LARGE WEIGHT DOES NOT YIELD AN IRREDUCIBLE BASE SH588

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Abstract. Answering a question of Juhász, Soukup and Szentmiklssy, we show that it is consistent that some first countable space of uncountable weight does not contain an uncountable subspace which has an irreducible base.

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§ 0. Introduction

For a topological space $X, w(X)$ is the minimal cardinality of a base for $X, \chi(p, X) = \min\{|u| : u$ is a neighbourhood base of $p\}$, and $\chi(X) = \sup\{\chi(p, X) : p \in X\}$.

In [**?**] the following problem was investigated: What makes a space have weight larger than its character? The notion of *irreducible base* was introduced, and it was proved [**?**, Lemma 2.6] that *if a topological space X has an irreducible base then $w(X) = |X| \cdot \chi(X)$.*
The following question was formulated:

Problem 0.1. *Does every first countable space of uncountable weight contain an uncountable subspace which has an irreducible base?*

We show that the answer is consistently NO. We thank Lajos Soukup for actually writing the paper.

{df:irred}

Definition 0.2. Let X be a topological space. A base $\mathscr{U}$ of X is called *irreducible* if it has an *irreducible decomposition* $\mathscr{U} = \bigcup\{\mathscr{U}_x : x \in X\}$, i.e, (i) and (ii) below hold:

(i) $\mathscr{U}_x$ is a neighbourhood base of x in X for each $x \in X$

(ii) for each $x \in X$ the family $\mathscr{U}_x^- = \bigcup\limits_{y \neq x} \mathscr{U}_y$ is not a base of X.

§ 1. The Theorem

Theorem 1.1. *There is a c.c.c. poset $\mathbb{P} = \langle P, \leq \rangle$ of size $\aleph_1$ such that in $\mathbf{V}^{\mathbb{P}}$ there is a first countable space $X = \langle \aleph_1, \tau \rangle$ of uncountable weight which does not contain an uncountable subspace which has an irreducible base.*

Proof. The elements of the poset $\mathbb{P}$ will be finite "approximations" of a base $\{U(\alpha, n) : \alpha < \omega_1, n < \omega\}$ of X.

We define the poset $\mathbb{P} = \langle P, \leq \rangle$ as follows. The underlying set of P consists of the triples $\langle A, n, U \rangle$ satisfying (P1)–(P3) below:

($P1$) $A \in [\omega_1]^{\aleph_0}, n \in \omega$ and U is a function, $U : A \times n \to \mathscr{P}(A)$,

($P2$) $\alpha \in \mathscr{U}(\alpha, j)$ and $U(\alpha, i) \subseteq U(\alpha, i-1)$ for each $\alpha \in A$ and $j < n, 0 < i < n$,

($P3$) if $\beta \in U(\alpha, i) \subseteq U(\beta, 0)$ for some $i < n$, then $\beta \leq \alpha$.

For $p \in P$ write $p = \langle A_p, n_p, U_p \rangle$. Let us remark that property (P3) will guarantee that $w(X) = \omega_1$.

Define the order $\leq$ on P as follows. For $p, q \in P$ we put q stronger than p, or q extends $p, q \leq p$ if:

(a) $A_p \subseteq A_q$,

(b) $n_p \leq n_q$,

(c) $U_p(\alpha, i) = U_q(\alpha, i) \cap A_p$ for each $\langle \alpha, i \rangle \in A_p \times n_p$,

(d) for each $\langle \alpha, i \rangle, \langle \beta, j \rangle \in A_p \times n_p$,

 ($d1$) if $U_p(\alpha, i) \cap U_p(\beta, j) = \emptyset$ then $U_q(\alpha, i) \cap U_q(\beta, j) = \emptyset$,

 ($d2$) if $U_p(\alpha, i) \subseteq U_p(\beta, j)$ then $U_q(\alpha, i) \subseteq U_q(\beta, j)$.

We say that the conditions $p_0 = \langle A_0, n_0, U_0 \rangle$ and $p_1 = \langle A_1, n_1, U_1 \rangle$ are *twins* iff $n_0 = n_1, |A_0| = |A_1|$ and denoting by σ the unique $<_{\mathrm{On}}$-preserving bijection between A_0 and A_1 we have

($I1$) $\sigma \restriction (A_0 \cap A_1) = \mathrm{id}_{A_0 \cap A_1}$,

($I2$) σ is an isomorphism between p_0 and p_1, i.e. for each $\alpha \in A_0$ and $i < n_0$ we have $U_1(\sigma(\alpha), i) = \sigma'' U_0(\alpha, i)$.

We say that σ is the *twin function* between p_0 and p_1. Define the *smashing function* σ of p_0 and p_1 as follows: $\underline{\sigma} = \sigma^{-1} \cup \mathrm{id}_{A_0}$. The function σ^* defined by the formula $\sigma^* = \sigma \cup \sigma^{-1}$ is called the *exchange function* of p_0 and p_1. The rest of the proof is broken into a series of claims. □

The burden of the proof is to verify the next lemma.

{lm:twins}

Amalgamation Lemma 1.2. *Assume that $p_0 = \langle A_0, n_0, U_0 \rangle$ and $p_1 = \langle A_1, n_1, U_1 \rangle$ are twins, $A_0 \cap A_1 < A_0 \setminus A_1 < A_1 \setminus A_0, \xi_0 \in A_0 \setminus A_1, \xi_1 = \sigma(\xi_0)$, where σ is the twin function between p_0 and p_1, and let $k < m < n_0$. Then p_0 and p_1 have a common extension $p = \langle A, n, U \rangle$ in P such that:*

$$(*) \qquad \xi_0 \in U(\xi_1, m) \subseteq U(\xi_1, k) \subseteq U(\xi_0, k).$$

Proof. Write $n = n_0 = n_1, D = A_0 \cap A_1$ and $A^* = A_0 \cup A_1$. Unfortunately we can not assume that $A = A^*$ because in this case we can not guarantee (P3) for p. So we need to add further elements to A^* to get a large enough A as follows. Choose a set $B \subseteq \omega_1 \setminus A^*$ of cardinality $|A^* \times n|$ and fix a bijection ρ between $A^* \times n$ and B. We will take $A = A^* \cup B$. To simplify the notation we will write $\langle \alpha, i \rangle$ for $\rho(\alpha, i)$, for all $\alpha \in A^*$ and $i < n$, i.e. we identify the elements of B and of $A^* \times n$.

The idea of the proof is the following: for each $\langle \alpha, i \rangle \in A^* \times n$ we put the element $\langle \alpha, i \rangle$ into $U(\alpha, i)$. On the other hand, we try to keep $U(\alpha, i)$ small, so we put $\langle \beta, j \rangle$ into $U(\alpha, i)$ if and only if we can "derive" from the property (d2) that $U(\beta, j) \subseteq U(\alpha, i)$ should hold in any condition $p = \langle A, n, U \rangle$ which is a common extension of p_0 and p_1 and which satisfies $(*)$.

The condition p will be constructed in two steps. First we construct a condition $p' = \langle A, n, U' \rangle$ extending both p_0 and p_1. This p' can be considered as the minimal amalgamation of p_0 and p_1. Then, in the second step, we carry out small modifications on the function U', namely we increase its value on certain places to guarantee $(*)$.

Now we carry out our construction. For $\varepsilon < 2$ and $\langle \beta, j \rangle \in A_\varepsilon \times n$ let

$$V_\varepsilon(\beta, j) = \{\langle \alpha, i \rangle \in A_\varepsilon \times n : U_\varepsilon(\alpha, i) \subseteq U_\varepsilon(\beta, j)\} \tag{0.1}$$

and

$$W_\varepsilon(\beta, j) = \{\langle \alpha, i \rangle \in A_{1-\varepsilon} \times n : \begin{array}{l} \exists \langle \gamma, l \rangle \in D \times n \\ U_{1-\varepsilon}(\alpha, i) \subseteq U_{1-\varepsilon}(\gamma, l) \wedge U_\varepsilon(\gamma, l) \subseteq U_\varepsilon(\beta, j)\}. \end{array} \tag{0.2}$$

If we want to define p' in such a way that p' extends p_0, p_1, then (d2) implies that $U'(\alpha, i) \subseteq U'(\beta, j)$ should hold whenever $\langle \alpha, i \rangle \in V(\beta, j) \cup W(\beta, j)$.

Now we are ready to define the function U'. For $\varepsilon < 2$, $\beta \in A_\varepsilon$ and $j < n$ let

$$U'(\beta, j) = U_\varepsilon(\beta, j) \cup U_{1-\varepsilon}(\sigma^*(\beta), j) \cup V_\varepsilon(\beta, j) \cup W_\varepsilon(\beta, j). \tag{0.3}$$

For $\langle \alpha, i \rangle \in A^* \times n$ and $j < n$ let

$$U'(\langle \alpha, i \rangle, j) = \{\langle \alpha, i \rangle\}. \tag{0.4}$$

Let us remark that $U'(\delta, j)$ is well-defined even for $\delta \in A_0 \cap A_1$. Indeed, in this case $\sigma^*(\delta) = \delta$ and $V_\varepsilon(\delta, j) = W_{1-\varepsilon}(\delta, j)$, and so

$$U'(\delta, j) = U_0(\delta, j) \cup U_1(\delta, j) \cup V_0(\delta, j) \cup V_1(\delta, j).$$

Now put

$$p' = \langle A, n, U' \rangle.$$

{cl:push}

Claim 1.3. *If $\alpha \in U'(\beta, j)$ so $\alpha \in A^*$ then $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$.*

Indeed, if $\beta \in A_\varepsilon$ and $j < n$ then $U'(\beta, j) \cap A^* = U_\varepsilon(\beta, j) \cup U_{1-\varepsilon}(\sigma(\beta), j)$.

{cl:push2}

Claim 1.4. *If $\langle \alpha, i \rangle \in U'(\beta, j)$ then $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$.*

Proof of the Claim. Assume that $\beta \in A_\varepsilon$. If $\langle \alpha, i\rangle \in V_\varepsilon(\beta, j)$ then $\alpha \in U_\varepsilon(\alpha, i) \subseteq U_\varepsilon(\beta, j)$ and $U_\varepsilon(\beta, j) \subseteq U'(\beta, j)$. So we have $\alpha \in U'(\beta, j)$ which implies $\sigma(\alpha) \in$ {cl:push} $U_0(\sigma(\beta), j)$ by Claim 1.3.

If $\langle \alpha, i\rangle \in W_{1-\varepsilon}(\beta, j)$ then for some $\langle \gamma, l\rangle \in D \times n$ we have $U_{1-\varepsilon}(\alpha, i) \subseteq U_{1-\varepsilon}(\gamma, l) \wedge U_\varepsilon(\gamma, l) \subseteq U_\varepsilon(\beta, j)$. Thus $\alpha \in U_\varepsilon(\beta, j) \subseteq U'(\beta, j)$, which implies {cl:push} $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j)$ by Claim 1.3. □

{cl:p'1}

Claim 1.5. $p' \in P$.

Proof of claim 1.5. Proof of the claim 1.5. (P1) and (P2) clearly hold, so we need to check only (P3). {cl:p'1}

Assume on the contrary that (P3) fails for p'. Since $U'(\langle \nu, s\rangle, j) = \{\langle \nu, s\rangle\}$ by (0.4) for each $\langle \nu, s\rangle \in B$ and $j < n$, we can assume that some $\alpha < \beta$ from A^* and $i < n$ witness that (P3) fails, i.e. $\beta \in U'(\alpha, i) \subseteq U'(\beta, 0)$. Then $\underline{\sigma}(\beta) \in U_0(\underline{\sigma}(\alpha), i) \subseteq U'(\underline{\sigma}(\beta), 0)$ by Claim 1.3. Since p_0 satisfies (P3) it follows that $\underline{\sigma}(\beta) \le \underline{\sigma}(\alpha)$, and {cl:push} so $\alpha \in A_0 \setminus A_1$ and $\beta \in A_1 \setminus A_0$. Consider the element $u = \langle \alpha, i\rangle \in A \setminus A^*$. Then $u \in U'(\alpha, i)$ and so $u \in U'(\beta, 0)$ as well. By the definition of $U'(\beta, 0)$ this means that $\langle \alpha, i\rangle \in W_1(\beta, 0)$, that is, there is $\langle \gamma, l\rangle \in D \times n$ such that $U_0(\alpha, i) \subseteq U_0(\gamma, l)$ and $U_1(\gamma, l) \subseteq U_1(\beta, j)$. Thus

$$\underline{\sigma}(\beta) \in U_0(\alpha, i) \subseteq U_0(\gamma, l) \subseteq U_0(\underline{\sigma}(\beta), 0) \tag{0.5}$$

by Claim 1.3. Thus $\underline{\sigma}(\beta) \in U_0(\gamma, l) \subseteq U_0(\underline{\sigma}(\beta), 0)$ and so $\underline{\sigma}(\beta) \le \gamma$ because p_0 {cl:push} satisfies (P3). But this is a contradiction because $\gamma \in D = A_0 \cap A_1, \sigma(\beta) \in A_0 \setminus A_1$ and we assumed that $(A_0 \cap A_1) < (A_0 \setminus A_1)$. □

{cl:p'}

Claim 1.6. *p' is stronger than p_0, p_1.*

Proof of claim 1.6. Proof of claim 1.6. Conditions (a) and (b) are clear. {cl:p'}

To check (c) assume that $\alpha \in A_\varepsilon$ and $i \in n$. By (0.3),

$$U'(\alpha, i) \cap A_\varepsilon = (U_\varepsilon(\alpha, i) \cup U_{1-\varepsilon}(\sigma^*(\alpha), i)) \cap A_\varepsilon = \\ U_\varepsilon(\alpha, i) \cup (U_{1-\varepsilon}(\sigma^*(\alpha), i) \cap A_\varepsilon) = U_\varepsilon(\alpha, i)$$

because $U_{1-\varepsilon}(\sigma^*(\alpha), i) = \sigma[U_\varepsilon(\alpha, i)]$.

To check (d1) assume that $\beta, \gamma \in A_\varepsilon$ and $j, k < n$ such that $U'(\beta, j) \cap U'(\gamma, k) \neq \emptyset$. Fix $x \in U'(\beta, j) \cap U'(\gamma, k)$.

Then

$$\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j) \cap U_0(\underline{\sigma}(\gamma), k)$$

by Claim 1.3 if $x = \alpha \in A^*$, and by Claim 1.4 if $x = \langle \alpha, i\rangle \in A \setminus A^*$. {cl:push2}

If $\varepsilon = 0$ then $\underline{\sigma}(\beta) = \beta$ and $\underline{\sigma}(\gamma) = \gamma$, so $\underline{\sigma}(\alpha) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$.

If $\varepsilon = 1$ then $\underline{\sigma}(\beta) = \underline{\sigma}(\beta)$ and $\underline{\sigma}(\gamma) = \sigma(\gamma)$, and so $\sigma^*(\underline{\sigma}(\alpha)) \in U_\varepsilon(\beta, j) \cap U_\varepsilon(\gamma, k)$.

Finally to check (d2) assume that $\beta, \gamma \in A_\varepsilon$ and $j, k < n$ such that $U_\varepsilon(\beta, j) \subseteq U_\varepsilon(\gamma, k)$.

Then clearly

$$U_{1-\varepsilon}(\beta, j) = \sigma[U_\varepsilon(\beta, j)] \subseteq \sigma[U_\varepsilon(\gamma, k)] = U_{1-\varepsilon}(\gamma, k),$$

moreover, $V_\varepsilon(\beta,j) \subseteq V_\varepsilon(\gamma,k)$ by (0.1), and $W_\varepsilon(\beta,j) \subseteq W_\varepsilon(\gamma,k)$ by (0.2), and so $U'(\beta,j) \subseteq U'\varepsilon(\gamma,k)$ by (3). $\square$

Now carry out the promised modification of U' to obtain U as follows, recalling that $\xi_0 \in \mathscr{U}_0(\xi_0,j) = \mathscr{U}_0(\sigma^*(\xi_1),j) \subseteq \mathscr{U}'(\xi_1,j)$, so the "problem" is about "$\mathscr{U}(\xi_1,k) \subseteq \mathscr{U}(\xi_0,k)$".

If $z \in A$ and $j < n$ let

$$U(z,j) = \begin{cases} U'(z,j) \cup U'(\xi_1,k) & \text{if } U_0(\xi_0,k) \subseteq U_0(z,j) \text{ so } z \in A_0, \\ U'(z,j) & \text{otherwise.} \end{cases}$$

Put

$$p = \langle A, n, U\rangle.$$

If $U_0(\xi_0,k) \subseteq U_0(z,j)$ then $U_1(\xi_1,k) \subseteq U_1(\sigma(z),j) \subseteq U'(z,j)$ and $W_1(\xi_1,k) \subseteq V_0(\xi_0,k) \subseteq U'(z,j)$.

So

$$U(z,j) \setminus U'(z,j) \subseteq V_1(\xi_1,k). \tag{0.6}$$

Hence

$$U(z,j) = \begin{cases} U'(z,j) \cup V_1(\xi_1,k) & \text{if } U_0(\xi_0,k) \subseteq U_0(z,j) \text{ so } z \in A_0, \\ U'(z,j) & \text{otherwise.} \end{cases} \tag{0.7}$$

{cl:push3} **Claim 1.7.** *If $\langle\alpha,i\rangle \in U(\beta,j)$ then $\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta),j)$.*

Indeed, if $\langle\alpha,i\rangle \in U(\beta,j)$ then by (6), (7) we have $\langle\alpha,i\rangle \in U'(\beta,j)$ or $\langle\alpha,i\rangle \in U'(\sigma(\beta),j)$, and now apply Claim 1.4.

{cl:push2} {cl:p} **Claim 1.8.** $p \in P$.

{cl:p} *Proof of claim 1.8. Proof of claim 1.8.* (P1) and (P2) clearly hold, so we need to check (P3) only.

Assume on the contrary that (P3) fails for p. Since $U(\langle\nu,s\rangle,j) = \{\langle\nu,s\rangle\}$ for each $\langle\nu,s\rangle \in A \setminus A^*$ and $j < n$ we can assume that there are $\alpha < \beta$ from A^* and $i < n$ witness that (P3) fails, i.e.

$$\beta \in U(\alpha,i) \subseteq U(\beta,0). \tag{0.8}$$

Then by 1.8 and the definitions of $\mathscr{U}_0, \mathscr{U}$ we have $\underline{\sigma}(\beta) \in U_0(\underline{\sigma}(\alpha),i) \subseteq U(\underline{\sigma}(\beta),0)$.

But p_0 satisfies (P3) so $\underline{\sigma}(\beta) \le \underline{\sigma}(\alpha)$, and so $\alpha \in A_0 \setminus A_1$ and $\beta \in A_1 \setminus A_0$. Thus $U_0(\beta,j)$ is undefined, and so

$$U(\beta,0) = U'(\beta,0) \text{ and } U(\alpha,i) \setminus U'(\alpha,i) \subseteq A \setminus A^*. \tag{0.9}$$

by (0.7). So (0.8) yields

$$\beta \in U'(\alpha,i) \subseteq U'(\beta,0).$$

However this is a contradiction because p' satisfies (P3). $\square$

{cl:p2}

Claim 1.9. *p is stronger than p_0, p_1.*

Proof. Clauses (a) and (b) are trivial. Clause (c) also holds because p' is stronger than p_ε and $(U(\alpha,i) \setminus U'(\alpha,i)) \cap A_\varepsilon = \emptyset$ by (0.6).

To check (d1) assume that $\beta, \gamma \in A_\varepsilon$ and $j, k < n$ such that $U(\beta,j) \cap U(\gamma,k) \neq \emptyset$. Pick $x \in U(\beta,j) \cap U(\gamma,k)$.

Then

$$\underline{\sigma}(\alpha) \in U_0(\underline{\sigma}(\beta), j) \cap U_0(\underline{\sigma}(\gamma), k)$$

by Claim 1.3 if $x = \alpha \in A^*$, and by Claim 1.7 if $x = \langle \alpha, i\rangle \in A \setminus A^*$. {cl:push3}

If $\varepsilon = 0$ then $\underline{\sigma}(\beta) = \beta$ and $\underline{\sigma}(\gamma) = \gamma$, so $\underline{\sigma}(\alpha) \in U_\varepsilon(\beta,j) \cap U_\varepsilon(\gamma,k)$ hence $\mathscr{U}_\varepsilon(\beta,j) \cap \mathscr{U}_\varepsilon(\gamma,k) \neq \emptyset$ as promised in (d1).

If $\varepsilon = 1$ then $\underline{\sigma}(\beta) = \sigma^*(\beta)$ and $\underline{\sigma}(\gamma) = \sigma^*(\gamma)$, and so $\sigma^*(\underline{\sigma}(\alpha)) \in U_\varepsilon(\beta,j) \cap U_\varepsilon(\gamma,k)$ hence $\mathscr{U}_\varepsilon(\beta,j) \cap \mathscr{U}_\varepsilon(\gamma,k) \neq \emptyset$ as promised in (d1).

Finally to check (d2) assume that $\beta, \gamma \in A_\varepsilon$ and $i, j < n$ such that $U_\varepsilon(\beta,i) \subseteq U_\varepsilon(\gamma,j)$. Since $p' \leq p_\varepsilon$ we have $U'(\beta,i) \subseteq U'(\gamma,j)$. If $U(\beta,i) = U'(\beta,i)$, we are done. So we can assume that ($\beta \in A_0$ and) $U(\beta,i) = U'(\beta,i) \cup V(\xi_1,k)$ hence $U_0(\xi_0,k) \subseteq U_0(\beta,i)$.

If $\gamma \in A_1 \setminus A_0$ <u>then</u> $\varepsilon = 1$ and $\mathscr{U}_0(\beta,i) \subseteq \mathscr{U}_0(\underline{\sigma}(\gamma),j)$ hence $\mathscr{U}_0(\xi_0,k) \subseteq \mathscr{U}_0(\beta,i) \subseteq \mathscr{U}_0(\sigma^*(\gamma),j)$ hence $\mathscr{U}_1(\xi_1,k) = \mathscr{U}_1(\sigma^*(\xi_0),k) \subseteq \mathscr{U}_0(\gamma,j)$ hence $V_1(\xi_1,k) \subseteq \mathscr{U}'(\gamma,j) = \mathscr{U}(\gamma,1)$ as promised.

So without loss of generality $\gamma \in A_0$, so recalling $\beta \in A_0$ without loss of generality $\varepsilon = 0$. But then $U_0(\xi_0,k) \subseteq U_0(\gamma,j)$ and so $U(\gamma,j) = U'(\gamma,j) \cup V(\xi_1,k)$, and so $U(\beta,i) \subseteq U(\gamma,j)$. $\square$

Since p satisfies $(*)$, the amalgamation lemma is proved.

Using the amalgamation lemma it is easy to complete the proof of the theorem.

By standard Δ-system argument, any uncountable set of conditions contains two elements, p_0 and p_1, which are twins. So, by Lemma 1.2, they have a common extension p. So $\mathbb{P}$ satisfies c.c.c. {lm:twins}

If $\mathscr{G}$ is a generic filter, for $\alpha < \omega_1$ and $i < \omega$ put

$$U(\alpha,i) = \cup\{U_p(\alpha,i) : p \in \mathscr{G}, \alpha \in A_p, i < n_p\}, \tag{0.10}$$

and let $\mathscr{U}_\alpha = \{U(\alpha,i) : i < \omega\}$ be the base of the point α in $X = \langle \omega_1, \tau\rangle$.

For any countable set $\mathscr{W}$ of open sets for X there is $\gamma < \omega_1$ such that: if $k < m < \omega, \alpha < \omega_1, u \in \mathscr{W}$ and $\mathscr{U}(\alpha,k,m) \subseteq u \subseteq \mathscr{U}(\alpha,k)$ then for some $\beta < \alpha_*$ we have $\mathscr{U}(\beta,m) \subseteq u \subseteq \mathscr{U}(\beta,k)$ so by (P3), $\mathscr{W}$ is not a base of X. So $w(X) = \aleph_1$.

Finally we show that X does not contain an uncountable subspace which has an irreducible base.

Assume on the contrary that

$$r \Vdash \text{ the subspace } \dot{Y} = \{\dot{y}_\xi : \xi < \omega_1\} \text{ has an irreducible base } \mathscr{B}$$
$$\text{and } \{\dot{\mathscr{B}}_{y_\xi} : \xi < \omega_1\} \text{ is an irreducible decomposition of } \dot{\mathscr{B}}\text{"}.$$

We can assume that $r \Vdash$ "$\dot{y}_\xi \geq \check{\xi}$".

For each $\xi < \omega_1$ pick a condition r_ξ and $k_\xi \in \omega$ such that

$$r_\xi \Vdash \text{“ if } V \in \mathscr{B} \text{ with } \dot{y}_\xi \in V \subseteq U(\dot{y}_\xi, \check{k}_\xi) \text{ then } V \in \mathscr{B}_{y_\xi}\text{”.} \tag{0.11}$$

For each $\xi < \omega_1$ pick a condition $p_\xi \le r_\xi$, an ordinal $\alpha_\xi \ge \xi$, a name $\dot{V}_\xi$ and a natural number $m_\xi < \omega$ such that $\alpha_\xi \in A_{p_\xi}$ and

$$p_\xi \Vdash \dot{y}_\xi = \check{\alpha}_\xi, \dot{V}_\xi \in \dot{\mathscr{B}}_{\alpha_\xi} \text{ and } U(\check{\alpha}_\xi, \check{m}_\xi) \subseteq \dot{V}_\xi \subseteq U(\check{\alpha}_\xi, \check{k}_\xi). \tag{0.12}$$

By standard argument find $I \in [\omega_1]^{\aleph_1}$ such that:

- (i) $m_\xi = m$ and $k_\xi = k$ for each $\xi \in I$,
- (ii) the sequence $\{\alpha_\xi : \xi \in I\}$ is strictly increasing,
- (iii) the conditions $\{p_\xi : \xi \in I\}$ are pairwise twins,
- (iv) $\sigma_{\xi,\eta}(\alpha_\xi) = \alpha_\eta$ for $\{\xi, \eta\} \in [I]^2$, where $\sigma_{\xi,\eta}$ is the twin function for p_ξ, p_ζ.

Pick $\xi < \eta$ from I. By the Amalgamation Lemma there is a common extension p of p_ξ and p_η such that

$$\alpha_\xi \in U_p(\alpha_\eta, m) \wedge U_p(\alpha_\eta, k) \subseteq U_p(\alpha_\xi, k). \tag{0.13}$$

Then, by (d2),

$$p \Vdash \text{“}\check{\alpha}_\xi \in U(\check{\alpha}_\eta, \check{m}) \wedge U(\check{\alpha}_\eta, \check{k}) \subseteq U(\check{\alpha}_\xi, \check{k})\text{”.} \tag{0.14}$$

Then, by (0.12),

$$p \Vdash \text{“}\dot{V}_\eta \in \mathscr{B}_{\alpha_\eta} \text{ and } \check{\alpha}_\xi \in U(\check{\alpha}_\eta, \check{m}) \subseteq \dot{V}_\eta \subseteq U(\check{\alpha}_\eta, \check{k}) \subseteq U(\check{\alpha}_\xi, \check{k})\text{”} \tag{0.15}$$

which contradicts (0.11).

This completes the proof of the Theorem. □

References

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