)$ then it has one in ${}^*\mathcal{G}^\theta_S(\mathbb{Q}, r)$.

For a set $S \subseteq \theta$ let $S^\perp = \{2\alpha : \alpha \in S\} \cup \{2\alpha + 1 : \alpha \in \theta \setminus S\}$. (Plainly $S^\perp \subseteq \theta$ as θ is limit.)

Claim A.1.2.1. *For each set $S \subseteq \theta$ the games $\mathcal{G}^\theta_S(\mathbb{Q}, r)$ and ${}^*\mathcal{G}^\theta_{S^\perp}(\mathbb{Q}, r)$ are equivalent [i.e.,* COM/INC *has a winning strategy in $\mathcal{G}^\theta_S(\mathbb{Q}, r)$ if and only if it has one in ${}^*\mathcal{G}^\theta_{S^\perp}(\mathbb{Q}, r)$].*

Proof of the claim. Look at the definitions of the games and the set $S^\perp$. □

Claim A.1.2.2. *Suppose that $S_0, S_1 \subseteq \theta$ are such that for every non–successor ordinal $\delta < \theta$ we have*

(a) $\delta \in S_0 \;\equiv\; \delta \in S_1$,
(b) $(\exists^\infty n \in \omega)(\delta + n \in S_0)$, $(\exists^\infty n \in \omega)(\delta + n \notin S_0)$, $(\exists^\infty n \in \omega)(\delta + n \in S_1)$, *and* $(\exists^\infty n \in \omega)(\delta + n \notin S_1)$.

Then the games ${}^*\mathcal{G}^\theta_{S_0}(\mathbb{Q}, r)$ and ${}^*\mathcal{G}^\theta_{S_1}(\mathbb{Q}, r)$ are equivalent.

Proof of the claim. Should be clear once you realize that finitely many successive moves by the same player may be interpreted as one move. □

Now we may finish the proof of A.1.2(2). Let $S' \subseteq S \subseteq \theta$ (and θ be limit). Let

$$S^* = \{\delta \in S^\perp : \delta \text{ is not a successor }\} \cup \{\delta \in (S')^\perp : \delta \text{ is a successor }\}.$$

Note that $(S')^\perp \subseteq S^*$ and the sets $S^*, S^\perp$ satisfy the demands (a), (b) of A.1.2.2. Consequently, by A.1.2.1 and A.1.2.2:

Player COM has a winning strategy in $\mathcal{G}^\theta_{S'}(\mathbb{Q}, r)$ $\Rightarrow$
Player COM has a winning strategy in ${}^*\mathcal{G}^\theta_{(S')^\perp}(\mathbb{Q}, r)$ $\Rightarrow$
Player COM has a winning strategy in ${}^*\mathcal{G}^\theta_{S^*}(\mathbb{Q}, r)$ $\Rightarrow$
Player COM has a winning strategy in ${}^*\mathcal{G}^\theta_{S^\perp}(\mathbb{Q}, r)$ $\Rightarrow$
Player COM has a winning strategy in $\mathcal{G}^\theta_S(\mathbb{Q}, r)$.

□

Proposition A.1.3. *Assume κ is a regular cardinal and $\theta \leq \kappa$. Suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a $(<\kappa)$–support iteration of $(<\theta)$–complete ((θ, S)–strategically complete, strategically $(<\theta)$–complete, respectively) forcing notions. Then $\mathbb{P}_\gamma$ is $(<\theta)$–complete ((θ, S)–strategically complete, strategically $(<\theta)$–complete, respectively).*

Proof. Easy: remember that union of less than κ sets of size less than κ is of size $<\kappa$, and use A.1.2(3). □

Note that if we pass from a $(<\lambda)$–complete forcing notion $\mathbb{Q}$ to its completion $\hat{\mathbb{Q}}$ we may loose $(<\lambda)$–completeness. However, a large amount of the completeness is preserved.

Proposition A.1.4. *Suppose that $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}'$.*

(1) *If $\mathbb{Q}$ is $(<\lambda)$–complete (or just $(<\lambda)$–strategically complete) then $\mathbb{Q}'$ is $(<\lambda)$–strategically complete.*
(2) *If $\mathbb{Q}'$ is $(<\lambda)$–strategically complete then so is $\mathbb{Q}$.*
(3) *Similarly, in (1),(2) for (λ, S)–complete.*

Proof. 1) We describe a winning strategy for player COM in the game $\mathcal{G}^\lambda_\emptyset(\mathbb{Q}', r)$ $(r \in \mathbb{Q}')$, such that it tells player COM to choose elements of $\mathbb{Q}$ only. So

at stage $i < \lambda$ of the play, COM chooses the $<^*_\chi$–first condition $q_i \in \mathbb{Q}$ stronger than $p_i \in \mathbb{Q}'$ chosen by INC right before.

This strategy is a winning one, as at a limit stage $i < \lambda$ of the play, the sequence $\langle q_j : j < i\rangle$ has an upper bound in $\mathbb{Q}$ (remember $\mathbb{Q}$ is $(<\lambda)$–complete).

2) Even easier. □

Definition A.1.5. (1) By $\mathfrak{D}_{<\kappa,<\lambda}(\mu^*)$ we will denote the collection of all families $\mathcal{S} \subseteq [\mu^*]^{<\kappa}$ such that for every large enough regular cardinal χ, for some $x \in \mathcal{H}(\chi)$ we have

if $x \in N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\|N\| < \kappa$, $N^{<\lambda} \subseteq N$ and $N \cap \kappa$ is an ordinal, then $N \cap \mu^* \in \mathcal{S}$ (compare with A.1.7).

If $\lambda = \aleph_0$ then we may omit it.

(2) By $\mathfrak{D}^\alpha_{<\kappa,<\lambda}(\mu^*)$ we will denote the collection of all sets $\hat{\mathcal{S}}$ such that

$$\hat{\mathcal{S}} \subseteq \big\{\bar{a} = \langle a_i : i \le \alpha\rangle : \text{ the sequence } \bar{a} \text{ is increasing continuous and each } a_i \text{ is from } [\mu^*]^{<\kappa}\big\}$$

and for every large enough regular cardinal χ, for some $x \in \mathcal{H}(\chi)$ we have:

if $\bar{N} = \langle N_i : i \le \alpha\rangle$ is an increasing continuous sequence of models such that $x \in N_0$ and for each $i < j \le \alpha$: $N_i \prec N_j \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\langle N_\xi : \xi \le j\rangle \in N_{j+1}$, $\|N_j\| < \kappa$, $N_j \cap \kappa \in \kappa$ and

$$j \text{ is non-limit} \quad \Rightarrow \quad {N_j}^{<\lambda} \subseteq N_j,$$

then $\langle N_i \cap \mu^* : i \le \alpha\rangle \in \hat{\mathcal{S}}$.

(3) For a family $\mathfrak{D} \subseteq \mathcal{P}(\mathcal{X})$ (say $\mathcal{X} = \bigcup\limits_{X \in \mathfrak{D}} X$) let $\mathfrak{D}^+$ stand for the family of all $\mathcal{S} \subseteq \mathcal{X}$ such that

$$(\forall \mathcal{C} \in \mathfrak{D})(\mathcal{S} \cap \mathcal{C} \neq \emptyset).$$

[So $\mathfrak{D}^+$ is the collection of all $\mathfrak{D}$–positive subsets of $\mathcal{X}$.]

(4) For $\mathcal{S} \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ we define $\mathfrak{D}^\alpha_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}]$ like $\mathfrak{D}^\alpha_{<\kappa,<\lambda}(\mu^*)$ above, except that its members $\hat{\mathcal{S}}$ are subsets of

$$\big\{\bar{a} = \langle a_i : i \le \alpha\rangle : \bar{a} \text{ is increasing continuous and for each } i \le \alpha, a_i \in [\mu^*]^{<\kappa} \text{ and if } i \text{ is not limit then } a_i \in \mathcal{S}\big\},$$

and, naturally, we consider only those sequences $\bar{N} = \langle N_i : i \le \alpha\rangle$ for which

$$i \le \alpha \text{ is non-limit} \quad \Rightarrow \quad N_i \cap \mu^* \in \mathcal{S}.$$

As λ is determined by κ in our present case we may forget to mention it.

Remark A.1.6. (1) These are normal filters in a natural sense.
(2) Concerning $\mathfrak{D}^\alpha_{<\kappa,<\lambda}(\mu^*)$, we may not distinguish $\bar{a}_0$, $\bar{a}_1$ which are similar enough (e.g. see A.1.16 below).

(3) Remember: our case is GCH, $\lambda=\mathrm{cf}(\lambda)$, $\kappa=\lambda^+$ and $\alpha=\lambda$.

Definition A.1.7. Assume $\mathcal{S}\subseteq[\mu^*]^{\leq\lambda}$.

(1) Let χ be a large enough regular cardinal. We say that an elementary submodel N of $(\mathcal{H}(\chi),\in,<^*_\chi)$ is $(\lambda,\mathcal{S})$*–good* if

$$\|N\|=\lambda,\quad N^{<\lambda}\subseteq N,\quad \text{ and }\ N\cap\mu^*\in\mathcal{S}.$$

(2) We say that a forcing notion $\mathbb{Q}$ is *strongly* $\mathcal{S}$*–complete* if for most (see 0.20) $(\lambda,\mathcal{S})$–good elementary submodels N of $(\mathcal{H}(\chi),\in,<^*_\chi)$ such that $\mathbb{Q}\in N$ and for each $\mathbb{Q}$–generic over N increasing sequence $\bar{p}=\langle p_i:i<\lambda\rangle\subseteq\mathbb{Q}\cap N$ there is an upper bound of $\bar{p}$ in $\mathbb{Q}$.
[Recall that a sequence $\bar{p}=\langle p_i:i<\lambda\rangle\subseteq\mathbb{Q}\cap N$ is $\mathbb{Q}$*–generic over* N if for every open dense subset $\mathcal{I}$ of $\mathbb{Q}$ from N for some $i<\lambda$, $p_i\in\mathcal{I}$.]

(3) Let $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ be $(\lambda,\mathcal{S})$–good. For a forcing notion $\mathbb{Q}$, a set $S\subseteq\lambda$ and a condition $r\in\mathbb{Q}\cap N$ we define a game $\mathcal{G}_{N,S}(\mathbb{Q},r)$ like the game $\mathcal{G}^{\lambda+1}_S(\mathbb{Q},r)$ with an additional requirement that during a play all choices below λ have to be done from N, i.e. $p_i,q_i\in N\cap\mathbb{Q}$ for all $i<\lambda$.
If $S=\emptyset$ then we may omit it.

(4) Let $\bar{S}:\mathcal{S}\longrightarrow\mathcal{P}(\lambda)$. We say that a forcing notion $\mathbb{Q}$ is $(\mathcal{S},\bar{S})$*–complete* if for most $(\lambda,\mathcal{S})$–good models N, for every condition $r\in N\cap\mathbb{Q}$ the player COM has a winning strategy in the game $\mathcal{G}_{N,\bar{S}(N\cap\mu^*)}(\mathbb{Q},r)$.
If $\bar{S}(a)=\emptyset$ for each $a\in\mathcal{S}$ then we write $\mathcal{S}$*–complete.* (In both cases we may add "strategically".) If $\bar{S}(a)=S$ for each $a\in\mathcal{S}$, then we write $(\mathcal{S},S)$–complete.

Remark A.1.8. In the use of *most* in A.1.7 (and later) we do not mention explicitly the witness x for it. And in fact, normally it is not necessary. If χ_1,χ are large enough, $2^{<\chi_1}<\chi$ (so $\mathcal{H}(\chi_1)\in\mathcal{H}(\chi)$), $\mathcal{S},\mathbb{Q},\ldots\in N$, then there is a witness in $\mathcal{H}(\chi_1)$ and, without loss of generality, $\chi_1\in N$ and therefore there is such a witness in N. Consequently we may forget it.

Remark A.1.9. (1) The most popular choice of μ^* is κ; then $\mathcal{S}\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ if and only if the set $\{\delta<\kappa:\mathrm{cf}(\delta)=\lambda\ \&\ \delta\in\mathcal{S}\}$ is stationary. So $\mathcal{S}$ "becomes" a stationary subset of κ.
(2) Also here we have obvious monotonicities and implications.

Proposition A.1.10. *Suppose that $\mathcal{S}\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ and a forcing notion $\mathbb{Q}$ is $\mathcal{S}$–complete. Then the forcing with $\mathbb{Q}$ adds no new λ–sequences of ordinals (or, equivalently, of elements of* $\mathbf{V}$*) and $\Vdash_{\mathbb{Q}}$ "$\mathcal{S}\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$".*

Proof. Standard; compare with the proof of A.1.13. □

Proposition A.1.11. (1) *Let $\mathcal{S}\subseteq[\mu^*]^{\leq\lambda}$. If a forcing notion $\mathbb{Q}$ is strongly $\mathcal{S}$–complete and is $(<\lambda)$–complete, then it is $\mathcal{S}$–complete.*
(2) *If a forcing notion $\mathbb{Q}$ is strongly $\mathcal{S}$–complete and is S–strategically complete, then $\mathbb{Q}$ is $(\mathcal{S},S)$–complete.*

Strong $\mathcal{S}$–completeness is preserved if we pass to the completion of a forcing notion.

Proposition A.1.12. *Suppose that $\mathcal{S}\subseteq[\mu^*]^{\leq\lambda}$ and $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}'$. Then*

(1) $\mathbb{Q}'$ *is strongly* $\mathcal{S}$*–complete if and only if* $\mathbb{Q}$ *is strongly* $\mathcal{S}$*–complete,*
(2) *similarly for* $(\mathcal{S},\bar{S})$*–completeness.*

Proof. 1) Assume $\mathbb{Q}'$ is strongly $\mathcal{S}$–complete and let $x'\in\mathcal{H}(\chi)$ be a witness for the "most" in the definition of this fact. Let $x=\langle x',\mathbb{Q}'\rangle$. Suppose that $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ is $(\lambda,\mathcal{S})$–good and $\mathbb{Q},x\in N$. Then $\mathbb{Q}',x'\in N$ too. Now suppose that $\bar{q}=\langle q_i:i<\lambda\rangle\subseteq\mathbb{Q}\cap N$ is an increasing $\mathbb{Q}$–generic sequence over N. Since $\mathbb{Q}$ is dense in $\mathbb{Q}'$, $\bar{q}$ is $\mathbb{Q}'$–generic over N and thus, as $\mathbb{Q}'$ is strongly $\mathcal{S}$–complete, it has an upper bound in $\mathbb{Q}'$ (and so in $\mathbb{Q}$).

Now suppose $\mathbb{Q}$ is strongly $\mathcal{S}$–complete with a witness $x\in\mathcal{H}(\chi)$ and let $x'=\langle x,\mathbb{Q}\rangle$. Let N be $(\lambda,\mathcal{S})$–good and $\mathbb{Q}',x'\in N$. So $\mathbb{Q},x\in N$. Suppose that $\bar{q}=\langle q_i:i<\lambda\rangle\subseteq\mathbb{Q}'\cap N$ is increasing and $\mathbb{Q}'$–generic over N. For each $i<\lambda$ choose a condition $p_i\in\mathbb{Q}\cap N$ and an ordinal $\varphi(i)<\lambda$ such that

$$q_i\le_{\mathbb{Q}'}p_i\le_{\mathbb{Q}'}q_{\varphi(i)}$$

(possible by the genericity of $\bar{q}$; remember that $\bar{q}$ is increasing). Look at the sequence

$$\langle p_i:i<\lambda\ \ \&\ \ (\forall j<i)(\varphi(j)<i)\rangle.$$

It is an increasing $\mathbb{Q}$–generic sequence over N, so it has an upper bound in $\mathbb{Q}$. But this upper bound is good for $\bar{q}$ in $\mathbb{Q}'$ as well.

2) Left to the reader. $\square$

Proposition A.1.13. *Suppose that* $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\gamma\rangle$ *is a* $(<\kappa)$*–support iteration,* $\mathcal{S}\in\mathbf{V}$*,* $\mathcal{S}\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$*.*

(1) *If for each* $\alpha<\gamma$

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{\mathbb{Q}}_\alpha\text{ is strongly }\mathcal{S}\text{–complete”},$$

then the forcing notion $\mathbb{P}_\gamma$ *is strongly* $\mathcal{S}$*–complete (and even each quotient* $\mathbb{P}_\beta/\mathbb{P}_\alpha$ *is strongly* $\mathcal{S}$*–complete for* $\alpha\le\beta\le\gamma$*).*
(2) *Similarly for* $(\mathcal{S},\bar{S})$*–completeness.*

Proof. 1) The proof can be presented as an inductive one (on γ), so then we assume that each $\mathbb{P}_\alpha$ $(\alpha<\gamma)$ is strongly $\mathcal{S}$–complete. However, the main use of the inductive hypothesis will be that it helps to prove that no new sequences of length λ are added (hence λ is not collapsed, so in $\mathbf{V}^{\mathbb{P}_\alpha}$ (for $\alpha<\gamma$) we may talk about $(\lambda,\mathcal{S})$–good models without worrying about the meaning of the definition if λ is not a cardinal, and $N[G_{\mathbb{P}_\alpha}]$ is $(\lambda,\mathcal{S})$–good).

For each $\alpha<\gamma$ and $p\in\mathbb{P}_\alpha$ fix a $\mathbb{P}_\alpha$–name $\underset{\sim}{f}^\alpha_p$ for a function from λ to $\mathbf{V}$ such that

if $p\Vdash_{\mathbb{P}_\alpha}$"there is a new function from λ to $\mathbf{V}$",
then $p\Vdash_{\mathbb{P}_\alpha}\underset{\sim}{f}^\alpha_p\notin\mathbf{V}$, and otherwise $p\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{f}^\alpha_p$ is constantly 0".

Let

$$\mathcal{I}_\alpha=\{p\in\mathbb{P}_\alpha:\ \text{ either } p\Vdash_{\mathbb{P}_\alpha}\text{“there is no new function from }\lambda\text{ to }\mathbf{V}\text{” or } p\Vdash_{\mathbb{P}_\alpha}\underset{\sim}{f}^\alpha_p\notin\mathbf{V}\ \}.$$

Clearly $\mathcal{I}_\alpha$ is an open dense subset of $\mathbb{P}_\alpha$. Let $\underset{\sim}{x}_\alpha$ (for $\alpha<\gamma$) be a $\mathbb{P}_\alpha$–name for a witness to the assumption that $\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{\mathbb{Q}}_\alpha$ is strongly $\mathcal{S}$–complete". Let

$$x=(\langle\underset{\sim}{x}_\alpha:\alpha<\gamma\rangle,\langle\bar{\mathbb{Q}}\rangle,\langle(\mathcal{I}_\alpha,\underset{\sim}{f}^\alpha_p):\alpha<\gamma\ \&\ p\in\mathbb{P}_\alpha\rangle).$$

Suppose that N is $(\lambda,\mathcal{S})$–good, $\mathbb{P}_\gamma, x\in N$ and $\bar{p}$ is a $\mathbb{P}_\gamma$–generic sequence over N. Note that $\bar{\mathbb{Q}}\in N$. We define a condition $r^*\in\mathbb{P}_\gamma$:
we let $\mathrm{dom}(r^*)=N\cap\gamma$ and we inductively define $r^*(\alpha)$ for $\alpha\in\mathrm{dom}(r^*)$ by

> if there is a $\mathbb{P}_\alpha$–name $\underset{\sim}{\tau}$ such that
>
> $$r^*{\restriction}\alpha\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{\tau}\in\underset{\sim}{\mathbb{Q}}_\alpha \text{ is an upper bound to } \langle p_i(\alpha): i<\lambda\rangle\text{”},$$
>
> then $r^*(\alpha)$ is the $<^*_\chi$–first such a name;
> if there is no $\underset{\sim}{\tau}$ as above, then $r^*(\alpha)=\mathbf{0}_{\underset{\sim}{\mathbb{Q}}_\alpha}$.

It should be clear that $r^*\in\mathbb{P}_\gamma$ (as $\|N\|=\lambda<\kappa$). What we have to do is to show that r^* is an upper bound to $\bar{p}$ in $\mathbb{P}_\gamma$. We do this by showing by induction on $\alpha\leq\gamma$ that

$(\otimes_\alpha)$ for each $i<\lambda$, $p_i{\restriction}\alpha\leq_{\mathbb{P}_\alpha} r^*{\restriction}\alpha$.

For $\alpha=0$ there is nothing to do.

For α limit this is immediate by the induction hypothesis.

If $\alpha=\beta+1$ and $\beta\notin N$ then we use the induction hypothesis and the fact that for each $i<\lambda$, $\mathrm{dom}(p_i)\subseteq\gamma\cap N$ (remember $p_i\in\mathbb{P}_\gamma\cap N$, $\lambda\subseteq N$ and $\|\mathrm{dom}(p_i)\|\leq\lambda$).

So we are left with the case $\alpha=\beta+1$, $\beta\in N$. Suppose that $G_\beta\subseteq\mathbb{P}_\beta$ is a generic filter over $\mathbf{V}$ such that $r^*{\restriction}\beta\in G_\beta$ (so necessarily $p_i{\restriction}\beta\in G_\beta$ for each $i<\lambda$). We will break the rest of the proof into several claims. Each of them has a very standard proof, but we will sketch the proofs for reader's convenience. Remember that we are in the case $\beta\in N$, so in particular $\mathbb{P}_\beta,\mathbb{P}_{\beta+1},\underset{\sim}{x}_\beta,\mathcal{I}_\beta\in N$ and $\langle p_i{\restriction}\beta: i<\lambda\rangle\subseteq N$ is a $\mathbb{P}_\beta$–generic sequence over N.

Claim A.1.13.1.

$$r^*{\restriction}\beta\Vdash_{\mathbb{P}_\beta}\text{“ there is no new function from } \lambda \text{ to } \mathbf{V}\text{ ”.}$$

Proof of the claim. Since $\mathcal{I}_\beta\in N$ is an open dense subset of $\mathbb{P}_\beta$ we know that $p_i{\restriction}\beta\in\mathcal{I}_\beta$ for some $i<\lambda$. If the condition $p_i{\restriction}\beta$ forces that "there is no new function from λ to $\mathbf{V}$", then we are done (as $r^*{\restriction}\beta\geq p_i{\restriction}\beta$). So suppose otherwise. Then $p_i{\restriction}\beta\Vdash_{\mathbb{P}_\beta}$“ $\underset{\sim}{f}^\beta_{p_i\restriction\beta}\notin\mathbf{V}$ ”. But, as $\beta,p_i\in N$, clearly $\beta,p_i{\restriction}\beta\in N$ and we have $\underset{\sim}{f}^\beta_{p_i\restriction\beta}\in N$ and therefore for each $\zeta<\lambda$ there is $j<\lambda$ such that the condition $p_j{\restriction}\beta$ decides the value of $\underset{\sim}{f}^\beta_{p_i\restriction\beta}(\zeta)$. Consequently the condition $r^*{\restriction}\beta$ decides all values of $\underset{\sim}{f}^\beta_{p_i\restriction\beta}$, so $r^*{\restriction}\beta\Vdash_{\mathbb{P}_\beta}\underset{\sim}{f}^\beta_{p_i\restriction\beta}\in\mathbf{V}$, a contradiction. ☐

Claim A.1.13.2. *$N[G_\beta]\cap\mathbf{V}=N$ (so $N[G_\beta]\cap\mu^*\in\mathcal{S}$).*

Proof of the claim. Suppose that $\underset{\sim}{\tau}\in N$ is a $\mathbb{P}_\beta$–name for an element of $\mathbf{V}$. As the sequence $\langle p_i{\restriction}\beta: i<\lambda\rangle$ is $\mathbb{P}_\beta$–generic over N, for some $i<\lambda$, the condition $p_i{\restriction}\beta$ decides the value of the name $\underset{\sim}{\tau}$. Since $p_i{\restriction}\beta\in N$ the result of the decision belongs to N (remember the elementarity of N) and hence $\underset{\sim}{\tau}^{G_\beta}\in N$. ☐

Claim A.1.13.3.

$$\|N[G_\beta]\|=\lambda,\quad N[G_\beta]^{<\lambda}\subseteq N[G_\beta]\quad \textit{and}\quad N[G_\beta]\prec(\mathcal{H}(\chi),\in,<^*_\chi)^{\mathbf{V}[G_\beta]}.$$

Consequently, $\mathbf{V}[G_\beta]\models$*“the model $N[G_\beta]$ is $(\lambda,\mathcal{S})$–good and $\underset{\sim}{x}^{G_\beta}_\beta\in N[G_\beta]$”.*

Proof of the claim. Names for elements of $N[G_\beta]$ are from N, so clearly $\|N[G_\beta]\|=\lambda=\|N\|$. It follows from 0.19 and A.1.13.1 that $N[G_\beta]^{<\lambda}\subseteq N[G_\beta]$. To check that $N[G_\beta]$ is an elementary submodel of $(\mathcal{H}(\chi),\in,<^*_\chi)$ (in $\mathbf{V}[G_\beta]$) we use the genericity of $\langle p_i{\restriction}\beta: i<\lambda\rangle$ and the elementarity of N: each existential formula of the language of forcing (with parameters from N) is decided by some $p_i{\restriction}\beta$. If the decision is positive, then there is in N a name for a witness for the formula. So we finish by the Tarski–Vaught criterion. $\square$

Claim A.1.13.4.

$$\mathbf{V}[G_\beta] \models \text{“}\langle p_i(\beta)^{G_\beta} : i < \lambda\rangle \textit{ is an increasing } \underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}\textit{–generic sequence over } N[G_\beta]\text{”}.$$

Proof of the claim. By the induction hypothesis, the condition $r^*{\restriction}\beta$ is stronger then all $p_i{\restriction}\beta$. Hence (by Definition 0.13), as $\bar{p}$ is increasing, the sequence $\langle p_i(\beta)^{G_\beta} : i < \lambda\rangle$ is increasing (in $\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}$). Suppose now that $\underset{\sim}{I} \in N$ is a $\mathbb{P}_\beta$–name for an open dense subset of $\underset{\sim}{\mathbb{Q}}_\beta$. Look at the set $\{p \in \mathbb{P}_{\beta+1} : p{\restriction}\beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \underset{\sim}{I}\}$. It is an open dense subset of $\mathbb{P}_{\beta+1}$ from N. But $\mathbb{P}_{\beta+1} \lessdot \mathbb{P}_\gamma$, so for some $i < \lambda$ we have

$$p_i{\restriction}\beta \Vdash_{\mathbb{P}_\beta} p_i(\beta) \in \underset{\sim}{I},$$

finishing the claim. $\square$

By A.1.13.3, A.1.13.4 (remember we assume $\Vdash_{\mathbb{P}_\beta}$ “$\underset{\sim}{\mathbb{Q}}_\beta$ is strongly $\mathcal{S}$–complete”) we conclude that, in $\mathbf{V}[G_\beta]$, the sequence $\langle p_i(\beta)^{G_\beta} : i < \lambda\rangle \subseteq \underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}$ has an upper bound (in $\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}$). Now, as G_β was an arbitrary generic filter containing $r^*{\restriction}\beta$ we conclude that there is a $\mathbb{P}_\beta$–name $\underset{\sim}{\tau}$ such that

$$r^*{\restriction}\beta \Vdash_{\mathbb{P}_\beta} \text{“}\underset{\sim}{\tau} \in \underset{\sim}{\mathbb{Q}}_\beta \text{ is an upper bound to } \langle p_i(\beta) : i < \lambda\rangle\text{”}.$$

Now look at the definition of $r^*(\beta)$.

2) Left to the reader. $\square$

Definition A.1.14. Let (of course, $\kappa = \lambda^+$, and) $\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ and $\hat{\mathcal{S}}_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$. Suppose that $\mathbb{Q}$ is a forcing notion and χ is a large enough regular cardinal.

(1) We say that a sequence $\bar{N} = \langle N_i : i \le \lambda\rangle$ is $(\lambda, \kappa, \hat{\mathcal{S}}_1, \mathbb{Q})$*–considerable* if $\bar{N}$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\lambda \cup \{\lambda, \kappa, \mathbb{Q}\} \subseteq N_0$, the sequence $\langle N_i \cap \mu^* : i \le \lambda\rangle$ is in $\hat{\mathcal{S}}_1$ and for each $i < \lambda$

$$\|N_i\| < \kappa \quad\text{ and }\quad \langle N_j : j \le i\rangle \in N_{i+1} \quad\text{ and}$$

$$i \text{ is non-limit} \quad\Rightarrow\quad (N_i)^{<\lambda} \subseteq N_i.$$

(2) For a $(\lambda, \kappa, \hat{\mathcal{S}}_1, \mathbb{Q})$–considerable sequence $\bar{N} = \langle N_i : i \le \lambda\rangle$ and a condition $r \in N_0 \cap \mathbb{Q}$, let $\mathcal{G}^*_{\bar{N}}(\mathbb{Q}, r)$ be the following game of two players, COM and INC:

the game lasts λ moves and during a play the players construct a sequence $\langle(p_i, \bar{q}_i) : i < \lambda\rangle$ such that each p_i is a condition from $\mathbb{Q}$ and $\bar{q}_i = \langle q_{i,\xi} : \xi < \lambda\rangle$ is an increasing λ–sequence of conditions from $\mathbb{Q}$ (we may identify it with its least upper bound in the completion $\hat{\mathbb{Q}}$) and at the stage $i < \lambda$ of the game:
the player COM chooses a condition $p_i \in N_{-1+i+1} \cap \mathbb{Q}$ such that

$$r \le p_i, \qquad (\forall j < i)(\forall \xi < \lambda)(q_{j,\xi} \le p_i),$$

and the player INC answers by choosing a $\leq_{\mathbb{Q}}$–increasing $\mathbb{Q}$–generic over N_{-1+i+1} sequence $\bar{q}_i = \langle q_{i,\xi} : \xi < \lambda \rangle \subseteq N_{-1+i+1} \cap \mathbb{Q}$ such that

$$p_i \leq q_{i,0}, \quad \text{and} \quad \bar{q}_i \in N_{-1+i+2}.$$

The player COM wins the play of $\mathcal{G}^*_{\bar{N}}(\mathbb{Q}, r)$ if the sequence $\langle p_i : i < \lambda \rangle$ constructed by him during the play has an upper bound in $\mathbb{Q}$.

(3) We say that the forcing notion $\mathbb{Q}$ is *basically* $(\mathcal{S}_0, \hat{\mathcal{S}}_1)$*–complete* if
 (α) $\mathbb{Q}$ is $(<\lambda)$–complete (see A.1.1(2)), and
 (β) $\mathbb{Q}$ is strongly $\mathcal{S}_0$–complete (see A.1.7(3)), and
 (γ) for most $(\lambda, \kappa, \hat{\mathcal{S}}_1, \mathbb{Q})$–considerable sequences $\bar{N} = \langle N_i : i \leq \lambda \rangle$, for every condition $r \in N_0 \cap \mathbb{Q}$, the player INC DOES NOT have a winning strategy in the game $\mathcal{G}^*_{\bar{N}}(\mathbb{Q}, r)$.

Remark A.1.15. (1) Why do we have "strongly $\mathcal{S}_0$–complete" in A.1.14(3)(β) and not "strategically $\mathcal{S}_0$–complete"? To help proving the preservation theorem.

(2) Note that if a forcing notion $\mathbb{Q}$ is strongly $\mathcal{S}_0$–complete and $(<\lambda)$–complete, and $\bar{N}$ is $(\lambda, \kappa, \hat{\mathcal{S}}_1, \mathbb{Q})$–considerable (and N_0 contains the witness for "most" in the definition of "strongly $\mathcal{S}_0$–complete"), then both players always have legal moves in the game $\mathcal{G}^*_{\bar{N}}(\mathbb{Q}, r)$. Moreover, if $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}'$ and $\mathbb{Q}' \in N_0$ and the player COM plays elements of $\mathbb{Q}$ only then both players have legal moves in the game $\mathcal{G}^*_{\bar{N}}(\mathbb{Q}', r)$.
[Why? Arriving at a stage i of the game, the player COM has to choose a condition $p_i \in N_{-1+i+1} \cap \mathbb{Q}$ stronger than all $q_{j,\xi}$ (for $j < i$, $\xi < \lambda$). If i is a limit ordinal, COM looks at the sequence $\langle p_j : j < i \rangle$ constructed by him so far. Since $(N_{i+1})^{<\lambda} \subseteq N_{i+1}$ we have that $\langle p_j : j < i \rangle \in N_{i+1}$ and, as $\mathbb{Q}$ is $(<\lambda)$–complete, this sequence has an upper bound in N_{i+1} (remember that N_{i+1} is an elementary submodel of $(\mathcal{H}(\chi), \in, <^*_\chi)$). This upper bound is good for $q_{j,\xi}$ ($j < i$, $\xi < \lambda$) too. If $i = i_0 + 1$ then the player COM looks at the sequence $\bar{q}_{i_0} \in N_{-1+i_0+2}$ only. It is $\mathbb{Q}$-generic over N_{-1+i_0+1}, $\mathbb{Q}$ is strongly $\mathcal{S}_0$–complete and N_{-1+i_0+1} is $(\lambda, \mathcal{S}_0)$–good. Therefore, there is an upper bound to $\bar{q}_{i_0}$, and by elementarity there is one in N_{-1+i_0+2}. Now, the player INC may always use the fact that $\mathbb{Q}$ is $(<\lambda)$–complete to build above p_i an increasing sequence $\bar{q}_i \subseteq \mathbb{Q} \cap N_{-1+i+1}$ which is generic over N_{-1+i+1}. Since $N_{-1+i+1} \in N_{-1+i+2}$, by elementarity there are such sequences in N_{-1+i+2}.
Concerning the "moreover" part note that the only difference is when COM is supposed to choose an upper bound to $\bar{q}_{i_0}$. But then it proceeds like in A.1.12 reducing the task to finding an upper bound to a sequence (generic over N_{-1+i_0+1}) of elements of $\mathbb{Q}$.]

Unfortunately, the amount of completeness demanded in A.1.14 is too large to capture the examples we have in mind (see the next section). Therefore we slightly weaken the demand A.1.14(3)(γ) (or rather we change the appropriate game a little). In definition A.1.16 below we formulate the variant of completeness which seems to be the right one for our case.

Definition A.1.16. Let $\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ and $\hat{\mathcal{S}}_1 \in (\mathfrak{D}^{\lambda}_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$. Let D be a function such that $\mathrm{dom}(D) = \hat{\mathcal{S}}_1$ and for every $\bar{a} \in \hat{\mathcal{S}}_1$

$$D(\bar{a}) = D_{\bar{a}} \quad \text{is a filter on } \lambda.$$

Let $\mathbb{Q}$ be a forcing notion.

(1) We say that an increasing continuous sequence $\bar{N} = \langle N_i : i \leq \lambda \rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ is $(\lambda, \kappa, \hat{\mathcal{S}}_1, D, \mathbb{Q})$*–suitable* if:
$\lambda \cup \{\lambda, \kappa, \mathbb{Q}\} \subseteq N_0$, $\|N_i\| < \kappa$, $\langle N_j : j \leq i \rangle \in N_{i+1}$ and there are $\bar{a} \in \hat{\mathcal{S}}_1$ and $X \in D_{\bar{a}}$ such that for each $i \in X$

$$(N_{i+1})^{<\lambda} \subseteq N_{i+1} \quad \& \quad N_{i+1} \cap \mu^* = a_{i+1}$$

(compare with A.1.14(1)); we can add $N_i \cap \mu^* = a_i$ if $D_{\bar{a}}$ is normal.
A pair $(\bar{a}, X)$ witnessing the last demand on $\bar{N}$ will be called *a suitable base for* $\bar{N}$.

(2) For a $(\lambda, \kappa, \hat{\mathcal{S}}_1, D, \mathbb{Q})$–suitable sequence $\bar{N} = \langle N_i : i \leq \lambda \rangle$, a suitable base $(\bar{a}, X)$ for $\bar{N}$ and a condition $r \in N_0$, let $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}, r)$ be the following game of two players, COM and INC:

The game lasts λ moves and during a play the players construct a sequence $\langle (p_i, \zeta_i, \bar{q}_i) : i < \lambda \rangle$ such that $\zeta_i \in X$, $p_i \in \mathbb{Q}$ and $\bar{q}_i = \langle q_{i,\xi} : \xi < \lambda \rangle \subseteq \mathbb{Q}$ in the following manner.
At the stage $i < \lambda$ of the game:
player COM chooses $\zeta_i \in X$ above all ζ_j chosen so far and then it picks a condition $p_i \in N_{\zeta_i+1} \cap \mathbb{Q}$ such that

$$r \leq p_i, \qquad (\forall j < i)(\forall \xi < \lambda)(q_{j,\xi} \leq p_i),$$

after this player INC answers choosing a $\leq_{\mathbb{Q}}$–increasing $\mathbb{Q}$–generic over N_{ζ_i+1} sequence $\bar{q}_i = \langle q_{i,\xi} : \xi < \lambda \rangle \subseteq N_{\zeta_i+1} \cap \mathbb{Q}$ such that

$$p_i \leq q_{i,0}, \quad \text{and} \quad \bar{q}_i \in N_{\zeta_i+2}.$$

The player COM wins the play of $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}, r)$ if $\{\zeta_i : i < \lambda\} \in D_{\bar{a}}$ and the sequence $\langle p_i : i < \lambda \rangle$ constructed by him during the play has an upper bound in $\mathbb{Q}$.
We sometimes, abusing our notation, let INC choose just the lub in $\hat{\mathbb{Q}}$ of $\bar{q}_i$.

(3) We say that the forcing notion $\mathbb{Q}$ is *really* $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$*–complete* if
 (α) $\mathbb{Q}$ is $(< \lambda)$–complete (see A.1.1(3)), and
 (β) $\mathbb{Q}$ is strongly $\mathcal{S}_0$–complete (see A.1.7(3)), and
 (γ) for most $(\lambda, \kappa, \hat{\mathcal{S}}_1, D, \mathbb{Q})$–suitable sequences $\bar{N} = \langle N_i : i \leq \lambda \rangle$, for every suitable basis $(\bar{a}, X)$ for $\bar{N}$ and all conditions $r \in N_0 \cap \mathbb{Q}$, the player INC DOES NOT have a winning strategy in the game $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}, r)$.

Remark A.1.17. If a forcing notion $\mathbb{Q}$ is strongly $\mathcal{S}_0$–complete and $(< \lambda)$–complete, and $\bar{N}$ is $(\lambda, \kappa, \hat{\mathcal{S}}_1, D, \mathbb{Q})$–suitable (witnessed by $(\bar{a}, X)$) then both players always have legal moves in the game $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}, r)$. Moreover, if $\mathbb{Q}$ is dense in $\mathbb{Q}'$, $\mathbb{Q}' \in N_0$ and COM plays elements of $\mathbb{Q}$ only, then both players have legal moves in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}', r)$
[Why? Like in A.1.15.]

Remark A.1.18. We may equivalently describe the game $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q},r)$ in the following manner. Let $\hat{\mathbb{Q}}$ be the completion of $\mathbb{Q}$.

> The play lasts λ moves during which the players construct a sequence $\langle p_i,q_i:i<\lambda\rangle$ such that $p_i\in N_{i+1}\cap(\mathbb{Q}\cup\{*\})$ (where $*\notin\mathbb{Q}$ is a fixed element of N_0), $q_i\in N_{i+2}\cap\hat{\mathbb{Q}}$.
> At the stage $i<\lambda$ of the game, COM chooses p_i in such a way that
> $$p_i\neq *\quad\Rightarrow\quad i\in X\ \&\ (\forall j<i)(q_j\leq_{\hat{\mathbb{Q}}} p_i),$$
> and INC answers choosing q_i such that
> if $p_i=*$, then q_i is the least upper bound of $\langle q_j:j<i\rangle$ in $\hat{\mathbb{Q}}$,
> if $p_i\neq *$, then $q_i\in N_{i+2}\cap\hat{\mathbb{Q}}$ is the least upper bound of a $\mathbb{Q}$–generic filter over N_{i+1} containing p_i.

The player COM wins if $\{i<\lambda:p_i\neq *\}\in D_{\bar{a}}$ and the sequence $\langle p_i:p_i\neq *\rangle$ has an upper bound.

There is no real difference between A.1.16(2) and the description given above. Here, instead of "jumping" player COM puts $*$ (which has the meaning of *I am waiting*) and it uses the existence of the least upper bounds to replace a generic sequence by its least upper bound.

Proposition A.1.19. *Suppose that $\mathcal{S}_0\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$, $\hat{\mathcal{S}}_1\in(\mathfrak{D}^{\lambda}_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$ and $D_{\bar{a}}$ is a filter on λ for $\bar{a}\in\hat{\mathcal{S}}_1$. Assume that $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}'$, $\bar{N}$ is $(\lambda,\kappa,\hat{\mathcal{S}}_1,D,\mathbb{Q})$–suitable (witnessed by $(\bar{a},X)$), $\mathbb{Q}'\in N_0$. Then for each $r\in\mathbb{Q}$:*

> *the player* COM *has a winning strategy in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q},r)$ (the player* INC *does not have a winning strategy in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q},r)$, respectively) if and only if*
> *it has a winning strategy in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}',r)$ (the player* INC *does not have a winning strategy in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}',r)$, resp.).*

Proof. Suppose that COM has a winning strategy in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q},r)$. We describe a winning strategy for him in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}',r)$ which tells him to play elements of $\mathbb{Q}$ only. The strategy is very simple. At each stage $i<\lambda$, COM replaces the sequence $\bar{q}_i\subseteq\mathbb{Q}'$ by a sequence $\bar{q}^*_i\subseteq\mathbb{Q}$ which has the same upper bounds in $\mathbb{Q}$ as $\bar{q}_i$, is increasing and generic over N_{ζ_i+1}. To do this he applies the procedure from the proof of A.1.12 (in N_{ζ_i+2}, of course). Then it may use his strategy from $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q},r)$. The converse implication is easy too: if the winning strategy of COM in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}',r)$ tells him to play ζ_i,p_i then he puts ζ_i and any element p^*_i of $\mathbb{Q}\cap N_{\zeta_i+1}$ stronger than p_i. Note that this might be interpreted as playing p_i followed by a sequence $p^*_i{}^\frown\bar{q}_i$. $\square$

Proposition A.1.20. *Suppose $\mathcal{S}_0\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ and $\hat{\mathcal{S}}_1\in(\mathfrak{D}^{\lambda}_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$ (and as usual in this section, $\kappa=\lambda^+$). Let $D_{\bar{a}}$ be the club filter of λ for each $\bar{a}\in\hat{\mathcal{S}}_1$. Then any really $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$–complete forcing notion preserves stationarity of $\mathcal{S}_0$, $\hat{\mathcal{S}}_1$ in the respective filters.*

A.2. Examples

Before we continue with the general theory let us present a simple example with the properties we are investigating. It is related to *guessing clubs*; remember that there are ZFC theorems saying that many times we can guess clubs (see [15, Ch. III, sections 1,2], [10]).

Hypothesis A.2.1. Assume $\lambda^{<\lambda} = \lambda$ and $\lambda^+ = \kappa$. Suppose that $\mathcal{S}_0 = S_0 \subseteq S^\kappa_\lambda$ is a stationary set such that $S \stackrel{\rm def}{=} S^\kappa_\lambda \setminus S_0$ is stationary too (but the definitions below are meaningful also when $S = \emptyset$). Let

$$\hat{\mathcal{S}}_1 = \big\{\bar{a} = \langle a_i : i \leq \lambda\rangle : \bar{a} \ \text{ is increasing continuous and for each } i \leq \lambda, \\ a_i \in \kappa \ \text{ and if } i \text{ is not limit then } a_i \in S_0\big\}.$$

[Check that $\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\kappa))^+$ and $\hat{\mathcal{S}}_1 \in \mathfrak{D}^\lambda_{<\kappa,<\lambda}[\mathcal{S}_0]$.]

Note that (provably in ZFC, see [15, Ch III, §2]) there is a sequence $\bar{C} = \langle C_\delta : \delta \in S\rangle$ satisfying for each $\delta \in S$:

$$C_\delta \text{ is a club of } \delta \text{ of order type } \lambda, \text{ and if } \alpha \in {\rm nacc}(C_\delta), \text{ then } {\rm cf}(\alpha) = \lambda$$

such that $\kappa \notin {\rm id}^p(\bar{C})$, i.e., for every club E of κ for stationary many $\delta \in S$, $\delta = \sup(E \cap {\rm nacc}(C_\delta))$, even $\{\alpha < \delta : \min(C_\delta \setminus (\alpha+1)) \in E\}$ is a stationary subset of δ. We can use this to show that some natural preservation of not adding bounded subsets of κ (or just not collapsing cardinals) necessarily fails, just considering the forcing notion killing the property of such $\bar{C}$. [Why? As in the result such $\bar{C}$ exists, but by iterating we could have dealt with all possible $\bar{C}$'s.] We will show that we cannot demand

$$\alpha \in {\rm nacc}(C_\delta) \quad \Rightarrow \quad {\rm cf}(\alpha) < \lambda,$$

that is, in some forcing extension preserving GCH there is no such $\bar{C}$. So, for $\bar{C}$ as earlier but with the above demand we want to add generically a club E of λ^+ such that

$$(\forall \delta \in S)(E \cap {\rm nacc}(C_\delta) \text{ is bounded in } \delta).$$

We will want our forcing to be quite complete. To get the consistency of no guessing clubs we need to iterate, which *is* our main theme.

Definition A.2.2. Let $\bar{C} = \langle C_\delta : \delta \in S\rangle$ be a sequence such that for every $\delta \in S$:

C_δ is a club of δ of order type λ, and
if $\alpha \in {\rm nacc}(C_\delta)$, then ${\rm cf}(\alpha) < \lambda$ (or at least $\alpha \notin S_0$).

We define a forcing notion $\mathbb{Q}^1_{\bar{C}}$ to add a desired club $E \subseteq \lambda^+$:

a condition in $\mathbb{Q}^1_{\bar{C}}$ is a closed subset e of λ^+ such that $\alpha_e \stackrel{\rm def}{=} \sup(e) < \lambda^+$ and

$$(\forall \delta \in S \cap (\alpha_e + 1))(e \cap {\rm nacc}(C_\delta) \text{ is bounded in } \delta),$$

the order $\leq_{\mathbb{Q}^1_{\bar{C}}}$ of $\mathbb{Q}^1_{\bar{C}}$ is defined by

$e_0 \leq_{\mathbb{Q}^1_{\bar{C}}} e_1$ if and only if e_0 is an initial segment of e_1.

It should be clear that $(\mathbb{Q}^1_{\bar{C}}, \leq_{\mathbb{Q}^1_{\bar{C}}})$ is a partial order. We claim that it is quite complete.

Proposition A.2.3. (1) *$\mathbb{Q}^1_{\bar{C}}$ is $(<\lambda)$–complete.*
(2) *$\mathbb{Q}^1_{\bar{C}}$ is strongly $\mathcal{S}_0$–complete.*

Proof. 1) Should be clear.

2) Suppose that $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ is $(\lambda,\mathcal{S}_0)$–good (see A.1.7) and $\mathbb{Q}^1_{\bar C}\in N$. Further suppose that $\bar e=\langle e_i:i<\lambda\rangle\subseteq\mathbb{Q}^1_{\bar C}\cap N$ is an increasing $\mathbb{Q}^1_{\bar C}$–generic sequence over N. Let $e\stackrel{\rm def}{=}\bigcup\limits_{i<\lambda}e_i\cup\{\sup(\bigcup\limits_{i<\lambda}e_i)\}$.

Claim A.2.3.1. $e\in\mathbb{Q}^1_{\bar C}$.

Proof of the claim. First note that as each e_i is the an end extension of all e_j for $j<i$, the set e is closed. Clearly $\alpha_e\stackrel{\rm def}{=}\sup(e)<\lambda^+$ (as each α_{e_i} is below λ^+). So what we have to check is that

$$(\forall\delta\in S\cap(\alpha_e+1))(e\cap{\rm nacc}(C_\delta)\text{ is bounded in }\delta).$$

Suppose that $\delta\in S\cap(\alpha_e+1)$. If $\delta<\alpha_e$ then for some $i<\lambda$ we have $\delta\leq\alpha_{e_i}$ and $e\cap\delta=e_i\cap\delta$ and therefore $e\cap{\rm nacc}(C_\delta)$ is bounded in δ. So a problem could occur only if $\delta=\alpha_e=\sup\limits_{i<\lambda}\alpha_{e_i}$, but we claim that it is impossible. Why? Let $\delta^*=N\cap\lambda^+$, so $\delta^*\in\mathcal{S}_0$ (as N is $(\lambda,\mathcal{S}_0)$–good) and therefore $\delta^*\neq\delta$ (as $S_0\cap S=\emptyset$). For each $\beta<\delta^*$ the set

$$\mathcal{I}_\beta\stackrel{\rm def}{=}\{q\in\mathbb{Q}^1_{\bar C}:q\setminus\beta\neq\emptyset\}$$

is open dense in $\mathbb{Q}^1_{\bar C}$ (note that if $q\in\mathbb{Q}^1_{\bar C}$, $q\setminus\beta=\emptyset$ then $q\leq q\cup\{\alpha_q,\beta+1\}\in\mathbb{Q}^1_{\bar C}$). Clearly $\mathcal{I}_\beta\in N$. Consequently, by the genericity of $\bar e$, $e_i\in\mathcal{I}_\beta$ for some $i<\lambda$ and thus $\alpha_{e_i}>\beta$. Hence $\sup\limits_{i<\lambda}\alpha_{e_i}\geq\delta^*$. On the other hand, as each e_i is in N we have $\alpha_{e_i}<\delta^*$ (for each $i<\lambda$) and hence $\delta^*=\sup\limits_{i<\lambda}\alpha_{e_i}=\delta$, a contradiction. □

Claim A.2.3.2. *For each $i<\lambda$, $e_i\leq e$.*

Proof of the claim. Should be clear. □

Now, by A.2.3.1+A.2.3.2, we are done. □

Proposition A.2.4. *For each $\bar a\in\hat{\mathcal{S}}_1$, let $D_{\bar a}$ be the club filter of λ (or any normal filter on λ). Then the forcing notion $\mathbb{Q}^1_{\bar C}$ is really $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$–complete.*

Proof. By A.2.3 we have to check demand A.1.16(3γ) only. So suppose that $\bar N=\langle N_i:i\leq\lambda\rangle$ is $(\lambda,\kappa,\hat{\mathcal{S}}_1,D,\mathbb{Q}^1_{\bar C})$–suitable and $(\bar a,X)$ is a suitable basis for $\bar N$ (and we may assume that X is a closed unbounded subset of λ). Let $r\in N_0$. We are going to describe a winning strategy for player COM in the game $\mathcal{G}^\heartsuit_{\bar N,D,X,\bar a}(\mathbb{Q}^1_{\bar C},r)$. There are two cases to consider here: $N_\lambda\cap\kappa\in S$ and $N_\lambda\cap\kappa\notin S$. The winning strategy for COM in $\mathcal{G}^\heartsuit_{\bar N,D,X,\bar a}(\mathbb{Q}^1_{\bar C},r)$ is slightly more complicated in the first case, so let us describe it only then. So we assume $N_\lambda\cap\kappa\in S$.

Arriving at the stage $i<\lambda$ of the game, COM chooses ζ_i according to the following rules:

if $i=0$, then it takes $\zeta_i=\min X$,

if $i=i_0+1$, then it takes

$$\zeta_i=\min\big\{j\in X:\zeta_{i_0}+1<j\quad\&\quad(N_{\zeta_{i_0}+1}\cap\kappa,N_j\cap\kappa)\cap C_{N_\lambda\cap\kappa}\neq\emptyset\big\},$$

if i is limit, then it lets $\zeta_i=\sup\limits_{j<i}\zeta_j$.

Note that as $C_{N_\lambda\cap\kappa}$ is unbounded in $N_\lambda\cap\kappa$ and X is a club of λ, the above definition is correct; i.e., the respective ζ_i exists, belongs to X and is necessarily above all ζ_j

chosen so far. Next COM plays p_i defined as follows. The first p_0 is just r. If $i>0$ then COM takes the first ordinal γ_i such that

$$\sup(N_{\zeta_i+1}\cap C_{N_\lambda\cap\kappa})<\gamma_i<N_{\zeta_i+1}\cap\kappa$$

and it puts

$$p_i=\bigcup_{\substack{j<i\\ \xi<\lambda}} q_{j,\xi}\cup\{N_{\bigcup\limits_{j<i}\zeta_j+1}\cap\kappa\}\cup\{\gamma_i\}.$$

Note that $\mathrm{cf}(N_{\zeta_i+1}\cap\kappa)=\lambda$ and $C_{N_\lambda\cap\kappa}$ has order type λ, so $C_{N_\lambda\cap\kappa}\cap N_{\zeta_i+1}\cap\kappa$ is bounded in $N_{\zeta_i+1}\cap\kappa$ and the γ_i above is well defined. Moreover, by arguments similar to that of A.2.3, one easily checks that $\bigcup\limits_{\substack{j<i\\ \xi<\lambda}} q_{j,\xi}\in\mathbb{Q}^1_{\bar{C}}$ and then easily $p_i\in\mathbb{Q}^1_{\bar{C}}$ and it is $\leq_{\mathbb{Q}^1_{\bar{C}}}$–stronger then all $q_{j,\xi}$ (for $j<i$, $\xi<\lambda$). Consequently, the procedure described above produces a legal strategy for COM in $\mathcal{G}^{\heartsuit}_{\bar{N},D,X,\bar{a}}(\mathbb{Q}^1_{\bar{C}},r)$. But why is this a winning strategy for COM? Suppose that $\langle(p_i,\zeta_i,\bar{q}_i):i<\lambda\rangle$ is the result of a play in which COM follows our strategy. First note that the sequence $\langle\zeta_i:i<\lambda\rangle$ is increasing continuous so it is a club of λ and thus $\{\zeta_i:i<\lambda\}\in D_{\bar{a}}$. Now, let $e=\bigcup\limits_{i<\lambda} p_i\cup\{N_\lambda\cap\kappa\}$. We claim that $e\in\mathbb{Q}^1_{\bar{C}}$. First note that it is a closed subset of λ^+ with $\sup e\stackrel{\rm def}{=}\alpha_e=N_\lambda\cap\kappa$. So suppose now that $\delta\in S\cap(\alpha_e+1)$. If $\delta<\alpha_e$, then necessarily $\delta<\alpha_{p_i}$ for some $i<\lambda$ and therefore $e\cap\mathrm{nacc}(C_\delta)=p_i\cap\mathrm{nacc}(C_\delta)$ is bounded in δ. The only danger may come from $\delta=N_\lambda\cap\kappa$. Thus assume that $\beta\in e$ and we ask where does β come from? If it is from $p_0\cup\bigcup\limits_{\xi<\lambda} q_{0,\xi}$ then we cannot say anything about it (this is the part of e that we do not control). But in all other instances we may show that $\beta\notin\mathrm{nacc}(C_{N_\lambda\cap\kappa})$. Why? If $\beta\in\bigcup\limits_{\xi<\lambda} q_{i,\xi}\setminus p_i$ for some $0<i<\lambda$, then by the choice of γ_i and p_i and the demand that $\bar{q}_i\subseteq N_{\zeta_i+1}$ we have that $\beta\notin C_{N_\lambda\cap\kappa}$. Similarly if $\beta=\gamma_i$. So the only possibility left is that $\beta=N_{\bigcup\limits_{j<i}\zeta_j+1}\cap\kappa$. If i is not limit then $\mathrm{cf}(N_{\bigcup\limits_{j<i}\zeta_j+1}\cap\kappa)=\lambda$ so $\beta\notin\mathrm{nacc}(C_{N_\gamma\cap\kappa})$. If i is limit then, by the choice of the ζ_j's we have $N_{\bigcup\limits_{j<i}\zeta_j+1}\cap\kappa\in\mathrm{acc}(C_{N_\gamma\cap\kappa})$ and we are clearly done.

Note that if $N_\lambda\cap\kappa\notin S$ then the winning strategy for COM is much simpler: choose successive elements of X as the ζ_j's and play natural bounds to sequences constructed so far. $\square$

Remark A.2.5. (1) Note that one cannot prove that the forcing notion $\mathbb{Q}^1_{\bar{C}}$ is basically $(\mathcal{S}_0,\hat{\mathcal{S}}_1)$–complete. The place in which a try to repeat the proof of A.2.4 fails is the limit case of $N_i\cap\kappa$. If we do not allow COM to make "jumps" (the choices of ζ_i) then it cannot overcome difficulties coming from the case exemplified by

$$C_{N_\lambda\cap\kappa}=\{N_{\omega\cdot i}\cap\kappa:i<\lambda\}.$$

(2) The instance $S=S^{\lambda^+}_\lambda$ is not covered here, but we will deal with it later.

The following forcing notion is used to get Pr_S (see 0.2).

Definition A.2.6. Let $\bar{C}=\langle C_\delta:\delta\in S\rangle$ be with C_δ a club of δ of order type λ and let $\bar{h}=\langle h_\delta:\delta\in S\rangle$ be a sequence such that $h_\delta:C_\delta\longrightarrow\lambda$ for $\delta\in S$. Further let $\bar{D}=\langle D_\delta:\delta\in S\rangle$ be such that each D_δ is a filter on C_δ.

(1) We define a forcing notion $\mathbb{Q}^2_{\bar{C},\bar{h}}$:
a condition in $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is a function $f:\alpha_f\longrightarrow\lambda$ such that $\alpha_f<\lambda^+$ and

$(\forall\delta\in S\cap(\alpha_f+1))(\{\beta\in C_\delta:h_\delta(\beta)=f(\beta)\}$ is a co-bounded subset of $C_\delta)$,

the order $\leq_{\mathbb{Q}^2_{\bar{C},\bar{h}}}$ of $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is the inclusion (extension).

(2) The forcing notion $\mathbb{Q}^{2,\bar{D}}_{\bar{C},\bar{h}}$ is defined similarly, except that we demand that a condition f satisfies

$$(\forall\delta\in S\cap(\alpha_f+1))(\{\beta\in C_\delta:h_\delta(\beta)=f(\beta)\}\in D_\delta).$$

Proposition A.2.7. *Let $D_{\bar{a}}$ be the club filter of λ for $\bar{a}\in\hat{\mathcal{S}}_1$. Then the forcing notion $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is really $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$–complete.*

Proof. This is parallel to A.2.4. It should be clear that $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is $(<\lambda)$–complete. The proof that it is strongly $\mathcal{S}_0$–complete goes like that of A.2.3(2), so what we need is the following claim.

Claim A.2.7.1. *For each $\beta<\lambda^+$ the set*

$$\mathcal{I}_\beta\stackrel{\rm def}{=}\{f\in\mathbb{Q}^2_{\bar{C},\bar{h}}:\beta\in{\rm dom}(f)\}$$

is open dense in $\mathbb{Q}^2_{\bar{C},\bar{h}}$.

Proof of the claim. Let $f\in\mathbb{Q}^2_{\bar{C},\bar{h}}$. We have to show that for each $\delta<\lambda^+$ there is a condition $f'\in\mathbb{Q}^2_{\bar{C},\bar{h}}$ such that $f\leq f'$ and $\delta\leq\alpha_{f'}$. Assume that for some $\delta<\lambda^+$ there is no suitable $f'\geq f$, and let δ be the first such ordinal (necessarily δ is limit). Choose an increasing continuous sequence $\langle\beta_\zeta:\zeta<{\rm cf}(\delta)\rangle$ cofinal in δ and such that $\beta_0=\alpha_f$ and $\beta_\zeta\in\delta\setminus S$ for $0<\zeta<{\rm cf}(\delta)$. For each $\zeta<{\rm cf}(\delta)$ pick a condition $f_\zeta\geq f$ such that $\alpha_{f_\zeta}=\beta_\zeta$ and let $f^*=f\cup\bigcup\limits_{\zeta<{\rm cf}(\beta)}f_{\zeta+1}{\restriction}[\beta_\zeta,\beta_{\zeta+1})$. If $\delta\notin S$ then easily $f^*\in\mathbb{Q}^2_{\bar{C},\bar{h}}$ is a condition stronger than f. Otherwise we take $f':\delta\longrightarrow\lambda$ defined by

$$f'(\xi)=\begin{cases} h_\delta(\xi) & \text{if } \xi\in C_\delta\setminus\alpha_f,\\ f^*(\xi) & \text{otherwise.}\end{cases}$$

Plainly, $f'\in\mathbb{Q}^2_{\bar{C},\bar{h}}$ and it is stronger than f. Thus in both cases we may construct a condition f' stronger than f and such that $\delta=\alpha_{f'}$, a contradiction. □

With A.2.7.1 in hands we may repeat the proof of A.2.3(2) with no substantial changes.

The proof that $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is really $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$–complete is similar to that of A.2.4. So let $\bar{N}$, $(\bar{a},X)$ and r be as there and suppose that $N_\lambda\cap\kappa\in S$. The winning strategy for COM tells it to choose ξ_i as in A.2.4 and play p_i defined as follows. The first p_0 is r. If $i>0$ then COM lets $p'_i=\bigcup\limits_{\substack{j<i\\ \xi<\lambda}}q_{j,\xi}$ (which clearly is a condition in $\mathbb{Q}^2_{\bar{C},\bar{h}}$) and chooses $p_i\in\mathbb{Q}^2_{\bar{C},\bar{h}}\cap N_{\zeta_i+1}$ such that

$$p'_i\leq p_i,\quad C_{N_\lambda\cap\kappa}\cap N_{\zeta_i+1}\subseteq{\rm dom}(p_i)\quad\text{and}$$
$$(\forall\beta\in C_{N_\lambda\cap\kappa}\cap N_{\zeta_i+1})(\alpha_{p'_i}<\beta\ \Rightarrow\ p_i(\beta)=h_{N_\lambda\cap\kappa}(\beta)).$$

Clearly this is a winning strategy for COM. $\square$

Remark A.2.8. (1) In fact, the proof of A.2.7 shows that the forcing notion $\mathbb{Q}^2_{\bar{C},\bar{h}}$ is basically $(\mathcal{S}_0,\hat{\mathcal{S}}_1)$–complete. The same applies to A.2.10.

(2) In A.2.6, A.2.7 we may consider $\bar{h}$ such that for some $h^*:\kappa\longrightarrow\kappa$, for each $\delta\in S$ we have

$$(\forall\alpha\in C_\delta)(h_\delta(\alpha)<h^*(\alpha)),$$

which does not put forward any significant changes.

(3) Why do we need h^* above at all? If we allow, e.g., h_δ to be constantly δ then clearly there is no function f with domain κ and such that $(\forall\delta\in S)(\delta>\sup\{\alpha\in C_\delta: f(\alpha)\neq h_\delta(\alpha)\})$ (by Fodor lemma). We may still ask if we could just demand $h_\delta: C_\delta\longrightarrow\delta$? Even this necessarily fails, as we may let $h_\delta(\alpha)=\min(C_\delta\setminus(\alpha+1))$. Then, if f is as above, the set $E=\{\delta<\kappa:\delta$ is a limit ordinal and $(\forall\alpha<\delta)(f(\alpha)<\delta)\}$ is a club of κ. Hence for some $\delta\in S$ we have:

$$\lambda^2<\delta=\sup(E\cap\delta)=\mathrm{otp}(E\cap\delta)$$

and we get an easy contradiction.

Another example of forcing notions which we have in mind when developing the general theory is related to the following problem. Let K be a λ–free Abelian group of cardinality κ. We want to make it a Whitehead group.

Definition A.2.9. Suppose that

(a) K_1 is a strongly κ–free Abelian group of cardinality κ, $\langle K_{1,\alpha}:\alpha<\kappa\rangle$ is a filtration of K_1 (i.e., it is an increasing continuous sequence of subgroups of K_1 such that $K_1=\bigcup\limits_{\alpha<\kappa}K_{1,\alpha}$ and each $K_{1,\alpha}$ is of size $<\kappa$),

$$\Gamma=\{\alpha<\kappa: K_1/K_{1,\alpha} \text{ is not } \lambda\text{–free }\},$$

(b) K_2 is an Abelian group extending $\mathbb{Z}$, $h: K_2\stackrel{\text{onto}}{\longrightarrow}K_1$ is a homomorphism with kernel $\mathbb{Z}$.

We define a forcing notion $\mathbb{Q}^3_{K_2,h}$:
a condition in $\mathbb{Q}^3_{K_2,h}$ is a homomorphism $g:K_{1,\alpha}\longrightarrow K_2$ such that $\alpha\in\kappa\setminus\Gamma$ and $h\circ g=\mathrm{id}_{K_{1,\alpha}}$,
the order $\leq_{\mathbb{Q}^3_{K_2,h}}$ of $\mathbb{Q}^3_{K_2,h}$ is the inclusion (extension).

Proposition A.2.10. *Let $D_{\bar{a}}$ be a club filter of λ for $\bar{a}\in\hat{\mathcal{S}}_1$. Assume $K_1,K_{1,\alpha},K_2$ and Γ are as in assumptions of A.2.9 and $\Gamma\subseteq S$. Then the forcing notion $\mathbb{Q}^3_{K_2,h}$ is really $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$–complete.*

Proof. Similar to the proofs of A.2.4 and A.2.7. $\square$

A.3. The iteration theorem

In this section we will prove the preservation theorem needed for **Case A**. Let us start with some explanations which (hopefully) will help the reader to understand what and why we do to get our result.

We would like to prove that if $\bar{\mathbb{Q}}=\langle\mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i:i<\gamma\rangle$ is a $(<\kappa)$–support iteration of suitably complete forcing notions, $(\mathcal{S}_0,\hat{\mathcal{S}}_1,D)$ are as in A.1.16, then:

if $\bar{N}=\langle N_i : i\leq\lambda\rangle$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi),\in,<^*_\chi)$, $\|N_i\|=\lambda$, $\lambda+1\subseteq N_i$, $(N_i)^{<\lambda}\subseteq N_i$ for non-limit i, and for some $\bar{a}\in\hat{\mathcal{S}}_1$ and $X\in D_{\bar{a}}$

$$(\forall i\in X)(N_i\cap\mu^*=a_i \quad \& \quad N_{i+1}\cap\mu^*=a_{i+1})$$

and $p\in\mathbb{P}_\gamma\cap N_0$,
then there is a condition $q\in\mathbb{P}_\gamma$ stronger than p and $(N_\lambda,\mathbb{P}_\gamma)$–generic.

For each $\underset{\sim}{\mathbb{Q}}_i$ we may get respective q, but the problem is with the iteration. We can start with increasing successively p to $p_i\in N_\lambda$ $(i<\lambda)$ and we can keep meeting dense sets due to $(<\lambda)$–completeness. But the main question is: *why* is there a limit? For each $\alpha\in\gamma\cap N_\lambda$ we have to make sure that the sequence $\langle p_i(\alpha) : i<\lambda\rangle$ has an upper bound in $\underset{\sim}{\mathbb{Q}}_\alpha$, but for this we need information which is a $\mathbb{P}_\gamma$–name which does not belong to N_λ, e.g., if $\underset{\sim}{\mathbb{Q}}_i$ is $\mathbb{Q}^2_{\underset{\sim}{\bar{C}},\underset{\sim}{\bar{h}}}$ we need to know $\underset{\sim}{C}_{N\cap\kappa},\underset{\sim}{h}_{N\cap\kappa}$. *But* for each i, the size of the information needed is $<\lambda$.

As the life in our context is harder than for proper forcing iterations, we have to go back to pre–proper tools and methods and we will use trees of names (see [12]). A tree of conditions is essentially a non-deterministic condition; in the limit we will show that *some* choice of a branch through the tree does the job.
[Note that one of the difficulties one meets here is that we cannot diagonalize over objects of type $\lambda\times\omega$ when $\lambda>\aleph_0$.]

Definition A.3.1. (1) A tree $(T,<)$ is *normal* if for each $t_0,t_1\in T$,
if $\{s\in T : s<t_0\}=\{s\in T : s<t_1\}$ has no last element,
then $t_0=t_1$.
(2) For an ordinal γ, $\mathrm{Tr}(\gamma)$ stands for the family of all triples

$$\mathcal{T}=(T^{\mathcal{T}},<^{\mathcal{T}},\mathrm{rk}^{\mathcal{T}})$$

such that $(T^{\mathcal{T}},<^{\mathcal{T}})$ is a normal tree and $\mathrm{rk}^{\mathcal{T}} : T^{\mathcal{T}}\longrightarrow\gamma+1$ is an increasing function.

We will keep the convention that $\mathcal{T}^x_y=(T^x_y,<^x_y,\mathrm{rk}^x_y)$. Sometimes we may write $t\in\mathcal{T}$ instead $t\in T^{\mathcal{T}}$ (or $t\in T$).

The main case and examples we have in mind are triples $(T,<,\mathrm{rk})$ such that for some $w\subseteq\gamma$ (where γ is the length of our iteration), T is a family of partial functions such that:

$$(\forall t\in T)(\mathrm{dom}(t)\text{ is an initial segment of }w\quad\text{and}\quad(\forall\alpha\in w)(t{\restriction}\alpha\in T));$$

the order is the inclusion and the function rk is given by

$$\mathrm{rk}(t)=\min\{\alpha\in w\cup\{\gamma\} : \mathrm{dom}(t)=\alpha\cap w\},$$

(see A.3.3). Here we can let $N_\lambda\cap\gamma=\{\alpha_\xi : \xi<\lambda\}$. Defining p_i we are thinking of why $\langle p_j(\alpha_\xi) : j<\lambda\rangle$ will have an upper bound. Now $\lambda\times\lambda$ *has* a diagonal.

Note: *starting to take care of α_ξ only after some time* is a reasonable strategy, so in stage $i<\lambda$ we care about $\{\alpha_\xi : \xi<i\}$ only.

But what does it mean to do it? We have to guess the relevant information which is a $\mathbb{P}_{\alpha_\xi}$–name and is not present.

What do we do? We cover *all* possibilities. So the tree $\mathcal{T}_\zeta$ will consist of objects t which are guesses on what is $\langle$*information for α_ε up to ζ^{th} stage*: $\varepsilon<\zeta\rangle$. Of course we should not inflate, e.g., $\langle p^\zeta_t : t\in\mathcal{T}_\xi\rangle\in N_\lambda$.

It is very nice to have *an open option* so that in stage λ we can choose the most convenient branch. But we need to go into all dense sets and then we have to pay an extra price for having an extra luggage. We need to put *all* the p_i's into a dense set (which is trivial for a single condition). What will help us in this task is the strong $\mathcal{S}_0$–completeness. Without this *big brother to pay our bills*, our scheme would have to fail: we do have some ZFC theorems which put restrictions on the possible iteration theorems.

Definition A.3.2. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma \rangle$ be a $(< \kappa)$–support iteration.

(1) We define

$$\begin{aligned}\mathrm{FTr}(\bar{\mathbb{Q}}) \stackrel{\mathrm{def}}{=} \big\{\bar{p} = \langle p_t : t \in T^{\mathcal{T}} \rangle : \mathcal{T} \in \mathrm{Tr}(\gamma),\ (\forall t \in T^{\mathcal{T}})(p_t \in \mathbb{P}_{\mathrm{rk}(t)}) \ \text{ and } \\ (\forall s, t \in T^{\mathcal{T}}))(s < t \ \Rightarrow \ p_s = p_t {\restriction} \mathrm{rk}(s))\big\},\end{aligned}$$

and

$$\begin{aligned}\mathrm{FTr}_{wk}(\bar{\mathbb{Q}}) \stackrel{\mathrm{def}}{=} \big\{\bar{p} = \langle p_t : t \in T^{\mathcal{T}} \rangle : \mathcal{T} \in \mathrm{Tr}(\gamma),\ (\forall t \in T^{\mathcal{T}})(p_t \in \mathbb{P}_{\mathrm{rk}(t)}) \ \text{ and } \\ (\forall s, t \in T^{\mathcal{T}}))(s < t \ \Rightarrow \ p_s \geq p_t {\restriction} \mathrm{rk}(s))\big\}.\end{aligned}$$

We may write $\langle p_t : t \in \mathcal{T} \rangle$. Abusing notation, we mean $\bar{p} \in \mathrm{FTr}_{wk}(\bar{\mathbb{Q}})$ (and $\bar{p} \in \mathrm{FTr}(\bar{\mathbb{Q}})$) determines $\mathcal{T}$ and we call it $\mathcal{T}^{\bar{p}}$ (or we may forget and write $\mathrm{dom}(\bar{p})$).

Adding primes to FTr, FTr_{wk} means that we allow $p_t(\beta)$ be (a $\mathbb{P}_\beta$–name for) an element of the completion $\hat{\underset{\sim}{\mathbb{Q}}}_\beta$ of $\underset{\sim}{\mathbb{Q}}_\beta$. Then p_t is an element of $\mathbb{P}'_{\mathrm{rk}(t)}$ — the $(< \kappa)$–support iteration of the completions $\hat{\underset{\sim}{\mathbb{Q}}}_\beta$ (see 0.18).

(2) If $\mathcal{T} \in \mathrm{Tr}(\gamma)$, $\bar{p}, \bar{q} \in \mathrm{FTr}'_{wk}(\bar{\mathbb{Q}})$, $\mathrm{dom}(\bar{p}) = \mathrm{dom}(\bar{q}) = T^{\mathcal{T}}$ then we let

$$\bar{p} \leq \bar{q} \quad \text{if and only if} \quad (\forall t \in T^{\mathcal{T}})(p_t \leq q_t).$$

(3) Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{Tr}(\gamma)$. We say that a surjection $f : T_2 \stackrel{\text{onto}}{\longrightarrow} T_1$ is *a projection* if for each $s, t \in T_2$
 (α) $s \leq_2 t \quad \Rightarrow \quad f(s) \leq_1 f(t)$, and
 (β) $\mathrm{rk}_2(t) \leq \mathrm{rk}_1(f(t))$.

(4) Let $\bar{p}^0, \bar{p}^1 \in \mathrm{FTr}'_{wk}(\bar{\mathbb{Q}})$, $\mathrm{dom}(\bar{p}^\ell) = \mathcal{T}_\ell$ ($\ell < 2$) and $f : T_1 \longrightarrow T_0$ be a projection. Then we will write $\bar{p}^0 \leq_f \bar{p}^1$ whenever for all $t \in T_1$
 (α) $p^0_{f(t)} {\restriction} \mathrm{rk}_1(t) \leq_{\mathbb{P}'_{\mathrm{rk}_1(t)}} p^1_t$, and
 (β) if $i < \mathrm{rk}_1(t)$, then

$$p^1_t {\restriction} i \Vdash_{\mathbb{P}_i} \text{“} p^0_{f(t)}(i) \neq p^1_t(i) \quad \Rightarrow \quad (\exists q \in \underset{\sim}{\mathbb{Q}}_i)(p^0_{f(t)}(i) \leq_{\hat{\underset{\sim}{\mathbb{Q}}}_i} q \leq_{\hat{\underset{\sim}{\mathbb{Q}}}_i} p^1_t(i))\text{”}.$$

The projections play the key role in the iteration lemma. Therefore, to make the presentation clearer we will restrict ourselves to the case we actually need.

You may think of γ as the length of the iteration, and let $\{\beta_\xi : \xi < \lambda\}$ list $N \cap \gamma$, $w = \{\beta_\xi : \xi < \alpha\}$. We are trying to build a generic condition for $(\mathbb{P}_\gamma, N)$ by approximating it by a sequence of trees of conditions. In the present tree we are at stage α. Now, for $t \in \mathcal{T}$, $t(i)$ is a guess on the information needed to construct a generic for $(N[G_{\mathbb{P}_i}], \underset{\sim}{\mathbb{Q}}_i[G_{\mathbb{P}_i}])$, more exactly the α–initial segment of it.

Definition A.3.3. Let γ be an ordinal.

(1) Suppose that $w \subseteq \gamma$ and α is an ordinal. We say that $\mathcal{T} \in \mathrm{Tr}(\gamma)$ is *a standard* $(w, \alpha)^\gamma$*–tree* if
 (α) $(\forall t \in T^{\mathcal{T}})(\mathrm{rk}^{\mathcal{T}}(t) \in w \cup \{\gamma\})$,

(β) if $t\in T^{\mathcal{T}}$, $\mathrm{rk}^{\mathcal{T}}(t)=\varepsilon$, then t is a sequence $\langle t_i : i\in w\cap\varepsilon\rangle$, where each t_i is a sequence of length α,

(γ) $<^{\mathcal{T}}$ is the extension (inclusion) relation.

[In (β) above we may demand that each t_i is a function with domain $[i,\alpha)$, $i<\alpha$, but we can use a default value $*$ below i, hence making such t_i into sequences of length α. Note that $T^{\mathcal{T}}$ determines $\mathcal{T}$ in this case; $\langle\rangle$ is the root of T.]

(2) Suppose that $w_0\subseteq w_1\subseteq\gamma$, $\alpha_0\leq\alpha_1$ and $\mathcal{T}=(T,<,\mathrm{rk})$ is a standard $(w_1,\alpha_1)^\gamma$–tree. We define *the projection* $\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T})$ *of* $\mathcal{T}$ *onto* (w_0,α_0) as $(T^*,<^*,\mathrm{rk}^*)$ such that:

$T^*=\{\langle t_i{\restriction}\alpha_0 : i\in w_0\cap\mathrm{rk}(t)\rangle : t=\langle t_i : i\in w_1\cap\mathrm{rk}(t)\rangle\in T\}$,

$<^*$ is the extension relation,

$\mathrm{rk}^*(\langle t_i{\restriction}\alpha_0 : i\in w_0\cap\mathrm{rk}(t)\rangle)=\min(w_0\cup\{\gamma\}\setminus\mathrm{rk}(t))$ for $t\in T$.

[Note that $\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T})$ is a standard $(w_0,\alpha_0)^\gamma$–tree.]

(3) If $w_0\subseteq w_1\subseteq\gamma$, $\alpha_0<\alpha_1$, $\mathcal{T}_1=(T_1,<_1,\mathrm{rk}_1)$ is a standard $(w_1,\alpha_1)^\gamma$–tree and $\mathcal{T}_0=(T_0,<_0,\mathrm{rk}_0)=\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T}_1)$, then the mapping

$$T_1\ni\langle t_i : i\in w_1\cap\mathrm{rk}_1(t)\rangle\ \longmapsto\ \langle t_i{\restriction}\alpha_0 : i\in w_0\cap\mathrm{rk}_1(t)\rangle\in T_0$$

is denoted by $\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}$ (or $\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}$).

[Note that $\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}$ is a projection from $\mathcal{T}_1$ onto $\mathcal{T}_0$.]

(4) We say that $\bar{\mathcal{T}}=\langle\mathcal{T}_\alpha : \alpha<\alpha^*\rangle$ is *a legal sequence of standard* γ*–trees* if for some $\bar{w}=\langle w_\alpha : \alpha<\alpha^*\rangle$ we have

(α) $\bar{w}$ is an increasing continuous sequence of subsets of γ,

(β) for each $\alpha<\alpha^*$, $\mathcal{T}_\alpha$ is a standard $(w_\alpha,\alpha)^\gamma$–tree,

(γ) if $\alpha<\beta<\alpha^*$, then $\mathcal{T}_\alpha=\mathrm{proj}^{(w_\alpha,\alpha)}_{(w_\beta,\beta)}(\mathcal{T}_\beta)$.

(5) For a legal sequence $\bar{\mathcal{T}}=\langle\mathcal{T}_\alpha : \alpha<\alpha^*\rangle$ of standard γ–trees, α^* a limit ordinal, we define the inverse limit $\overleftarrow{\lim}(\bar{\mathcal{T}})$ of $\bar{\mathcal{T}}$ as a triple

$$(T^{\overleftarrow{\lim}(\bar{\mathcal{T}})},<^{\overleftarrow{\lim}(\bar{\mathcal{T}})},\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})})$$

such that

(a) $T^{\overleftarrow{\lim}(\bar{\mathcal{T}})}$ consists of all sequences t such that

(i) $\mathrm{dom}(t)$ is an initial segment of $w\stackrel{\mathrm{def}}{=}\bigcup\limits_{\alpha<\alpha^*}w_\alpha$ (not necessarily proper),

(ii) if $i\in\mathrm{dom}(t)$, then t_i is a sequence of length α^*,

(iii) for each $\alpha<\alpha^*$, $\langle t_i{\restriction}\alpha : i\in w_\alpha\cap\mathrm{dom}(t)\rangle\in T_\alpha$,

(b) $<^{\overleftarrow{\lim}(\bar{\mathcal{T}})}$ is the extension relation,

(c) $\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t)=\min(w\cup\{\gamma\}\setminus\mathrm{dom}(t))$ for $t\in T^{\overleftarrow{\lim}(\bar{\mathcal{T}})}$.

[Note that it may happen that $T^{\overleftarrow{\lim}(\bar{\mathcal{T}})}=\{\langle\rangle\}$, however not if $\bar{\mathcal{T}}$ is continuous, see below.]

(6) A legal sequence of standard γ–trees $\bar{\mathcal{T}}=\langle\mathcal{T}_\alpha : \alpha<\alpha^*\rangle$ is *continuous* if $\mathcal{T}_\alpha=\overleftarrow{\lim}(\mathcal{T}_\beta : \beta<\alpha)$ for each limit $\alpha<\alpha^*$.

Proposition A.3.4. *Suppose that* $\alpha_0,\alpha_1,\alpha_2,\gamma$ *are ordinals such that* $\alpha_0\leq\alpha_1\leq\alpha_2$. *Let* $w_{\alpha_0}\subseteq w_{\alpha_1}\subseteq w_{\alpha_2}\subseteq\gamma$. *If* $\mathcal{T}_1$ *is a standard* $(w_1,\alpha_1)^\gamma$*–tree, then* $\mathcal{T}_0=$

$\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T}_1)$ *is a standard* $(w_0,\alpha_0)^\gamma$*–tree. Assume that for* $\ell<3$, $\mathcal{T}_\ell$ *are standard* $(w_\ell,\alpha_\ell)^\gamma$*–trees such that*

$$\mathcal{T}_0=\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T}_1)\quad \text{and}\quad \mathcal{T}_1=\mathrm{proj}^{(w_2,\alpha_2)}_{(w_1,\alpha_1)}(\mathcal{T}_2).$$

Then $\mathcal{T}_0=\mathrm{proj}^{(w_2,\alpha_2)}_{(w_0,\alpha_0)}(\mathcal{T}_2)$ *and* $\mathrm{proj}^{\mathcal{T}_2}_{\mathcal{T}_0}=\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}\circ\mathrm{proj}^{\mathcal{T}_2}_{\mathcal{T}_1}$.
Moreover, if $\bar p^\ell=\langle p^\ell_t:t\in T_\ell\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ *(for* $\ell<3$*) are such that* $\bar p^0\le_{\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}}\bar p^1$ *and* $\bar p^1\le_{\mathrm{proj}^{\mathcal{T}_2}_{\mathcal{T}_1}}\bar p^2$ *then* $\bar p^0\le_{\mathrm{proj}^{\mathcal{T}_2}_{\mathcal{T}_0}}\bar p^2$.

Proposition A.3.5. *Let* γ,α^* *be ordinals,* α^* *limit, and let* $\bar{\mathcal{T}}=\langle\mathcal{T}_\alpha:\alpha<\alpha^*\rangle$ *be a continuous legal sequence of standard* γ*–trees.*

(1) *The inverse limit* $\overleftarrow{\lim}(\bar{\mathcal{T}})$ *is a standard* $(\bigcup\limits_{\alpha<\alpha^*}w_\alpha,\alpha^*)^\gamma$*–tree and each* $\mathcal{T}_\alpha$ *is a projection of* $\overleftarrow{\lim}(\bar{\mathcal{T}})$ *onto* (w_α,α) *and the respective projections commute. (Here,* $w_\alpha\subseteq\gamma$ *is such that* $\mathcal{T}_\alpha$ *is a standard* $(w_\alpha,\alpha)^\gamma$*–tree)*
[So we do not cheat: $\overleftarrow{\lim}(\bar{\mathcal{T}})$ *is really the inverse limit of* $\bar{\mathcal{T}}$*.]*
(2) *If* $\lambda^{<\lambda}=\lambda$, $\alpha^*<\lambda$ *and* $\|T_\alpha\|\le\lambda$ *for each* $\alpha<\alpha^*$ *then* $\|T^{\overleftarrow{\lim}(\bar{\mathcal{T}})}\|\le\lambda$.
(3) *If* $\alpha^*<\lambda$, $\kappa=\lambda^+$, $\bar{\mathbb{Q}}=\langle\mathbb{P}_\xi,\underset{\sim}{\mathbb{Q}}_\xi:\xi<\gamma\rangle$ *is a* $(<\kappa)$*–support iteration of* $(<\lambda)$*–complete forcing notions and* $\bar p^\alpha=\langle p^\alpha_t:t\in\mathcal{T}_\alpha\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ *(for each* $\alpha<\alpha^*$*) are such that* $|T_\alpha|\le\lambda$ *for* $\alpha<\alpha^*$ *and*

$$\beta<\alpha<\alpha^*\quad\Rightarrow\quad\bar p^\beta\le_{\mathrm{proj}^{\mathcal{T}_\alpha}_{\mathcal{T}_\beta}}\bar p^\alpha$$

then there is $\bar p^{\alpha^*}=\langle p^{\alpha^*}_t:t\in\overleftarrow{\lim}(\bar{\mathcal{T}})\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ *such that*

$$(\forall\alpha<\alpha^*)(\bar p^\alpha\le_{\mathrm{proj}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}_{\mathcal{T}_\alpha}}\bar p^{\alpha^*}).$$

Proof. 1) Should be clear: just read the definitions.

2) It follows from the following inequalities:

$$\|T^{\overleftarrow{\lim}(\bar{\mathcal{T}})}\|\le\prod_{\alpha<\alpha^*}\|T_\alpha\|\le\lambda^{<\lambda}=\lambda.$$

3) For each $t\in\overleftarrow{\lim}(\bar{\mathcal{T}})$ we define a condition $p^{\alpha^*}_t\in\mathbb{P}'_\gamma$ as follows. Let $t^\alpha=\mathrm{proj}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}_{\mathcal{T}_\alpha}(t)$ (for $\alpha<\alpha^*$). We know that the sequence $\langle p^\alpha_{t^\alpha}{\restriction}\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t):\alpha<\alpha^*\rangle$ is increasing (remember $\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t)\le\mathrm{rk}_\alpha(t_\alpha)$ for each $\alpha<\alpha^*$) and $p^{\alpha^*}_t$ is supposed to be an upper bound to it (and $p^{\alpha^*}_t\in\mathbb{P}'_{\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t)}$). We define $p^{\alpha^*}_t$ quite straightforward. We let

$$\mathrm{dom}(p^{\alpha^*}_t)=\bigcup\{\mathrm{dom}(p^\alpha_{t^\alpha})\cap\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t):\alpha<\alpha^*\}$$

and next we inductively define $p^{\alpha^*}_t(i)$ for $i\in\mathrm{dom}(p^{\alpha^*}_t)$. Assume we have defined $p^{\alpha^*}_t{\restriction}i$ such that

$$(\forall\alpha<\alpha^*)(p^\alpha_{t^\alpha}{\restriction}i\le_{\mathbb{P}'_i}p^{\alpha^*}_t{\restriction}i).$$

Then (remembering our convention that if $i\notin\mathrm{dom}(p)$ then $p(i)=\mathbf{0}_{\underset{\sim}{\mathbb{Q}}_i}$)

$$p_t^{\alpha^*}{\restriction}i\Vdash\text{“ the sequence }\langle p^\alpha_{t^\alpha}(i):\alpha<\alpha^*\rangle\subseteq\hat{\underset{\sim}{\mathbb{Q}}}_i\text{ is }\leq_{\hat{\underset{\sim}{\mathbb{Q}}}_i}\text{–increasing and}$$
$$\alpha<\beta<\alpha^*\ \&\ p^\alpha_{t^\alpha}(i)\neq p^\beta_{t^\beta}(i)\quad\Rightarrow\quad(\exists q\in\underset{\sim}{\mathbb{Q}}_i)(p^\alpha_{t^\alpha}(i)\leq_{\hat{\underset{\sim}{\mathbb{Q}}}_i}q\leq_{\hat{\underset{\sim}{\mathbb{Q}}}_i}p^\beta_{t^\beta}(i))$$
$$\text{and }\underset{\sim}{\mathbb{Q}}_i\text{ is }(<\lambda)\text{–complete and }\alpha^*<\lambda\text{ ”.}$$

Hence we find a $\mathbb{P}'_i$–name $p_t^{\alpha^*}(i)$ (and we take the $<^*_\chi$–first such a name) such that

$$p_t^{\alpha^*}{\restriction}i\Vdash\text{“}p_t^{\alpha^*}(i)\in\hat{\underset{\sim}{\mathbb{Q}}}_i\text{ is the least upper bound of }\langle p^\alpha_{t^\alpha}(i):\alpha<\alpha^*\rangle\text{ in }\hat{\underset{\sim}{\mathbb{Q}}}_i\text{”.}$$

Now one easily checks that $p_t^{\alpha^*}\in\mathbb{P}'_{\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(t)}$. Consequently the condition $p_t^{\alpha^*}$ is as required. But why does $\langle p_t^{\alpha^*}:t\in\overleftarrow{\lim}(\bar{\mathcal{T}})\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$? We still have to argue that

$$(\forall s,t\in\overleftarrow{\lim}(\bar{\mathcal{T}}))(s<t\quad\Rightarrow\quad p_s^{\alpha^*}=p_t^{\alpha^*}{\restriction}\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(s)).$$

For this note that if $s<t$ are in $\overleftarrow{\lim}(\bar{\mathcal{T}})$ and s_α,t_α are their projections to $\mathcal{T}_\alpha$ then $s_\alpha\leq_\alpha t_\alpha$ and $p^\alpha_{s_\alpha}=p^\alpha_{t_\alpha}{\restriction}\mathrm{rk}_\alpha(s_\alpha)$ and $\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(s)\leq\mathrm{rk}_\alpha(s_\alpha)$. Thus clearly $\mathrm{dom}(p_s^{\alpha^*})=\mathrm{dom}(p_t^{\alpha^*})\cap\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(s)$. Next, by induction on $i\in\mathrm{dom}(p_t^{\alpha^*})\cap\mathrm{rk}^{\overleftarrow{\lim}(\bar{\mathcal{T}})}(s)$ we show that $p_s^{\alpha^*}(i)=p_t^{\alpha^*}(i)$. Assume we have proved that $p_t^{\alpha^*}{\restriction}i=p_s^{\alpha^*}{\restriction}i$ and look at the way we defined the respective values at i. We looked there at the sequences $\langle p^\alpha_{t_\alpha}(i):\alpha<\alpha^*\rangle$, $\langle p^\alpha_{s_\alpha}(i):\alpha<\alpha^*\rangle$ and we have chosen the $<^*_\chi$–first names for the least upper bounds to them. But $i<\mathrm{rk}_\alpha(s_\alpha)$ for all $\alpha<\alpha^*$, so the two sequences are equal and the choice was the same. □

Proposition A.3.6. *Assume that $\mathcal{S}_0\subseteq[\mu^*]^{\leq\lambda}$ and $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\gamma\rangle$ is a $(<\kappa)$–support iteration of $(<\lambda)$–complete strongly $\mathcal{S}_0$–complete forcing notions, and $\underset{\sim}{x}_\alpha$ (for $\alpha<\gamma$) are $\mathbb{P}_\alpha$–names such that*

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{x}_\alpha\textit{ witnesses the }\text{most}\textit{ in A.1.7(2) for }\underset{\sim}{\mathbb{Q}}_\alpha\text{”.}$$

Further suppose that

- (α) *$N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ is $(\lambda,\mathcal{S}_0)$–good (see A.1.7), $\langle\underset{\sim}{x}_\alpha:\alpha<\gamma\rangle,\alpha_0,\bar{\mathbb{Q}},\ldots\in N$,*
- (β) *$0\in w_0\subseteq w_1\in N\cap[\gamma]^{<\lambda}$, $\alpha_0<\lambda$ is an ordinal, $\alpha_1=\alpha_0+1$,*
- (γ) *$\mathcal{T}_0=(T_0,<_0,\mathrm{rk}_0)\in N$ is a standard $(w_0,\alpha_0)^\gamma$–tree, $\|T_0\|\leq\lambda$,*
- (δ) *$\bar{p}=\langle p_t:t\in T_0\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})\cap N$,*
- (ε) *$\mathcal{T}_1=(T_1,<_1,\mathrm{rk}_1)$ is such that*
  T_1 consists of all sequences $t=\langle t_i:i\in\mathrm{dom}(t)\rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w_1, and
  - *each t_i is a sequence of length α_1,*
  - *$t'\overset{\text{def}}{=}\langle t_i{\restriction}\alpha_0:i\in\mathrm{dom}(t)\cap w_0\rangle\in T_0$,*
  - *if $i\in\mathrm{dom}(t)\setminus w_0$, $\alpha<\alpha_0$ then $t_i(\alpha)=*$,*
  - *for some $j(t)\in\mathrm{dom}(t)\cup\{\gamma\}$,*
    $t_i(\alpha_0)$ is $*$ for every $i\in\mathrm{dom}(t)\setminus j(t)$, and for each $i\in\mathrm{dom}(t)\cap j(t)$ $t_i(\alpha_0)\in N$ is a $\mathbb{P}_i$–name for an element of $\underset{\sim}{\mathbb{Q}}_i$,
  - *$\mathrm{rk}_1(t)=\min(w_1\cup\{\gamma\}\setminus\mathrm{Dom}(t))$ and $<_1$ is the extension relation.*

Then

- (a) *$\mathcal{T}_1$ is a standard $(w_1,\alpha_1)^\gamma$–tree, $\|T_1\|=\lambda$,*
- (b) *$\mathcal{T}_0$ is the projection of $\mathcal{T}_1$ onto (w_0,α_0),*

(c) *there is* $\bar{q} = \langle q_t : t \in T_1 \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ *such that*

(i) $\bar{p} \leq_{\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}} \bar{q}$,

(ii) *if* $t \in T_1 \setminus \{\langle\rangle\}$ *and* $(\forall i \in \mathrm{dom}(t))(t_i(\alpha_0) \neq *)$, *then the condition* $q_t \in \mathbb{P}'_{\mathrm{rk}_1(t)}$ *is an upper bound in* $\mathbb{P}'_{\mathrm{rk}_1(t)}$ *of a* $\mathbb{P}_{\mathrm{rk}_1(t)}$*–generic sequence over* N, *and for every* $\beta \in \mathrm{dom}(q_t) = N \cap \mathrm{rk}_1(t)$, $q_t(\beta)$ *is (a name for) the least upper bound in* $\hat{\mathbb{Q}}_\beta$ *of the family of all* $r(\beta)$ *for* r *from the generic set (over* N*) generated by* q_t,

(iii) *if* $t \in T_1$, $t' = \mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}(t) \in T_0$, $i \in \mathrm{dom}(t)$ *and* $t_i(\alpha_0) \neq *$, *then*

$$q_t{\restriction}i \Vdash_{\mathbb{P}_i} \text{“}p_{t'}(i) \leq_{\hat{\mathbb{Q}}_i} t_i(\alpha_0) \ \Rightarrow\ t_i(\alpha_0) \leq_{\hat{\mathbb{Q}}_i} q_t(i)\text{”},$$

(iv) $q_{\langle\rangle} = p_{\langle\rangle}$ *and*
if $t \in T_1 \setminus \{\langle\rangle\}$ *and* $j(t) < \gamma$, *then* $q_t = q_{t{\restriction}j(t)} \cup p_{t'}{\restriction}[j(t), \mathrm{rk}_1(t))$, *where* $t' = \mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}(t) \in T_0$.

Proof. Clauses (a) and (b) should be clear.

(c) Let $\langle t_\zeta : \zeta < \lambda \rangle$ list with λ-repetitions all elements t of $T_1 \setminus \{\langle\rangle\}$ such that $(\forall i \in \mathrm{dom}(t))(t_i(\alpha_0) \neq *)$. For $\alpha \in w_1 \cup \{\gamma\}$ let $\langle \mathcal{I}^\alpha_\zeta : \zeta < \lambda \rangle$ enumerate all open dense subsets of $\mathbb{P}_\alpha$ from N. By induction on $\zeta < \lambda$ choose r_ζ such that

- $r_\zeta \in \mathbb{P}_{\mathrm{rk}_1(t_\zeta)} \cap N$,
- if $t' = \mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}(t_\zeta)$, then $p_{t'}{\restriction}\mathrm{rk}_1(t_\zeta) \leq_{\mathbb{P}'_{\mathrm{rk}_1(t_\zeta)}} r_\zeta$ and for $i \in \mathrm{dom}(t_\zeta)$

$$r_\zeta{\restriction}i \Vdash_{\mathbb{P}_i} \text{“}p_{t'}(i) \leq_{\hat{\mathbb{Q}}_i} (t_\zeta)_i(\alpha_0) \ \Rightarrow\ (t_\zeta)_i(\alpha_0) \leq_{\hat{\mathbb{Q}}_i} r_\zeta(i)\text{”},$$

- $r_\zeta \in \mathcal{I}^{\mathrm{rk}_1(t_\zeta)}_\xi$ for all $\xi \leq \zeta$,
- if $t \in T_1$, $\xi < \zeta$, $t \leq_1 t_\xi$, $t \leq_1 t_\zeta$ (e.g., $t = t_\xi = t_\zeta$), then $r_\xi{\restriction}\mathrm{rk}_1(t) \leq_{\mathbb{P}_{\mathrm{rk}_1(t)}} r_\zeta{\restriction}\mathrm{rk}_1(t)$.

Since we have assumed that all $\mathbb{Q}_\alpha$'s are (names for) $(<\lambda)$–complete forcing notions there are no difficulties in carrying out the above construction. [First, working in N, choose $r^*_\zeta \in \mathbb{P}_{\mathrm{rk}_1(t_\zeta)} \cap N$ satisfying the second and the fourth demand. How? Declare

$$\mathrm{dom}(r^*_\zeta) = [w_1 \cup \bigcup\{\mathrm{dom}(r_\xi) : \xi < \zeta\} \cup \mathrm{dom}(p_{\mathrm{proj}^{\mathcal{T}_1}_{\mathcal{T}_0}(t_\zeta)})] \cap \mathrm{rk}_1(t_\zeta)$$

and by induction on i define $r^*_\zeta(i)$ using the $(<\lambda)$–completeness of $\mathbb{Q}_i$ and taking care of the respective demands (similar to the choice of q_t done in detail below). Next use the $(<\lambda)$–completeness (see A.1.3) to enter all $\mathcal{I}^{\mathrm{rk}_1(t_\zeta)}_\xi$ for $\xi \leq \zeta$. Note that the sequence $\langle \mathcal{I}^{\mathrm{rk}_1(t_\zeta)}_\xi : \xi \leq \zeta \rangle$ is in N, so we may choose the respective $r_\zeta \geq r^*_\zeta$ in N.]

Now we may define $\bar{q} = \langle q_t : t \in T_1 \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$. If $t \in T_1$ is such that $j(t) < \mathrm{rk}_1(t)$ then q_t is defined from $q_{t{\restriction}j(t)}$ and $\bar{p}$ by demand **(c)(iv)**. So we have to define q_t for these $t \in T_1$ such that $(\forall i \in \mathrm{dom}(t))(t_i(\alpha_0) \neq *)$ (and $t \neq \langle\rangle$) only. So let $t \in T_1 \setminus \{\langle\rangle\}$ be of this type. Let

$$\mathrm{dom}(q_t) = \bigcup\{\mathrm{dom}(r_\zeta) : \zeta < \lambda \ \&\ t \leq_1 t_\zeta\} \cap \mathrm{rk}_1(t) \subseteq N$$

and by induction on $i \in \mathrm{dom}(q_t)$ we define $q_t(i)$ (a $\mathbb{P}_i$–name for a member of $\hat{\mathbb{Q}}_i$). So suppose that $i \in \mathrm{dom}(q_t)$ and we have defined $q_t{\restriction}i \in \mathbb{P}'_i$ in such a way that

$$(*) \qquad (\forall \zeta < \lambda)(t \leq_1 t_\zeta \quad \Rightarrow \quad r_\zeta{\restriction}i \leq_{\mathbb{P}'_i} q_t{\restriction}i).$$

Note that this demand implies that $q_t{\restriction}i \in \mathbb{P}'_i$ is an upper bound of a generic sequence in $\mathbb{P}_i$ over N (remember the choice of the r_ζ's, and that $i \in N$ and there are unboundedly many $\zeta < \lambda$ such that $\mathrm{rk}_1(t_\zeta) = \gamma$, and all open dense subsets of $\mathbb{P}_i$ from N appear in the list $\langle \mathcal{I}^\gamma_\zeta : \zeta < \lambda\rangle$) and therefore

$$q_t{\restriction}i \Vdash_{\mathbb{P}'_i} \text{"the model } N[\underset{\sim}{G}_{\mathbb{P}'_i}] \text{ is } (\lambda, \mathcal{S}_0)\text{–good"}$$

(remember 0.19). Look at the sequence $\langle r_\zeta(i) : t \leq_1 t_\zeta \ \&\ i \in \mathrm{dom}(r_\zeta) \cap \mathrm{rk}_1(t)\rangle$. By the last two demands of the choice of the r_ζ's we have

$$q_t{\restriction}i \Vdash_{\mathbb{P}_i} \text{"}\langle r_\zeta(i) : t \leq_1 t_\zeta \ \&\ i \in \mathrm{dom}(r_\zeta) \cap \mathrm{rk}_1(t)\rangle \text{ is an increasing } \underset{\sim}{\mathbb{Q}}_i\text{–generic sequence over } N[\underset{\sim}{G}_{\mathbb{P}_i}]\text{".}$$

Consequently we may use the fact that $\underset{\sim}{\mathbb{Q}}_i$ is (a name for) a strongly $\mathcal{S}_0$–complete forcing notion and $\underset{\sim}{x}_i \in N$, and we take $\underset{\sim}{q}_t(i)$ to be the $<^*_\chi$–first name for the least upper bound of this sequence in $\hat{\underset{\sim}{\mathbb{Q}}}_i$. So we can prove by induction on $\mathrm{rk}_2(t)$ that $(*)$ holds.

This completes the definition of $\bar{q}$. Checking that it is as required is straightforward. □

Theorem A.3.7. *Assume $\lambda^{<\lambda} = \lambda$, $\kappa = \lambda^+ = 2^\lambda \leq \mu^*$. Suppose that $\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$, $\hat{\mathcal{S}}_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$ and D is a function such that* $\mathrm{dom}(D) = \hat{\mathcal{S}}_1$ *and for every $\bar{a} \in \hat{\mathcal{S}}_1$*

$$D(\bar{a}) = D_{\bar{a}} \quad \textit{is a normal filter on } \lambda.$$

Further suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma\rangle$ is a $(<\kappa)$–support iteration such that for each $i < \gamma$

$$\Vdash_{\mathbb{P}_i} \text{"}\underset{\sim}{\mathbb{Q}}_i \textit{ is really } (\mathcal{S}_0, \hat{\mathcal{S}}_1, D)\textit{–complete with witness } \underset{\sim}{x}_i \textit{ for the } \text{most} \textit{ in A.1.16(3)}(\gamma)\text{".}$$

Then:

(a) *the forcing notion $\mathbb{P}_\gamma$ is $(<\lambda)$–complete and strongly $\mathcal{S}_0$–complete,*
(b) *if a sequence $\bar{N} = \langle N_i : i \leq \lambda\rangle$ is $(\lambda, \kappa, \hat{\mathcal{S}}_1, D, \mathbb{P}_\gamma)$–suitable (see A.1.16(1)) and $p \in \mathbb{P}_\gamma \cap N_0$, $\langle \underset{\sim}{x}_i : i < \gamma\rangle$, $\langle \mathcal{S}_0, \hat{\mathcal{S}}_1, D\rangle \in N_0$, then there is an $(N_\lambda, \mathbb{P}_\gamma)$–generic condition $q \in \mathbb{P}_\gamma$ stronger than p,*
(c) *the forcing notion $\mathbb{P}_\gamma$ is really $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$–complete.*

Proof. (a) It is a consequence of A.1.3 and A.1.13.

(b) Plainly, we may assume $\gamma \geq \lambda$ and $\mathbb{Q} \in N_0$. Let $(X, \bar{a})$ be a suitable basis for $\bar{N}$, so $\bar{a} \in \hat{\mathcal{S}}_1$, $X \in D_{\bar{a}}$ and

$$(\forall i \in X)((N_{i+1})^{<\lambda} \subseteq N_{i+1} \ \ \&\ \ N_{i+1} \cap \mu^* = a_{i+1}).$$

We may assume that all members of X are limit ordinals. Let $w_\lambda = N_\lambda \cap \gamma$ (so $\|w_\lambda\| = \lambda$). Choose an increasing continuous sequence $\langle w_\alpha : \alpha < \lambda\rangle$ such that $\bigcup\limits_{\alpha<\lambda} w_\alpha = w_\lambda$ and for each $\alpha < \lambda$

$$\|w_\alpha\| < \lambda, \quad w_\alpha \subseteq N_\alpha \cap \gamma, \quad 0 \in w_\alpha, \quad \text{and} \quad \text{if } \alpha \text{ is limit then } w_\alpha = w_{\alpha+1}$$

(so then $\langle w_\beta : \beta \leq \alpha\rangle \in N_{\alpha+1}$).

Now, by induction on $\alpha \leq \lambda$ we define a legal continuous sequence of standard γ–trees $\langle \mathcal{T}_\alpha : \alpha \leq \lambda\rangle$ for $\langle w_\alpha : \alpha \leq \lambda\rangle$ and a sequence $\langle \bar{p}^\alpha : \alpha < \lambda\rangle$ such that

$\bar{p}^\beta = \langle p^\beta_t : t \in T_\beta \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ and $\bar{p}^\beta \leq_{\mathrm{proj}^{\mathcal{T}_\alpha}_{\mathcal{T}_\beta}} \bar{p}^\alpha$ for each $\beta < \alpha < \lambda$ and $\mathcal{T}_\alpha, \bar{p}^\alpha \in N_{\alpha+1}$.

At stage $\alpha = 0$ of the construction:
T_0 consists of all sequences $t = \langle t_i : i \in \mathrm{dom}(t) \rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w_0 (not necessarily proper) and for each $i \in \mathrm{dom}(t)$, t_i is a sequence of length 0 (i.e., $\langle\rangle$),
$\mathrm{rk}_0(t) = \min(w_0 \cup \{\gamma\} \setminus \mathrm{dom}(t))$ and $<_0$ is the extension relation;
for each $t \in T_0$ we let $p^0_t = p {\restriction} \mathrm{rk}_0(t)$ and finally $\bar{p}^0 = \langle p^0_t : t \in T_0 \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$.
[Note that $\mathcal{T}_0 = (T_0, <_0, \mathrm{rk}_0) \in N_0$ is a standard $(w_0, 0)^\gamma$–tree, $\bar{p}^0 \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_0$.]

At stage $\alpha = \alpha_0 + 1$ of the construction:
We have defined a standard $(w_{\alpha_0}, \alpha_0)^\gamma$–tree $\mathcal{T}_{\alpha_0} \in N_{\alpha_0+1}$ and $\bar{p}^{\alpha_0} = \langle p^{\alpha_0}_t : t \in \mathcal{T}_{\alpha_0} \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_{\alpha_0+1}$. Now we consider two cases.
If $\alpha_0 \in X$ (so N_{α_0+1} is $(\lambda, \mathcal{S}_0)$–good), then we apply the procedure of A.3.6 inside N_{α_0+2} to $\mathcal{T}_{\alpha_0}$, $\bar{p}^{\alpha_0}$, $(w_{\alpha_0+1}, \alpha_0+1)$ and N_{α_0+1} (in place of $\mathcal{T}_0$, $\bar{p}$, (w_1, α_1) and N there) and we get a standard $(w_{\alpha_0+1}, \alpha_0+1)^\gamma$–tree $\mathcal{T}_{\alpha_0} \in N_{\alpha_0+2}$ and $\bar{p}^{\alpha_0+1} = \langle p^{\alpha_0+1}_t : t \in T_{\alpha_0+1} \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_{\alpha_0+2}$ satisfying the demands A.3.6(ε) and A.3.6(a)–(c).
If $\alpha_0 \notin X$, then we define $\mathcal{T}_{\alpha_0+1}$ as above but we cannot put any new genericity requirements on $\bar{p}^{\alpha_0+1}$, so we just let $p^{\alpha_0+1}_t = p^{\alpha_0}_{t'} {\restriction} \mathrm{rk}_{\alpha_0+1}(t)$ where $t' = \mathrm{proj}^{\mathcal{T}_{\alpha_0+1}}_{\mathcal{T}_{\alpha_0}}(t)$.
[Note that in both cases $\mathcal{T}_{\alpha_0+1} \in N_{\alpha_0+2}$ is a standard $(w_{\alpha_0+1}, \alpha_0+1)^\gamma$–tree, projection of $\mathcal{T}_{\alpha_0+1}$ onto (w_{α_0}, α_0) is $\mathcal{T}_{\alpha_0}$, $\bar{p}^{\alpha_0+1} \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_{\alpha_0+2}$ and $\bar{p}^{\alpha_0} \leq_{\mathrm{proj}^{\mathcal{T}_{\alpha_0+1}}_{\mathcal{T}_{\alpha_0}}} \bar{p}^{\alpha_0+1}$.]

At limit stage α of the construction:
We let $\mathcal{T}_\alpha = \overleftarrow{\lim}(\langle \mathcal{T}_\beta : \beta < \alpha \rangle) \in N_{\alpha+1}$ and we choose $\bar{p}^\alpha = \langle p^\alpha_t : t \in T_\alpha \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_{\alpha+1}$ applying A.3.5 in $N_{\alpha+1}$.
[Note that the corresponding inductive assumptions hold true.]

After the construction is carried out we may let $\mathcal{T}_\lambda = \overleftarrow{\lim}(\langle \mathcal{T}_\alpha : \alpha < \lambda \rangle)$. Then $\mathcal{T}_\lambda$ is a standard $(w_\lambda, \lambda)^\gamma$–tree, but no longer we have $\|T_\lambda\| \leq \lambda$.

Now, by induction on $\alpha \in w_\lambda \cup \{\gamma\}$ we choose conditions q_α and $\mathbb{P}_\alpha$–names $\underset{\sim}{X}_\alpha$, $\underset{\sim}{Y}_\alpha$ and $\underset{\sim}{t}_\alpha$ such that

(a) $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{t}_\alpha \in T_\lambda$ & $\mathrm{rk}_\lambda(\underset{\sim}{t}_\alpha) = \alpha$",
(b) $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{t}_\beta = \underset{\sim}{t}_\alpha {\restriction} \beta$" for $\beta < \alpha$,
(c) $q_\alpha \in \mathbb{P}_\alpha$, $\mathrm{dom}(q_\alpha) = w_\lambda \cap \alpha$,
(d) if $\beta < \alpha$ then $q_\beta = q_\alpha {\restriction} \beta$,
(e) $q_\alpha \Vdash_{\mathbb{P}_\alpha}$ "$p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\alpha)} {\restriction} \alpha \in \underset{\sim}{G}_{\mathbb{P}_\alpha}$" for each $i < \lambda$,
(f) for each $\beta < \alpha$

$q_\alpha \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{X}_\beta = \{ i < \lambda : (\underset{\sim}{t}_{\beta+1})_\beta(i) \neq * \} \in D_{\bar{a}}$ and the sequence $\langle \langle (i, (\underset{\sim}{t}_{\beta+1})_\beta(i)), p^{i+1}_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_{i+1}}(\underset{\sim}{t}_{\beta+1})}(\beta) \rangle : i < \lambda \ \&\ i \in \underset{\sim}{X}_\beta \rangle$
is a result of a play of the game
$\mathcal{G}^\heartsuit_{\langle N_i[G_\beta] : i \leq \lambda \rangle, D, \underset{\sim}{Y}_\beta, \bar{a}}(\underset{\sim}{\mathbb{Q}}_\beta, p^{i_0}_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_{i_0}}(\underset{\sim}{t}_{\beta+1})}(\beta))$,
[where $i_0 < \lambda$ is the first such that $\beta \in w_{i_0}$],
won by player COM",

(g) the condition q_α forces (in $\mathbb{P}_\alpha$) that
"the sequence $\langle N_i[\underset{\sim}{G}_{\mathbb{P}_\alpha}] : i \leq \lambda\rangle$ is $(\lambda,\kappa,\hat{\mathcal{S}}_1,D,\underset{\sim}{\mathbb{Q}}_\alpha)$–suitable and $\underset{\sim}{Y}_\alpha \in D_{\bar{a}}$ is such that $\underset{\sim}{Y}_\alpha \subseteq X$ and for every $i \in \underset{\sim}{Y}_\alpha$ we have

$$(N_{i+1}[\underset{\sim}{G}_{\mathbb{P}_\alpha}])^{<\lambda} \subseteq N_{i+1}[\underset{\sim}{G}_{\mathbb{P}_\alpha}] \quad \text{and} \quad N_{i+1}[\underset{\sim}{G}_{\mathbb{P}_\alpha}] \cap \mathbf{V} = N_{i+1}$$

and $i \in \underset{\sim}{X}_\xi$ for all $\xi \in \alpha \cap w_i$ (hence $N_\lambda[G_{\mathbb{P}_\alpha}] \cap \mathbf{V} = N_\lambda$)"

CASE 1: $\alpha = 0$.
We do not have much choice here: we let $q_0 = \emptyset$, $\underset{\sim}{t}_0 = \langle\rangle \in \mathcal{T}_\lambda$ and $\underset{\sim}{Y}_0 = X$. Note that clauses (a)–(e) and (g) are trivially satisfied (for (g) remember that $(\bar{a}, X)$ is a suitable basis for $\bar{N}$) and clause (f) is not relevant.

CASE 2: $\alpha = \beta + 1$.
Arriving at this stage we have defined $q_\beta, \underset{\sim}{t}_\beta, \underset{\sim}{Y}_\beta$ and $\underset{\sim}{X}_\xi$ for $\xi < \beta$, and we want to choose $q_{\beta+1}, \underset{\sim}{t}_{\beta+1}, \underset{\sim}{Y}_{\beta+1}$ and $\underset{\sim}{X}_\beta$.

Suppose that $G_\beta \subseteq \mathbb{P}_\beta$ is a generic filter over $\mathbf{V}$ such that $q_\beta \in G_\beta$. Then (by clause (g) at stage β) we have

$$\begin{aligned}\mathbf{V}[G_\beta] \models \quad & \text{"the sequence } \langle N_i[G_\beta] : i \leq \lambda\rangle \text{ is } (\lambda,\kappa,\hat{\mathcal{S}}_1,D,\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta})\text{–suitable}\\ & \text{and } (\bar{a}, \underset{\sim}{Y}_\beta^{G_\beta}) \text{ is a suitable base for it}\\ & \text{and } (\forall i \in \underset{\sim}{Y}_\beta^{G_\beta})(\forall \xi \in \beta \cap w_i)(i \in \underset{\sim}{X}_\xi^{G_\beta})\text{"}\end{aligned}$$

Let $i_0 = \min\{j < \lambda : \beta \in w_j\}$. We know that the player INC does not have any winning strategy in the game

$$\mathcal{G}^{\heartsuit}_{\langle N_i[G_\beta]: i\leq\lambda\rangle, D, \underset{\sim}{Y}_\beta^{G_\beta}\setminus(i_0+1), \bar{a}}\Big(\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}, \big(p^{i_0}_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_{i_0}}(\underset{\sim}{t}_\beta^{G_\beta})}(\beta)\big)^{G_\beta}\Big).$$

Now, using the interpretation of the game presented in A.1.18, we describe a strategy for player INC in this game.

The strategy is: *during a play* COM *constructs a sequence* $\bar{s} = \langle s(i) : i < \lambda\rangle$ *of elements of* $\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta} \cup \{*\}$*, those are his moves; let*

$$r_i \stackrel{\mathrm{def}}{=} \mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta^{G_\beta})^\frown \bar{s}{\restriction} i \in T_i$$

(more pedantically: $\mathrm{Dom}(r_i) = w_i \cap \alpha = \mathrm{Dom}(\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta^{G_\beta})) \cup \{\beta\}$, $r_i \in T_i$, $\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta^{G_\beta}) \subseteq r_i$, $r_i(\beta) = \bar{s}{\restriction} i$*),*
and at the stage $i < \lambda$ *of the game* INC *answers with* $\big(p^{i+1}_{r_{i+1}}(\beta)\big)^{G_\beta}$.

We have to argue that the strategy described above is a legal one, i.e., that it always tells INC to play legal moves (assuming that COM plays according to the rules of the game). For this we show by induction on $i < \lambda$ that really $r_i \in T_i$ and that if $s(i) \neq *$, then $\big(p^{i+1}_{r_{i+1}}(\beta)\big)^{G_\beta} \in N_{i+2}[G_\beta] \cap \hat{\underset{\sim}{\mathbb{Q}}}_\beta^{G_\beta}$ is the least upper bound of a $\underset{\sim}{\mathbb{Q}}_\beta^{G_\beta}$–generic filter over $N_{i+1}[G_\beta]$ (to which $s(i)$ belongs) and if $s(i) = *$, then $\big(p^{i+1}_{r_{i+1}}\big)^{G_\beta}$ is the least upper bound of conditions played by INC so far.

First note that $s(i) = *$ for all $i \leq i_0$ and therefore $r_{i_0+1} \in T_{i_0+1}$ (just look at the successor stage of the construction of the $\mathcal{T}_\alpha$'s; remember that $\mathrm{dom}(\underset{\sim}{t}_\beta^{G_\beta}) = w_\lambda \cap \beta$, so adding $*$'s at level β is allowed by A.3.6(ε)). Note that $p^{i_0+1}_{r_{i_0+1}}(\beta) = p^{i_0}_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_{i_0}}(\underset{\sim}{t}_\beta^{G_\beta})}(\beta)$ (remember A.3.6(c)(iv)).

If $i < \lambda$ is a limit ordinal above i_0, and we already know that $r_j \in T_j$ for each $j < i$, then $r_i \in \mathcal{T}_i = \overleftarrow{\lim}(\langle \mathcal{T}_j : j < i\rangle)$ as clearly

$$j_0 < j_1 < i \quad \Rightarrow \quad \mathrm{proj}^{\mathcal{T}_{j_1}}_{\mathcal{T}_{j_0}}(r_{j_1}) = r_{j_0}.$$

Note that, by the limit stage of the construction of the $\mathcal{T}_\alpha$'s and A.3.5(3) (actually by the construction there), the condition $p^i_{r_i}(\beta)$ is the least upper bound of $\langle p^j_{r_j}(\beta) : j < i\rangle$ in $\hat{\mathbb{Q}}^{G_\beta}_\beta$.

Suppose now that we have $r_i \in T_i$, $i_0 < i < \lambda$ and the player COM plays $s(i)$. If $s(i) = *$ then easily $r_{i+1} \in T_{i+1}$ as adding stars at "top levels" does not make any problems (compare the case of i_0). Moreover, as there, we have then

$$p^{i+1}_{r_{i+1}}(\beta) = p^i_{\mathrm{proj}^{\mathcal{T}_{i+1}}_{\mathcal{T}_i}(r_{i+1})}(\beta) = p^i_{r_i}(\beta).$$

If $s(i) \neq *$, then $s(i) \in N_{i+1}[G_\beta] \cap \mathbb{Q}^{G_\beta}_\beta$ is a condition stronger than all conditions played by INC so far, and thus it is stronger than $p^i_{r_i}(\beta)$. Moreover, in this case we necessarily have $i \in Y^{G_\beta}_\beta$, so i is limit and therefore $w_i = w_{i+1}$. Hence $(\forall \xi \in w_{i+1} \cap \beta)(i \in X^{G_\beta}_\xi)$. By clause (f) for q_β we conclude that $(\forall \xi \in w_{i+1})((t^{G_\beta}_\beta)_\xi(i) \neq *)$. Therefore, if we look at the way $\mathcal{T}_{i+1}$ was constructed, we see that there is no collision in adding $s(i)$ at the top (i.e., it is allowed by A.3.6(ε)). Thus $r_{i+1} \in T_{i+1}$ and by A.3.6(c)(ii) we know that $p^{i+1}_{r_{i+1}}(\beta) \in N_{i+2}[G_\beta] \cap \hat{\mathbb{Q}}^{G_\beta}_\beta$ is the least upper bound of a $\mathbb{Q}^{G_\beta}_\beta$–generic sequence over $N_{i+1}[G_\beta]$ to which $s(i)$ belongs (the last is due to A.3.6(c)(iii)).

Thus we have proved that the strategy presented above is a legal strategy for INC. It cannot be the winning one, so there is a play $\bar{s} = \langle s(i) : i < \lambda\rangle$ (we give the moves of COM only) in which COM wins. Let $t_\alpha = t^{G_\beta}_\beta{}^\frown \bar{s}$; pedantically, $\mathrm{Dom}(t_\alpha) = \mathrm{Dom}(t^{G_\beta}_\beta) \cup \{\beta\}$, $t^{G_\beta}_\beta \subseteq t_\alpha$, $t_\alpha(\beta) = \bar{s}$. We have actually proved that $t_\alpha \in \mathcal{T}_\lambda = \overleftarrow{\lim}(\langle \mathcal{T}_i : i < \lambda\rangle)$. It should be clear that $\mathrm{rk}_\lambda(t_\alpha) = \alpha$ and $t^{G_\beta}_\beta = t_\alpha{\restriction}\beta$. Further let $q_\alpha(\beta) \in \mathbb{Q}^{G_\beta}_\beta$ be any upper bound of $\bar{s}$ in $\mathbb{Q}^{G_\beta}_\beta$ (there is one as COM wins) and X_β be the set $\{i < \lambda : s(i) \neq *\} \in D_{\bar{a}}$. Note that then $q_\alpha(\beta)$ is stronger than all $p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(t_\alpha)}(\beta)$ (as these are answers of the player INC; see above). Lastly, if we let $Y_\alpha = X_\beta$ then we have

$$\begin{aligned} q_\alpha(\beta) \Vdash_{\mathbb{Q}^{G_\beta}_\beta} \quad & \text{“the sequence } \langle N_i[G_\beta][G_{\mathbb{Q}^{G_\beta}_\beta}] : i \leq \lambda\rangle \text{ is suitable and } (\bar{a}, Y_\alpha) \\ & \text{is a suitable base for it and } (\forall i \in Y_\alpha)(\forall \xi \in \alpha \cap w_i)(i \in X^{G_\beta}_\xi)\text{”} \end{aligned}$$

(compare the arguments in the proof of A.1.13). This is everything we need: as G_β was any generic filter containing q_β we may take names t_α, X_β, Y_α for the objects defined above and the name for $q_\alpha(\beta)$ and conclude that $q_\beta{}^\frown q_\alpha(\beta)$ forces that they are as required.

Case 3: α is a limit ordinal.
Arriving at this stage we have defined $q_\beta, t_\beta, Y_\beta$ and X_β for $\beta \in \alpha \cap w_\lambda$ and we are going to define q_α, t_α and Y_α. The first two objects to be defined are determined by clauses (a)–(d). The only possible problem that may appear here is that we want t_α to be (a name for) an element of T_λ and thus of $\mathbf{V}$. But by A.3.7(a) and A.1.10 + A.1.11 we know that the forcing with $\mathbb{P}_\alpha$ adds no new sequences

of length $<\kappa$ of elements of $\mathbf{V}$ (remember $\kappa=\lambda^+$). Therefore the sequence $\langle \underset{\sim}{t}_\beta:\beta\in w_\lambda\cap\alpha\rangle$ is a $\mathbb{P}_\alpha$–name for a sequence *from* $\mathbf{V}$ and its limit $\underset{\sim}{t}_\alpha$ is forced to be in $\mathcal{T}_\lambda$. Now we immediately get that q_α, $\underset{\sim}{t}_\alpha$ satisfy demands (a)–(f) (for (e) note that $\mathrm{dom}(p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\alpha)})\subseteq w_\lambda$ and

($\boxtimes$) for each $\beta\in\alpha\cap w_\lambda$ and $i<\lambda$ we have $\mathrm{dom}(p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta)})\subseteq w_\lambda$ and

$$p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\alpha)}\restriction \mathrm{rk}_i(\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta))=p^i_{\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta)}\qquad\text{and}\qquad \mathrm{rk}_i(\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_i}(\underset{\sim}{t}_\beta))\geq\beta,$$

hence we may use the clause (e) from stages $\beta<\alpha$). Finally we let

$$\underset{\sim}{Y}_\alpha \stackrel{\mathrm{def}}{=} \{i<\lambda: i \text{ is limit and } (\forall\xi\in w_i\cap\alpha)(i\in\underset{\sim}{X}_\xi) \text{ and } \alpha\in w_i\}.$$

We have to check that the demand (g) is satisfied. Suppose that $G_\alpha\subseteq\mathbb{P}_\alpha$ is a generic filter over $\mathbf{V}$ containing q_α. The sequence $\langle \underset{\sim}{X}^{G_\alpha}_\xi:\xi\in w_\lambda\cap\alpha\rangle$ is a sequence of length $<\kappa$ of elements of $\mathbf{V}$, and the forcing with $\mathbb{P}_\alpha$ adds no new such sequences. Consequently

$$\langle \underset{\sim}{X}^{G_\alpha}_\xi:\xi\in w_\lambda\cap\alpha\rangle\in\mathbf{V}.$$

If for $j<\lambda$ we let $Z_j=\bigcap\limits_{\xi\in w_j\cap\alpha}\underset{\sim}{X}^{G_\alpha}_\xi$ we will have

$$\langle Z_j:j<\lambda\rangle\in\mathbf{V},\qquad\text{and}\qquad (\forall j<\lambda)(Z_j\in D_{\bar a})$$

(as the filter $D_{\bar a}$ is λ–complete) and therefore (by the normality of $D_{\bar a}$)

$$\underset{\sim}{Y}^{G_\alpha}_\alpha\supseteq\mathop{\triangle}_{j<\lambda}Z_j=\{i<\lambda: i\text{ is limit and }(\forall j<i)(i\in Z_j)\}\in D_{\bar a}.$$

Next note that $(\bar a,\underset{\sim}{Y}^{G_\alpha}_\alpha)$ is a suitable basis for the sequence $\langle N_i[G_\alpha]:i\leq\lambda\rangle$. Why? Suppose that $i\in\underset{\sim}{Y}^{G_\alpha}_\alpha$ and let $t=\mathrm{proj}^{\mathcal{T}_\lambda}_{\mathcal{T}_{i+1}}(\underset{\sim}{t}^{G_\alpha}_\alpha)$. By the choice of the w_i's we know that $w_{i+1}=w_i$ (remember i is limit). Since $\alpha\in w_i$ we have $\mathrm{rk}_{i+1}(t)=\alpha$ and since $i\in\bigcap\limits_{\xi\in w_i\cap\alpha}\underset{\sim}{X}^{G_\alpha}_\xi$ we have $t_\xi(i)\neq *$ for each $\xi\in w_i\cap\alpha=w_{i+1}\cap\alpha$. So look now at the way we defined $\bar p^{i+1}$: we were at the case when $\bar p^{i+1}_t$ was given by A.3.6(c)(ii). In particular, the condition $p^{i+1}_t\in\mathbb{P}'_{\mathrm{rk}_{i+1}(t)}\cap N_{i+2}$ generates a $\mathbb{P}_{\mathrm{rk}_{i+1}(t)}$–generic filter over N_{i+1}. We know already that $q_\alpha,\underset{\sim}{t}_\alpha$ satisfy (e) (or use just ($\boxtimes$)) and therefore $p^{i+1}_t\in G_\alpha$. This is enough to conclude that

$$N_{i+1}[G_\alpha]\prec(\mathcal{H}(\chi),\in,<^*_\chi),\quad (N_{i+1}[G_\alpha])^{<\lambda}\subseteq N_{i+1}[G_\alpha],\quad N_{i+1}[G_\alpha]\cap\mathbf{V}=N_{i+1},$$

(like in A.1.13) and therefore to finish the construction.

To finish the proof of this case of the theorem note that our demands on conditions q_α imply that each of them is $(N_\lambda,\mathbb{P}_\alpha)$–generic, so in particular q_γ is as required.

(c) The proof is similar to that of case (b) (and is not seriously used). $\square$

Theorem A.3.8. *Assume $\lambda^{<\lambda}=\lambda$, $\kappa=\lambda^+=2^\lambda\leq\mu^*$. Suppose that $\mathcal{S}_0\in(\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$, $\hat{\mathcal{S}}_1\in(\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$. Let $\bar{\mathbb{Q}}=\langle\mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i:i<\gamma\rangle$ be a $(<\kappa)$–support iteration such that for each $i<\gamma$*

$$\Vdash_{\mathbb{P}_i}\text{“}\underset{\sim}{\mathbb{Q}}_i\textit{ is basically }(\mathcal{S}_0,\hat{\mathcal{S}}_1)\textit{–complete”}.$$

Then the forcing notion $\mathbb{P}_\gamma$ is basically $(\mathcal{S}_0,\hat{\mathcal{S}}_1)$–complete.

Proof. Similar to the proof of A.3.7 (but easier) and not used in our examples, so we do not give details. □

A.4. The Axiom

Definition A.4.1. Suppose that $\lambda^{<\lambda} = \lambda$, $\kappa = \lambda^+ = 2^\lambda \le \mu^*$ and θ is a regular cardinal. Let $\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$, $\hat{\mathcal{S}}_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$ and let D be a function from $\hat{\mathcal{S}}_1$ such that each $D_{\bar a}$ is a normal filter on λ. Let $\mathrm{Ax}^\kappa_\theta(\mathcal{S}_0, \hat{\mathcal{S}}_1)$, *the forcing axiom for* $(\mathcal{S}_0, \hat{\mathcal{S}}_1)$ *and* θ, be the following sentence:

> If $\mathbb{Q}$ is a really $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$–complete forcing notion of size $\le \kappa$ and $\langle \mathcal{I}_i : i < i^* < \theta \rangle$ is a sequence of dense subsets of $\mathbb{Q}$, then there exist a directed set $H \subseteq \mathbb{Q}$ such that
> $$(\forall i < i^*)(H \cap \mathcal{I}_i \neq \emptyset).$$

Theorem A.4.2. *Assume that* $\lambda, \kappa = \mu^*, \theta$ *and* $(\mathcal{S}_0, \hat{\mathcal{S}}_1, D)$ *are as in A.4.1 and*

$$\kappa < \theta = \mathrm{cf}(\theta) \le \mu = \mu^\kappa$$

(e.g.,

- ($\circledast$) $S_0 \subseteq S^\lambda_\kappa$, $S_1 = S^\lambda_\kappa \setminus S_0$ *are stationary subsets of* κ, $\mathcal{S}_0 = S_0$, $\hat{\mathcal{S}}_1 = \{\bar a : \bar a$ *is an increasing continuous sequence of ordinals,* $a_0 \in S_0$, $a_{i+1} \in S_0$, $a_\lambda \in S_1$ $\}$*).*

Then there is a forcing notion $\mathbb{P}$ *of cardinality* μ *such that*

- (α) $\mathbb{P}$ *satisfies the* κ^+*–cc,*
- (β) $\Vdash_{\mathbb{P}}$ "$\mathcal{S}_0 \in (\mathfrak{D}_{<\kappa,<\lambda}(\mu^*))^+$ & $\hat{\mathcal{S}}_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$ ", *and even more:*
- (β^+) *if* $\hat{\mathcal{S}}^*_1 \subseteq \hat{\mathcal{S}}_1$ *is such that* $\hat{\mathcal{S}}^*_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$, *then* $\Vdash_{\mathbb{P}} \hat{\mathcal{S}}^*_1 \in (\mathfrak{D}^\lambda_{<\kappa,<\lambda}(\mu^*)[\mathcal{S}_0])^+$,
- (γ) $\Vdash_{\mathbb{P}} \mathrm{Ax}^\kappa_\theta(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$,
- (δ) *if* ($\circledast$), *then all stationary subsets of* κ *are preserved.*

Proof. It is parallel to B.8.2 which is later done elaborately. □

Case: B

While **Case D** (see the introduction; κ inaccessible, S has stationary many inaccessible members) may be treated similarly to **Case A**, we need to refine our machinery to deal with **Case B**. Our prototype here is κ *is the first strongly inaccessible cardinal*, however the tools developed in this part will be applicable to cases **A**, **C**, **D** too (and other strong inaccessibles in **Case B**, of course).

Our Assumptions 2. κ is a strongly inaccessible cardinal and $\mu^* \ge \kappa$ is a regular cardinal.
These assumptions will be kept in the present part (unless otherwise stated) and we may forget to remind the reader of them.

There are two main difficulties which one meets when dealing with the present case. First problem, a more general one, is that $(<\mu)$–completeness is not reasonable even for $\mu = \aleph_1$. Why? As we would like to force the Uniformization Property for $\langle S_\delta : \delta \in S\rangle$, where $S \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ is stationary not reflecting. The second problem is related to closure properties of models we consider. In **Case A**, when $\kappa = \lambda^+$, the demand $N^{<\lambda} \subseteq N$ was reasonable. If κ is Mahlo, $\|N\| = N \cap \kappa$ is an inaccessible cardinal $< \kappa$, then the demand $N^{<N\cap\kappa} \subseteq N$ is reasonable too. However, if κ is the first inaccessible this does not work. (Note that these models are parallel of *countable* $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ of the case $\kappa = \aleph_1$.) To handle these problems we will use exclusively sequences $\bar{N} = \langle N_i : i \leq \alpha\rangle$ of models and all action will take place at limit stages only. For example, we will have completeness for $\bar{N} = \langle N_i : i \leq \omega\rangle$ by looking at N_ω, BUT the equivalence class $\bar{N}/\approx$ will be important too, where for two sequences $\bar{N}$, $\bar{N}'$ of length ω we write $\bar{N} \approx \bar{N}'$ if

$$(\forall n \in \omega)(\exists m \in \omega)(N_n \subseteq N'_m) \quad \text{and} \quad (\forall n \in \omega)(\exists m \in \omega)(N'_n \subseteq N_m)$$

B.5. More on completeness of forcing notions

In this section we introduce more notions of completeness of forcing notions. In some sense we will generalize and develop the notions introduced in section A.1.

Definition B.5.1. (1) Let $\bar{N} = \langle N_i : i \leq \alpha\rangle$ be a sequence of models and $\bar{a} = \langle a_i : i \leq \alpha\rangle$ be a sequence of elements of $[\mu^*]^{<\kappa}$. We say that $\bar{N}$ *obeys* $\bar{a}$ *with an error* $n \in \omega$ if

$$(\forall i < \alpha)(a_i \subseteq N_i \cap \mu^* \subseteq a_{i+n}).$$

When we say $\bar{N}$ *obeys* $\bar{a}$ we mean *with some error* $n \in \omega$.

(2) By $\mathfrak{C}_{<\kappa}(\mu^*)$ we will denote the collection of all sets $\hat{\mathcal{E}}$ such that

$$\hat{\mathcal{E}} \subseteq \big\{\bar{a} = \langle a_i : i \leq \alpha\rangle : \text{ the sequence } \bar{a} \text{ is increasing continuous, } \alpha < \kappa \text{ and } (\forall i \leq \alpha)(a_i \in [\mu^*]^{<\kappa} \ \&\ a_i \cap \kappa \in \kappa)\big\},$$

and for every regular large enough cardinal χ, for every $x \in \mathcal{H}(\chi)$ and a regular cardinal $\theta < \kappa$ there are $\bar{N}$ and $\bar{a}$ such that

(a) $\bar{N} = \langle N_i : i \leq \theta\rangle$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $x \in N_0$ and

$$(\forall i < \theta)(\bar{N}\restriction(i+1) \in N_{i+1} \quad \& \quad \|N_i\| < \kappa),$$

(b) $\bar{a} = \langle a_i : i \leq \theta\rangle \in \hat{\mathcal{E}}$,

(c) $\bar{N}$ obeys $\bar{a}$.

(3) If $\bar{a} \in \hat{\mathcal{E}}$, $\bar{N}$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $(\forall i+1 < \ell g(\bar{N}))(\bar{N}\restriction(i+1) \in N_{i+1} \ \&\ \|N_i\| < \kappa)$ and $\bar{N}$ obeys $\bar{a}$ (with error n, respectively), then we say that $(\bar{N}, \bar{a})$ is *an* $\hat{\mathcal{E}}$*–complementary pair* (*an* $(\hat{\mathcal{E}}, n)$*–complementary pair*, respectively).

(4) We say that a family $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is *closed* if for every sequence $\bar{a} = \langle a_i : i \leq \alpha\rangle \in \hat{\mathcal{E}}$ and ordinals β, γ such that $\beta + \gamma \leq \alpha$ we have

$$\langle a_{\beta+i} : i \leq \gamma\rangle \in \hat{\mathcal{E}}$$

(or, in other words, $\hat{\mathcal{E}}$ is closed under both initial and end segments).

Remark B.5.2. (1) Definition B.5.1 is from [14, §1].

(2) The exact value of the error n in B.5.1(2) is not important at all, we may consider here several other variants as well.
(3) Note that $N_i, \|N_i\| \in N_{i+1}$. Sometimes we may add to B.5.1(1) a requirement that $2^{\|N_i\|} \subseteq a_{i+n}$ (saying then that $\bar{N}$ *strongly obeys* $\bar{a}$). Note that this naturally occurs for strongly inaccessible κ, as we demand that $\bar{a} \in \hat{\mathcal{E}} \quad \Rightarrow \quad a_i \cap \kappa \in \kappa$. So then $2^{\|N_i\|} \in a_{i+n}$, but $a_{i+n} \cap \kappa \in \kappa$ so we have $2^{\|N_i\|} \subseteq a_{i+n}$.

In this situation, *if* $\chi_1 < \chi$ are large enough, $\chi_1 \in N_0$ and for non-limit i, N_i' is the closure of $N_i \cap \mathcal{H}(\chi_1)$ under Skolem functions and sequences of length $\leq \|N_i\|$, and for limit i, $N_i' = N_i \cap \mathcal{H}(\chi_1)$ *then* the sequence $\langle N_i' : i \leq \alpha\rangle$ will have closure properties and will obey $\bar{a}$ (as $N_i \in N_{i+1}$, $\mathcal{H}(\chi_1) \in N_{i+1}$ imply $N_i' \in N_{i+1}$ and so $N_i' \subseteq N_{i+n}$).
(4) The presence of "regular $\theta < \kappa$" in B.5.1(2) is not accidental; it will be of special interest when κ is a successor of a singular strong limit cardinal, as then $\theta = \mathrm{cf}(\theta) < \kappa = \mu^+$ implies $\theta < \mu$.

Definition B.5.3. Let $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ and let $\mathbb{Q}$ be a forcing notion.

(1) Let $\bar{N} = \langle N_i : i \leq \delta\rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$, $\mathbb{Q} \in N_0$ and $\bar{p} = \langle p_i : i < \delta\rangle$ be an increasing sequence of conditions from $\mathbb{Q} \cap N_\delta$, $n \in \omega$. We say that $\bar{p}$ *is* $(\bar{N}, \mathbb{Q})^n$*–generic* if for each $i < \delta$

$$\bar{p}\restriction(i+1) \in N_{i+1} \quad \text{and} \quad p_{i+n} \in \bigcap\{\mathcal{I} \in N_i : \mathcal{I} \text{ is an open dense subset of } \mathbb{Q}\}.$$

When we say that $\bar{p}$ *is* $(\bar{N}, \mathbb{Q})^*$*–generic* we mean that it is $(\bar{N}, \mathbb{Q})^n$–generic for some $n \in \omega$. We may say then that $\bar{p}$ is $(\bar{N}, \mathbb{Q})^*$–generic with an error n.
(2) We say that $\mathbb{Q}$ *is complete for* $\hat{\mathcal{E}}$ if for large enough χ, for some $x \in \mathcal{H}(\chi)$ the following condition is satisfied:
$(\circledast)^{\hat{\mathcal{E}}}_x$ *if*
 (a) $(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair (see B.5.1(3)), $\bar{a} \in \hat{\mathcal{E}}$, $\bar{N} = \langle N_i : i \leq \delta\rangle$, $\mathbb{Q}, x \in N_0$, and
 (b) $\bar{p}$ is an increasing $(\bar{N}, \mathbb{Q})^*$–generic sequence,
 then $\bar{p}$ has an upper bound in $\mathbb{Q}$.
(3) We say that a forcing notion $\mathbb{Q}$ is *strongly complete for* $\hat{\mathcal{E}}$ if it is complete for $\hat{\mathcal{E}}$ and does not add sequences of ordinals of length $< \kappa$.

Remark B.5.4. (1) The x in definition B.5.3(2) is the way to say "for most", compare with 0.20.
(2) In the present applications, we will have $\mu^* = \kappa$ and a stationary set $S \subseteq \kappa$ such that

$$\hat{\mathcal{E}}^{\mathrm{c}}_S \stackrel{\mathrm{def}}{=} \big\{\bar{a} : \bar{a} \text{ an increasing sequence of ordinals from } \kappa \setminus S \text{ of length } < \kappa \text{ with the last element from } S\big\}$$

will be in $\mathfrak{C}_{<\kappa}(\mu^*)$. The forcing notions will be complete for $\hat{\mathcal{E}}^{\mathrm{c}}_S$, so the iteration will add no new sequences of length $< \kappa$ (see B.5.6 below). On S the behavior will be more interesting, as there we shall be doing the uniformization. Thus the pair $(\hat{\mathcal{E}}^{\mathrm{c}}_S, S)$ corresponds to the pair $(\mathcal{S}_0, \hat{\mathcal{S}}_1)$ from the previous part (on Case A).

For example, if $C_\delta\subseteq\delta=\sup(C_\delta)$, $\mathrm{otp}(C_\delta)=\mathrm{cf}(\delta)$, $(\forall\delta\in S)(\mathrm{cf}(\delta)<\delta)$ and $h_\delta:C_\delta\longrightarrow 2$ then

$$\mathbb{Q}=\{g:\ \text{for some }\alpha<\kappa,\ \ g:\alpha\longrightarrow 2\ \text{ and } (\forall\delta\in(\alpha+1)\cap S)(\forall\gamma\in C_\delta\text{ large enough})(g(\gamma)=h_\delta(\gamma))\}$$

is such a forcing (but we need that S is not reflecting or $\langle C_\delta:\delta\in S\rangle$ is somewhat free, so that for each $\alpha<\kappa$ there are $g\in\mathbb{Q}$ with $\mathrm{dom}(g)=\alpha$).

(3) If we want to have S reflecting on a stationary set though still "thin", then things are somewhat more complicated, but manageable, see later.

Proposition B.5.5. *Suppose that $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\mathbb{Q}$ is a forcing notion.*

(1) *Assume $(\bar a,\bar N)$ is an $(\hat{\mathcal{E}},n_1)$–complementary pair, $\bar a\in\hat{\mathcal{E}}$, $\bar N=\langle N_i:i\leq\delta\rangle$, $\mathbb{Q}\in N_0$. If $\bar p\subseteq\mathbb{Q}\cap N_\delta$ is $(\bar N,\mathbb{Q})^{n_2}$–generic (see B.5.3(1)) and $q\in\mathbb{Q}$ is an upper bound of $\bar p$ in $\mathbb{Q}$, then*

$$q\Vdash_{\mathbb{Q}}\text{ “}(\langle N_i[\underset{\sim}{G}_{\mathbb{Q}}]:i\leq\delta\rangle,\bar a)\text{ is an }(\hat{\mathcal{E}},n_1+n_2+1)\text{–complementary pair”.}$$

(2) *If $\mathbb{Q}$ is strongly complete for $\hat{\mathcal{E}}$, then $\Vdash_{\mathbb{Q}}\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$.*

Proof. 1) Since $\bar p$ is $(\bar N,\mathbb{Q})^{n_2}$–generic, for each $i<\delta$ and every $\mathbb{Q}$–name $\underset{\sim}{\tau}\in N_i$ for an element of $\mathbf{V}$, the condition p_{i+n_2} decides the value of $\underset{\sim}{\tau}$ and the decision belongs to N_{i+n_2+1} (remember $p_{i+n_2}\in N_{i+n_2+1}$). Now, by standard arguments (like in the proofs of A.1.13.2 and A.1.13.3) we conclude that for each $i<\delta$

$$p_{i+n_2+1}\Vdash_{\mathbb{Q}}\quad\text{“}N_i[\underset{\sim}{G}_{\mathbb{Q}}]\cap\mathbf{V}\subseteq N_{i+n_2+1}\text{ and }N_i[\underset{\sim}{G}_{\mathbb{Q}}]\prec(\mathcal{H}(\chi),\in,<^*_\chi)^{\mathbf{V}[\underset{\sim}{G}_\beta]}\text{ and } \langle N_j[\underset{\sim}{G}_{\mathbb{Q}}]:j\leq i\rangle\in N_{i+1}[\underset{\sim}{G}_{\mathbb{Q}}]\text{”}.$$

Since $a_{i+n_2+1}\subseteq N_{i+n_2+1}\subseteq a_{i+n_2+1+n_1}$ (for $i<\delta$) we get

$$q\Vdash_{\mathbb{Q}}\text{ “}(\langle N_i[\underset{\sim}{G}_{\mathbb{Q}}]:i\leq\delta\rangle,\bar a)\text{ is an }(\hat{\mathcal{E}},n_1+n_2+1)\text{–complementary pair”.}$$

2) Suppose that $p\Vdash_{\mathbb{Q}}\underset{\sim}{x}\in\mathcal{H}(\chi)$ and let $\theta<\kappa$ be a regular cardinal. Since $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$ we can find an $(\hat{\mathcal{E}},n_1)$–complementary pair $(\bar N,\bar a)$ such that $\ell g(\bar N)=\ell g(\bar a)=\theta+1$ and $(p,\underset{\sim}{x},\mathbb{Q},\hat{\mathcal{E}})\in N_0$. Now, by induction on $i<\theta$, we define an $(\bar N,\mathbb{Q})^1$–generic sequence $\bar p=\langle p_i:i<\theta\rangle$:

$p_i\in N_{i+1}\cap\mathbb{Q}$ is the $<^*_\chi$–first element q of $\mathbb{Q}$ such that
(i)$_i$ $p\leq q$ and $(\forall j<i)(p_j\leq q)$,
(ii)$_i$ $q\in\bigcap\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{Q}\text{ is open dense}\}$.

To show that this definition is correct we have to prove that, for each $i<\theta$, there is a condition $q\in\mathbb{Q}$ satisfying (i)$_i$+(ii)$_i$ and $\bar p{\restriction}i\in N_{i+1}$. Note that once we know this, we are sure that the $<^*_\chi$–first condition with these properties is in N_{i+1} and therefore $\bar p{\restriction}(i+1)\in N_{i+1}$ too.

There are no problems for $i=0$, so suppose that $i=i_0+1$ and we have already defined $\bar p{\restriction}i_0\in N_{i_0+1}$, and $p_{i_0}\in N_{i_0+1}$, and hence $\bar p{\restriction}(i_0+1)\in N_{i_0+1}\prec N_{i_0+2}$. The forcing notion $\mathbb{Q}$ does not add new sequences of ordinals of length $<\kappa$ and $\|N_{i_0+1}\|<\kappa$. Therefore we find a condition $q\in\mathbb{Q}$ stronger than p_{i_0} and such that q decides all $\mathbb{Q}$–names for ordinals from N_{i_0+1} (i.e., $q\in\bigcap\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{Q}$ is open dense$\}$).

Suppose now that we have arrived to a limit stage i and we have defined $\bar p{\restriction}i$. Since $\langle N_j:j\leq i\rangle\in N_{i+1}$ we know that $\bar p{\restriction}i\in N_{i+1}$ (as all the parameters needed for the definition of $\bar p{\restriction}i$ are in N_{i+1} and we have no freedom left). Note that

$\bar{a}\restriction(i+1) \in \hat{\mathcal{E}}$ (as $\hat{\mathcal{E}}$ is closed), $(\bar{a}\restriction(i+1), \bar{N}\restriction(i+1))$ is an $(\hat{\mathcal{E}}, n_1)$–complementary pair and the sequence $\bar{p}\restriction i$ is $(\bar{N}\restriction(i+1), \mathbb{Q})^1$–generic. Since $\mathbb{Q}$ is strongly complete for $\hat{\mathcal{E}}$ we conclude that there is an upper bound to $\bar{p}\restriction i$ in $\mathbb{Q}$. Now it should be clear that such an upper bound p_i satisfies $(\mathrm{i})_i$+$(\mathrm{ii})_i$ (remember that $\bar{N}$ is increasing continuous).

Now look at the sequence $\bar{p} = \langle p_i : i < \theta\rangle$. Immediately by its definition we see that $\bar{p}$ is $(\bar{N}\restriction(i+1), \mathbb{Q})^1$–generic. Since $\mathbb{Q}$ is strongly complete for $\hat{\mathcal{E}}$ we can find an upper bound $q \in \mathbb{Q}$ of $\bar{p}$. Now, by the first part of the proposition, we conclude that

$$q \Vdash_{\mathbb{Q}} \text{“}(\langle N_i[G_{\mathbb{Q}}] : i \le \delta\rangle, \bar{a}) \text{ is an } (\hat{\mathcal{E}}, n_1+2)\text{–complementary pair”},$$

which finishes the proof. □

Theorem B.5.6. *Suppose that $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma\rangle$ is a $(<\kappa)$–support iteration such that for each $i < \gamma$*

$$\Vdash_{\mathbb{P}_i} \text{“the forcing notion } \underset{\sim}{\mathbb{Q}}_i \text{ is strongly complete for } \hat{\mathcal{E}}\text{”}.$$

Then $\mathbb{P}_\gamma$ is strongly complete for $\hat{\mathcal{E}}$.

Proof. We prove the theorem by induction on γ.

Case 1: $\gamma = 0$.
There is nothing to do in this case.

Case 2: $\gamma = \beta + 1$.
By the induction hypothesis we know that $\mathbb{P}_\beta$ is strongly complete for $\hat{\mathcal{E}}$ and therefore, by B.5.5, $\Vdash_{\mathbb{P}_\beta} \hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$.
Clearly the composition of two forcing notions not adding new sequences of length $<\kappa$ of ordinals does not add such sequences. Thus what we have to prove is that $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$ is complete for $\hat{\mathcal{E}}$ (i.e., B.5.3(2)).

Let $y \in \mathcal{H}(\chi)$ be the witness for “$\mathbb{P}_\beta$ is complete for $\hat{\mathcal{E}}$” and let $\underset{\sim}{x}$ be a $\mathbb{P}_\beta$–name for the witness for “$\underset{\sim}{\mathbb{Q}}_\beta$ is complete for $\hat{\mathcal{E}}$”. We are going to show that the composition $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$ satisfies the condition $(\circledast)^{\hat{\mathcal{E}}}_{\langle y, \underset{\sim}{x}, \hat{\mathcal{E}}, \mathbb{P}_{\beta+1}\rangle}$ of B.5.3(2). So suppose that

(a) $(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair (with an error, say, n_1), $y, \underset{\sim}{x}, \hat{\mathcal{E}}, \mathbb{P}_{\beta+1} \in N_0$, $\ell g(\bar{N}) = \ell g(\bar{a}) = \delta + 1$,
(b) $\bar{p} = \langle p_i : i < \delta\rangle$ is an increasing $(\bar{N}, \mathbb{P}_{\beta+1})^{n_2}$–generic sequence.

It should be clear that the sequence $\langle p_i\restriction\beta : i < \delta\rangle$ is $(\bar{N}, \mathbb{P}_\beta)^{n_2}$–generic. Therefore, as $\mathbb{P}_\beta$ is complete for $\hat{\mathcal{E}}$ and $y \in N_0$, we can find a condition $q^* \in \mathbb{P}_\beta$ stronger than all $p_i\restriction\beta$ (for $i < \delta$). By B.5.5(1) we know that

$$q^* \Vdash_{\mathbb{P}_\beta} \text{“}(\langle N_i[\underset{\sim}{G}_{\mathbb{P}_\beta}] : i \le \delta\rangle, \bar{a}) \text{ is an } (\hat{\mathcal{E}}, n_1+n_2+1)\text{–complementary pair”}.$$

Moreover

$$q^* \Vdash_{\mathbb{P}_\beta} \text{“}\langle p_i(\beta) : \beta < \delta\rangle \text{ is an increasing } (\langle N_i[\underset{\sim}{G}_{\mathbb{P}_\beta}] : i \le \delta\rangle, \underset{\sim}{\mathbb{Q}}_\beta)^{n_2}\text{-generic sequence”}.$$

[Why? Like in A.1.13.4, if $\underset{\sim}{\mathcal{I}} \in N_i$ is a $\mathbb{P}_\beta$–name for an open dense subset of $\underset{\sim}{\mathbb{Q}}_\beta$ then the set

$$\{p \in \mathbb{P}_{\beta+1} : p\restriction\beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \underset{\sim}{\mathcal{I}}\} \in N_i$$

is open dense in $\mathbb{P}_{\beta+1}$; now use the choice of q^*.] Consequently, we can find a $\mathbb{P}_\beta$–name $\underset{\sim}{\tau}$ for an element of $\underset{\sim}{\mathbb{Q}}_\beta$ such that

$$q^* \Vdash_{\mathbb{P}_\beta} \text{“}(\forall i<\delta)(p_i(\beta) \leq_{\underset{\sim}{\mathbb{Q}}_\beta} \underset{\sim}{\tau})\text{”}.$$

Let $q = q^* \cup \{(\beta, \underset{\sim}{\tau})\}$. Clearly $q \in \mathbb{P}_{\beta+1}$ is an upper bound of $\bar{p}$.

CASE 3: γ is a limit ordinal.
Let $\underset{\sim}{x}_\beta$ (for $\beta < \gamma$) be a $\mathbb{P}_\beta$–name for the witness for $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is complete for $\hat{\mathcal{E}}$". Let $x = \langle \langle \underset{\sim}{x}_\beta : \beta < \gamma\rangle, \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma\rangle\rangle$.

Claim B.5.6.1. *Suppose that $(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair, $\ell g(\bar{N}) = \ell g(\bar{a}) = \delta + 1$, δ is a limit ordinal and $x \in N_0$. Further assume that $\bar{p} = \langle p_i : i < \delta\rangle \subseteq \mathbb{P}_\gamma$ is an increasing sequence of conditions from $\mathbb{P}_\gamma$ such that*

(a) $(\forall i < \delta)(\bar{p}{\restriction}(i+1) \in N_{i+1})$, *and*
(b) *for every $\beta \in \gamma \cap N_\delta$ there are $n < \omega$ and $i_0 < \delta$ such that*

$$(\forall i \in [i_0, \delta))(p_{i+n}{\restriction}\beta \in \bigcap\{\mathcal{I} \in N_i : \mathcal{I} \textit{ is an open dense subset of } \mathbb{P}_\beta\}).$$

Then the sequence $\bar{p}$ has an upper bound in $\mathbb{P}_\gamma$.
[Note: we do not put any requirements on meeting dense subsets of $\mathbb{P}_\gamma$!]

Proof of the claim. We define a condition $q \in \mathbb{P}_\gamma$. First we declare that $\mathrm{dom}(q) = N_\delta \cap \gamma$ and next we choose $q(\beta)$ by induction on $\beta \in N_\delta \cap \gamma$ in such a way that $(\forall i < \delta)(p_i{\restriction}\beta \leq_{\mathbb{P}_\beta} q{\restriction}\beta)$. So suppose that we have defined $q{\restriction}\beta \in \mathbb{P}_\beta$, $\beta \in \gamma \cap N_\delta$. Let $n \in \omega$ and $i_0 < \delta$ be given by the assumption (b) of the claim for $\beta + 1$. We may additionally demand that $\beta \in N_{i_0}$. (Note that n, $i_0' = \min(\{i : i_0 \leq i,\ \beta \in N_i\})$ are good for β too, remember $\mathbb{P}_\beta \lessdot \mathbb{P}_{\beta+1}$.) Since $\hat{\mathcal{E}}$ is closed we know that $(\bar{N}{\restriction}[i_0,\delta], \bar{a}{\restriction}[i_0,\delta])$ is an $\hat{\mathcal{E}}$–complementary pair and the sequence $\langle p_i{\restriction}\beta : i_0 \leq i < \delta\rangle$ is $(\bar{N}{\restriction}[i_0,\delta], \mathbb{P}_\beta)^n$–generic. Consequently, by B.5.5(1), we get

$$q{\restriction}\beta \Vdash_{\mathbb{P}_\beta} \text{“}(\bar{N}[\underset{\sim}{G}_{\mathbb{P}_\beta}]{\restriction}[i_0,\delta], \bar{a}{\restriction}[i_0,\delta]) \text{ is an } \hat{\mathcal{E}}\text{–complementary pair”}.$$

Moreover, like in the previous case, the condition $q{\restriction}\beta$ forces (in $\mathbb{P}_\beta$) that

$$\text{“}\langle p_i(\beta) \colon i_0 \leq i < \delta\rangle \text{ is an increasing } (\bar{N}[\underset{\sim}{G}_{\mathbb{P}_\beta}]{\restriction}[i_0,\delta], \underset{\sim}{\mathbb{Q}}_\beta)^n\text{-generic sequence”}.$$

Thus, as $\underset{\sim}{x}_\beta \in N_{i_0}$ and $\underset{\sim}{\mathbb{Q}}_\beta$ is a name for a forcing notion which is complete for $\hat{\mathcal{E}}$ with the witness $\underset{\sim}{x}_\beta$, we find a $\mathbb{P}_\beta$–name $q(\beta)$ such that

$$q{\restriction}\beta \Vdash_{\mathbb{P}_\beta} \text{“}(\forall i<\delta)(p_i(\beta) \leq_{\underset{\sim}{\mathbb{Q}}_\beta} q(\beta))\text{”}.$$

Now we finish the proof of the claim noting that if $\beta \in \gamma \cap N_\delta$ is limit and for each $\alpha \in \beta \cap N_\delta$, $q{\restriction}\alpha$ is an upper bound to $\langle p_i{\restriction}\alpha : i < \delta\rangle$ then $q{\restriction}\beta$ is an upper bound of $\langle p_i{\restriction}\beta : i < \delta\rangle$ (remember $\mathrm{dom}(p_i) \subseteq N_\delta$ for each $i < \delta$). □

Claim B.5.6.2. *Suppose that $M \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\|M\| < \kappa$, $x \in M$ and $p \in \mathbb{P}_\gamma$. Then there is a condition $q \in \mathbb{P}_\gamma$ stronger than p and such that*

$$(\forall \beta \in M \cap \gamma)(q{\restriction}\beta \in \bigcap\{\mathcal{I} \in M : \mathcal{I} \textit{ is an open dense subset of } \mathbb{P}_\beta\}).$$

Proof of the claim. Let $\theta = \mathrm{cf}(\mathrm{otp}(M \cap \gamma))$ and let $\langle \gamma_i : i \leq \theta\rangle$ be an increasing continuous sequence such that $\gamma_0 = 0$, $\gamma_\theta = \sup(M \cap \gamma)$ and $\gamma_i \in M \cap \gamma$ (for non-limit $i < \theta$). As $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$, we find $\bar{N} = \langle N_i : i \leq \theta\rangle$ and $\bar{a} = \langle a_i : i \leq \theta\rangle \in \hat{\mathcal{E}}$ such that $\langle \gamma_i : i \leq \theta\rangle, x, p \in N_0$ and $(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair and $M \subseteq N_0$.

The last demand may seem to be too strong, but we use the fact that $\hat{\mathcal{E}}$ is closed and

$$M \in N' \prec N'' \prec (\mathcal{H}(\chi), \in, <^*_\chi) \ \&\ \sup(N' \cap \kappa) \subseteq N'' \quad \Rightarrow \quad M \subseteq N'.$$

(Alternatively, first we take an $\hat{\mathcal{E}}$–complementary pair $(\bar{N}^*, \bar{a}^*)$ such that $\ell g(\bar{N}) = \ell g(\bar{a}) = \|M\|^+ + 1$ and $\langle \gamma_i : i \leq \theta\rangle, x, p, M \in N^*_0$. Next look at the model $N^*_{\|M\|+1}$ – it contains all ordinals below $\|M\|$, M and $\|M\|$. Hence $M \subseteq N^*_{\|M\|+1}$. Take $\bar{N} = \bar{N}^*{\restriction}[\|M\|+1, \|M\|+\theta]$ and $\bar{a} = \bar{a}^*{\restriction}[\|M\|+1, \|M\|+\theta]$.)

Next, by induction on $i \leq \theta$, we define a sequence $\langle p_i : i \leq \theta\rangle \subseteq \mathbb{P}_\gamma$:

$p_i \in \mathbb{P}_\gamma$ is the $<^*_\chi$–first element q of $\mathbb{P}_\gamma$ such that
(i)$_i$ $p{\restriction}\gamma_i \leq_{\mathbb{P}_{\gamma_i}} q{\restriction}\gamma_i$ and $(\forall j < i)(p_j{\restriction}\gamma_i \leq_{\mathbb{P}_{\gamma_i}} q{\restriction}\gamma_i)$,
(ii)$_i$ $q{\restriction}\gamma_i \in \bigcap\{\mathcal{I} \in N_i : \mathcal{I} \subseteq \mathbb{P}_{\gamma_i}$ is open dense$\}$,
(iii)$_i$ $q{\restriction}[\gamma_i, \gamma) = p{\restriction}[\gamma_i, \gamma)$.

We have to show that this definition is correct and for this we prove by induction on $i \leq \theta$ that there is a condition $q \in \mathbb{P}_{\gamma_i}$ satisfying (i)$_i$–(iii)$_i$ and $\bar{p}{\restriction}i \in N_{i+1}$. By the way p_i's are defined we will have that then $\bar{p}{\restriction}(i+1) \in N_{i+1}$ for $i < \theta$.

If i is not limit (and we have p_j for $j < i$) then there is no problem in finding the respective condition q once one realizes that, by the inductive hypothesis of the theorem, the forcing notion $\mathbb{P}_{\gamma_i}$ does not add new sequences of length $< \kappa$ of ordinals and $\|N_i\| < \kappa$. So we just pick up a condition in $\mathbb{P}_{\gamma_i}$ stronger than the (respective restriction of the) previous condition (if there is any) and which decides all names for ordinals from N_i. This takes care of (i)$_i$ and (ii)$_i$. Next we extend our condition to a condition in $\mathbb{P}_\gamma$ as the requirement (iii)$_i$ demands. Arriving to a limit stage i we use Claim B.5.6.1. So we have defined $\bar{p}{\restriction}i$ and by the way it was defined we know that $\bar{p}{\restriction}i \in N_{i+1}$ (as all parameters are there). Since $\hat{\mathcal{E}}$ is closed we know that $(\bar{N}{\restriction}(i+1), \bar{a}{\restriction}(i+1))$ is an $\hat{\mathcal{E}}$–complementary pair. Now apply B.5.6.1 to γ_i, $\mathbb{P}_{\gamma_i}$, $\bar{p}{\restriction}i$, $\bar{N}{\restriction}(i+1)$ and $\bar{a}{\restriction}(i+1)$ in place of γ, $\mathbb{P}_\gamma$, $\bar{p}$, $\bar{N}$, and $\bar{a}$ there. Note that the assumptions are satisfied: for (b) use the fact that i is limit, so if $\beta < \gamma_i$ then for some $j < i$ we have $\beta < \gamma_j$ and now this j works as i_0 there with $n = 1$. Consequently the sequence $\bar{p}{\restriction}i$ has an upper bound in $\mathbb{P}_{\gamma_i}$. Now, similarly as in the non-limit case, we can find a condition $q \in \mathbb{P}_\gamma$ (stronger than this upper bound) satisfying (i)$_i$–(iii)$_i$.

Now look at the condition $p_\theta \in \mathbb{P}_\gamma$. If $\beta \in M \cap \gamma$ and $i < \theta$ is such that $\beta < \gamma_i$ then $p_i{\restriction}\gamma_i$ decides all $\mathbb{P}_{\gamma_i}$–names from N_i for ordinals. But $M \subseteq N_0$, $p_i{\restriction}\gamma_i \leq_{\mathbb{P}_{\gamma_i}} p_\theta{\restriction}\gamma_i$ and $\mathbb{P}_\beta \lessdot \mathbb{P}_{\gamma_i}$. Hence $p_\theta{\restriction}\beta \in \bigcap\{\mathcal{I} \in M : \mathcal{I} \subseteq \mathbb{P}_\beta$ is open dense$\}$. As p_θ is stronger than p, this finishes the proof of the claim. □

Claim B.5.6.3. *$\mathbb{P}_\gamma$ is complete for $\hat{\mathcal{E}}$.*

Proof of the claim. We are going to show that $\mathbb{P}_\gamma$ satisfies the condition $(\circledast)^{\hat{\mathcal{E}}}_{\langle x, \hat{\mathcal{E}}\rangle}$ of B.5.3(2). So suppose that $(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair, $\bar{N} = \langle N_i : i \leq \delta\rangle$, $x, \hat{\mathcal{E}} \in N_0$ and $\bar{p} = \langle p_i : i < \delta\rangle$ is an increasing $(\bar{N}, \mathbb{P}_\gamma)^{n_1}$–generic sequence. For $i < \delta$ let

$$\mathcal{I}^*_i \stackrel{\text{def}}{=} \big\{q \in \mathbb{P}_\gamma : (\forall \beta \in N_i \cap \gamma)(q{\restriction}\beta \in \bigcap\{\mathcal{I} \in N_i : \mathcal{I} \subseteq \mathbb{P}_\beta \text{ is open dense in } \mathbb{P}_\beta\})\big\}.$$

Note that Claim B.5.6.2 says that each $\mathcal{I}^*_i$ is an open dense subset of $\mathbb{P}_\gamma$. Clearly $\mathcal{I}^*_i$ is in N_{i+1}, as it is defined from N_i. Hence, for each $i < \delta$, $p_{i+1+n_1} \in \mathcal{I}^*_i$. Now look at the assumptions of Claim B.5.6.1: both (a) and (b) there are satisfied (for the second note that if $\beta\in N_\delta\cap\gamma$ then we may take $i_0<\delta$ large enough so that $\beta\in N_{i_0}$ and let $n=n_1+1$). Thus we may conclude that $\bar p$ has an upper bound in $\mathbb{P}_\gamma$. $\square$

Claim B.5.6.4. *Forcing with $\mathbb{P}_\gamma$ does not add new sequences of length $<\kappa$ of ordinals.*

Proof of the claim. First note that for a forcing notion $\mathbb{P}$, "not adding new sequences of length θ of ordinals" is equivalent to "not adding new sequences of length θ of elements of $\mathbf{V}$". Next note that, for a forcing notion $\mathbb{P}$, if θ is the first ordinal such that for some $\mathbb{P}$–name $\underset{\sim}{\tau}$ and a condition $p\in\mathbb{P}$ we have

$$p\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\tau}:\theta\longrightarrow\mathbf{V}\quad\text{and}\quad\underset{\sim}{\tau}\notin\mathbf{V}\text{”},$$

then $\mathrm{cf}(\theta)=\theta$. [Why? Clearly such a θ has to be limit; if $\mathrm{cf}(\theta)<\theta$ then take an increasing cofinal in θ sequence $\langle\zeta_i:i<\mathrm{cf}(\theta)\rangle$ and look at $\langle\underset{\sim}{\tau}\restriction\zeta_i:i<\mathrm{cf}(\theta)\rangle$. Each $\underset{\sim}{\tau}\restriction\zeta_i$ is forced to be in $\mathbf{V}$, so the sequence of them is in $\mathbf{V}$ too – a contradiction.] Consequently it is enough to prove that for every regular cardinal $\theta<\kappa$, forcing with $\mathbb{P}_\gamma$ does not add new sequences of length θ of elements of $\mathbf{V}$. So suppose that, for $i<\theta$, $\underset{\sim}{\tau}_i$ is a $\mathbb{P}_\gamma$–name for an element of $\mathbf{V}$, and $p\in\mathbb{P}_\gamma$. Take an $\hat{\mathcal{E}}$–complementary pair $(\bar N,\bar a)$ such that $\bar N=\langle N_i:i\leq\theta\rangle$ and $x,p,\langle\underset{\sim}{\tau}_i:i<\theta\rangle\in N_0$ (exists as $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$). Now, by induction on $i\leq\theta$, define a sequence $\langle p_i:i\leq\theta\rangle\subseteq\mathbb{P}_\gamma$:

$p_i\in\mathbb{P}_\gamma$ is the $<^*_\chi$–first element q of $\mathbb{P}_\gamma$ such that
(i)$_i$ $p\leq_{\mathbb{P}_\gamma} q$ and $(\forall j<i)(p_j\leq_{\mathbb{P}_\gamma} q)$,
(ii)$_i$ if $\beta\in N_i\cap\gamma$ then $q\restriction\beta\in\bigcap\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{P}_\beta$ is open dense$\}$,
(iii)$_i$ q decides the value of $\underset{\sim}{\tau}_i$ (when $i<\theta$).

Checking that this definition is correct is straightforward (compare with the proof of B.5.6.2). At successor stages $i<\theta$ we use B.5.6.2 to show that there is a condition $q'\in\mathbb{P}_\gamma$ satisfying (i)$_i$+(ii)$_i$ and next we extend it to a condition q deciding the value of $\underset{\sim}{\tau}_i$. At limit stages $i\leq\theta$ we know, by the definition of $\bar p\restriction i$, that for each $j\leq i$, $\bar p\restriction j\in N_{j+1}$. Moreover, we may apply B.5.6.1 to $\bar N\restriction(i+1)$, $\bar a\restriction(i+1)$ and $\bar p\restriction i$ to conclude that $\bar p\restriction i$ has an upper bound $q'\in\mathbb{P}_\gamma$. Now take $q\geq q'$ which decides the value of $\underset{\sim}{\tau}_i$ (if $i<\theta$) – it satisfies the demands (i)$_i$–(iii)$_i$.

Finally look at the condition $p_\theta\in\mathbb{P}_\gamma$: it forces values to all $\underset{\sim}{\tau}_i$ (for $i<\theta$) and so $p_\theta\Vdash_{\mathbb{P}_\gamma}\langle\underset{\sim}{\tau}_i:i<\theta\rangle\in\mathbf{V}$, finishing the proof of the claim and thus that of the theorem. $\square$

$\square$

Definition B.5.7. (1) Let $\mathfrak{C}^-_{<\kappa}(\mu^*)$ be the family of all subsets of

$$\big\{\bar a=\langle a_i:i\leq\alpha\rangle:\ \text{the sequence } \bar a \text{ is increasing continuous,}\\ \alpha<\kappa\ \text{ and } (\forall i\leq\alpha)(a_i\in[\mu^*]^{<\kappa}\ \&\ a_i\cap\kappa\in\kappa)\big\}.$$

(2) Let $\bar M=\langle M_i:i\leq\alpha\rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi),\in,<^*_\chi)$, $\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1\in\mathfrak{C}^-_{<\kappa}(\mu^*)$. We say that $\bar M$ *is ruled by* $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ if
 (a) $\bar M\restriction(i+1)\in M_{i+1}$, $\|M_i\|<\kappa$ and $2^{\|M_i\|}+1\subseteq M_{i+1}$ for all $i<\alpha$,
 (b) $\langle M_i\cap\mu^*:i\leq\alpha\rangle\in\hat{\mathcal{E}}_1$,
 (c) for each $i<\alpha$ (and we allow $i=-1$) there is an $\hat{\mathcal{E}}_0$–complementary pair $(\bar N^i,\bar a^i)$ such that

(α) $\ell g(\bar{N}^i) = \ell g(\bar{a}^i) = \delta_i + 1$, $\mathrm{cf}(\delta_i) > 2^{\|M_i\|}$ and, for simplicity, δ_i is additively indecomposable,

(β) $\bar{M}\restriction(i+1) \in N^i_0$, $N^i_{\delta_i} = M_{i+1}$ and

(γ) $\|N^i_\varepsilon\|^{2^{\|M_i\|}} + 1 \subseteq N^i_{\varepsilon+1}$.

The sequence $\langle \bar{N}^i : i < \alpha\rangle$ given by the clause (c) above will be called *an $\hat{\mathcal{E}}_0$–approximation to $\bar{M}$*.

(3) $\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ is the family of all pairs $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ such that $\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1 \in \mathfrak{C}^-_{<\kappa}(\mu^*)$, $\hat{\mathcal{E}}_0$ is closed and for every large enough regular cardinal χ, for every $x \in \mathcal{H}(\chi)$ there is a sequence $\bar{M}$ ruled by $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ and such that $x \in M_0$ and every end segment of $\bar{M}$ is ruled by $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ (follows if $\hat{\mathcal{E}}_1$ is closed under end segments).

Remark B.5.8. (1) Condition B.5.7(2)(c) is the replacement for

$$\|N_{i+1}\| = \lambda \quad \text{and} \quad (N_{i+1})^{<\lambda} \subseteq N_{i+1}$$

in **Case A**. Here, there are no natural closed candidates for M_{i+1}, as in that case. So we use a relative candidate.

(2) In B.5.7(2)(c)(γ) we may put stronger demands (if required in applications). For example one may consider a demand that $\|N^i_\varepsilon\|^{h^*(\|M_i\|)} + 1 \subseteq N^i_{\varepsilon+1}$, for some function $h^* : \kappa \longrightarrow \kappa$.

(3) Note that if $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ then necessarily $\hat{\mathcal{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$.
[Why? If $\theta = \mathrm{cf}(\theta) < \kappa$, $x = \langle \theta, y\rangle$ then $\ell g(\bar{N}^i) > \theta$.]

(4) Note that in examples there is no need to assume that $\hat{\mathcal{E}}_1$ is closed under end segments as "complete for $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$" (see B.5.9) is preserved, as this just restricts the choice of the "bad guy" INC of i_0 (and so p) to those in the end segment.

Definition B.5.9. Let $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ and let $\mathbb{Q}$ be a forcing notion.

(1) For a sequence $\bar{M} = \langle M_i : i \le \delta\rangle$ ruled by $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ with an $\hat{\mathcal{E}}_0$–approximation $\langle \bar{N}^i : i < \delta\rangle$ and a condition $r \in \mathbb{Q}$ we define a game $\mathcal{G}^{\spadesuit}_{\bar{M},\langle \bar{N}^i : i<\delta\rangle}(\mathbb{Q}, r)$ between two players COM and INC.

The play lasts δ moves during which the players construct a sequence $\langle i_0, p, \langle p_i, \bar{q}_i : i_0 - 1 \le i < \delta\rangle\rangle$ such that $i_0 < \delta$ is non-limit, $p \in M_{i_0} \cap \mathbb{Q}$, $p_i \in M_{i+1} \cap \mathbb{Q}$, $\bar{q}_i = \langle q_{i,\varepsilon} : \varepsilon < \delta_i\rangle \subseteq \mathbb{Q}$ (where $\delta_i + 1 = \ell g(\bar{N}^i)$).

The player INC first decides what is $i_0 < \delta$ and then it chooses a condition $p \in \mathbb{Q} \cap M_{i_0}$ stronger than r. Next, at the stage $i \in [i_0 - 1, \delta)$ of the game, COM chooses $p_i \in \mathbb{Q} \cap M_{i+1}$ such that

$$p \le_{\mathbb{Q}} p_i \quad \text{and} \quad (\forall j < i)(\forall \varepsilon < \delta_j)(q_{j,\varepsilon} \le_{\mathbb{Q}} p_i),$$

and INC answers choosing an increasing sequence $\bar{q}_i = \langle q_{i,\varepsilon} : \varepsilon < \delta_i\rangle$ such that $p_i \le_{\mathbb{Q}} q_{i,0}$ and $\bar{q}_i$ is $(\bar{N}^i\restriction[\alpha, \delta_i], \mathbb{Q})^*$–generic for some $\alpha < \delta$.

The player COM wins if it has always legal moves and the sequence $\langle p_i : i < \delta\rangle$ has an upper bound.

(2) We say that the forcing notion $\mathbb{Q}$ is *complete for* $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ if

(a) $\mathbb{Q}$ is strongly complete for $\hat{\mathcal{E}}_0$ and

(b) for a large enough regular χ, for some $x\in\mathcal{H}(\chi)$, for every sequence $\bar{M}$ ruled by $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ with an $\hat{\mathcal{E}}_0$–approximation $\langle\bar{N}^i:i<\delta\rangle$ and such that $x\in M_0$ and for any condition $r\in\mathbb{Q}\cap M_0$, the player INC DOES NOT have a winning strategy in the game $\mathcal{G}^{\spadesuit}_{\bar{M},\langle\bar{N}^i:i<\delta\rangle}(\mathbb{Q},r)$.

Proposition B.5.10. *Assume*

(a) $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$,

(b) $\mathbb{Q}$ *is a forcing notion complete for* $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$.

Then $\Vdash_{\mathbb{Q}}$ " $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ ".

Proof. Straightforward (and not used in this paper). □

B.6. Examples for an inaccessible cardinal κ

Let us look at a variant of the examples presented in section A.2 relevant for our present case. (Remember B.5.8(4).)

Hypothesis B.6.1. Assume that κ is a strongly inaccessible cardinal, $S\subseteq\kappa$ is a stationary set and $\bar{C}=\langle C_\delta:\delta\in S\rangle$ is such that for each $\delta\in S$:

C_δ is a club of δ such that $\mathrm{otp}(C_\delta)<\delta$, moreover for simplicity $\mathrm{otp}(C_\delta)<\min(C_\delta)$, $\mathrm{nacc}(C_\delta)\subseteq\kappa\setminus S$ and
if $\alpha\in\mathrm{nacc}(C_\delta)$, then $\mathrm{cf}(\alpha)>2^{\max(\alpha\cap C_\delta)}$ and $S\cap\alpha$ is not stationary,
if $\alpha\in\mathrm{acc}(C_\delta)\cap S$, then $C_\alpha=C_\delta\cap\alpha$.

[Note that if S does not reflect, then we can ask that the assumption of the second demand never occurs, hence the second demand holds trivially].
Further we assume that $\bar{C}$ guesses clubs, i.e.,

if $E\subseteq\kappa$ is a club,
then the set $\{\delta\in S:C_\delta\subseteq E\}$ is stationary.

Moreover we demand that for every club $E\subseteq\kappa$, the set $\kappa\setminus S$ contains arbitrarily long (but $<\kappa$) increasing continuous sequences from E.

Definition B.6.2. Let $\kappa,S,\bar{C}$ be as in Hypothesis B.6.1 and let $\mu^*=\kappa$.

(1) Define

$$\hat{\mathcal{E}}^S_0=\big\{\bar{\alpha}=\langle\alpha_i:i\leq\gamma\rangle:\ \bar{\alpha}\text{ is an increasing continuous sequence of ordinals from }\kappa\setminus S,\quad\gamma<\kappa\big\}$$

$$\hat{\mathcal{E}}^{S,\bar{C}}_1=\big\{\bar{\beta}':\ \bar{\beta}'\text{ is an end segment (not necessarily proper) of }\bar{\beta}^\frown\langle\delta\rangle,\text{ for some }\delta\in S\text{ and }\bar{\beta}\text{ is the increasing enumeration of }C_\delta\big\}.$$

(2) Suppose that $\bar{A}=\langle A_\delta:\delta\in S\rangle$, $\bar{h}=\langle h_\delta:\delta\in S\rangle$ and $\mathrm{cf}(\theta)=\theta<\kappa$ are such that for each $\delta\in S$:

$$A_\delta\subseteq\delta,\quad\|A_\delta\|<\theta,\quad h_\delta:A_\delta\longrightarrow\theta,\quad\text{and }\sup(A_\delta)=\delta$$

(so $\mathrm{cf}(\delta)<\theta$; we may omit the last demand as only $\bar{A}{\restriction}S'$, for $S'=\{\delta\in S:\delta=\sup A_\delta\}$, affects the forcing). We define a forcing notion $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$:

a condition in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is a function $g:\beta\longrightarrow\theta$ (for some $\beta<\kappa$) such that

$$(\forall\delta\in S\cap(\beta+1))(\{\xi\in A_\delta:h_\delta(\xi)\neq g(\xi)\}\text{ is bounded in }\delta),$$

the order $\leq_{\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}}$ of $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is the inclusion (extension).

(3) For $\bar{A}$, $\bar{h}$ and θ as above and $\alpha<\kappa$ we let

$$\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha \stackrel{\rm def}{=} \{g\in\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}:\alpha\in{\rm dom}(g)\}.$$

Remark B.6.3. One of the difficulties in handling the forcing notion $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is that the sets $\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha$ do not have to be dense in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$. Of course, if this happens then the generic object is not what we expect it to be. However, if the set S is not reflecting and $\delta\in S\ \Rightarrow\ S\cap{\rm acc}(C_\delta)=\emptyset$, then each $\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha$ is dense in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ and even weaker conditions are enough for this. One of them is the following:

(∗) $\langle A_\delta:\delta\in S\rangle$ is κ–free, i.e., for every $\alpha<\kappa$ there is a function g such that ${\rm dom}(g)=S\cap\alpha$ and $g(\delta)<\delta$ and the sets $\langle A_\delta\setminus g(\delta):\delta\in S\cap\alpha\rangle$ are pairwise disjoint.

We can of course weaken it further demanding that $\langle A_\delta:\delta\in S\cap\alpha\rangle$ has uniformization. (So if we force inductively on all κ's this may be reasonable, or we may ask uniformization just for our h_δ's.)

Proposition B.6.4. $(\hat{\mathcal{E}}^S_0,\hat{\mathcal{E}}^{S,\bar{C}}_1)\in\mathfrak{C}^\spadesuit_{<\kappa}(\mu^*)$.

Proof. Immediately from its definition we get that $\hat{\mathcal{E}}^S_0$ is closed. Suppose now that χ is a sufficiently large regular cardinal and $x\in\mathcal{H}(\chi)$. First construct an increasing continuous sequence $\bar{W}=\langle W_j:j<\kappa\rangle$ of elementary submodels of $(\mathcal{H}(\chi),\in,<^*_\chi)$ such that $x\in W_0$ and for each $j<\kappa$:

$$\|W_j\|<\kappa,\quad\text{and}\quad W_j\cap\kappa=\|W_j\|,\quad\text{and}\quad \bar{W}\restriction(j+1)\in W_{j+1}.$$

Note that then, for each $j<\kappa$, we have $2^{\|W_j\|}+1\subseteq W_{j+1}$. Clearly the set $E=\{W_j\cap\kappa:j<\kappa\text{ is limit}\}$ is a club of κ and so ${\rm acc}(E)$ is a club of κ as well. Thus, by our assumptions on $\bar{C}$ (see B.6.1), we find $\delta\in S$ such that $C_\delta\subseteq{\rm acc}(E)$ (then, of course, $\delta\in{\rm acc}(E)$ too). Let $\bar{M}=\langle M_i:i\leq{\rm otp}(C_\delta)\rangle$ be the increasing enumeration of

$$\big\{W_j:j<\kappa\ \&\ W_j\cap\kappa\in C_\delta\cup\{\delta\}\big\}.$$

Fix $i<{\rm otp}(C_\delta)$. Let $j'<j<\kappa$ be such that $W_{j'}=M_i$ and $W_j=M_{i+1}$, and let $\alpha=M_{i+1}\cap\kappa=W_j\cap\kappa$. Then $\alpha\in{\rm nacc}(C_\delta)\cap{\rm acc}(E)$ and, by B.6.1, $\alpha\notin S$ and the set S does not reflect at α. Consequently we find a club C^i of α disjoint from $S\cap\alpha$. Let $\bar{N}^i=\langle N^i_\varepsilon:\varepsilon\leq\delta_i\rangle$ be the increasing enumeration of

$$\big\{W_\xi:j'<\xi\leq j\ \&\ W_\xi\cap\kappa\in C^i\cup\{\alpha\}\big\}.$$

(Note that the set above is non-empty as $\alpha\in{\rm acc}(E)$; passing to a cofinal subsequence we may demand that δ_i is additively indecomposable.) We claim that the sequence $\bar{M}$ is ruled by $(\hat{\mathcal{E}}^S_0,\hat{\mathcal{E}}^{S,\bar{C}}_1)$ and $\langle\bar{N}^i:i<{\rm otp}(C_\delta)\rangle$ is an $\hat{\mathcal{E}}^S_0$–approximation to $\bar{M}$. For this we have to check the demands of B.5.7(2). By the choice of the W_j's we have that the clause (a) there is satisfied. As $\langle M_i\cap\kappa:i\leq{\rm otp}(C_\delta)\rangle$ enumerates $C_\delta\cup\{\delta\}$ we get the demand (b) there. For the clause (c), fix $i<{\rm otp}(C_\delta)$ and look at the way we defined $\bar{N}^i=\langle N^i_\varepsilon:\varepsilon\leq\delta_i\rangle$. For each $\varepsilon\leq\delta_i$, $N^i_\varepsilon\cap\kappa\in C^i\cup\{\alpha\}\subseteq\kappa\setminus S$. Hence $(\bar{N}^i,\langle N^i_\varepsilon\cap\kappa:\varepsilon\leq\delta_i\rangle)$ is an $\hat{\mathcal{E}}^S_0$–complementary pair. Moreover,

$${\rm cf}(\delta_i)={\rm cf}(\alpha)>2^{\max(\alpha\cap C_\delta)}=2^{M_i\cap\kappa}=2^{\|M_i\|}$$

(by B.6.1) and δ_i is additively indecomposable. This verifies (c)(α). The clauses (c)(β) and (c)(γ) should be clear by the choice of the W_j's and that of $\bar{N}^i$. □

Proposition B.6.5. *Suppose that $\bar{A}$, $\bar{h}$, θ are as in B.6.2(2) and for each $\alpha<\kappa$ the set $\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha$ (see B.6.2(3)) is dense in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ (e.g., S does not reflect). Then the forcing notion $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is complete for $(\hat{\mathcal{E}}^S_0,\hat{\mathcal{E}}^{S,\bar{C}}_1)$.*

Proof. We break the proof to three steps checking the requirements of B.5.9(2).

Claim B.6.5.1. *$\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is complete for $\hat{\mathcal{E}}^S_0$.*

Proof of the claim. Suppose that $(\bar{N},\bar{a})$ is an $\hat{\mathcal{E}}^S_0$–complementary pair, $\bar{N}=\langle N_i: i\leq\delta\rangle$ and $\bar{p}=\langle p_i:i\leq\delta\rangle\subseteq\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ is an increasing $(\bar{N},\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}})^n$–generic sequence. Let $p\stackrel{\rm def}{=}\bigcup\limits_{i<\delta}p_i$. Note that p is a function from $\mathrm{dom}(p)=\bigcup\limits_{i<\delta}\mathrm{dom}(p_i)$ to θ. Moreover, as the sets $\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha$ are dense in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ (and $\mathcal{I}^{\bar{A},\bar{h},\theta}_\alpha\in N_i$ if $\alpha\in N_i\cap\kappa$), we have $N_i\cap\kappa\subseteq\mathrm{dom}(p_{i+n})\subseteq N_{i+n+1}$. Hence

$$\mathrm{dom}(p)=\bigcup_{i<\delta}N_i\cap\kappa=N_\delta\cap\kappa\in\kappa.$$

Note that $N_\delta\cap\kappa\notin S$ (by the definition of $\hat{\mathcal{E}}^S_0$). Suppose that $\alpha\in S\cap(\mathrm{dom}(p)+1)$, so $\alpha\in\mathrm{dom}(p)$. Then for some $i<\delta$ we have $\alpha\in\mathrm{dom}(p_i)$ and, as $p_i\in\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$, the set $\{\xi\in A_\alpha: h_\alpha(\xi)\neq p(\xi)=p_i(\xi)\}$ is bounded in α. This shows that $p\in\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ and clearly it is an upper bound of $\bar{p}$. □

Claim B.6.5.2. *Forcing with $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ does not add new sequences of length $<\kappa$ of ordinals.*

Proof of the claim. Suppose that $\zeta<\kappa$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$–name for a function from ζ to ordinals, $p\in\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$. Take an increasing continuous sequence $\bar{W}=\langle W_j:j<\kappa\rangle$ of elementary submodels of $(\mathcal{H}(\chi),\in,<^*_\chi)$ such that $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}},p,\underset{\sim}{\tau}\in W_0$, $\zeta+1\subseteq W_0$ and for each $j<\kappa$

$$\|W_j\|<\kappa,\quad\text{ and }\quad W_j\cap\kappa=\|W_j\|,\quad\text{ and }\quad\bar{W}{\restriction}(j+1)\in W_{j+1}.$$

Look at the club $E=\{W_j\cap\kappa:j<\kappa\}$. By the last assumption of B.6.1 we find an increasing continuous sequence $\langle j_\xi:\xi\leq\zeta\rangle$ such that $\{W_{j_\xi}\cap\kappa:\xi\leq\zeta\}\cap S=\emptyset$. Now we build inductively an increasing sequence $\langle p_\xi:\xi\leq\zeta\rangle$ of conditions from $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ such that $p\leq_{\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}}p_0$ and for each $\xi<\zeta$:

(1) $p_\xi\in W_{j_{\xi+1}}$,
(2) p_ξ forces a value to $\underset{\sim}{\tau}(\xi)$, and
(3) $W_{j_\xi}\cap\kappa\subseteq\mathrm{dom}(p_\xi)$.

There are no problems with carrying out the construction. At a non-limit stage ξ, we may easily choose a condition p_ξ in $W_{j_{\xi+1}}$ stronger than the condition chosen before (if any) and such that $W_{j_\xi}\cap\kappa\subseteq\mathrm{dom}(p_\xi)$ (remember that $\mathcal{I}^{\bar{A},\bar{h},\theta}_{W_{j_\xi}\cap\kappa}\in W_{j_{\xi+1}}$ is a dense subset of $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$) and p_ξ decides the value of $\underset{\sim}{\tau}(\xi)$. Arriving at a limit stage $\xi\leq\zeta$ we take the union of conditions chosen so far and we note that it is a condition in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ as

$$\mathrm{dom}(\bigcup_{i<\xi}p_i)=\bigcup_{i<\xi}\mathrm{dom}(p_i)=\bigcup_{i<\xi}W_{j_i}\cap\kappa=W_{j_\xi}\cap\kappa\notin S.$$

Now proceed as in the successor case. Finally look at the condition p_ζ – it decides the value of $\underset{\sim}{\tau}$ (and is stronger than p). □

Claim B.6.5.3. *Assume that $\bar{M} = \langle M_i : i \le \delta\rangle$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ ruled by $(\hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^{S,\bar{C}}_1)$ with an $\hat{\mathcal{E}}^S_0$–approximation $\langle \bar{N}^i : i < \delta\rangle$ and such that $S, \hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^{S,\bar{C}}_1, \bar{A}, \bar{h}, \theta, \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}} \in M_0$. Let $r \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}} \cap M_0$. Then the player* COM *has a winning strategy in the game $\mathcal{G}^{\spadesuit}_{\bar{M},\langle \bar{N}^i : i<\delta\rangle}(\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}, r)$.*

Proof of the claim. First, we are going to describe a strategy for player COM in the game $\mathcal{G}^{\spadesuit}_{\bar{M},\langle \bar{N}^i : i<\delta\rangle}(\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}, r)$, and then we will show that it is a winning one.

Since $\langle M_i : i \le \delta\rangle \in \hat{\mathcal{E}}^{S,\bar{C}}_1$ and for each $\alpha \in S$, $\mathrm{otp}(C_\alpha) < \alpha$ (see B.6.1) we know that $\delta = \mathrm{otp}(C_{M_\delta\cap\kappa}) < M_\delta \cap \kappa \in S$. Recall $\mathrm{otp}(C_\delta) < \min(C_\delta)$. Let

$$Z \stackrel{\text{def}}{=} \bigcup\{A_{M_i\cap\kappa} :\ i \le \delta\ \ \&\ \ M_i \cap \kappa \in S\}.$$

Note that $\|Z\| \le \delta \cdot \theta < \|M_{i_0}\|$. By induction on $i \le \theta^+$ choose an increasing continuous sequence $\langle Z_i : i \le \theta^+\rangle$ of subsets of κ such that $Z_0 = Z$ and $Z_{i+1} = Z_i \cup \bigcup\{A_\alpha : \alpha \in S\ \&\ \alpha = \sup(Z_i \cap \alpha)\}$. Clearly $\|Z_i\| \le \delta\cdot\theta\cdot\|i\|$ for each $i \le \theta^+$ and if $\alpha = \sup(\alpha \cap Z_{\theta^+})$ then $A_\alpha \subseteq Z_{\theta^+}$. So as $\|A_\alpha\| \le \theta$ we have

$$\alpha \in S\ \&\ \alpha = \sup(Z_{\theta^+} \cap \alpha) \quad\Rightarrow\quad A_\alpha \subseteq Z_{\theta^+}.$$

Now, in his first move, player INC chooses non-limit $i_0 < \delta$ and $p \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}} \cap M_{i_0}$ stronger than r. We have assumed that each $\mathcal{I}^{\bar{A},\bar{h},\theta}_\xi$ (for $\xi < \kappa$) is dense in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$, so we have a condition $p^+ \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ stronger than p and such that $M_\delta \cap \kappa \in \mathrm{dom}(p^+)$. In the next steps, the strategy for COM will have the property that for each $i \ge i_0 - 1$ it says COM to play a condition $p_i \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ such that

$(\boxdot)_i$ $Z_{\theta^+} \cap M_{i+1} \subseteq \mathrm{dom}(p_i)$ and $p_i{\restriction}Z_{\theta^+} = p^+{\restriction}(Z_{\theta^+} \cap M_{i+1})$.

So, first the player COM chooses a condition $p_{i_0-1} \in M_{i_0} \cap \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ stronger than p and such that

$$Z_{\theta^+} \cap M_{i_0} \subseteq \mathrm{dom}(p_{i_0}) \quad \text{and} \quad p_{i_0}{\restriction}(Z_{\theta^+} \cap M_{i_0}) = p^+{\restriction}(Z_{\theta^+} \cap M_{i_0}).$$

Why is it possible? We know that

$$\|Z_{\theta^+} \cap M_{i_0}\| \le \theta^+ < M_0 \cap \kappa \le \|M_{i_0}\| < \mathrm{cf}(\delta_{i_0-1})$$

(where $\delta_{i_0-1} + 1 = \ell g(\bar{N}^{i_0-1})$) and therefore $Z_{\theta^+} \cap M_{i_0} \subseteq N^{i_0-1}_\varepsilon$ for some $\varepsilon < \delta_{i_0}$. Taking possibly larger ε we may have $\mathrm{dom}(p) \subseteq N^{i_0-1}_\varepsilon$ too. Let $p' \in N^{i_0-1}_{\varepsilon+1} \cap \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ be such that $p \le p'$ and $N^{i_0-1}_\varepsilon \cap \kappa \subseteq \mathrm{dom}(p')$. Let

$$p_{i_0} = p'{\restriction}(\mathrm{dom}(p') \setminus Z_{\theta^+} \cup p^+{\restriction}Z_{\theta^+} \cap N^{i_0-1}_\varepsilon.$$

Note that $p_{i_0} : \mathrm{dom}(p_{i_0}) \longrightarrow \theta$ is a well defined function such that $p_{i_0} \in N^{i_0-1}_{\varepsilon+1}$ (for the last remember B.5.7(2)(c)(γ): we are sure that $Z_{\theta^+} \cap N^{i_0-1}_\varepsilon \in N^{i_0-1}_{\varepsilon+1}$ and $p^+{\restriction}(Z_{\theta^+} \cap N^{i_0-1}_\varepsilon) \in N^{i_0-1}_{\varepsilon+1}$, as $\|N^{i_0-1}_\varepsilon\|^{\delta\cdot\theta^+} + 1 \subseteq N^{i_0-1}_{\varepsilon+1}$). Finally, to check that p_{i_0} is a condition in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ suppose that $\gamma \in S \cap (\mathrm{dom}(p_{i_0-1}) + 1)$. If $A_\gamma \subseteq Z_{\theta^+}$ then $p_{i_0}{\restriction}A_\gamma = p^+{\restriction}A_\gamma$ and the requirement of B.6.2(2) is satisfied. If A_γ is not contained in Z_{θ^+} then necessarily $Z_{\theta^+} \cap \gamma$ is bounded in γ and we use the fact that $p_{i_0-1}{\restriction}(A_\gamma \setminus Z_{\theta^+}) = p'{\restriction}(A_\gamma \setminus Z_{\theta^+})$, $p' \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$.

At a stage $i \in [i_0, \delta)$ of the game the player COM applies a similar procedure, but first it looks at the union $p^*_i = \bigcup\limits_{j<i} \bigcup\limits_{\varepsilon<\delta_j} q_{j,\varepsilon}$ of all conditions played by his opponent so far. If i is not limit then, directly from B.6.5.1, we know that p^*_i is a condition in $\mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ (stronger than r). But what if i is limit? In this case the demands $(\boxdot)_j$ for $j < i$ help. The only possible trouble could come from $A_{M_i\cap\kappa}$ when $M_i \cap \kappa \in S$. But then the set Z_{θ^+} contains $A_{M_i\cap\kappa}$ and, by $(\boxdot)_j$ for $j < i$, $p^*_i \restriction A_{M_i\cap\kappa} = p^+ \restriction A_{M_i\cap\kappa}$. This implies that the set

$$\{\xi \in A_{M_i\cap\kappa} : h_{M_i\cap\kappa}(\xi) \neq p^*_i(\xi)\}$$

is bounded in $M_i \cap \kappa$. Hence easily $p^*_i \in \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$. Next, player COM extends the condition p^*_i to $p_i \in N^i_{\varepsilon+1} \cap \mathbb{Q}^{S,\theta}_{\bar{A},\bar{h}}$ (for some $\varepsilon < \delta_i$) such that the demand $(\boxdot)_i$ is satisfied, applying a procedure similar to the one described for getting p_{i_0}.

Why is the strategy described above a winning strategy? Suppose that $\langle p_i : i_0 - 1 \leq i < \delta\rangle$ is a sequence constructed by COM during a play in which it uses this strategy. As it is an increasing sequence of conditions and $\bigcup\limits_{i<\delta} \mathrm{dom}(p_i) = M_\delta \cap \kappa$, the only thing we should check is that the set

$$\{\xi \in A_{M_\delta\cap\kappa} : h_{M_\delta\cap\kappa}(\xi) \neq (\bigcup_{i<\delta} p_i)(\xi)\}$$

is bounded in $M_\delta \cap \kappa$. But by the choice of $Z \subseteq Z_{\theta^+}$, and by keeping the demand $(\boxdot)_i$ (for $i < \delta$) we know that

$$\{\xi \in A_{M_\delta\cap\kappa} : h_{M_\delta\cap\kappa}(\xi) \neq (\bigcup_{i<\delta} p_i)(\xi)\} \subseteq \{\xi \in A_{M_\delta\cap\kappa} : h_{M_\delta\cap\kappa}(\xi) \neq p^+(\xi)\},$$

so the choice of p^+ works.

This finishes the proof of the claim and that of the proposition. □

□

Now, let us turn to the applications for Abelian groups (i.e., the forcing notions needed for 0.11). We continue to use Hypothesis B.6.1.

Definition B.6.6. Assume that G is a strongly κ–free Abelian group and $h : H \stackrel{\mathrm{onto}}{\longrightarrow} G$ is a homomorphism onto G with kernel K of cardinality $< \kappa$. We define a forcing notion $\mathbb{P}_{h,H,G}$:

a condition in $\mathbb{P}_{h,H,G}$ is a function q such that

(a) $\mathrm{dom}(q)$ is a subgroup of G of size $< \kappa$,

(b) $G/\mathrm{dom}(q)$ is κ–free,

(c) q is a lifting for $\mathrm{dom}(q)$ and $h : H \longrightarrow G$;

the order $\leq_{\mathbb{P}_{h,H,G}}$ of $\mathbb{P}_{h,H,G}$ is the inclusion (extension).

Hypothesis B.6.7. Let $\bar{G} = \langle G_i : i < \kappa\rangle$ be a filtration of G, $\Gamma[G] \subsetneq S$ (modulo the club filter on κ). So, $\gamma[\bar{G}] \subseteq S$ and without loss of generality, $\gamma[\bar{G}]$ is a set of limit ordinals. Let $h^{-1}[G_i] = H_i$.

Proposition B.6.8. *For each $\alpha < \kappa$ the set*

$$\mathbb{P}^*_{h,H,G} \stackrel{\mathrm{def}}{=} \{q \in \mathbb{P}_{h,H,G} : (\exists i < \kappa)(\mathrm{dom}(q) = G_{i+1} \ \&\ i \geq \alpha\}$$

is dense in $\mathbb{P}_{h,H,G}$.

Proof. Let $q \in \mathbb{P}_{h,H,G}$ and let $i < \kappa$ be such that $\mathrm{dom}(q) \subseteq G_i$ and $i \geq \alpha$. Then $G/\mathrm{dom}(q)$ is κ–free and so $G_{i+1}/\mathrm{dom}(q)$ is free. Now consider the mapping $x \mapsto x + \mathrm{dom}(q) : G_{i+1} \longrightarrow G_{i+1}/\mathrm{dom}(q)$. So by 0.8 we get that $G_{i+1} = \mathrm{dom}(q) + L$ for some free L ($L \cong G_{i+1}/\mathrm{dom}(q)$). Consequently, there is a lifting f of L and now $\langle f, q\rangle$ extends q and it is in $\mathbb{P}'_{h,H,G}$. □

Proposition B.6.9. *The forcing notion $\mathbb{P}_{h,H,G}$ is strongly complete for $\hat{\mathcal{E}}^S_0$.*

Proof. The two parts, not adding bounded subsets of κ and completeness for $\hat{\mathcal{E}}^S_0$, are similar to those for uniformization, so we do just the second.

Assume now that χ is a regular large enough cardinal, $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\bar{N} = \langle N_i : i \leq \delta\rangle$, $\bar{N}\restriction(i+1) \in N_{i+1}$, $N_i \cap \kappa \in \kappa$ is limit, $\bar{N}$ obeys $\bar{a} \in \hat{\mathcal{E}}^S_0$ and $\bar{p} = \langle p_i : i < \delta\rangle$ is generic for $\bar{N}$ with error n, let γ_i be such that $\mathrm{dom}(p_i) = G_{\gamma_i+1}$ if possible, zero otherwise (no big lost if we assume that always the first possibility occurs). In particular, $p_i \in N_{i+1}$ and as γ_i is computable from $\bar{G}, p_i$ we know that $\gamma_i \in N_{i+1}$.

Let $\beta_i = \sup(N_i \cap \kappa)$ (so the sequence $\langle \beta_i : i \leq \delta\rangle$ is increasing continuous). Note that

$$p_{i+n} \in \bigcap \{\mathcal{I} \in N_{i+1} : \mathcal{I} \subseteq \mathbb{P}'_{h,H,G} \text{ is open dense}\}$$

and $N_i, \beta_i \in N_{i+1}$. Moreover, the set

$$\mathcal{I}_{\beta_i} = \{q \in \mathbb{P}'_{h,H,G} : \mathrm{dom}(q) \supseteq G_{\beta_i} \text{ and } \mathrm{Dom}(q) = \gamma + 1 \text{ for some ordinal } \gamma\}$$

is open dense in $\mathbb{P}'_{h,H,G}$. So $p_{i+n} \in \mathcal{I}_{\beta_i+1} \in N_{i+1}$ and $\gamma_{i+n} > \beta_i$. Now, $\mathrm{dom}(\bigcup\limits_{i<\delta} p_i) = G_{\bigcup\limits_{i<\delta}(\gamma_i+1)}$ and $\bigcup\limits_{i<\delta}(\gamma_i + 1) = \bigcup\limits_{i<\delta} \beta_i = N_\delta \cap \kappa$. Since $N_\delta \cap \kappa \notin S$ and $S \supseteq \Gamma[G]$ we conclude $N_\delta \cap \kappa \notin \Gamma[G]$, and thus $G_{N_\delta\cap\kappa+1}/G_{N_\delta\cap\kappa}$ is free. So we can complete to a condition. □

Proposition B.6.10. *The forcing notion $\mathbb{P}_{h,H,G}$ is complete for $(\hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^S_1)$.*

Proof. Suppose that $\bar{M} = \langle M_i : i \leq \delta\rangle$ is ruled by $(\hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^S_1)$. So $M_i \cap \kappa = a_i$ and $M_{i+1} = \bigcup\limits_{\zeta < \mathrm{cf}(a_{i+1})} N^i_\zeta$ and $(\bar{N}^i, \bar{b}^i)$ is an $\hat{\mathcal{E}}^S_0$–complementary pair, $\bar{b}^i \in \hat{\mathcal{E}}^S_0$ (also for $i = -1$).

We are dealing with the case $\delta \in M_0$. Recall:

Claim B.6.10.1. *There is G^+_i such that $G_i \subseteq G^+_i \subseteq G_{i+1}$, $\|G^+_i\| \leq \|G_i\| + \aleph_0$ and G_{i+1}/G^+_i is free. (Of course if $i \notin S$ is non-limit then G^+_i/G_i is free.)*

Proof of the claim. Since G_{i+1} is free we may fix a basis $\langle x_{i,\varepsilon} : \varepsilon < \varepsilon_{i+1}\rangle$ of it. Choose $A_i \subseteq \varepsilon_{i+1}$ such that $\|A_i\| \leq \|G_i\|$ and $G_i \subseteq \langle\{x_{i,\varepsilon} : \varepsilon \in A_i\}\rangle_G$ (and call the last group G^+_i). Then G_{i+1}/G^+_i is freely generated by $\{x_{i,\varepsilon} + G^+_i : \varepsilon \in \varepsilon_{i+1} \setminus A_i\}$. The claim is proved. □

Let $H^+_\alpha = h^{-1}[G^+_\alpha]$ and wlog $\langle G_i, G^+_i : i < \kappa\rangle \in M_0$.

Thus if $i < j$ then G_j/G^+_i is free. All action will be in $G^+_{a_i}/G_{a_i}$ for limit $i \leq \delta$. Necessarily a_i is a singular cardinal of small cofinality ($\leq \delta < a_0$). [Remember $a_i = M_i \cap \kappa$ and $\sup(M_i \cap \kappa)$ is a limit cardinal. Why? If not then there is a cardinal λ such that $\lambda < \sup(M_i \cap \kappa) < \lambda^+$, so there is $\gamma \in M_i \cap \kappa$ such that $\lambda < \gamma < \sup(M_i \cap \kappa) < \lambda^+$. Hence $\lambda^+ = \|\gamma\|^+ \in M_i$, a contradiction.]

We may have "a difficulty" in defining $p{\restriction}G^+_{a_i}$, so we should "think" about it earlier. This will mean defining $p{\restriction}G_{a_{j+1}}$, $j<i$. The player COM can give only a condition in M_{j+1}, and we will arrange that our "prepayments" are of "size" a_j (so bounded in M_{j+1} and thus included in some N^j_ζ, $\zeta<\mathrm{cf}(a_{j+1})$; they will even belong to it).

Let $r\in\mathbb{P}'_{h,H,G}\cap M_0$. [Remember: $G_{a_0}/\mathrm{dom}(r)$ is free, so there is a lifting.] Let INC choose non-limit $i_0<\delta$ and $p'_0\in M_{i_0}\cap\mathbb{P}_{h,H,G}$ above p, and $\bar q_0=\langle q_{i_0,\zeta}:\zeta<\delta_{i_0-1}\rangle$ *generic* for some end segment of $\bar N_{i_0-1}$.

We choose by induction on $i\leq\delta$ models $\mathcal{B}_i\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ such that

- $\bar G,\bar M,\langle\bar N^i:i\leq\delta\rangle,\langle H_i:i\leq\delta\rangle,\ldots\in\mathcal{B}_i$,
- the sequence $\langle\mathcal{B}_i:i\leq\delta\rangle$ is increasing (but not continuous),
- $\|\mathcal{B}_i\|=a_i$, $a_i+1\subseteq\mathcal{B}_i$ and $\langle\mathcal{B}_j:j<i\rangle\in\mathcal{B}_i$,
- $\mathcal{B}_i\cap M_j\in M_{j+1}$ if $i<j$.

(But see for additional requirements later.)

The rest of the moves are indexed by $i\in[i_0-1,\delta)$ and in the $i^{\rm th}$ move COM chooses $p_i\in M_i$ and INC plays $\bar q^i=\langle q^i_\zeta:\zeta<\delta_i\rangle$ as in the definition of the game.

Now COM will choose on a side also $f_i\in\mathbb{P}_{h,H,G}$ for $i\in[i_0-1,\delta)$ such that additionally:

$(*)_1$ $f_i\in\mathbb{P}_{h,H,G}$ is a function with domain $\mathcal{B}_i\cap G^+_{a_\delta}$, increasing with i,
$(*)_2$ $a_i\subseteq\mathrm{dom}(f_i)$,
$(*)_3$ $f_i{\restriction}a_{i+1}=f_i{\restriction}(a_{i+1}\cap\mathcal{B}_i)$ belongs to $\mathbb{P}_{h,H,G}$ and is below p_i.

Note that

$(\oplus)$ $\mathcal{B}_j\cap M_{j+1}\cap G^+_{a_\delta}$ is a subset of $G_{M_{j+1}\cap\kappa}$ of cardinality $\|M_j\|<\mathrm{cf}(\delta_j)$, hence it belongs to M_{j+1}.

For $i=i_0-1$ let $f_{i_0-1}\in\mathbb{P}_{h,H,G}$ be above p and have domain $\mathcal{B}_{i_0-1}\cap G^+_{a_\delta}$, and let $p_{i_0-1}=f_0{\restriction}(\mathcal{B}_{i_0-1}\cap G^+_{a_\delta}\cap M_{i_0})$. Clearly $[M_{i_0}]^{2^{\|M_{i_0-1}\|}}\subseteq M_{i_0}$, and $\|\mathcal{B}_{i_0-1}\|=a_i$ and p_{i_0-1} is a function extending p, its domain belongs to M_{i_0} and it is a subgroup of $G^+_{a_\delta}$. Consequently, p_{i_0-1} a lifting and is in M_{i_0}. By manipulating bases (or see [11]) we have

- $\mathrm{Dom}(p)\subseteq\mathrm{Dom}(f_i)\subseteq G^+_{a_\delta}$,
- $G^+_{a_\delta}/\mathrm{Dom}(f_i)=G^+_{a_\delta}/(G^+_{a_\delta}\cap\mathcal{B}_{i_0-1})$ is free as $G^+_{a_\delta}$ is free and $G^+_{a_\delta}\in\mathcal{B}_{i_0-1}$,
- $\mathrm{Dom}(f_i)/\mathrm{Dom}(p_i)$ is free as it is equal to $G^+_{a_\delta}\cap\mathcal{B}_{i_0-1}/G^+_{a_\delta}\cap\mathcal{B}_{i_0-1}\cap M_{i_0})$ and $G^+_{a_\delta}\cap M_{i_0}\subseteq G^+_{a_\delta}$ and they belong to $\mathcal{B}_{i_0-1}$, and $G^+_{a_\delta}/(G^+_{a_\delta}\cap M_{i_0})$ is free as $M_{i_0}\cap\kappa\notin S$, so κ–free.

For $i=j+1\geq i_0$ we have f_j, p_j and $\bar q'_j=\langle q^j_\zeta:\zeta<\delta_j\rangle$. Let $q'_j=\bigcup\limits_{\zeta<\delta_j}q_\zeta$. So as $\mathrm{dom}(q'_j)=a_{j+1}=a_i\notin S$ (by the choice of $\hat{\mathcal{E}}_1$), clearly $q'_j\in\mathbb{P}_{h,H,G}$. We have to find $p_i\in\mathbb{P}_{h,H,G}\cap M_{i+1}$ above q'_j and $f_j{\restriction}M_{i+1}$ (and then choose f_i). Clearly the domains of q'_j, $f_j{\restriction}M_{i+1}$ are pure subgroups, $\mathrm{Dom}(q'_j)=G\cap M_i=G_{M_i\cap\kappa}=G_{a_i}$ and p_i, $f_j{\restriction}M_j$ agree on their intersection (which is $\mathcal{B}_j\cap M_{j+1}$). Hence there is a common extension p'_i, a homomorphism from $G_{a_i}+(\mathcal{B}_j\cap M_{i+1})$ to H, which clearly is a lifting. Does $p'_i\in\mathbb{P}_{h,H,G}$? For this it suffices to show that the group $G^+_{a_\delta}/\mathrm{dom}(p'_i)$ is free. But $G^+_{a_\delta}/G^+_{a_{j+1}}$ is free, hence $(G^+_{a_\delta}/G_{a_{j+1}})/\mathcal{B}_j\cap(G^+_{a_\delta}/G_{a_{j+1}})$ is free (see [11]). Therefore $G^+_{a_\delta}/(\mathcal{B}_j\cap G^+_{a_\delta}+G_{a_{j+1}})$ is free. Also $(\mathcal{B}_j\cap G^+_{a_\delta}+G_{a_{j+1}})/(\mathcal{B}_j\cap G_{a_{i+1}}+G_{a_{j+1}})$ is free (see [11]). Together, $G^+_{a_\delta}/(\mathcal{B}_j\cap G_{a_{i+1}}+G_{a_{j+1}})$ is free as required.

We are left with the case of limit i. Let $q'_i = \bigcup\{q'_j : i_0 - 1 \le j < i\}$. Then q'_i is a lifting for G_{a_i}. Now clearly $f'_i = \bigcup\{f_j : i_0 - 1 \le j < i\}$ is a lifting for $G^+_{a_\delta} \cap \bigcup\limits_{j<i} \mathcal{B}_j$, also $G^+_{a_\delta}/\mathrm{dom}(f'_i)$ is free (see [11]) and $G^+_{a_i} \in \mathcal{B}_0$, $\|G^+_{a_i}\| = \|G_{a_i}\| \subseteq \bigcup\limits_{j<i} \mathcal{B}_j$. Hence $G^+_{a_i} \subseteq \bigcup\limits_{j<i} \mathcal{B}_j$ and therefore $G^+_{a_i} \subseteq \mathrm{dom}(f'_i)$ and we can proceed to define p_i as above.

Having finished the play, again $\bigcup\{f_j : i_0 - 1 \le j < \delta\} \in \mathbb{P}_{h,H,G}$ (as in the limit case) is an upper bound as required. □

Remark B.6.11. In this section, we can replace $\hat{\mathcal{E}}_1$ by any $\hat{\mathcal{E}}^S_1$ defined below (or any subset which is rich enough):

$$\hat{\mathcal{E}}^S_1 = \big\{\bar{\alpha} = \langle \alpha_i : i \le \delta\rangle : \quad \bar{\alpha} \text{ is an increasing continuous sequence of ordinals from } \kappa, \quad a_{i+1} \notin S,\ \mathrm{cf}(a_{i+1}) > a_i \text{ and } S \cap a_{i+1} \text{ not stationary}\big\}.$$

B.7. The iteration theorem for inaccessible κ

In this section we prove the preservation theorem needed for our present case. Like in **Case A**, we will use trees of conditions. So, our way to prove the iteration theorem will be parallel to that of **Case A**.

Proposition B.7.1. *Assume that $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle$ is a $(<\kappa)$–support iteration of forcing notions which are strongly complete for $\hat{\mathcal{E}}$. Let $\mathcal{T} = (T, <, \mathrm{rk})$ be a standard $(w, \alpha_0)^\gamma$–tree (see A.3.3), $\|T\| < \kappa$, $w \subseteq \gamma$, α_0 an ordinal, and let $\bar{p} = \langle p_t : t \in T\rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$. Suppose that $\mathcal{I}$ is an open dense subset of $\mathbb{P}_\gamma$.* Then *there is $\bar{q} = \langle q_t : t \in T\rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ such that $\bar{p} \le \bar{q}$ and for each $t \in T$*

1. *$q_t \in \{q{\restriction}\mathrm{rk}(t) : q \in \mathcal{I}\}$, and*
2. *for each $\alpha \in \mathrm{dom}(q_t)$, either $q_t(\alpha) = p_t(\alpha)$ or $\Vdash_{\mathbb{P}_\alpha} q_t(\alpha) \in \underset{\sim}{\mathbb{Q}}_\alpha$ (not just in the completion $\hat{\underset{\sim}{\mathbb{Q}}}_\alpha$).*

Proof. Let $\langle t_i : i < i(*)\rangle$ be an enumeration of T such that

$$(\forall i, j < i(*))(t_i < t_j \quad \Rightarrow \quad i < j).$$

We are proving the proposition by induction on $i(*)$.

Case 1: $i(*) = 1$.
In this case $T = \{\langle\rangle\}$ and we have to choose $q_{\langle\rangle}$ only, but this is easy, as the set $\{q{\restriction}\mathrm{rk}(\langle\rangle) : q \in \mathcal{I}\}$ is open dense in $\mathbb{P}_{\mathrm{rk}(\langle\rangle)}$.

Case 2: $i(*) = i_0 + 1 > 1$.
Let $T^* = \{t_i : i < i_0\}$ and let $\mathcal{T}^* = \mathcal{T}{\restriction}T^*$. Then T^* is a standard $(w, \alpha_0)^\gamma$–tree to which we may apply the inductive hypothesis. Consequently we find $\langle q^*_t : t \in T^*\rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ such that for each $t \in T^*$:

1. $p_t \le q^*_t \in \{q{\restriction}\mathrm{rk}(t) : q \in \mathcal{I}\}$, and
2. for each $\alpha \in \mathrm{dom}(q^*_t)$, either $q^*_t(\alpha) = p_t(\alpha)$ or $\Vdash_{\mathbb{P}_\alpha} q^*_t(\alpha) \in \underset{\sim}{\mathbb{Q}}_\alpha$.

Let $q^0 = \bigcup\{q^*_s : s < t_{i_0}\}$ (note that $s < t_{i_0} \Rightarrow s \in T^*$; and also $s_1 < s_2 < t_{i_0}$ implies $q^*_{s_1} = q^*_{s_2}{\restriction}\mathrm{rk}'(s_1)$, hence easily $q^0 \in \mathbb{P}'_{\mathrm{rk}(t_{i_0})}$). Clearly q^0 and $p_{t_{i_0}}$ are compatible (actually q^0 is stronger then the suitable restriction of $p_{t_{i_0}}$) and therefore we may find a condition $q_{t_{i_0}} \in \mathbb{P}_{\mathrm{rk}(t_{i_0})}$ (note: no primes now) such that $q_{t_{i_0}} \in$

$\{q{\restriction}\mathrm{rk}(t_{i_0}) : q\in\mathcal{I}\}$ and $q_{t_{i_0}}$ stronger than both q^0 and $p_{t_{i_0}}$. Next, for each $t\in T^*$ let

$$q_t \stackrel{\rm def}{=} q_{t_{i_0}}{\restriction}\mathrm{rk}(t\cap t_{i_0})\cup q^*_t{\restriction}[\mathrm{rk}(t\cap t_{i_0}),\gamma)\geq q^*_t\geq p_t.$$

One easily checks that $\bar q=\langle q_t : t\in T\rangle$ is as required.

Case 3: $i(*)$ is a limit ordinal.
Let $\theta=\mathrm{cf}(i(*))$ and let $\langle i_\zeta:\zeta\leq\theta\rangle$ be an increasing continuous sequence, $i_0=0$, $i_\theta=i(*)$. For $\alpha<\gamma$, let $\underset{\sim}{x}_\alpha$ be a $\mathbb{P}_\alpha$–name for a witness that $\underset{\sim}{\mathbb{Q}}_\alpha$ is (forced to be) strongly complete for $\hat{\mathcal{E}}$ and let $x=\langle \underset{\sim}{x}_\alpha:\alpha<\gamma\rangle$. Take an $\hat{\mathcal{E}}$–complementary pair $(\bar N,\bar a)$ of length θ such that $\langle i_\zeta:\zeta<\theta\rangle,\bar p,\bar{\mathbb{Q}},\hat{\mathcal{E}},x,T\in N_0$ and $\|T\|\subseteq N_0$ (exists as $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$ is closed: first take a complementary pair of length $\|T\|^+$ and then restrict it to the interval $[\|T\|+1,\|T\|+\theta]$).

By induction on $\zeta\leq\theta$ we define a sequence $\langle\bar q^\zeta:\zeta\leq\theta\rangle$:

$\bar q^\zeta=\langle q^\zeta_t : t\in T\rangle$ is the $<^*_\chi$–first sequence $\bar r=\langle r_t:t\in T\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ such that

- (i)$_\zeta$ for every $t\in T$: $p_t\leq r_t$ and $(\forall\xi<\zeta)(q^\xi_t\leq r_t)$ and if $\alpha\in\mathrm{dom}(r_t)$, $p_t(\alpha)\neq r_t(\alpha)$, then $r_t(\alpha)$ is a name for an element of $\underset{\sim}{\mathbb{Q}}_\alpha$ (not the completion),
- (ii)$_\zeta$ if $i_\zeta\leq i<i(*)$ and $\sup\{\mathrm{rk}(t_j):j<i_\zeta\ \&\ t_j<t_i\}\leq\alpha<\mathrm{rk}(t_i)$, then $r_{t_i}(\alpha)=p_{t_i}(\alpha)$,
- (iii)$_\zeta$ if $i<i_\zeta$, then
$$r_{t_i}\in\bigcap\big\{\mathcal{J}\in N_\zeta:\mathcal{J}\subseteq\mathbb{P}_{\mathrm{rk}(t_i)}\text{ is open dense}\big\}.$$

To show that this definition is correct we have to prove that arriving at a stage $\zeta\leq\theta$ of the construction we may find $\bar r$ satisfying (i)$_\zeta$–(iii)$_\zeta$. Note that once we know that we may define $\bar q^\xi$ for $\xi\leq\zeta$, we are sure that $\langle\bar q^\xi:\xi\leq\zeta\rangle\in N_{\zeta+1}$ (remember $\bar N{\restriction}(\zeta+1)\in N_{\zeta+1}$). Similarly, arriving at a limit stage $\zeta<\theta$ we are sure that $\langle\bar q^\xi:\xi<\zeta\rangle\in N_{\zeta+1}$.

Stage $\zeta=0$.
Look at $\bar r=\bar p$: as $i_0=0$, the clause (iii)$_0$ is empty and (i)$_0$, (ii)$_0$ are trivially satisfied.

Stage $\zeta=\xi+1$.
Let $T^*=\{t_i:i<i_\zeta\}$, $\bar p^*=\langle q^\xi_t:t\in T^*\rangle$. We may apply the inductive hypothesis to T^*, p^* and

$$\mathcal{I}^*\stackrel{\rm def}{=}\bigcap\big\{\mathcal{J}\in N_\zeta:\mathcal{J}\subseteq\mathbb{P}_\gamma\text{ is open dense}\big\}$$

(remember $i_\zeta<i(*)$ and $\mathbb{P}_\gamma$ does not add new $<\kappa$–sequences of ordinals, see B.5.6, so $\mathcal{I}^*$ is open dense). Consequently we find $\bar s=\langle s_t:t\in T^*\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ such that for each $t\in T^*$:

- $q^\xi_t\leq s_t\in\{q{\restriction}\mathrm{rk}(t):q\in\mathcal{I}^*\}$, and
- for each $\alpha\in\mathrm{dom}(s_t)$, either $q^\xi_t(\alpha)=s_t(\alpha)$ or $\Vdash_{\mathbb{P}_\alpha} s_t(\alpha)\in\underset{\sim}{\mathbb{Q}}_\alpha$.

For $t\in T\setminus T^*$ let $\alpha_t=\sup\{\mathrm{rk}(t_i):i<i_\zeta\ \&\ t_i<t\}$. Note that, for $t\in T\setminus T^*$, $\bigcup\{s_{t_i}:i<i_\zeta\ \&\ t_i<t\}$ is a condition in $\mathbb{P}'_{\alpha_t}$ stronger than $q^\xi_t{\restriction}\alpha_t$. So let

$$r_t=\bigcup\{s_{t_i}:i<i_\zeta\ \&\ t_i<t\}\cup q^\xi_t{\restriction}[\alpha_t,\gamma)=\bigcup\{s_{t_i}:i<i_\zeta\ \&\ t_i<t\}\cup p_t{\restriction}[\alpha_t,\gamma)$$

for $t\in T\setminus T^*$ and $r_t=s_t$ for $t\in T^*$. It should be clear that $\bar r=\langle r_t:t\in T\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ satisfies the demands (i)$_\zeta$–(iii)$_\zeta$.

Stage ζ is a limit ordinal.
As we noted before, we know that $\langle \bar{q}^\varepsilon : \varepsilon \le \xi \rangle \in N_{\xi+1}$ for each $\xi < \zeta$. Hence, as $T \subseteq N_0$ (remember $\|T\| \subseteq N_0$ and $T \in N_0$), we have $\langle q^\varepsilon_t : \varepsilon \le \xi \rangle \in N_{\xi+1}$ for each $t \in T$ and $\xi < \zeta$. Fix $i < i_\zeta$ and let $\xi < \zeta$ be such that $i < i_\xi$. Look at the sequence $\langle q^\varepsilon_{t_i} : \xi \le \varepsilon < \zeta \rangle$. By the choice of $\bar{q}^\varepsilon$ (see demands (i)$_\varepsilon$ and (iii)$_\varepsilon$) we have that it is an increasing $(\bar{N}{\restriction}[\xi,\zeta), \mathbb{P}_{\mathrm{rk}(t_i)})^*$–generic sequence (note no primes; if we are not in $\underset{\sim}{\mathbb{Q}}_\alpha$, then the value is fixed). By B.5.6 the forcing notion $\mathbb{P}_{\mathrm{rk}(t_i)}$ is complete for $\hat{\mathcal{E}}$ (and N_ξ contains the witness), so $\langle q^\varepsilon_{t_i} : \xi \le \varepsilon < \zeta \rangle$ has an upper bound in $\mathbb{P}_{\mathrm{rk}(t_i)}$. Moreover, for each $\alpha < \mathrm{rk}(t_i)$, if $q \in \mathbb{P}_\alpha$ is an upper bound of $\langle q^\varepsilon_{t_i}{\restriction}\alpha : \varepsilon < \zeta \rangle$, then

$$q \Vdash_{\mathbb{P}_\alpha} \text{“the sequence } \langle q^\varepsilon_{t_i}(\alpha) : \varepsilon < \zeta \rangle \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\alpha\text{”.}$$

Now, for $t \in T$ we may let $\mathrm{dom}(r_t) = \bigcup\limits_{\varepsilon<\zeta} \mathrm{dom}(q^\varepsilon_t)$ and define inductively $r_t(\alpha)$ for $\alpha \in \mathrm{dom}(r_t)$ by

if $(\forall \varepsilon < \zeta)(q^\varepsilon_t(\alpha) = p_t(\alpha))$, then $r_t(\alpha) = p_t(\alpha)$, and otherwise
$r_t(\alpha)$ is the $<^*_\chi$–first $\mathbb{P}_\alpha$–name for an element of $\underset{\sim}{\mathbb{Q}}_\alpha$ such that

$$r_t{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} (\forall \varepsilon < \zeta)(q^\varepsilon_t(\alpha) \le_{\underset{\sim}{\mathbb{Q}}_\alpha} r_t(\alpha)).$$

It is a routine to check that $\bar{r} = \langle r_t : t \in T \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ and it satisfies (i)$_\zeta$–(iii)$_\zeta$.

Thus our definition is correct and we may look at the sequence $\bar{q}^\theta$. Since $\mathcal{I} \in N_0$ it should be clear that it is as required. This finishes the inductive proof of the proposition. $\square$

Our next proposition corresponds to A.3.6. However, note that the meaning of $*$'s is slightly different now. The difference comes from another type of the game involved and it will be more clear in the proof of theorem B.7.3 below.

Proposition B.7.2. *Assume that $\hat{\mathcal{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a $(<\kappa)$–support iteration and $x = \langle \underset{\sim}{x}_\alpha : \alpha < \gamma \rangle$ is such that*

$$\Vdash_{\mathbb{P}_\alpha} \text{“} \underset{\sim}{\mathbb{Q}}_\alpha \textit{ is strongly complete for } \hat{\mathcal{E}} \textit{ with witness } \underset{\sim}{x}_\alpha \text{”}$$

(for $\alpha < \gamma$). Further suppose that

(α) *$(\bar{N}, \bar{a})$ is an $\hat{\mathcal{E}}$–complementary pair, $\bar{N} = \langle N_i : i \le \delta \rangle$, and $x, \hat{\mathcal{E}}, \bar{\mathbb{Q}} \in N_0$,*

(β) *$\mathcal{T} = (T, <, \mathrm{rk}) \in N_0$ is a standard $(w, \alpha_0)^\gamma$–tree, $w \subseteq \gamma \cap N_0$, $\|w\| < \mathrm{cf}(\delta)$, α_0 is an ordinal, $\alpha_1 = \alpha_0 + 1$, $0 \in w$,*

(γ) *$\bar{p} = \langle p_t : t \in T \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}}) \cap N_0$, $w \in N_0$, (of course $\alpha_0 \in N_0$),*

(δ) *$\|N_i\|^{\|w\|+\|T\|} \subseteq N_{i+1}$ for each $i < \delta$,*

(ε) *for $i \le \delta$, $\mathcal{T}_i = (T_i, <_i, \mathrm{rk}_i)$ is such that T_i consists of all sequences $t = \langle t_\zeta : \zeta \in \mathrm{dom}(t) \rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w, and*
- *each t_ζ is a sequence of length α_1,*
- *$\langle t_\zeta{\restriction}\alpha_0 : \zeta \in \mathrm{dom}(t) \rangle \in T$,*
- *for each $\zeta \in \mathrm{dom}(t)$, either $t_\zeta(\alpha_0) = *$ or $t_\zeta(\alpha_0) \in N_i$ is a $\mathbb{P}_\zeta$–name for an element of $\underset{\sim}{\mathbb{Q}}_\zeta$ and if $t_\zeta(\alpha) \ne *$ for some $\alpha < \alpha_0$, then $t_\zeta(\alpha_0) \ne *$,*

$\mathrm{rk}_i(t) = \min(w \cup \{\gamma\} \setminus \mathrm{dom}(t))$ and $<_i$ is the extension relation.

Then

(a) *each $\mathcal{T}_i$ is a standard $(w, \alpha_1)^\gamma$–tree, $\|T_i\| \le \|T\| \cdot \|N_i\|^{\|w\|}$, and if $i < \delta$ then $T_i \in N_{i+1}$,*

(b) *$\mathcal{T}$ is the projection of each $\mathcal{T}_i$ onto (w, α_0),*

(c) *there is* $\bar{q} = \langle q_t : t \in T_\delta \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ *such that*

(i) $\bar{p} \leq_{\mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}}} \bar{q}$,

(ii) *if* $t \in T_\delta \setminus \{\langle\rangle\}$ *then the condition* $q_t \in \mathbb{P}'_{\mathrm{rk}_\delta(t)}$ *is an upper bound of an* $(\bar{N}{\restriction}[i_0,\delta], \mathbb{P}_{\mathrm{rk}_\delta(t)})^*$*–generic sequence (where* $i_0 < \delta$ *is such that* $t \in T_{i_0}$*), and for every* $\beta \in \mathrm{dom}(q_t) = N_\delta \cap \mathrm{rk}_\delta(t)$*,* $q_t(\beta)$ *is a name for the least upper bound in* $\hat{\mathbb{Q}}_\beta$ *of an* $(\bar{N}[G_\beta]{\restriction}[\xi,\delta), \mathbb{Q}_\beta)^*$*–generic sequence (for some* $\xi < \delta$*),*
[Note that, by B.5.5, the first part of the demand on q_t *implies that if* $i_0 \leq \xi$ *then* $q_t{\restriction}\beta$ *forces that* $(\bar{N}[G_\beta]{\restriction}[\xi,\delta], \bar{a}{\restriction}[\xi,\delta])$ *is an* $\hat{\mathcal{S}}$*–complementary pair.]*

(iii) *if* $t \in T_\delta$*,* $t' = \mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}}(t) \in T$*,* $\zeta \in \mathrm{dom}(t)$ *and* $t_\zeta(\alpha_0) \neq *$*, then*

$$q_t{\restriction}\zeta \Vdash_{\mathbb{P}_\zeta} \text{“}p_{t'}(\zeta) \leq_{\hat{\mathbb{Q}}_\zeta} t_\zeta(\alpha_0) \ \Rightarrow\ t_\zeta(\alpha_0) \leq_{\hat{\mathbb{Q}}_\zeta} q_t(\zeta)\text{”},$$

(iv) $q_{\langle\rangle} = p_{\langle\rangle}$.

Proof. Clauses (a) and (b) should be clear.

(c) One could try to use directly B.7.1 for $\bigcap\{\mathcal{I} \in N_\delta : \mathcal{I} \subseteq \mathbb{P}_\gamma \text{ open dense}\}$ and suitably “extend” $\bar{p}$ (see, e.g., the successor case below). However, this would not guarantee the demand (ii). This clause is the reason for the assumption that $\|w\| < \mathrm{cf}(\delta)$.

By induction on $i < \delta$ we define a sequence $\langle \bar{q}^i : i < \delta \rangle$:

$\bar{q}^i = \langle q^i_t : t \in T_i \rangle$ is the $<^*_\chi$–first sequence $\bar{r} = \langle r_t : t \in T_i \rangle \in \mathrm{FTr}'(\bar{\mathbb{Q}})$ such that

(i)$_i$ $\bar{p} \leq_{\mathrm{proj}^{\mathcal{T}_i}_{\mathcal{T}}} \bar{r}$ and $(\forall j < i)(\forall t \in T_j)(q^j_t \leq_{\mathbb{P}'_{\mathrm{rk}(t)}} r_t)$,

(ii)$_i$ if $t = \langle t_\zeta : \zeta \in \mathrm{dom}(t) \rangle \in T_i$ and $t' = \mathrm{proj}^{\mathcal{T}_i}_{\mathcal{T}}(t) \in T$, then

- $(\forall \alpha \in \mathrm{dom}(r_t))(p_{t'}(\alpha) = r_t(\alpha) \ \text{ or } \ \Vdash_{\mathbb{P}_\alpha} r_t(\alpha) \in \mathbb{Q}_\alpha)$, and
- $r_t \in \bigcap\{\mathcal{I} \in N_i : \mathcal{I} \subseteq \mathbb{P}_{\mathrm{rk}_i(t)}$ is open dense$\}$, and
- for every $\zeta \in \mathrm{dom}(t)$ such that $t_\zeta(\alpha_0) \neq *$,

$$r_t{\restriction}\zeta \Vdash_{\mathbb{P}_\zeta} \text{“}p_{t'}(\zeta) \leq_{\mathbb{Q}_\zeta} t_\zeta(\alpha_0) \quad\Rightarrow\quad t_\zeta(\alpha_0) \leq_{\mathbb{Q}_\zeta} r_t(\zeta)\text{”},$$

(iii)$_i$ $r_{\langle\rangle} = p_{\langle\rangle}$.

We have to verify that this definition is correct, i.e., that for each $i < \delta$ there is an $\bar{r}$ satisfying (i)$_i$–(iii)$_i$. So suppose that we arrive to a non-limit stage $i < \delta$ and we have defined $\langle \bar{q}^j : j < i \rangle$. Note that necessarily $\langle \bar{q}^j : j < i \rangle \in N_i$ (remember i is non-limit). Let $i = j+1$ and, if $j = -1$, let $q^{-1}_t = p_{\mathrm{proj}^{\mathcal{T}_0}_{\mathcal{T}}(t)}$ for $t \in T_0$ and let $T_{-1} = \{\langle\rangle\}$. For $t \in T_i$ we define $s_t \in \mathbb{P}_{\mathrm{rk}_i(t)}$ as follows.

- If $t \in T_j$, then $s_t = q^j_t$.
- If $t \in T_i \setminus T_j$ and $\zeta^* \in w$ is the first such that $t{\restriction}(\zeta^*+1) \notin T_j$, then we let $\mathrm{dom}(s_t) = \mathrm{dom}(q^j_{t{\restriction}\zeta^*}) \cup \mathrm{dom}(p_{t'}) \cup \mathrm{dom}(t)$, where $t' = \mathrm{proj}^{\mathcal{T}_i}_{\mathcal{T}}(t)$. Next we define $s_t(\zeta)$ by induction on $\zeta \in \mathrm{dom}(s_t)$:
  if $\zeta \in \mathrm{dom}(s_t) \cap \zeta^*$, then $s_t(\zeta) = q^j_{t{\restriction}\zeta^*}(\zeta)$,
  if $\zeta \in \mathrm{dom}(t) \setminus \zeta^*$ and $t_\zeta(\alpha_0) \neq *$, then $s_t(\zeta)$ is the $<^*_\chi$–first $\mathbb{P}_\zeta$–name for an element of $\mathbb{Q}_\zeta$ such that

$$s_t{\restriction}\zeta \Vdash_{\mathbb{P}_\zeta} \text{“}p_{t'}(\zeta) \leq s_t(\zeta) \quad\text{and}\quad p_{t'}(\zeta) \leq t_\zeta(\alpha_0) \ \Rightarrow\ t_\zeta(\alpha_0) \leq s_t(\zeta)\text{”}$$

and otherwise it is $p_{t'}(\zeta)$.

It should be clear that $\bar{s}=\langle s_t : t\in T_i\rangle\in \mathrm{FTr}'(\bar{\mathbb{Q}})$. Now we apply B.7.1 to T_i, $\bar{s}$ and

$$\mathcal{I}^*\stackrel{\rm def}{=}\bigcap\big\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{P}_\gamma \text{ open dense}\big\}$$

and we find $\bar{r}=\langle r_t : t\in T_i\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ such that $\bar{s}\leq\bar{r}$ and for each $t\in T_i$

$$r_t\in\{q{\restriction}\mathrm{rk}_i(t)\colon q\in\mathcal{I}^*\}\quad\text{and}\quad(\forall\alpha\in\mathrm{dom}(r_t))(s_t(\alpha)=r_t(\alpha)\text{ or }\Vdash_{\mathbb{P}_\alpha} r_t(\alpha)\in\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha).$$

One easily checks that this $\bar{r}$ satisfies demands $(\mathrm{i})_i$–$(\mathrm{iii})_i$.

Now suppose that we have successfully defined $\bar{q}^j$ for $j<i$, $i<\delta$ limit ordinal. Fix $t\in\bigcup\limits_{j<i}T_j$, say $t\in T_{j_0}$, $j_0<i$. We know that $T_{j_0}\subseteq N_{j_0+1}$ (remember the assumption (δ) and the assertion (a)) and that for each $j<i$, $\langle\bar{q}^\varepsilon:\varepsilon\leq j\rangle\in N_{j+1}$. Consequently,

$$(\forall j\in[j_0,i))(\langle q^\varepsilon_t : j_0\leq\varepsilon\leq j\rangle\in N_{j+1}).$$

By the demand $(\mathrm{ii})_\varepsilon$ we have that $\langle q^\varepsilon_t : j_0\leq\varepsilon<i\rangle$ is an $(\bar{N}{\restriction}[j_0,i],\mathbb{P}_{\mathrm{rk}_{j_0}(t)})^*$–generic sequence. As $\mathbb{P}_{\mathrm{rk}_{j_0}(t)}$ is complete for $\hat{\mathcal{E}}$ (see B.5.6) and N_0 contains all witnesses we conclude that the sequence $\langle q^\varepsilon_t : j_0\leq\varepsilon<i\rangle$ has an upper bound in $\mathbb{P}_{\mathrm{rk}_{j_0}(t)}$. Moreover, if $\alpha<\mathrm{rk}_{j_0}(t)$, and $q\in\mathbb{P}_\alpha$ is an upper bound of the sequence $\langle q^\varepsilon_t{\restriction}\alpha : j_0\leq\varepsilon<i\rangle$, then

$$q\Vdash_{\mathbb{P}_\alpha}\text{“}\langle q^\varepsilon_t(\alpha) : j_0\leq\varepsilon<i\rangle\text{ has an upper bound in }\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha\text{”}$$

(see the proof of B.5.6). Now we let $\mathrm{dom}(s_t)=\bigcup\{\mathrm{dom}(q^\varepsilon_t) : j_0\leq\varepsilon<i\}$ and we define inductively

$s_t(\alpha)$ is the $<^*_\chi$–first $\mathbb{P}_\alpha$–name for an element of $\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha$ such that

$$s_t{\restriction}\alpha\Vdash_{\mathbb{P}_\alpha}(\forall\varepsilon\in[j_0,i))(q^\varepsilon_t(\alpha)\leq_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha} s_t(\alpha)).$$

This defines $\bar{s}^0=\langle s_t : t\in\bigcup\limits_{j<i}T_j\rangle$. Clearly $\bigcup\limits_{j<i}T_j$ is a standard $(w,\alpha_1)^\gamma$–tree and $\bar{s}\in\mathrm{FTr}'(\bar{\mathbb{Q}})$. Now suppose that $t\in T_i\setminus\bigcup\limits_{j<i}T_j$ and let ζ^* be the first such that $t{\restriction}\zeta^*\notin\bigcup\limits_{j<i}T_j$ (so necessarily $\mathrm{dom}(t)\cap\zeta^*$ is cofinal in ζ^* and $\mathrm{cf}(\mathrm{otp}(\mathrm{dom}(t)\cap\zeta^*))=\mathrm{cf}(i)$). Then $\bigcup\{s_{t\restriction\zeta} : \zeta<\zeta^*\}\in\mathbb{P}_{\zeta^*}$. Now define

$$s_t=\bigcup\{s_{t\restriction\zeta} : \zeta<\zeta^*\}\cup p_{t'}{\restriction}[\zeta^*,\gamma),$$

where $t'=\mathrm{proj}^{\mathcal{T}_i}_{\mathcal{T}}(t)$. Note that $\bar{s}=\langle s_t : t\in T_i\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ and if $t\in T_i$, $\alpha\in\mathrm{dom}(s_t)$, then either $s_t(\alpha)=p_{t'}(\alpha)$ or $\Vdash_{\mathbb{P}_\alpha} s_t(\alpha)\in\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha$. Now we proceed like in the successor case: we apply B.7.1 to $\bar{s}$, T_i and

$$\mathcal{I}^*\stackrel{\rm def}{=}\bigcap\big\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{P}_\gamma\text{ open dense}\big\},$$

and as a result we get $\bar{r}=\langle r_t : t\in T_i\rangle\in\mathrm{FTr}'(\bar{\mathbb{Q}})$ such that for each $t\in T_i$:

$$s_t\leq r_t\in\{q{\restriction}\mathrm{rk}_i(t) : q\in\mathcal{I}^*\}\quad\text{and}$$
$$(\forall\alpha\in\mathrm{dom}(r_t))(s_t(\alpha)=r_t(\alpha)\quad\text{or}\quad\Vdash_{\mathbb{P}_\alpha} r_t(\alpha)\in\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha).$$

Now one easily checks that $\bar{r}$ satisfies the requirements $(\mathrm{i})_i$–$(\mathrm{iii})_i$.

Thus our definition is the legal one and we have the sequence $\langle\bar{q}^i : i<\delta\rangle$. We define $\bar{q}=\bar{q}^\delta$ similarly to $\bar{s}$ from the limit stages $i<\delta$, but we replace “the $<^*_\chi$–first upper bound in $\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha$” by “the least upper bound in $\hat{\mathbb{Q}}_\alpha$”. So suppose

$t\in T_\delta$. Since $\|w\|<{\rm cf}(\delta)$ we know that $t\in T_{j_0}$ for some $j_0<\delta$. We declare ${\rm dom}(q_t)=\bigcup\{{\rm dom}(q^\varepsilon_t):j_0\leq\varepsilon<\delta\}$ and inductively define $q_t(\alpha)$ for $\alpha\in{\rm dom}(q_t)$:

$q_t(\alpha)$ is the $<^*_\chi$–first $\mathbb{P}_\alpha$–name such that

$$q_t{\restriction}\alpha\Vdash_{\mathbb{P}_\alpha}\text{“}q_t(\alpha)\text{ is the least upper bound of the sequence } \langle q^\varepsilon_t(\alpha):j_0\leq\varepsilon<\delta\rangle \text{ in } \hat{\mathbb{Q}}_\alpha\text{”}.$$

Like in the limit case of the construction, the respective upper bounds exist, so $\bar{q}=\langle q_t:t\in T_\delta\rangle$ is well defined. Checking that it has the required properties is straightforward. □

Theorem B.7.3. *Suppose that $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ (so $\hat{\mathcal{E}}_0\in\mathfrak{C}_{<\kappa}(\mu^*)$) and $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\mathbb{Q}_\alpha:\alpha<\gamma\rangle$ is a $(<\kappa)$–support iteration such that for each $\alpha<\kappa$*

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\mathbb{Q}_\alpha \textit{ is complete for } (\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\text{”}.$$

Then

(a) $\Vdash_{\mathbb{P}_\gamma}(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$, *moreover*

(b) $\mathbb{P}_\gamma$ *is complete for* $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$.

Proof. We need only part (a) of the conclusion, so we concentrate on it. Let χ be a large enough regular cardinal, x be a name for an element of $\mathcal{H}(\chi)$ and $p\in\mathbb{P}_\gamma$. Let x_α be a $\mathbb{P}_\alpha$–name for the witness that $\mathbb{Q}_\alpha$ is (forced to be) complete for $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$, and let $\bar{x}=\langle x_\alpha:\alpha<\gamma\rangle$. Since $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ we find $\bar{M}=\langle M_i:i\leq\delta\rangle$ which is ruled by $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ with an $\hat{\mathcal{E}}_0$–approximation $\langle\bar{N}^i:-1\leq i<\delta\rangle$ and such that $p,\bar{\mathbb{Q}},x,\bar{x},\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1\in M_0$ (see B.5.7). Let $\bar{N}^i=\langle N^i_\varepsilon:\varepsilon\leq\delta_i\rangle$ and let $\bar{a}^i\in\hat{\mathcal{E}}_0$ be such that $(\bar{N}^i,\bar{a}^i)$ is an $\hat{\mathcal{E}}_0$–complementary pair. Let $w_i=\{0\}\cup\bigcup\limits_{j<i}(\gamma\cap M_j)$ (for $i\leq\delta$). By the demands of B.5.7 we know that $\|w_i\|<{\rm cf}(\delta_i)$.

By induction on $i\leq\delta$ we define standard $(w_i,i)^\gamma$–trees $\mathcal{T}_i\in M_{i+1}$ and $\bar{p}^i=\langle p^i_t: t\in T_i\rangle\in{\rm FTr}'(\bar{\mathbb{Q}})\cap M_{i+1}$ such that $\|T_i\|\leq\|M_i\|^{\|w_i\|}\leq\|M_{i+1}\|$, and if $j<i\leq\delta$ then $\mathcal{T}_j={\rm proj}^{(w_i,i+1)}_{(w_j,j+1)}(\mathcal{T}_i)$ and $\bar{p}^j\leq_{{\rm proj}^{\mathcal{T}_i}_{\mathcal{T}_j}}\bar{p}^i$.

Case 1: $i=0$.

Let T^*_0 consist of all sequences $\langle t_\zeta:\zeta\in{\rm dom}(t)\rangle$ such that ${\rm dom}(t)$ is an initial segment of w_0 and $t_\zeta=\langle\rangle$ for $\zeta\in{\rm dom}(t)$. Thus T^*_0 is a standard $(w_0,0)^\gamma$–tree, $\|T^*_0\|=\|w_0\|$. For $t\in T^*_0$ let $p^{*0}_t=p{\restriction}{\rm rk}^*_0(t)$. Clearly the sequence $\bar{p}^{*0}=\langle p^{*0}_t: t\in T^*_0\rangle$ is in ${\rm FTr}'(\bar{\mathbb{Q}})\cap N^{-1}_0$. Apply B.7.2 to $\hat{\mathcal{E}}_0,\bar{\mathbb{Q}},\bar{N}^{-1},\mathcal{T}^*_0,w_0$ and $\bar{p}^{*0}$ (note that $\|N^{-1}_\varepsilon\|^{\|w_0\|}\leq\|N^{-1}_{\varepsilon+1}\|^{\|N^{-1}_0\|}$ for $\varepsilon<\delta_0$). As a result we get a $(w_0,1)^\gamma$–tree $\mathcal{T}_0$ (the one called $\mathcal{T}_{\delta_0}$ there) and $\bar{p}^0=\langle p^0_t:t\in T_0\rangle\in{\rm FTr}'(\bar{\mathbb{Q}})\cap M_1$ (the one called $\bar{q}$ there) satisfying clauses B.7.2(ε), B.7.2(c)(i)–(iv) and such that $\|T_0\|\leq \|N^{-1}_{\delta_0}\|^{\|w_0\|}=\|M_0\|^{\|w_0\|}=\|M_0\|$ (remember ${\rm cf}(\delta_0)>2^{\|M_0\|}$). So, in particular, if $t\in T_0$, $\zeta\in{\rm dom}(t)$ then $t_\zeta(0)\in M_1$ is either $*$ or a $\mathbb{P}_\zeta$–name for an element of $\mathbb{Q}_\zeta$.

Moreover, we additionally require that $(\mathcal{T}_0,\bar{p}^0)$ is the $<^*_\chi$–first with all these properties, so $\mathcal{T}_0,\bar{p}^0\in M_1$.

Case 2: $i=i_0+1$.

We proceed similarly to the previous case. Suppose we have defined $\mathcal{T}_{i_0}$ and $\bar{p}^{i_0}$ such that $\mathcal{T}_{i_0},\bar{p}^{i_0}\in M_{i_0+1}$, $\|T_{i_0}\|\leq\|M_{i_0+1}\|$. Let $\mathcal{T}^*_i$ be a standard $(w_i,i_0)^\gamma$–tree such that

T_i^* consists of all sequences $\langle t_\zeta : \zeta \in \mathrm{dom}(t)\rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w_i and

$$\langle t_\zeta : \zeta \in \mathrm{dom}(t) \cap w_{i_0}\rangle \in T_{i_0} \quad \text{and} \quad (\forall \zeta \in \mathrm{dom}(t) \setminus w_{i_0})(\forall j < i_0)(t_\zeta(j) = *).$$

Thus $\mathcal{T}_{i_0} = \mathrm{proj}^{(w_i,i)}_{(w_{i_0},i_0)}(\mathcal{T}_i^*)$ and $\|T_i^*\| \leq \|M_i\|$. Let $p^{*i}_t = p^{i_0}_{t'}{\restriction}\mathrm{rk}^*_i(t)$ for $t \in T_i^*$, $t' = \mathrm{proj}^{\mathcal{T}_i}_{\mathcal{T}_{i_0}}(t)$. Now apply B.7.2 to $\hat{\mathcal{E}}_0$, $\bar{\mathbb{Q}}$, $\bar{N}^{i_0}$, $\mathcal{T}_i^*$, w_i and $\bar{p}^{*i}$ (check that the assumptions are satisfied). So we get a standard $(w_i, i_0+1)^\gamma$–tree $\mathcal{T}_i$ and a sequence $\bar{p}^i$ satisfying B.7.2(ε), B.7.2(c)(i)–(iv), and we take the $<^*_\chi$–first pair $(\mathcal{T}_i, \bar{p}^i)$ with these properties. In particular we will have $\|T_i\| \leq \|M_{i_0}\| \cdot \|N^{i_0}_{\delta_i}\|^{\|M_{i_0}\|} = \|M_{i_0+1}\|$, and $\bar{p}^i, \mathcal{T}_i \in M_{i+1}$.

Case 3: i is a limit ordinal.
Suppose we have defined $\mathcal{T}_j$, $\bar{p}^j$ for $j < i$ and we know that $\langle(\mathcal{T}_j, \bar{p}^j) : j < i\rangle \in M_{i+1}$ (this is the consequence of taking "the $<^*_\chi$–first such that…"). Let $\mathcal{T}_i^* = \overleftarrow{\lim}(\langle \mathcal{T}_j : j < i\rangle)$. Now, for $t \in T_i^*$ we would like to define p^{*i}_t as the limit of $p^j_{\mathrm{proj}^{\mathcal{T}_i^*}_{\mathcal{T}_j}(t)}$. However, our problem is that we do not know if the limit exists. Therefore we restrict ourselves to these t for which the respective sequence has an upper bound. To be more precise, for $t \in \mathcal{T}_i^*$ we apply the following procedure.

($\otimes$) Let $t^j = \mathrm{proj}^{\mathcal{T}_i^*}_{\mathcal{T}_j}(t)$ for $j < i$. Try to define inductively a condition $p^{*i}_t \in \mathbb{P}_{\mathrm{rk}^*_i(t)}$ such that $\mathrm{dom}(p^{*i}_t) = \bigcup\{\mathrm{dom}(p^j_{t^j}) \cap \mathrm{rk}^*_i(t) : j < i\}$. Suppose we have successfully defined $p^{*i}_t{\restriction}\alpha$, $\alpha \in \mathrm{dom}(p^{*i}_t)$, in such a way that $p^{*i}_t{\restriction}\alpha \geq p^j_{t^j}{\restriction}\alpha$ for all $j < i$. We know that

$$p^{*i}_t{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{"the sequence } \langle p^j_{t^j}(\alpha) : j < i\rangle \text{ is } \leq_{\hat{\underset{\sim}{\mathbb{Q}}}_\alpha}\text{–increasing"}.$$

So now, if there is a $\mathbb{P}_\alpha$–name $\underset{\sim}{\tau}$ for an element of $\underset{\sim}{\mathbb{Q}}_\alpha$ such that

$$p^{*i}_t{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} (\forall j < i)(p^j_{t^j}(\alpha) \leq_{\hat{\underset{\sim}{\mathbb{Q}}}_\alpha} \underset{\sim}{\tau}),$$

then we $p^{*i}_t(\alpha)$ be the $\mathbb{P}_\alpha$-name of the lub of $\langle p^j_{t^j}(\alpha) : j < i\rangle$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\alpha$ and we continue. If there is no such $\underset{\sim}{\tau}$ then we decide that $t \notin \mathcal{T}_i^+$ and we stop the procedure.

Now, let $\mathcal{T}_i^+$ consist of those $t \in T_i^*$ for which the above procedure resulted in a successful definition of $p^{*i}_t \in \mathbb{P}_{\mathrm{rk}^*_i(t)}$. It might be not clear at the moment if T_i^+ contains anything more than $\langle\rangle$, but we will see that this is the case. Note that

$$\|T_i^+\| \leq \|T_i^*\| \leq \prod_{j<i} \|T_j\| \leq \prod_{j<i} \|M_j\| \leq 2^{\|M_i\|} \leq \|N^i_2\|.$$

Moreover, for $\varepsilon > 2$ we have $\|N^i_\varepsilon\|^{\|w_i\| + \|T_i^+\|} \leq \|N^i_\varepsilon\|^{\|N^i_2\|} \subseteq N^i_{\varepsilon+1}$ and $\mathcal{T}_i^+, \bar{p}^{*i} \in M_{i+1}$. Let $\mathcal{T}_i = \mathcal{T}_i^*$, $\bar{p}^i = \bar{p}^{*i}$ (this time there is no need to take the $<^*_\chi$–first pair as the process leaves no freedom).

After the construction is carried out we continue in a similar manner as in A.3.7 (but note slightly different meaning of the $*$'s here).

So we let $\mathcal{T}_\delta = \overleftarrow{\lim}(\langle \mathcal{T}_i : i < \delta\rangle)$. It is a standard $(w_\delta, \delta)^\gamma$–tree. By induction on $\alpha \in w_\delta \cup \{\gamma\}$ we choose $q_\alpha \in \mathbb{P}'_\alpha$ and a $\mathbb{P}_\alpha$–name $\underset{\sim}{t}_\alpha$ such that

(a) $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{t}_\alpha \in T_\delta$ & $\mathrm{rk}_\delta(\underset{\sim}{t}_\alpha) = \alpha$", and let $i^\alpha_0 = \min\{i < \delta : \alpha \in M_i\} < \delta$,

(b) $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{t}_\beta = \underset{\sim}{t}_\alpha{\restriction}\beta$" for $\beta < \alpha$,

(c) $\mathrm{dom}(q_\alpha)=w_\delta\cap\alpha$,
(d) if $\beta<\alpha$ then $q_\beta=q_\alpha\restriction\beta$,
(e) $p^i_{\mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}_i}(\underset{\sim}{t}_\alpha)}$ is well defined and $p^i_{\mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}_i}(\underset{\sim}{t}_\alpha)}\restriction\alpha\leq q_\alpha$ for each $i<\delta$,
(f) for each $\beta<\alpha$

$q_\alpha\Vdash_{\mathbb{P}_\alpha}$ "$(\forall i<\delta)((\underset{\sim}{t}_{\beta+1})_\beta(i)=*\ \Leftrightarrow\ i<i^\beta_0)$ and the sequence $\langle i^\beta_0,p^{i^\beta_0}_{\mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}_{i^\beta_0}}(\underset{\sim}{t}_{\beta+1})}(\beta),\langle(\underset{\sim}{t}_{\beta+1})_\beta(i),p^i_{\mathrm{proj}^{\mathcal{T}_\delta}_{\mathcal{T}_i}(\underset{\sim}{t}_{\beta+1})}(\beta):\ i^\beta_0\leq i<\delta\rangle\rangle$ is a result of a play of the game $\mathcal{G}^\spadesuit_{\bar{M}[\underset{\sim}{G}_\beta],\langle\bar{N}^i[\underset{\sim}{G}_\beta]:i<\delta\rangle}(\underset{\sim}{\mathbb{Q}}_\beta,\mathbf{0}_{\underset{\sim}{\mathbb{Q}}_\beta})$, won by player COM",

(g) the condition q_α forces (in $\mathbb{P}_\alpha$) that "the sequence $\bar{M}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]\restriction[i_\alpha,\delta]$ is ruled by $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ and $\langle\bar{N}^i[\underset{\sim}{G}_{\mathbb{P}_\alpha}]:i^\alpha_0\leq i<\delta\rangle$ is its $\hat{\mathcal{E}}_0$–approximation".

(Remember: $\hat{\mathcal{E}}_1$ is closed under end segments.) This is done completely parallely to the last part of the proof of A.3.7.

Finally look at the condition q_γ and the clause (g) above. $\square$

Proposition B.7.4. *Suppose that $\mu^*=\kappa$ and $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$ is closed. Let $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\gamma\rangle$ be a $(<\kappa)$–support iteration such that for each $\alpha<\gamma$*

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{\mathbb{Q}}_\alpha\text{ is strongly complete for }\hat{\mathcal{E}}\text{ and }\|\underset{\sim}{\mathbb{Q}}_\alpha\|\leq\kappa\text{”.}$$

Then $\mathbb{P}_\gamma$ satisfies κ^+–cc (even more: it satisfies the κ^+–Knaster condition).

Proof. For $\alpha<\gamma$ choose $\mathbb{P}_\alpha$–names $\underset{\sim}{x}_\alpha$ and $\underset{\sim}{h}_\alpha$ such that

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{x}_\alpha\text{ witnesses that }\underset{\sim}{\mathbb{Q}}_\alpha\text{ is complete for }\hat{\mathcal{E}}\text{ and }\underset{\sim}{h}_\alpha:\underset{\sim}{\mathbb{Q}}_\alpha\longrightarrow\kappa\text{ is one–to–one”.}$$

Since $\hat{\mathcal{E}}\in\mathfrak{C}_{<\kappa}(\mu^*)$, for each $p\in\mathbb{P}_\gamma$ we find an $\hat{\mathcal{E}}$–complementary pair $(\bar{N}^p,\bar{a}^p)$ such that $\bar{N}^p=\langle N^p_i:i\leq\omega\rangle$ and $p,\bar{\mathbb{Q}},\hat{\mathcal{E}},\langle\underset{\sim}{x}_\alpha:\alpha<\gamma\rangle,\langle\underset{\sim}{h}_\alpha:\alpha<\gamma\rangle\in N^p_0$. Next choose an increasing sequence $\bar{q}^p=\langle q^p_i:i<\omega\rangle$ of conditions from $\mathbb{P}_\gamma$ such that for each $i<\omega$:

(α) $p\leq q^p_0$, $\quad\bar{q}^p\restriction(i+1)\in N^p_{i+1}$,
(β) $q^p_i\in\bigcap\{\mathcal{I}\in N_i:\mathcal{I}\subseteq\mathbb{P}_\gamma\text{ open dense}\}$.

[Why is this possible? Remember B.5.6 and particularly B.5.6.4.] So the condition q^p_i is generic over N_i in the weak sense of clause (β), and therefore it decides the values of $\underset{\sim}{h}_\alpha(q^p_j(\alpha))$ for each $j<i$, $\alpha\in\mathrm{dom}(q^p_j)$ (remember: if $j<i$ then $q^p_j\in N_i$ and thus $\mathrm{dom}(q^p_j)\subseteq N_i$). Let $\varepsilon^{p,\alpha}_j<\kappa$ be such that for each $i>j$ (remember $\bar{q}^p$ is increasing)

$$q^p_i\restriction\alpha\Vdash_{\mathbb{P}_\alpha}\underset{\sim}{h}_\alpha(q^p_j(\alpha))=\varepsilon^{p,\alpha}_j.$$

Suppose now that $\langle p^\zeta:\zeta<\kappa^+\rangle\subseteq\mathbb{P}_\gamma$. For $\zeta<\kappa^+$ let $A_\zeta=\bigcup\limits_{i<\omega}\mathrm{dom}(q^{p^\zeta}_i)$ (so $A_\zeta\in[\gamma]^{<\kappa}$). Applying the Δ–system lemma (remember κ is strongly inaccessible) we find $\mathcal{X}\subseteq\kappa^+$, $\|\mathcal{X}\|=\kappa^+$ such that $\{A_\zeta:\zeta\in\mathcal{X}\}$ forms a Δ–system and for each $\zeta,\xi\in\mathcal{X}$:

- $\|A_\zeta\|=\|A_\xi\|$,

- if $\alpha \in A_\zeta \cap A_\xi$ then

$$\min\{i<\omega : \alpha \in \mathrm{dom}(q_i^{p^\zeta})\} = \min\{i<\omega : \alpha \in \mathrm{dom}(q_i^{p^\xi})\},$$

  and call it i_α, and for each $i<\omega$

$$\mathrm{otp}(\alpha \cap \mathrm{dom}(q_i^{p^\zeta})) = \mathrm{otp}(\alpha \cap \mathrm{dom}(q_i^{p^\xi})) \quad \text{and} \quad \varepsilon_i^{p^\zeta,\alpha} = \varepsilon_i^{p^\xi,\alpha}$$

  (the last for $i \geq i_\alpha$).

We are going to show that for each $\xi, \zeta \in \mathcal{X}$ the conditions p^ζ, p^ξ are compatible. To this end we define a common upper bound r of p^ζ, p^ξ. First we declare that

$$\mathrm{dom}(r) = A_\zeta \cup A_\xi$$

and then we inductively define $r(\alpha)$ for $\alpha \in \mathrm{dom}(r)$:

if $\alpha \in A_\zeta$ then $r(\zeta)$ is a $\mathbb{P}_\alpha$–name such that

$$r{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{"}r(\alpha) \text{ is the upper bound of } \langle q_i^{p^\zeta}(\alpha) : i<\omega\rangle \text{ with the minimal value of } \underset{\sim}{h}_\alpha(r(\alpha))\text{"}$$

and otherwise (i.e. if $\alpha \in A_\xi \setminus A_\zeta$) it is a $\mathbb{P}_\alpha$–name such that

$$r{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{"}r(\alpha) \text{ is the upper bound of } \langle q_i^{p^\xi}(\alpha) : i<\omega\rangle \text{ with the minimal value of } \underset{\sim}{h}_\alpha(r(\alpha))\text{"}.$$

By induction on $\alpha \in \mathrm{dom}(r) \cup \{\gamma\}$ we show that

$$q_i^{p^\zeta}{\restriction}\alpha \leq_{\mathbb{P}_\alpha} r{\restriction}\alpha \quad \text{and} \quad q_i^{p^\xi}{\restriction}\alpha \leq_{\mathbb{P}_\alpha} r{\restriction}\alpha \quad \text{for all } i<\omega.$$

Note that, by B.5.5, this implies that the respective upper bounds exist and thus $r(\alpha)$ is well defined then. There is nothing to do at non-successor stages, so suppose that we have arrived to a stage $\alpha = \beta+1$.

If $\beta \in A_\zeta$ then, by the definition of $r(\beta)$, we have

$$r{\restriction}\beta \Vdash_{\mathbb{P}_\beta} (\forall i<\omega)(q_i^{p^\zeta}(\beta) \leq r(\beta)).$$

Similarly if $\beta \in A_\xi \setminus A_\zeta$ and we consider $q_i^{p^\xi}(\beta)$. Trivially, no problems can happen if $\beta \in A_\zeta \setminus A_\xi$ and we consider $q_i^{p^\xi}(\beta)$ or if $\beta \in A_\xi \setminus A_\zeta$ and we consider $q_i^{p^\zeta}(\beta)$. So the only case we may worry about is that $\beta \in A_\zeta \cap A_\xi$ and we want to show that $r(\beta)$ is (forced to be) stronger than all $q_i^{p^\xi}(\beta)$. But note: by the inductive hypothesis we know that $r{\restriction}\beta$ is an upper bound to both $\langle q_i^{p^\xi}{\restriction}\beta : i<\omega\rangle$ and $\langle q_i^{p^\zeta}{\restriction}\beta : i<\omega\rangle$ and therefore

$$r{\restriction}\beta \Vdash_{\mathbb{P}_\beta} \text{" } \underset{\sim}{h}_\beta(q_i^{p^\xi}(\beta)) = \varepsilon_i^{p^\xi,\beta} \ \& \ \underset{\sim}{h}_\beta(q_j^{p^\zeta}(\beta)) = \varepsilon_j^{p^\zeta,\beta} \text{ "},$$

whenever $i,j<\omega$ are such that $\beta \in \mathrm{dom}(q_i^{p^\xi})$, $\beta \in \mathrm{dom}(q_j^{p^\zeta})$. But now, by the choice of $\mathcal{X}$ we have:

$$\beta \in \mathrm{dom}(q_i^{p^\xi}) \quad \Leftrightarrow \quad \beta \in \mathrm{dom}(q_i^{p^\zeta}), \qquad \text{and } \varepsilon_i^{p^\zeta,\beta} = \varepsilon_i^{p^\xi,\beta}.$$

Since $\underset{\sim}{h}_\beta$ is (forced to be) a one–to–one function, we conclude that

$$r{\restriction}\beta \Vdash_{\mathbb{P}_\beta} (\forall i<\omega)(q_i^{p^\zeta}(\beta) = q_i^{p^\xi}(\beta)),$$

so taking care of the ζ's side we took care of the ξ's side as well. This finishes the proof of the proposition. □

B.8. The Axiom and its applications

Definition B.8.1. Suppose that $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ and θ is a regular cardinal. Let $\mathrm{Ax}^\kappa_\theta(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$, *the forcing axiom for* $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ *and* θ, be the following sentence:

> If $\mathbb{Q}$ is a complete for $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$ forcing notion of size $\leq\kappa$ and $\langle\mathcal{I}_i: i<i^*<\theta\rangle$ is a sequence of dense subsets of $\mathbb{Q}$,
> then there exists a directed set $H\subseteq\mathbb{Q}$ such that
> $$(\forall i<i^*)(H\cap\mathcal{I}_i\neq\emptyset).$$

Theorem B.8.2. *Assume that* $\mu^*=\kappa$, $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ *and*

$$\kappa<\theta=\mathrm{cf}(\theta)\leq\mu=\mu^\kappa.$$

Then there is a strongly complete for $\hat{\mathcal{E}}_0$ forcing notion $\mathbb{P}$ of cardinality μ such that

(α) $\mathbb{P}$ *satisfies the* κ^+*–cc,*

(β) $\Vdash_{\mathbb{P}}(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ *and even more:*

(β^+) *if* $\hat{\mathcal{E}}^*_0\subseteq\hat{\mathcal{E}}_0,\hat{\mathcal{E}}^*_1\subseteq\hat{\mathcal{E}}_1$ *are such that* $(\hat{\mathcal{E}}^*_0,\hat{\mathcal{E}}^*_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ *then* $\Vdash_{\mathbb{P}}(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}^*_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$,

(γ) $\Vdash_{\mathbb{P}}\mathrm{Ax}^\kappa_\theta(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$.

Proof. The forcing notion $\mathbb{P}$ will be the limit of a $(<\kappa)$–support iteration $\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\alpha^*\rangle$ (for some $\alpha^*<\mu^+$) such that

(a) for each $\alpha<\alpha^*$

$$\Vdash_{\mathbb{P}_\alpha}\text{“}\underset{\sim}{\mathbb{Q}}_\alpha\text{ is a partial order on }\kappa\text{ complete for }(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\text{”}.$$

By B.7.4 we will be sure that $\mathbb{P}=\mathbb{P}_{\alpha^*}$ satisfies κ^+–cc. Applying B.7.3 we will see that $\Vdash_{\mathbb{P}_{\alpha^*}}(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ (also $\mathbb{P}_{\alpha^*}$ is complete for $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)$). The iteration $\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\alpha^*\rangle$ will be built by a bookkeeping argument, but we do not determine in advance its length α^*.

Before we start the construction, note that if $\mathbb{Q}$ is a κ^+–cc forcing notion of size $\leq\mu$ then there are at most μ $\mathbb{Q}$–names for partial orders on κ (up to isomorphism). Why? Remember $\mu^\kappa=\mu$ and each $\mathbb{Q}$–name for a poset on κ is described by a κ–sequence of maximal antichains of $\mathbb{Q}$. By a similar argument we will know that each $\mathbb{P}_\alpha$ has a dense subset of size $\leq\mu$ (for $\alpha\leq\alpha^*$). Consequently there are, up to an isomorphism, at most μ $\mathbb{P}_{\alpha^*}$–names for partial orders on κ.

Let $\mathfrak{K}$ consist of all $(<\kappa)$–support iterations $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\alpha_0\rangle$ of length $<\mu^+$ satisfying the demand **(a)** above (with α_0 in place of α^*). Elements of $\mathfrak{K}$ are naturally ordered by

$$\bar{\mathbb{Q}}^0\leq_{\mathfrak{K}}\bar{\mathbb{Q}}^1\quad\text{ if and only if }\quad\bar{\mathbb{Q}}^0=\bar{\mathbb{Q}}^1\restriction\ell g(\bar{\mathbb{Q}}^0).$$

Note that every $\leq_{\mathfrak{K}}$–increasing sequence of length $<\mu^+$ has the least upper bound in $(\mathfrak{K},\leq_{\mathfrak{K}})$. By what we said before, we know that if $\langle\mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\alpha_0\rangle\in\mathfrak{K}$, then $\mathbb{P}_{\alpha_0}$ contains a dense subset of size $\leq\mu$, satisfies κ^+–cc and forces that $(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\in\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$. Moreover,

($\circledast_{\mathfrak{K}}$) if $\bar{\mathbb{Q}}^0=\langle\mathbb{P}^0_\alpha,\underset{\sim}{\mathbb{Q}}^0_\alpha:\alpha<\alpha_0\rangle\in\mathfrak{K}$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}^0_{\alpha_0}$–name for a forcing notion on κ then

($\oplus_1$) *either* there is no $\bar{\mathbb{Q}}^1=\langle\mathbb{P}^1_\alpha,\underset{\sim}{\mathbb{Q}}^1_\alpha:\alpha<\alpha_1\rangle\in\mathfrak{K}$ such that $\bar{\mathbb{Q}}^0\leq_{\mathfrak{K}}\bar{\mathbb{Q}}^1$ and

$$\Vdash_{\mathbb{P}^1_{\alpha_1}}\text{“}\underset{\sim}{\mathbb{Q}}\text{ is complete for }(\hat{\mathcal{E}}_0,\hat{\mathcal{E}}_1)\text{”}$$

($\oplus_2$) *or* there is $\bar{\mathbb{Q}}^1 = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\alpha : \alpha < \alpha_1 \rangle \in \mathfrak{K}$ such that $\bar{\mathbb{Q}}^0 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}^1$ and

$$\Vdash_{\mathbb{P}^1_{\alpha_1}} \text{"there is a directed set } H \subseteq \underset{\sim}{\mathbb{Q}} \text{ which meets all dense subsets of } \underset{\sim}{\mathbb{Q}} \text{ from } \mathbf{V}^{\mathbb{P}^0_{\alpha_0}}\text{"}.$$

[Why? Suppose that ($\oplus_1$) fails and it is exemplified by $\bar{\mathbb{Q}}^1$. Take $\bar{\mathbb{Q}}^1 * \underset{\sim}{\mathbb{Q}}$.]

Consequently, as $\mathfrak{K}$ is closed under increasing $< \mu^+$–sequences, we have

($\circledast^+_{\mathfrak{K}}$) for every $\bar{\mathbb{Q}} \in \mathfrak{K}$ there is $\bar{\mathbb{Q}}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\alpha : \alpha < \alpha_0 \rangle \in \mathfrak{K}$ such that $\bar{\mathbb{Q}} \prec_{\mathfrak{K}} \bar{\mathbb{Q}}^0$ and for every $\mathrm{Lim}(\bar{\mathbb{Q}})$–name $\underset{\sim}{\mathbb{Q}}$ for a forcing notion on κ one of the following conditions occurs:

($\oplus_1$) there is no $\bar{\mathbb{Q}}^1 = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\alpha : \alpha < \alpha_1 \rangle \in \mathfrak{K}$ such that $\bar{\mathbb{Q}}^0 \prec_{\mathfrak{K}} \bar{\mathbb{Q}}^1$ and

$$\Vdash_{\mathbb{P}^1_{\alpha_1}} \text{"}\underset{\sim}{\mathbb{Q}} \text{ is complete for } (\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)\text{"}$$

($\oplus^+_2$) $\Vdash_{\mathbb{P}^0_{\alpha_0}}$ " there is a directed set $H \subseteq \underset{\sim}{\mathbb{Q}}$ which meets all dense subsets of $\underset{\sim}{\mathbb{Q}}$ from $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}})}$ ".

[Why? Remember that there is at most μ $\mathrm{Lim}(\bar{\mathbb{Q}})$–names for partial orders on κ.]

Using these remarks we may build our iteration in the following way. We choose a $\leq_{\mathfrak{K}}$–increasing continuous sequence $\langle \bar{\mathbb{Q}}^\zeta : \zeta \leq \theta^+ \rangle \subseteq \mathfrak{K}$ such that

(b) for every $\zeta < \theta^+$, $\bar{\mathbb{Q}}^{\zeta+1}$ is given by ($\circledast^+_{\mathfrak{K}}$) for $\bar{\mathbb{Q}}^\zeta$.

Now it is a routine to check that $\mathbb{P} = \mathbb{P}^{\theta^+}_{\alpha_{\theta^+}}$ is as required. □

In B.8.3 below remember about our main case: $S^* \subseteq \kappa$ is stationary co-stationary and $\hat{\mathcal{E}}_0$ consists of all increasing continuous sequences $\bar{a} = \langle a_i : i \leq \alpha \rangle$ such that $a_i \in \kappa \setminus S^*$ (for $i \leq \alpha$). In this case the forcing notion $\mathbb{R}$ is the standard way to make the set S^* non-stationary (by adding a club of κ; a condition gives an initial segment of the club). Since forcing with $\mathbb{R}$ preserves stationarity of subsets of $\kappa \setminus S^*$, the conclusion of B.8.3 below gives us

($*$) in $\mathbf{V}^{\mathrm{Lim}\bar{\mathbb{Q}}}$, every stationary set $S \subseteq \kappa \setminus S^*$ reflects in some inaccessible.

Proposition B.8.3. *Suppose that $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ (so $\hat{\mathcal{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$), $\mu^* = \kappa$ (for simplicity) and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a $(< \kappa)$–support iteration such that for each $\alpha < \kappa$*

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\underset{\sim}{\mathbb{Q}}_\alpha \text{ is complete for } (\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \text{ and } \|\underset{\sim}{\mathbb{Q}}_\alpha\| \leq \kappa.\text{"}$$

Further assume that:

(a) *$\hat{\mathcal{E}}_0$ is reasonably closed: it is closed under subsequences and if $\bar{a} = \langle a_i : i \in \delta \rangle \in \hat{\mathcal{E}}_0$ and $\bar{b}^i = \langle b^i_\alpha : \alpha \leq \alpha_i \rangle \in \hat{\mathcal{E}}_0$ are such that $b^i_0 = a_i$, $b^i_{\alpha_i} = a_{i+1}$ (for $i < \delta$), then the concatenation of all $\bar{b}^i$ (for $i < \delta$) belongs to $\hat{\mathcal{E}}_0$ [e.g., $\hat{\mathcal{E}}_0$ is derived from $S \subseteq \kappa$ like in B.6.2],*

(b) $\mathbb{R} = (\hat{\mathcal{E}}_0, \lhd)$,

(c) *in $\mathbf{V}^{\mathbb{R}}$ and even in $\mathbf{V}^{\mathbb{R} * \mathrm{Cohen}_\kappa}$, κ is a weakly compact cardinal (or just: stationary subsets of κ reflect in inaccessibles).*

Then, in $\mathbf{V}^{\mathbb{P}_\gamma * \mathbb{R}}$, κ is weakly compact (or just: stationary subsets of κ reflect in inaccessibles).

Proof. First note that the forcing with $\mathbb{R}$ does not add new sequences of length $< \kappa$ of ordinals. [Why? Suppose that $\underset{\sim}{x}$ is an $\mathbb{R}$–name for a function from θ to $\mathbf{V}$, $\theta < \kappa$ is a regular cardinal and $r \in \mathbb{R}$. Take an $\hat{\mathcal{E}}_0$–complementary pair $(\bar{N}, \bar{a})$ such that

$\bar{N} = \langle N_i : i \leq \theta \rangle$ and $r, \underset{\sim}{x} \in N_0$ and the error is, say, n. Now build inductively an increasing sequence $\langle r_i : i \leq \theta \rangle \subseteq \mathbb{R}$ such that for every $i \leq \theta$:

- $r_0 = r$, the condition r_{i+1} decides the value of $\underset{\sim}{x}(i)$,
- if $i = \gamma + k + 1$, γ is a non-successor, $k < \omega$ then $r_i \in N_{\gamma+(2k+2)(n+1)}$ and if $r_i = \langle a^i_\xi : \xi \leq \alpha_i \rangle$ then $a^i_{\alpha_i} = a_{\gamma+(2k+1)(n+1)}$,
- if $i < \theta$ is limit then $\langle r_j : j < i \rangle \in N_{i+1}$ and r_i is the least upper bound of $\langle r_j : j < i \rangle$ (so $r_i \in N_{i+1}$).

The construction is straightforward. If we have defined $r_i \in N_{\gamma+(2k+2)(n+1)}$, then we first take the $<^*_\chi$–first condition $r^*_i = \langle a^*_\xi : \xi \leq \alpha^* \rangle$ stronger than r_i and deciding the value of $\underset{\sim}{x}(i)$ (so $r^*_i \in N_{\gamma+(2k+2)(n+1)}$). We know that

$$a^*_{\alpha^*} \subseteq a_{\gamma+(2k+2)(n+1)+n} \in N_{\gamma+(2k+2)(n+1)+2n+1}.$$

Let $r_{i+1} = r^*_i {}^\frown \langle a_{\gamma+(2k+2)(n+1)+n} \rangle$. Clearly $r_{i+1} \in N_{\gamma+(2k+4)(n+1)}$. By the choice of "the $<_\chi$–first" conditions we are sure that, arriving to a limit stage $i < \theta$, we have $\langle r_j : j < i \rangle \in N_{i+1}$. Now use the assumption (a) on $\hat{\mathcal{E}}_0$ to argue that the sequence $\langle r_i : i < \theta \rangle$ has a least upper bound r_θ – clearly this condition decides the name $\underset{\sim}{x}$.]

Without loss of generality we may assume that, for each $\alpha < \gamma$

$$\Vdash_{\mathbb{P}_\alpha} \text{"} \underset{\sim}{\mathbb{Q}}_\alpha \text{ is a partial order on } \kappa \text{"}.$$

For a forcing notion $\mathbb{Q}$ let $\check{\mathbb{Q}}$ stand for the completion of $\mathbb{Q}$ with respect to increasing $<\kappa$–sequences (i.e., it is like $\hat{\mathbb{Q}}$ but we consider only increasing sequences of length $<\kappa$). Note that $\mathbb{Q}$ is dense in $\check{\mathbb{Q}}$ and if $\|\mathbb{Q}\| \leq \kappa$, then $\|\check{\mathbb{Q}}\| \leq \kappa$ (κ is strongly inaccessible!). Now, let $\langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\alpha : \alpha < \gamma \rangle$ be the iteration of the respective $<\kappa$–completions of the $\underset{\sim}{\mathbb{Q}}_\alpha$'s. Thus each $\mathbb{P}_\alpha$ is a dense subset of $\mathbb{P}'_\alpha$ (see 0.18). We may assume that each $\underset{\sim}{\check{\mathbb{Q}}}_\alpha$ is a $\mathbb{P}_\alpha$–name for a partial order on $\kappa + \kappa$ (for $\alpha < \gamma$). Now, for $\alpha \leq \gamma$, let

$$\mathbb{P}''_\alpha = \big\{ p \in \mathbb{P}'_\alpha : \begin{array}{l} \text{there is a sequence } \langle \bar{p}^\beta : \beta \in \mathrm{dom}(p) \rangle \text{ such that for some } \delta < \kappa, \\ \text{each } \bar{p}^\beta = \langle p^\beta_\zeta : \zeta < \delta \rangle \text{ is a } \delta\text{–sequence of ordinals} < \kappa \text{ and} \\ p(\beta) \text{ is (the } \mathbb{P}'_\beta\text{–name of) the minimal (as an ordinal)} \\ \text{least upper bound of } \bar{p}^\beta \text{ by } \leq_{\underset{\sim}{\mathbb{Q}}_\alpha} \big\}. \end{array}$$

Claim B.8.3.1. *For each $\alpha \leq \gamma$, $\mathbb{P}''_\alpha$ is a dense subset of $\mathbb{P}'_\alpha$.*

Proof of the claim. Let $p \in \mathbb{P}'_\alpha$. By B.5.6 we know that $\mathbb{P}_\alpha$ is strongly complete for $\hat{\mathcal{E}}_0$. Let $(\bar{N}, \bar{a})$ be an $\hat{\mathcal{E}}_0$–complementary pair such that $\bar{N} = \langle N_i : i < \omega \rangle$ and $p, \bar{\mathbb{Q}}, \mathbb{P}'_\alpha, \hat{\mathcal{E}}_0 \ldots \in N_0$. Take an increasing sequence $\langle q_i : i < \omega \rangle \subseteq \mathbb{P}_\alpha$ such that $q_i \in N_{i+1}$ is generic over N_i and such that $p \leq_{\mathbb{P}'_\alpha} q_0$. Now let $q \in \mathbb{P}'_\alpha$ be defined by $\mathrm{dom}(q) = N_\omega \cap \alpha$ and:

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \begin{array}{l} \text{"}q(\beta) \text{ is the minimal (as an ordinal) least upper bound in } \underset{\sim}{\check{\mathbb{Q}}}_\beta \\ \text{of the sequence } \langle q_i(\beta) : i < \omega \rangle\text{".} \end{array}$$

By B.5.6.3 (actually by its proof) we know that the above definition is correct. Now it a routine to check that $q \in \mathbb{P}''_\alpha$ is as required, finishing the proof of the claim. □

One could ask what is the point of introducing $\mathbb{P}''_\alpha$. The main difference between $\mathbb{P}'_\alpha$ and $\mathbb{P}''_\alpha$ is that in the first, $q(\beta)$ is a least upper bound of an increasing sequence of conditions from $\underset{\sim}{\mathbb{Q}}_\alpha$, but we know the name for the sequence only. In $\mathbb{P}''_\alpha$, we have the representation of $q(\alpha)$ as the least upper bound of a sequence of ordinals from $\mathbf{V}$! This is of use if we look at the iteration in different universes. If we look at $\bar{\mathbb{Q}}$ (defined as an iteration in $\mathbf{V}$) in $\mathbf{V}^{\mathbb{R}}$, then it does not have to be an iteration anymore: let $\alpha < \gamma$. Forcing with $\mathbb{R}$ may add new maximal antichains in $\mathbb{P}_\alpha$ thus creating new names for elements of $\underset{\sim}{\mathbb{Q}}_\alpha$. However

Claim B.8.3.2. *For each $\alpha \le \gamma$, in $\mathbf{V}^{\mathbb{R}}$, $\langle \mathbb{P}''_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \gamma, \beta < \gamma\rangle$ is a $(<\kappa)$–support iteration.*

Proof of the claim. Easy induction on α. □

Claim B.8.3.3. *For each $\alpha < \gamma$*

$$\Vdash_{\mathbb{P}_\alpha * \mathbb{R}} \text{“}\check{\underset{\sim}{\mathbb{Q}}}_\alpha \textit{ is isomorphic to } \mathrm{Cohen}_\kappa\text{”}.$$

Proof of the claim. Working in $\mathbf{V}^{\mathbb{P}_\alpha}$ choose an increasing continuous sequence $\bar{N} = \langle N_i : i < \kappa\rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\underset{\sim}{\mathbb{Q}}_\alpha \in N_0$ and for each $i < \kappa$:

$$\bar{N}{\restriction}(i+1) \in N_{i+1}, \quad N_i \cap \kappa \in \kappa, \quad \text{and} \quad \|N_i\| < \kappa.$$

Now, passing to $\mathbf{V}^{\mathbb{P}_\alpha * \mathbb{R}}$, we can find an increasing continuous sequence $\bar{j} = \langle j_\zeta : \zeta < \kappa\rangle \subseteq \kappa$ such that

$$(\forall \varepsilon < \kappa)(\langle N_{j_\zeta} \cap \kappa : \zeta \le \varepsilon\rangle \in \hat{\mathcal{E}}_0).$$

[Why? Forcing with $\mathbb{R}$ adds an increasing continuous sequence $\bar{\beta} = \langle \beta_\zeta : \zeta < \kappa\rangle$ such that $\bar{\beta}{\restriction}(\zeta+1) \in \hat{\mathcal{E}}_0$ for each $\zeta < \kappa$. Now let $\bar{j}$ be the increasing enumeration of $\{j < \kappa : N_j \cap \kappa = j \ \&\ (\exists \zeta < \kappa)(j = \beta_{\omega\cdot\zeta})\}$; remember that $\hat{\mathcal{E}}_0$ is closed under subsequences.]

Now, for $p \in \check{\underset{\sim}{\mathbb{Q}}}_\alpha$ let

$$j(p) = \sup\big\{j < \kappa : \begin{array}{l} j = 0 \text{ or } j \in \{j_\zeta : \zeta < \kappa\} \text{ and} \\ p \in \bigcap\{\mathcal{I} \in N_j : \mathcal{I} \subseteq \underset{\sim}{\mathbb{Q}}_\alpha \text{ is open dense in } V^{\mathbb{P}_\alpha}\}\end{array}\big\}$$

and $k(p) = \min\{j < \kappa : p \in N_j\}$. Now we finish notifying that

(1) if $\bar{p} = \langle p_\varepsilon : \varepsilon < \delta\rangle$ is increasing in $\check{\underset{\sim}{\mathbb{Q}}}_\alpha$ and such that $(\forall \varepsilon < \delta)(k(p_\varepsilon) < j(p_{\varepsilon+1}))$, then the sequence $\bar{p}$ has an upper bound in $\check{\underset{\sim}{\mathbb{Q}}}_\alpha$;

(2) for every $j < \kappa$ the set $\{p \in \check{\underset{\sim}{\mathbb{Q}}}_\alpha : j(p) > j\}$ is open dense in $\check{\underset{\sim}{\mathbb{Q}}}_\alpha$.

This finishes the proof of the claim and the proposition. □

Alternatively, first prove that wlog $\gamma < \kappa^+$ and then show that $\mathbb{P}'_\gamma$ becomes κ–Cohen in $\mathbf{V}^{\mathbb{R}}$. □

Conclusion B.8.4. Assume that

- $\mathbf{V}_0 \models \kappa$ is weakly compact and GCH holds (for simplicity),
- $\mathbf{V}_1$ is a generic extension of $\mathbf{V}_0$ making “κ weakly compact” indestructible by Cohen_κ (any member of κ–Cohen),
- $\mathbf{V}_2 = \mathbf{V}_1^{\mathbb{R}_0}$, where $\mathbb{R}_0$ adds a stationary non-reflecting subset S^* of κ by initial segments.

Further, in $\mathbf{V}_2$, let $\hat{\mathcal{E}}_0 = \hat{\mathcal{E}}_0[S^*]$, $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_1[S^*]$ be as in the main example for the current case (see, e.g., B.6.2), both in $\mathbf{V}_2$. Suppose that $\bar{\mathbb{Q}}$ is a $(<\kappa)$–support iteration of forcing notions on κ, say of length γ^*, complete for $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$. Let $\mathbf{V}_3 =$

$\mathbf{V}_2^{\mathrm{Lim}(\bar{\mathbb{Q}})}$ and let $\mathbb{R}$ be the forcing notion killing stationarity of S^* in $\mathbf{V}_3$, but also in $\mathbf{V}_2$; see B.8.3. Then

$$\mathbf{V}_2^{\mathbb{R}} = \mathbf{V}_1^{\mathbb{R}_0 * \mathbb{R}} = \mathbf{V}_1^{\mathrm{Cohen}} \models \text{“}\kappa \text{ is weakly compact indestructible by Cohen”},$$

and in $\mathbf{V}_2^{\mathbb{R}}$, the forcing notion $\mathrm{Lim}(\bar{\mathbb{Q}})$ is adding κ–Cohen. Consequently in $\mathbf{V}_3^{\mathbb{R}} = (\mathbf{V}_2^{\mathbb{R}})^{\lim(\bar{\mathbb{Q}})}$, $\lim(\bar{\mathbb{Q}})$ is adding Cohens and hence κ is weakly compact in $\mathbf{V}_3^{\mathbb{R}}$.

Conclusion B.8.5. (1) Let $\mathbf{V} = \mathbf{L}$ and let κ be a weakly compact cardinal, $\chi^\kappa = \chi$. Then for some forcing notion $\mathbb{P}$ we have, in $\mathbf{V}^{\mathbb{P}}$:
- (a) there are almost free Abelian groups in κ,
- (b) all almost free Abelian groups in κ are Whitehead.

(2) If $\mathbf{V} \models \mathrm{GCH}$ then $\mathbf{V}^{\mathbb{P}} \models \mathrm{GCH}$.

(3) We can add:
- (c) the forcing does not collapse any cardinals nor changes cofinalities, and it makes $2^\kappa = \chi$, $\chi = \|\mathbb{P}\|$,
- (d) for some stationary subset S^* of κ which is non reflecting and has stationary intersection with S^κ_θ for every regular *theta* $<$ *kappa* we have
  - every stationary subset of $\kappa \setminus S^*$ reflects in some inaccessible,
  - letting $\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1$ be defined from S^* as above, we have $\mathrm{Ax}^\kappa_{\kappa^+}(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$,
  - if $\kappa < \theta = \mathrm{cf}(\theta) \leq \chi$ then we can add $\mathrm{Ax}^\kappa_\theta(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$.

If κ is κ–Cohen indestructible weakly compact cardinal (or every stationary set reflects) then we may add:
- (e) the forcing adds no bounded subsets to κ.

Proof. 1) Let $\mathbf{V}_0 = \mathbf{V}$ and let $\mathbf{V}_1, \mathbf{V}_2, \mathbb{R}_0$ be defined as in B.8.4, just $\mathbb{R}_0$ adds a non-reflecting stationary subset of $\{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$. Working in $\mathbf{V}_2$ define $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \alpha^* \rangle$, $\alpha^* < \chi^+$ be as in the proof of the consistency of $\mathrm{Ax}^\kappa_\theta(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ in B.8.2. The desired universe is $\mathbf{V}_3 = \mathbf{V}_2^{\mathbb{P}_{\alpha^*}}$.

Clearly, as every step of the construction is a forcing extension, we have $\mathbf{V}_3 = \mathbf{V}^{\mathbb{P}}$ for some forcing notion $\mathbb{P}$. The forcing notion $\mathbb{R}_0 \in \mathbf{V}_1$ adds a non-reflecting stationary subset S to κ. As $\mathbb{P}_{\alpha^*}$ preserves $(\hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^S_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ (by B.7.3) the set S is stationary also in $\mathbf{V}_3$. Since $(\forall \delta \in S)(\mathrm{cf}(\delta) = \aleph_0)$ we may use S to build an almost free Abelian group in κ, so clause (a) holds. Let us prove the demand (b).

Suppose that G is an almost free Abelian group in κ with a filtration $\bar{G} = \langle G_i : i < \kappa \rangle$. Thus the set $\gamma(\bar{G}) = \{i < \kappa : G/G_i \text{ is not } \kappa\text{–free }\}$ is stationary. Now we consider two cases.

Case 1: the set $\gamma(\bar{G}) \setminus S$ is stationary.
By B.8.3 we know that after forcing with $\mathbb{R}$ (defined as there) the cardinal κ is still weakly compact (or just all its stationary subsets reflect in inaccessibles). But this forcing preserves the stationarity of $\gamma(\bar{G}) \setminus S$ (and generally any stationary subset of κ disjoint from S, as S does not reflect). Consequently, in $\mathbf{V}_3$, the set

$$\Gamma' = \{\kappa' : \begin{array}{l} \kappa' \text{ is strongly inaccessible and} \\ (\gamma(\bar{G}) \setminus S) \cap \kappa' \text{ is a stationary subset of } \kappa' \end{array}$$

is stationary in κ. Hence for some $\kappa' \in \Gamma'$ we have $(\forall i < \kappa')(\|G_i\| < \kappa')$ and therefore the filtration $\langle G_i : i < \kappa' \rangle$ of $G_{\kappa'}$ shows that $G_{\kappa'}$ is not free, contradicting “G is almost free in κ”.

Case 2: the set $\gamma(\bar{G}) \setminus S$ is not stationary.
By renaming, wlog $\gamma(\bar{G}) \subseteq S$. We shall prove that G is Whitehead. So let H be an Abelian group extending $\mathbb{Z}$ and let $h : H \stackrel{\text{onto}}{\longrightarrow} G$ be a homomorphism such that $\mathrm{Ker}(h) = \mathbb{Z}$. By B.6.10 the forcing notion $\mathbb{P} = \mathbb{P}_{h,H,G}$ is well defined and it is complete for $(\hat{\mathcal{E}}^S_0, \hat{\mathcal{E}}^S_1)$ and has cardinality κ (and for each $\alpha < \kappa$ the set $\mathcal{I}_\alpha = \{p \in \mathbb{P} : G_\alpha \subseteq p\}$ is dense in $\mathbb{P}$). Since $\mathbf{V}_3 \models \mathrm{Ax}^\kappa_\theta$, there is a directed set $\mathcal{G} \subseteq \mathbb{P}$ such that $\mathcal{G} \cap \mathcal{I}_\alpha \neq \emptyset$ for each $\alpha < \kappa$. Thus $f = \bigcup \mathcal{G}$ is a lifting as required (and G is Whitehead).

2) Implicit in the proof above. $\square$

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