There may be no nowhere dense ultrafilter

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Abstract

We show the consistency of ZFC + "there is no NWD-ultrafilter on ω", which means: for every non-principal ultrafilter $\mathcal{D}$ on the set of natural numbers, there is a function f from the set of natural numbers to the reals, such that for every nowhere dense set A of reals, $\{n : f(n) \in A\} \notin \mathcal{D}$. This answers a question of van Douwen, which was put in more general context by Baumgartner.

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0 Introduction

We prove here the consistency of "there is no NWD-ultrafilter on ω" (non-principal, of course). This answers a question of van Douwen [vD81] which appears as question 31 of [B6]. Baumgartner [B6] considers the question which he dealt more generally with J-ultrafilter where

Definition 0.1 *1. An ultrafilter $\mathcal{D}$, say on ω, is called a J-ultrafilter where J is an ideal on some set X (to which all singletons belong, to avoid trivialities)* if *for every function $f : \omega \longrightarrow X$ for some $A \in \mathcal{D}$ we have $f''(A) \in J$.*

2. The NWD–ultrafilters are the J-ultrafilters for $J = \{B \subseteq \mathcal{Q} : B$ is nowhere dense$\}$ ($\mathcal{Q}$ is the set of all rationals; we will use an equivalent version, see 2.4).

This is also relevant for the consistency of "every (non-trivial) c.c.c. σ-centered forcing notion adds a Cohen real", see [Sh:F151].

The most natural approach to a proof of the consistency of "there is no NWD–ultrafilter" was to generalize the proof of CON(there is no P-point) (see [Sh:b, VI, §4] or [Sh:f, VI, §4]), but I (and probably others) have not seen how.

We use an idea taken from [Sh 407], which is to replace the given maximal ideal I on ω by a quotient; moreover, we allow ourselves to change the quotient. In fact, the forcing here is simpler than the one in [Sh 407]. A related work is Goldstern Shelah [GoSh 388].

We similarly may consider the consistency of "no α–ultrafilter" for limit $\alpha < \omega_1$ (see [B6] for definition and discussion of α–ultrafilters). This question and the problems of preservation of ultrafilters and distinguishing existence properties of ultrafilters will be dealt with in a subsequent work [Sh:F187].

In §3 we note that any ultrafilter with property M (see Definition 3.2) is an NWD–ultrafilter, hence it is consistent that there is no ultrafilter (on ω) with property M.

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1 The basic forcing

In Definition 1.2 below we define the forcing notion $\mathbb{Q}^1_{I,h}$ which will be the one used in the proof of the main result 3.1. The other forcing notion defined below, $\mathbb{Q}^2_{I,h}$, is a relative of $\mathbb{Q}^1_{I,h}$. Various properties are much easier to check for $\mathbb{Q}^2_{I,h}$, but unfortunately it does not do the job. The reader interested in the main result of the paper only may concentrate on $\mathbb{Q}^1_{I,h}$.

Definition 1.1 *Let I be an ideal on ω containing the family $[\omega]^{<\omega}$ of finite subsets of ω.*

1. *We say that an equivalence relation E is an I–*equivalence relation *if:*

   (a) $\mathrm{dom}(E) \subseteq \omega$,

   (b) $\omega \setminus \mathrm{dom}(E) \in I$,

   (c) *each E-equivalence class is in I.*

2. *For I-equivalence relations E_1, E_2 we write $E_1 \leq E_2$* if

   (i) $\mathrm{dom}(E_2) \subseteq \mathrm{dom}(E_1)$,

   (ii) $E_1 \restriction \mathrm{dom}(E_2)$ *refines* E_2,

   (iii) $\mathrm{dom}(E_2)$ *is the union of a family of E_1-equivalence classes.*

Definition 1.2 *Let I be an ideal on ω to which all finite subsets of ω belong and let $h : \omega \longrightarrow \omega$ be a non-decreasing function. Let $\ell \in \{1, 2\}$. We define a forcing notion $\mathbb{Q}^{\ell}_{I,h}$ (if $h(n) = n$ we may omit it) intended to add $\langle y^n_i : i < h(n), n < \omega\rangle$, $y^n_i \in \{-1, 1\}$. We use x^n_i as variables.*

1. $p \in \mathbb{Q}^{\ell}_{I,h}$ *if and only if* $p = (H, E, A) = (H^p, E^p, A^p)$ *and*

   (a) *E is an I–equivalence relation on* $\mathrm{dom}(E) \subseteq \omega$,

   (b) $A = \{n \in \mathrm{dom}(E) : n = \min(n/E)\}$,

   (c) *if $\ell = 1$,* then *H is a function with range $\subseteq \{-1, 1\}$ and domain*

   $$B^p_1 = \big\{x^n_i : \begin{array}{l} i < h(n) \text{ and } n \in \omega \setminus \mathrm{dom}(E) \text{ or} \\ n \in \mathrm{dom}(E) \text{ and } i \in [h(\min(n/E)), h(n))\big\}, \end{array}$$

   (d) *if $\ell = 2$,* then

   (α) *H is a function on $\mathrm{dom}(H) = B^p_2 \cup B^p_3$, where*

   $$\begin{array}{ll} B^p_2 = \{x^m_i : m \in \omega,\ A^p \cap (m+1) = \emptyset,\ i < h(m)\} & \text{and} \\ B^p_3 = \{x^m_i : m \in \mathrm{dom}(E^p) \setminus A^p \text{ or } m \notin \mathrm{dom}(E^p) \text{ but } & A^p \cap m \neq \emptyset, \\ & i < h(m)\}, \end{array}$$

   (β) *for $x^m_i \in B^p_3$, $H(x^m_i)$ is a function of the variables $\{x^n_j : (n, j) \in w_p(m, i)\}$ to $\{-1, 1\}$, where*

   $$w_p(m) = w_p(m, i) = \{(\ell, j) : \ell \in A^p \cap m \text{ and } j < h(\ell)\},$$

   for $n \in A^p$ we stipulate $H^p(x^n_i) = x^n_i$ and

   (γ) *$H \restriction B^p_2$ is a function to $\{-1, 1\}$.*

   (e) *if $\ell = 2$ and $x^n_i \in B^p_3$, $n^* = \min(n/E^p) < n$ and $y^m_i \in \{-1, 1\}$ for $m \in A^p \cap n^*$, $i < h(m)$ and $z^n_j \in \{-1, 1\}$ for $j < h(n^*)$* then *for some $y^{n^*}_j \in \{-1, 1\}$ for $j < h(n^*)$ we have*

   $$j < h(n^*) \quad \Rightarrow \quad z^n_j = (H^p(x^n_j))(\ldots, y^m_i, \ldots)_{(m,i) \in w_p(n,j)}.$$

When it can not cause any confusion, or we mean "for both $\ell = 1$ and $\ell = 2$", we omit the superscript ℓ.

2. *Defining functions like $H(x_i^m)$, $x_i^m \in B_3^p$ (when $\ell = 2$), we may allow to use dummy variables. In particular, if $H^p(x_i^m)$ is $-1, 1$ we identify it with constant functions with this value.*

3. *We say that a function $f : \{x_i^n : i < h(n),\ n < \omega\} \longrightarrow \{-1, 1\}$ satisfies a condition $p \in \mathbb{Q}^\ell_{I,h}$ if:*

   (a) $f(x_i^n) = H^p(x_i^n)$ when $x_i^n \in B_1^p$ *and* $\ell = 1$*, or* $x_i^n \in B_2^p$ *and* $\ell = 2$*,*

   (b) $f(x_i^n) = H^p(x_i^n)(\ldots, f(x_j^m), \ldots)_{(m,j)\in w_p(n,i)}$ when $\ell = 2$ *and* $x_i^n \in B_3^p$*,*

   (c) $f(x_i^n) = f(x_i^{\min(n/E^p)})$ when $\ell = 1$, $n \in \mathrm{dom}(E^p)$ *and* $i < h(\min(n/E^p))$.

4. *The partial order $\leq = \leq_{\mathbb{Q}^\ell_{I,h}}$ is defined by $p \leq q$ if and only if:*

   (α) $E^p \leq E^q$*,*

   (β) *every function $f : \{x_i^n : i < h(n),\ n < \omega)\} \longrightarrow \{-1, 1\}$ satisfying q satisfies p.*

Proposition 1.3 $(\mathbb{Q}^\ell_{I,h}, \leq_{\mathbb{Q}^\ell_{I,h}})$ *is a partial order.* ■

Remark 1.4 We may reformulate the definition of the partial orders $\leq_{\mathbb{Q}^\ell_{I,h}}$, making them perhaps more direct. Thus, in particular, if $p, q \in \mathbb{Q}^1_{I,h}$ then $p \leq_{\mathbb{Q}^1_{I,h}} q$ if and only if the demand (α) of 1.2(4) holds and

$(\beta)^*$ for each $x_i^n \in B_1^q$:

- **(i)** if $x_i^n \in B_1^p$ then $H^q(x_i^n) = H^p(x_i^n)$,
- **(ii)** if $n \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q)$, $i < h(\min(n/E^p))$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$,
- **(iii)** if $n \in \mathrm{dom}(E^q) \setminus A^p$, $\min(n/E^p) > \min(n/E^q)$ and $h(\min(n/E^q)) \leq i < h(\min(n/E^p))$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$.

The corresponding reformulation for the forcing notion $\mathbb{Q}^2_{I,h}$ is more complicated, but it should be clear too.

One may wonder why we have h in the definition of $\mathbb{Q}^\ell_{I,h}$ and we do not fix that e.g. $h(n) = n$. This is to be able to describe nicely what is the forcing notion $\mathbb{Q}^\ell_{I,h}$ below a condition p like. The point is that $\mathbb{Q}^\ell_{I,h}\restriction\{q : q \geq p\}$ is like $\mathbb{Q}^\ell_{I,h}$ but we replace I by its quotient and we change the function h. More precisely:

Proposition 1.5 *If $p \in \mathbb{Q}^\ell_{I,h}$ and $A^p = \{n_k : k < \omega\}$, $n_k < n_{k+1}$, $h^* : \omega \longrightarrow \omega$ is $h^*(k) = h(n_k)$ and $I^* = \{B \subseteq \omega : \bigcup\limits_{k \in B} (n_k/E) \in I\}$* then *$\mathbb{Q}^\ell_{I,h} \restriction \{q : p \leq_{\mathbb{Q}^\ell_{I,h}} q\}$ is isomorphic to $\mathbb{Q}^\ell_{I^*,h^*}$.*

Proof   Natural. ■

Definition 1.6 *We define a $\mathbb{Q}_{I,h}$–name $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_n : n < \omega \rangle$ by:*
$\underset{\sim}{\eta}_n$ is a sequence of length $h(n)$ of members of $\{-1, 1\}$ such that

$$\underset{\sim}{\eta}_n[G_{\mathbb{Q}_{I,h}}](i) = 1 \quad \Leftrightarrow \quad (\exists p \in G_{\mathbb{Q}_{I,h}})(H^p(x^n_i) = 1).$$

[Note that in both cases $\ell = 1$ and $\ell = 2$, if $H^p(x^n_i) = 1$, $x^n_i \in \mathrm{dom}(H^p)$ and $q \geq p$ then $H^q(x^n_i) = 1$; remember 1.2(2).]

Proposition 1.7

1. *If $n < \omega$, $A^p \cap (n+1) = \emptyset$* then *$p \Vdash$ "$\underset{\sim}{\eta}_n = \langle H^p(x^n_i) : i < h(n) \rangle$".*
2. *For each $n < \omega$ the set $\{p \in \mathbb{Q}_{I,h} : A^p \cap (n+1) = \emptyset\}$ is dense in $\mathbb{Q}_{I,h}$.*
3. *If $p \in \mathbb{Q}_{I,h}$ and $a \subseteq A^p$ is finite or at least $\bigcup\limits_{n \in a} (n/E^p) \in I$, and*

$$f : \{x^n_i : i < h(n) \text{ and } n \in a\} \longrightarrow \{-1, 1\},$$

   then *for some unique q which we denote by $p^{[f]}$, we have:*

   (a) *$p \leq q \in \mathbb{Q}_{I,h}$,*
   (b) *$E^q = E^p \restriction \bigcup\{n/E^p : n \in A \setminus a\}$,*
   (c) *for $n \in a$, $i < h(n)$ we have $H^q(x^n_i)$ is $f(x^n_i)$.*

Proof   Straight. ■

Definition 1.8

1. *$p \leq_n q$ (in $\mathbb{Q}_{I,h}$)* if *$p \leq q$ and:*

$$k \in A^p \ \&\ |A^p \cap k| < n \quad \Rightarrow \quad k \in A^q.$$

2. *$p \leq^*_n q$* if *$p \leq q$ and:*

$$k \in A^p \ \&\ |A^p \cap k| < n \quad \Rightarrow \quad k \in A^q \ \&\ k/E^p = k/E^q.$$

3. *$p \leq^\otimes_n q$* if *$p \leq_{n+1} q$ and:*

$$n > 0 \quad \Rightarrow \quad p \leq^*_n q \qquad \textit{and} \quad \mathrm{dom}(E^q) = \mathrm{dom}(E^p).$$

4. *For a finite set $\mathbf{u} \subseteq \omega$ we let $\mathrm{var}(\mathbf{u}) \overset{\mathrm{def}}{=} \{x^n_i : i < h(n),\ n \in \mathbf{u}\}$.*

Proposition 1.9 *1. If $p \le q$, $\mathbf{u}$ is a finite initial segment of A^p and $A^q \cap \mathbf{u} = \emptyset$,* then *for some unique $f : \{x^n_i : i < h(n)$ and $n \in \mathbf{u}\} \longrightarrow \{-1,1\}$ we have $p \le p^{[f]} \le q$ (where $p^{[f]}$ is from 1.7(3)).*

2. If $p \in \mathbb{Q}^\ell_{I,h}$ and $\mathbf{u}$ is a finite initial segment of A^p then

$(*)_1$ *$f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\}$ implies $p \le p^{[f]}$ and*

$$p^{[f]} \Vdash \text{“} (\forall n \in \mathbf{u})(\forall i < h(n))(\underset{\sim}{\eta}_n(i) = f(x^n_i))\text{”},$$

$(*)_2$ *the set $\{p^{[f]} : f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\}\}$ is predense above p (in $\mathbb{Q}^\ell_{I,h}$).*

3. $\le_n$ is a partial order on $\mathbb{Q}^\ell_{I,h}$, and $p \le_{n+1} q \;\Rightarrow\; p \le_n q$. Similarly for $<^*_n$ and $<^\otimes_n$. Also

$$p \le^\otimes_n q \quad\Rightarrow\quad p \le^*_n q \quad\Rightarrow\quad p \le_n q \quad\Rightarrow\quad p \le q.$$

4. If $p \in \mathbb{Q}^\ell_{I,h}$, $\mathbf{u}$ is a finite initial segment of A^p, $|\mathbf{u}| = n$ and

$$f : \{x^n_i : i < h(n) \text{ and } n \in \mathbf{u}\} \longrightarrow \{-1,1\} \quad \text{and} \quad p^{[f]} \le q \in \mathbb{Q}^\ell_{I,h},$$

then *for some $r \in \mathbb{Q}^\ell_{I,h}$ we have $p \le^*_n r \le q$, $r^{[f]} = q$.*

5. If $p \in \mathbb{Q}^2_{I,h}$, $\mathbf{u}$ is a finite initial segment of A^p, $|\mathbf{u}| = n+1$ and

$$f : \{x^n_i : i < h(n) \text{ and } n \in \mathbf{u}\} \longrightarrow \{-1,1\} \quad \text{and} \quad p^{[f]} \le q,$$

then *for some $r \in \mathbb{Q}^2_{I,h}$ we have $p <^\otimes_n r \le q$ and $r^{[f]} = q$.*

PROOF 1) Define $f : \{x^n_i : i < h(n) \text{ and } n \in \mathbf{u}\} \longrightarrow \{-1,1\}$ by:

$$f(x^n_i) \text{ is the constant value of } H^q(x^n_i)$$

(it is a constant function by 1.2(1)(e), 1.2(1)(f(γ))).
2) By 1.7 and 1.9(1).
3) Check.
4) First let us define the required condition r in the case $\ell = 1$. So we let

$$\begin{aligned}
&\mathrm{dom}(E^r) = \bigcup_{n \in \mathbf{u}} (n/E^p) \cup \mathrm{dom}(E^q),\\
&E^r = \big\{(n_1,n_2) : n_1 \, E^q \, n_2 \text{ or for some } n \in \mathbf{u} \text{ we have: } \{n_1,n_2\} \subseteq (n/E^p)\big\},\\
&A^r = \mathbf{u} \cup A^q
\end{aligned}$$

(note that if $n_1 \, E^q \, n_2$ then $n_1 \notin \mathbf{u}$). Next, for $x^n_i \in B^r_1$ (where B^r_1 is given by 1.2(1)(e)) we define

$$H^r(x^n_i) = \begin{cases} H^q(x^n_i) & \text{if } n \notin \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^q),\\ H^p(x^n_i) & \text{if } n \in \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^p). \end{cases}$$

It should be clear that $r = (H^r, E^r, A^r) \in \mathbb{Q}^1_{I,h}$ is as required.

If $\ell = 2$ then we define r in a similar manner, but we have to be more careful defining the function H^r. Thus E^r and A^r are defined as above, B^r_2, B^r_3 and $w_r(m,i)$ for $x^m_i \in B^r_3$ are given by 1.2(1)(f). Note that $B^r_2 = B^p_2$ and $B^r_3 \subseteq B^p_3$. Next we define:

if $x^m_i \in B^r_2$ then $H^r(x^m_i) = H^p(x^m_i)$,
if $x^m_i \in B^r_3$, $m \cap A^r \subseteq \mathbf{u}$ then $H^r(x^m_i) = H^p(x^m_i)$,
if $x^m_i \in B^r_3$ and $\min(\mathrm{dom}(E^q)) < m$ then

$$\begin{array}{l} H^r(x^m_i)(\ldots, x^k_j, \ldots)_{(k,j) \in w_r(m,i)} = \\ H^p(x^m_i)(x^k_j, H^q(x^{k'}_{j'})(\ldots, x^{k''}_{j''}, \ldots)_{(k'',j'') \in w_q(k',j')}))_{\substack{(k,j) \in w_r(m,i) \\ (k',j') \in w_p(m,i) \setminus w_r(m,i)}} . \end{array}$$

Note that if $(k', j') \in w_p(m,i) \setminus w_r(m,i)$, $x^m_i \in B^r_3$ then $k' \in A^p \setminus (\mathbf{u} \cup A^q)$ and $w_q(k', j') \subseteq w_r(m,i)$.

5) Like the proof of (4). Let $n^* = \max(\mathbf{u})$. Put $\mathrm{dom}(E^r) = \mathrm{dom}(E^p)$ and declare that $n_1 \, E^r \, n_2$ if one of the following occurs:

(a) for some $n \in \mathbf{u} \setminus \{n^*\}$ we have $\{n_1, n_2\} \subseteq (n/E^p)$, *or*

(b) $n_1 \, E^q \, n_2$ (so $n \in \mathbf{u} \Rightarrow \neg n \, E^p \, n_1$), *or*

(c) $\{n_1, n_2\} \subseteq B$, where

$$B \stackrel{\mathrm{def}}{=} n^*/E^p \cup \bigcup \{m/E^p : m \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q),\ \min(m/E^p) > n^*\}.$$

We let $A^r = \mathbf{u} \cup A^q$ (in fact A^r is defined from E^r). Finally the function H^r is defined exactly in the same manner as in (4) above (for $\ell = 2$). ■

Corollary 1.10 *If $p \in \mathbb{Q}^\ell_{I,h}$, $n < \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^\ell_{I,h}$–name of an ordinal,* then *there are $\mathbf{u}, q$ and $\bar{\alpha} = \langle \alpha_f : f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\} \rangle$ such that:*

(a) $p \le^*_n q \in \mathbb{Q}^\ell_{I,h}$,

(b) $\mathbf{u} = \{\ell \in A^p : |\ell \cap A^p| < n\}$,

(c) *for $f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\}$ we have $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$",*

(d) $q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\}\}$" *(which is a finite set).*

Proof Let $k = \prod\limits_{\ell \in \mathbf{u}} 2^{h(\ell)}$. Let $\{f_\ell : \ell < k\}$ enumerate ${}^{\mathrm{var}(\mathbf{u})}\{-1,1\}$. By induction on $\ell \le k$ define r_ℓ, α_{f_ℓ} such that:

$$r_0 = p, \quad r_\ell \le^*_n r_{\ell+1} \in \mathbb{Q}^\ell_{I,h}, \quad r^{[f_\ell]}_{\ell+1} \Vdash_{\mathbb{Q}^\ell_{I,h}} \text{"}\underset{\sim}{\tau} = \alpha_{f_\ell}\text{"}.$$

The induction step is by 1.9(4). Now $q = r_k$ and $\langle \alpha_f : f \in {}^{\mathrm{var}(\mathbf{u})}\{-1,1\} \rangle$ are as required. ■

Corollary 1.11 *If $\ell = 2$ then in 1.10(a) we may require $p \leq_n^{\otimes} q \in \mathbb{Q}^{\ell}_{I,h}$.*

PROOF Similar: just use 1.9(5) instead of 1.9(4). ■

Definition 1.12 *Let I be an ideal on ω containing $[\omega]^{<\omega}$ and let E be an I–equivalence relation.*

1. *We define a game $GM_I(E)$ between two players. The game lasts ω moves. In the $n^{\rm th}$ move the first player chooses an I-equivalence relation E^1_n such that*
$$E^1_0 = E, \qquad [n > 0 \quad \Rightarrow \quad E^2_{n-1} \leq E^1_n],$$
and the second player chooses an I-equivalence relation E^2_n such that $E^1_n \leq E^2_n$. In the end, the second player wins if
$$\bigcup \{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I$$
(otherwise the first player wins).
2. *For a countable elementary submodel N of $(\mathcal{H}(\chi), \in, <^*)$ such that $I, E \in N$ we define a game $GM^N_I(E)$ in a similar manner as $GM_I(E)$, but we demand additionally that the relations played by both players are from N (i.e. $E^1_n, E^2_n \in N$ for $n \in \omega$).*

Proposition 1.13 1. *Assume that I is a maximal (non-principal) ideal on ω and E is an I–equivalence relation. Then the game $GM_I(E)$ is not determined. Moreover, for each countable $N \prec (\mathcal{H}(\chi), \in, <^*)$ such that $I, E \in N$ the game $GM^N_I(E)$ is not determined.*

2. *For the conclusion of (1) it is enough to assume that $\mathcal{P}(\omega)/I \models$ ccc.*

PROOF 1) As each player can imitate the other's strategy.
2) Easy, too, and will not be used in this paper. ■

Proposition 1.14 1. *Let $p \in \mathbb{Q}^{\ell}_{I,h}$. Suppose that the first player has no winning strategy in $GM_I(E^p)$.* Then *in the following game Player I has no winning strategy:*

in the $n^{\rm th}$ move,

Player I chooses a $\mathbb{Q}^{\ell}_{I,h}$-name $\underset{\sim}{\tau}_n$ of an ordinal and

Player II chooses $p_n, \mathbf{u}_n, w_n$ such that: w_n is a set of $\leq \prod\limits_{\ell \in \mathbf{u}_n} 2^{h(\ell)}$ ordinals, $p \leq p_n \leq^*_n p_{n+1}$, $p_n \leq_{n+1} p_{n+1}$, $\mathbf{u}_n$ a finite initial segment of A^{p_n} with n elements and $p_n \Vdash$ "$\underset{\sim}{\tau}_n \in w_n$", moreover
$$f \in {}^{\mathrm{var}(\mathbf{u}_n)}\{-1, 1\} \quad \Rightarrow \quad p_n^{[f]} \text{ forces a value to } \underset{\sim}{\tau}_n.$$

In the end, the second player wins if for some $q \geq p$ we have

$$q \Vdash \text{“}(\forall n \in \omega)(\underset{\sim}{\tau}_n \in w_n) \text{ ”}.$$

We can let Player II choose $k_n < \omega$ and demand $|\mathbf{u}_n| \leq k_n$, and in the end Player II wins if $\liminf\langle k_n : n < \omega\rangle < \omega$ *or there is q as above.*

2. *Let $p \in \mathbb{Q}^\ell_{I,h}$ and let N be a countable elementary submodel of $(\mathcal{H}(\chi), \in, <^*)$ such that $p, I, h \in N$. If the first player has no winning strategy in $GM^N_I(E^p)$ then Player I has no winning strategy in the game like above but with restriction that $\underset{\sim}{\tau}_n, p_n \in N$.*

PROOF 1) As in [Sh 407, 1.11, p.436].
Let $\mathbf{St}_p$ be a strategy for Player I in the game from 1.14. We shall define a strategy $\mathbf{St}$ for the first player in $GM_I(E^p)$ during which the first player, on a side, plays a play of the game from 1.14, using $\mathbf{St}_p$, with $\langle p_\ell : \ell < \omega\rangle$ and he also chooses $\langle q_\ell : \ell < \omega\rangle$.

Then, as $\mathbf{St}$ cannot be a winning strategy in $GM_I(E)$, in some play in which the first player uses his strategy $\mathbf{St}$ he loses, and then $\langle p_\ell : \ell < \omega\rangle$ will have an upper bound as required.

In the n^{th} move (so $E^1_\ell, E^2_\ell, q_\ell, p_\ell, \mathbf{u}_\ell, w_\ell$ for $\ell < n$ are defined), the first player in addition to choosing E^1_n chooses $q_n, p_n, \mathbf{u}_n$, such that:

(a) $p = p_{-1} \leq q_0 = p_0$, $p_n \in \mathbb{Q}^\ell_{I,h}$, $q_n \in \mathbb{Q}^\ell_{I,h}$,

(b) $p_n \leq^*_n p_{n+1} \in \mathbb{Q}^\ell_{I,h}$,

(c) $\mathbf{u}_0$ is $\emptyset$,

(d) $\mathbf{u}_{n+1} = \mathbf{u}_n \cup \{\min(A^{q_{n+1}} \setminus \mathbf{u}_n)\}$, so $|\mathbf{u}_{n+1}| = n + 1$,

(e) $E^1_0 = E^p$, $E^1_{n+1} = E^{p_n} \restriction \big(\mathrm{dom}(E^{p_n}) \setminus \bigcup\limits_{i\in\mathbf{u}_n} i/E^{p_n}\big)$,

(f) q_n is defined as follows:

- (f_0) if $n = 0$ then $E^{q_n} = E^2_0$,
- (f_1) if $n > 0$ then $\mathrm{dom}(E^{q_n}) = \mathrm{dom}(E^{p_{n-1}})$ and $x\, E^{q_n}\, y$ if and only if
  either $x\, E^2_n\, y$,
  or for some $k \in \mathbf{u}_{n-1}$ we have $x, y \in k/E^{p_{n-1}}$'
  or $x, y \in \big(\mathrm{dom}(E^1_n) \setminus \mathrm{dom}(E^2_n)\big) \cup \min(\mathrm{dom}(E^2_n))/E^2_n$,
- (f_2) H^{q_n} is such that $p_{n-1} \leq q_n$,

(g) $p_n \leq^*_n q_{n+1} \leq^*_{n+1} p_{n+1}$, $p_n \leq_{n+1} q_{n+1}$ (so $p_n \leq_{n+1} p_{n+1}$),

(h) if $f \in {}^{\mathrm{var}(\mathbf{u}_n)}\{-1, 1\}$ then $p^{[f]}_n$ forces a value to $\underset{\sim}{\tau}_n$.

In the first move, when $n = 0$, the first player plays $E^1_0 = E^p$ (as the rules of the game require, according to (e)). The second player answers choosing an I–equivalence relation $E^2_0 \geq E^1_0$. Now, on a side, Player I starts to play the game of 1.14 using his strategy $\mathbf{St}_p$. The strategy says him to play a name $\underset{\sim}{\tau}_0$ of an ordinal. He defines q_0 by (f) (so $q \in \mathbb{Q}^\ell_{I,h}$ is a condition stronger than p and such that $E^{q_0} = E^2_0$) and chooses a condition $p_0 \geq q_0$ deciding the value of the name $\underset{\sim}{\tau}_0$, say $p_0 \Vdash \underset{\sim}{\tau}_0 = \alpha$. He pretends that the second player answered (in the game of 1.14) by: p_0, $\mathbf{u}_0 = \emptyset$, $w_0 = \{\alpha\}$. Next, in the play of $GM_I(E^p)$, he plays $E^1_1 = E^{p_0}$ as declared in (e).
Now suppose that we are at the $(n+1)^{\rm th}$ stage of the play of $GM_I(E^p)$, the first player has played E^1_{n+1} already and on a side he has played the play of the game 1.14 as defined by (a)–(h) and $\mathbf{St}_p$ (so in particular he has defined a condition p_n and $E^1_{n+1} = E^{p_n}\restriction\big(\mathrm{dom}(E^{p_n}) \setminus \bigcup\limits_{i\in\mathbf{u}_n} i/E^{p_n}\big)$ and $\mathbf{u}_n$ is the set of the first n elements of A^{p_n}). The second player plays an I–equivalence relation $E^2_{n+1} \geq E^1_{n+1}$. Now the first player chooses (on a side, pretending to play in the game of 1.14): a name $\underset{\sim}{\tau}_{n+1}$ given by the strategy $\mathbf{St}_p$, a condition $q_{n+1} \in \mathbb{Q}^\ell_{I,h}$ determined by (f) (check that (g) is satisfied), $\mathbf{u}_{n+1}$ as in (d) and a condition $p_{n+1} \in \mathbb{Q}^\ell_{I,h}$ satisfying (g), (h) (the last exists by 1.10). Note that, by (g) and 1.9, the condition p_{n+1} determines a suitable set w_{n+1}. Thus, Player I pretends that his opponent in the game of 1.14 played $p_{n+1}, \mathbf{u}_{n+1}, w_{n+1}$ and he passes to the actual game $GM_I(E^p)$. Here he plays E^1_{n+2} defined by (e).

The strategy $\mathbf{St}$ described above cannot be the winning one. Consequently, there is a play in $GM_I(E^p)$ in which Player I uses $\mathbf{St}$, but he looses. During the play he constructed a sequence $\langle (p_n, \mathbf{u}_n, w_n) : n \in \omega\rangle$ of legal moves of Player II in the game of 1.14 against the strategy $\mathbf{St}_p$. Let $E^q = \lim\limits_{n<\omega} E^{p_n}$ (i.e. $\mathrm{dom}(E^q) = \bigcap\limits_{n<\omega} \mathrm{dom}(E^{p_n})$, $x\, E^q\, y$ if and only if for every large enough n, $x\, E^{p_n}\, y$) and let $H^q(x^m_i)$ will be $H^{p_n}(x^m_i)$ for any large enough n (it is eventually constant). It follows from the demand (g) that E^q-equivalence classes are in I. Moreover, $\mathrm{dom}(E^1_{n+1}) \setminus \mathrm{dom}(E^2_{n+1}) \subseteq k/E^q$, where k is the $(n+1)^{\rm th}$ member of A^q. Therefore

$$\begin{aligned}&\omega \setminus \mathrm{dom}(E^q) = \omega \setminus \bigcap_{n\in\omega} \mathrm{dom}(E^{p_n}) \subseteq \\ &\omega \setminus \mathrm{dom}(E^{p_0}) \cup \bigcup\{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I\end{aligned}$$

(remember, Player I lost in $GM_I(E^p)$). Now it should be clear that $q \in \mathbb{Q}^\ell_{I,h}$ and it is stronger than every p_n (even $p_n \leq^*_n q$). Hence Player II wins the corresponding play of 1.14, showing that $\mathbf{St}_p$ is not a winning strategy.

2) The same proof. ■

Proposition 1.15 *If in 1.14 we assume $\ell = 2$ and demand $p_n \leq^\otimes_n p_{n+1}$ instead $p_n \leq^*_n p_{n+1}$ then Player II has a winning strategy.*

Proof Using 1.11, the second player can find suitable conditions p_n (in the game of 1.14) such that $p_n \leq^\otimes_{n+1} p_{n+1}$. But note that the partial orders $\leq^\otimes_n$

have the fusion property, so the sequence $\langle p_n : n < \omega\rangle$ will have an upper bound in $\mathbb{Q}^2_{I,h}$. ■

Remark 1.16 *We could have used $<^{\otimes}_n$ also in [Sh 407].*

Definition 1.17 (see [Sh:f, VI, 2.12, A-F]) *1. A forcing notion $\mathbb{P}$ has the PP-property if:*

($\otimes^{PP}$) for every $\eta \in {}^\omega\omega$ *from* $\mathbf{V}^{\mathbb{P}}$ *and a strictly increasing* $x \in {}^\omega\omega \cap \mathbf{V}$ there is *a closed subtree* $T \subseteq {}^{<\omega}\omega$ *such that:*

(α) $\eta \in \lim(T)$*, i.e.* $(\forall n < \omega)(\eta \restriction n \in T)$,

(β) $T \cap {}^n\omega$ *is finite for each* $n < \omega$,

(γ) *for arbitrarily large* n *there are* k*, and* $n < i(0) < j(0) < i(1) < j(1) < \ldots < i(k) < j(k) < \omega$ *and for each* $\ell \leq k$*, there are* $m(\ell) < \omega$ *and* $\eta^{\ell,0}, \ldots, \eta^{\ell,m(\ell)} \in T \cap {}^{j(\ell)}\omega$ *such that* $j(\ell) > x(i(\ell) + m(\ell))$ *and*

$$(\forall \nu \in T \cap {}^{j(k)}\omega)(\exists \ell \leq k)(\exists m \leq m(\ell))(\eta^{\ell,m} \trianglelefteq \nu).$$

2. We say that a forcing notion $\mathbb{P}$ has the strong PP–property if

($\oplus^{sPP}$) *for every function* $g : \omega \longrightarrow \mathbf{V}$ *from* $\mathbf{V}^{\mathbb{P}}$ *there exist a set* $B \in [\omega]^{\aleph_0} \cap \mathbf{V}$ *and a sequence* $\langle w_n : n \in B\rangle \in \mathbf{V}$ *such that for each* $n \in B$

$$|w_n| \leq n \qquad \text{and} \qquad g(n) \in w_n.$$

Remark 1.18 Of course, if a proper forcing notion has the strong PP–property then it has the PP–property.

Conclusion 1.19 *Assume that for each $p \in \mathbb{Q}^\ell_{I,h}$ and for each countable $N \prec (\mathcal{H}(\chi), \in, <^*)$ such that $p, I, h \in N$, the first player has no winning strategy in $GM^N_I(E^p)$ (e.g. if I is a maximal ideal). Then*

(*) $\mathbb{Q}^\ell_{I,h}$ *is proper, α-proper, strongly α-proper for every $\alpha < \omega_1$, is ${}^\omega\omega$-bounding and it has the PP-property, even the strong PP–property.* ■

By [Sh:f, VI, 2.12] we know

Theorem 1.20 *Suppose that $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : j < \alpha, i \leq \alpha\rangle$ is a countable support iteration such that*

$$\Vdash_{\mathbb{P}_j} \text{“} \underset{\sim}{\mathbb{Q}}_j \textit{ is proper and has the PP-property”}.$$

Then $\mathbb{P}_\alpha$ has the PP-property. ■

2 NWD ultrafilters

A subset A of the set $\mathcal{Q}$ of rationals is *nowhere dense* (NWD) if its closure (in $\mathcal{Q}$) has empty interior. Remember that the rationals are equipped with the order topology and both "closure" and "interior" refer to this topology. Of course, as $\mathcal{Q}$ is dense in the real line, we may consider these operations on the real line and get the same notion of nowhere dense sets. For technical reasons, in forcing considerations we prefer to work with ${}^{\omega}2$ instead of the real line. So naturally we want to replace rationals by ${}^{<\omega}2$. But what are nowhere dense subsets of ${}^{<\omega}2$ then? (One may worry about the way we "embed" ${}^{<\omega}2$ into ${}^{\omega}2$.) Note that we have a natural lexicographical ordering $<_{\ell x}$ of ${}^{<\omega}2$:

$\eta <_{\ell x} \nu$ if and only if
either there is $\ell < \omega$ such that $\eta\restriction\ell = \nu\restriction\ell$ and $\eta(\ell) < \nu(\ell)$
or $\eta^\frown\langle 1\rangle \trianglelefteq \nu$
or $\nu^\frown\langle 0\rangle \trianglelefteq \eta$.

Clearly $({}^{<\omega}2, <_{\ell x})$ is a linear dense order without end-points (and consequently it is order–isomorphic to the rationals). Now, we may talk about nowhere dense subsets of ${}^{<\omega}2$ looking at this ordering only, but we may relate this notion to the topology of ${}^{\omega}2$ as well.

Proposition 2.1 *For a set $A \subseteq {}^{<\omega}2$ the following conditions are equivalent:*

1. *A is nowhere dense,*
2. $(\forall\eta \in {}^{<\omega}2)(\exists\nu \in {}^{<\omega}2)[\eta \trianglelefteq \nu\ \&\ (\forall\rho \in {}^{<\omega}2)(\nu \trianglelefteq \rho\ \Rightarrow\ \rho \notin A)]$,
3. *the set*
$$A^* \stackrel{\rm def}{=} \{\eta \in {}^{\omega}2 : (\forall n \in \omega)(\exists\nu \in A)(\eta\restriction n \trianglelefteq \nu)\}$$
is nowhere dense (in the product topology of ${}^{\omega}2$),
4. *there is a sequence $\langle\eta_n : n < \omega\rangle$ such that for each $n < \omega$*

   (i)$_n$ $\eta_n : [n, \ell_n) \longrightarrow 2$ *for some $\ell_n > n$ and*

   (ii)$_n$ $(\forall\rho \in A)(\eta_n \not\subseteq \rho)$,
5. *there is a sequence $\langle\eta_n : n < \omega\rangle$ such that for each $n < \omega$ condition* **(i)**$_n$ *(see above) holds and*

   (ii)$^*_n$ $(\forall\nu \in {}^{n}2)(\{\rho \in {}^{<\omega}2 : \nu \cup \eta_n \trianglelefteq \rho\} \cap A = \emptyset)$,
6. *there are $B \in [\omega]^{\aleph_0}$ and $\langle\eta_n : n \in B\rangle$ such that for each $n \in B$ the conditions* **(i)**$_n$, **(ii)**$_n$ *above are satisfied.*

Proof 1. $\Rightarrow$ 2. Suppose $A \subseteq {}^{<\omega}2$ is nowhere dense but for some sequence $\eta \in {}^{<\omega}2$, for every $\nu \in {}^{<\omega}2$ extending η there is $\rho \in A$ such that $\nu \trianglelefteq \rho$. Look at the interval $(\eta^\frown\langle 0\rangle, \eta^\frown\langle 1\rangle)_{<_{\ell x}}$ (of $({}^{<\omega}2, <_{\ell x})$). We claim that A is dense in this interval. Why? Suppose

$$\eta^\frown\langle 0\rangle \leq_{\ell x} \eta^*_0 <_{\ell x} \eta^*_1 \leq_{\ell x} \eta^\frown\langle 1\rangle.$$

Assume $\ell g(\eta_0^*) \leq \ell g(\eta_1^*)$. Take $\nu \stackrel{\rm def}{=} \eta_1^* {}^\frown\langle 0\rangle$. By the definition of the order $<_{\ell x}$ we have then

$$\eta_0^* <_{\ell x} \nu^\frown\langle 0\rangle <_{\ell x} \nu^\frown\langle 1\rangle <_{\ell x} \eta_1^* \qquad \text{and} \qquad \eta \vartriangleleft \nu.$$

By our assumption we find $\rho \in A$ such that $\nu^\frown\langle 0,1\rangle \trianglelefteq \rho$. Then

$$\nu^\frown\langle 0\rangle <_{\ell x} \rho <_{\ell x} \nu^\frown\langle 1\rangle \qquad \text{and hence} \qquad \rho \in (\eta_0^*, \eta_1^*)_{<_{\ell x}}.$$

Similarly if $\ell g(\eta_1^*) \leq \ell g(\eta_0^*)$.

2. $\Rightarrow$ 3. Should be clear if you remember that sets

$$[\nu] \stackrel{\rm def}{=} \{\eta \in {}^\omega 2 : \nu \vartriangleleft \eta\} \qquad\qquad (\text{for } \nu \in {}^{<\omega}2)$$

constitute the basis of the topology of ${}^\omega 2$.

3. $\Rightarrow$ 4. Suppose A^* is nowhere dense in ${}^\omega 2$. Let $n < \omega$. Considering all elements of 2^n build (e.g. inductively) a function $\eta_n^* : [n, \ell_n^*) \longrightarrow 2$ such that $n < \ell_n^*$ and

$$(\forall \nu \in 2^n)([\nu^\frown\eta_n^*] \cap A^* = \emptyset).$$

This means that for each $\nu \in 2^n$ the set $\{\rho \in A : \nu^\frown\eta_n^* \trianglelefteq \rho\}$ is finite (otherwise use König lemma to construct an element of A^* in $[\nu^\frown\eta_n^*]$). Taking sufficiently large $\ell_n > \ell_n^*$ and extending η_n^* to η_n with domain $[n, \ell_n)$ we get that $(\forall \rho \in A)(\eta_n \not\subseteq \rho)$ (as required).

4. $\Rightarrow$ 5. $\Rightarrow$ 6. Read the conditions.

6. $\Rightarrow$ 1. Let B, $\langle \eta_n : n \in B\rangle$ be as in 6. Suppose $\nu_0, \nu_1 \in {}^{<\omega}2$, $\nu_0 <_{\ell x} \nu_1$. Assume $\ell g(\nu_0) \leq \ell g(\nu_1) = m$. Take any $n \in B \setminus (m+1)$ and let $\nu = \nu_1{}^\frown\langle\underbrace{0,\ldots,0}_{n-m}\rangle^\frown\eta_n$. We know that no element of A extends ν. But this implies that the interval $(\nu^\frown\langle 0\rangle, \nu^\frown\langle 1\rangle)_{<_{\ell x}}$ is disjoint from A (and is contained in the interval $(\nu_0, \nu_1)_{<_{\ell x}}$). Similarly if $\ell g(\nu_1) \leq \ell g(\nu_0)$. ■

Lemma 2.2 *Let $n, k^* < \omega$. Assume that $\bar\nu^k = \langle \nu_i^k : n \leq i < i_k\rangle$ for $k < k^* < \omega$, $n \leq i_k < \omega$, $\nu_i^k \in \bigcup\limits_{j \geq i} {}^{[i,j)}2$ and $w_k \subseteq [n, i_k)$, $|w_k| \geq k^*$ and:*

$$\textit{if } k < k^*,\ m_1 < m_2 \textit{ are in } w_k \textit{ then } \max\mathrm{dom}(\nu_{m_1}^k) < m_2.$$

Lastly let

$$i(*) = \max\{\sup\mathrm{dom}(\nu_i^k) + 1 : k < k^* \textit{ and } i \in (n, i_k)\}.$$

Then *we can find $\rho \in {}^{[n,i(*))}2$ such that:*

$$(\forall k < k^*)(\exists i \in w_k)(\nu_i^k \subseteq \rho).$$

Proof By induction on k^* (for all possible other parameters). For $k^* = 0, 1$ it is trivial.
Let $n^0_k = \min(w_k)$ and $n^1_k = \min(w_k \setminus (n^0_k + 1))$. Let $\ell < k^*$ be with minimal n^1_ℓ. Apply the induction hypothesis with n^1_ℓ, $\bar{\nu}^k = \langle \bar{\nu}^k_i : n^1_\ell \leq i < i_k \rangle$ for $k < k^*$, $k \neq \ell$ and $\langle w_k \setminus n^1_\ell : k < k^*, k \neq \ell \rangle$ here standing for n, $\bar{\nu}^k$ for $k < k^*$, $\langle w_k : k < k^* \rangle$ there and get $\rho_1 \in {}^{[n^1_\ell, i(*))}2$. Note that $w_k \setminus n^1_\ell \supseteq w_k \setminus n^1_k$ has at least $|w_k| - 1$ elements. Let $\rho \in {}^{[n,i(*))}2$ be such that $\rho_1 \subseteq \rho$ and $\bar{\nu}^\ell_{n^0_\ell} \subseteq \rho$. ■

Proposition 2.3 *Assume that $\mathbb{R}$ is a proper forcing notion with the PP-property. Then*

($\oplus^{\rm nwd}$) *for every nowhere dense set $A \subseteq {}^{<\omega}2$ in $\mathbf{V}^{\mathbb{R}}$ there is a nowhere dense set $A^* \subseteq {}^{<\omega}2$ in $\mathbf{V}$ such that $A \subseteq A^*$.*

Proof Let $A \in \mathbf{V}^{\mathbb{R}}$ be a nowhere dense subset of ${}^{<\omega}2$. Thus, in $\mathbf{V}^{\mathbb{R}}$, we can, for each $n < \omega$, choose $\nu_n \in \bigcup\limits_{\ell \geq n} {}^{[n,\ell)}2$ such that:

$$(\forall \nu \in {}^n2)(\forall \rho \in {}^{<\omega}2)(\nu^\frown \nu_n \trianglelefteq \rho \ \Rightarrow\ \rho \notin A).$$

So $\langle \nu_n : n < \omega \rangle \in \mathbf{V}^{\mathbb{R}}$ is well defined. Next for each n we choose an integer $\ell_n \in (n, \omega)$, a sequence $\eta_n \in {}^{[n,\ell_n)}2$ and a set $w_n \subseteq [n, \ell_n)$ such that:

- $|w_n| > n$,
- $(\forall m \in w_n)(\nu_m \subseteq \eta_n)$, so in particular $(\forall m \in w_n)(\max \mathrm{dom}(\nu_m) < \ell_n)$, and
- for any $m_1 < m_2$ from w_n we have $\max \mathrm{dom}(\nu_{m_1}) < m_2$.

So $\bar{w} = \langle w_n : n < \omega \rangle$, $\bar{\eta} = \langle \eta_n : n < \omega \rangle \in \mathbf{V}^{\mathbb{R}}$ are well defined.

Since $\mathbb{R}$ has the PP-property it is ${}^\omega\omega$-bounding, and hence there is a strictly increasing $x \in {}^\omega\omega \cap \mathbf{V}$ such that $(\forall n \in \omega)(\ell_n < x(n))$. Applying the PP-property of $\mathbb{R}$ to x and the function $n \mapsto (\eta_n, w_n)$ we can find $\langle \langle V^n_\ell : \ell \leq k_n \rangle : n < \omega \rangle$ in $\mathbf{V}$ and $\langle \langle (i_\ell(n), j_\ell(n)) : \ell \leq k_n \rangle : n < \omega \rangle$ in $\mathbf{V}$ such that:

(a) $i_0(n) < j_0(n) < i_1(n) < j_1(n) < \ldots < i_{k_n}(n) < j_{k_n}(n)$,

(b) $j_{k_n}(n) < i_0(n+1)$ for $n < \omega$,

(c) $x(i_\ell(n)) < j_\ell(n)$,

(d) $V^n_\ell \subseteq \{(\eta, w) : \eta \in {}^{[i_\ell(n), j_\ell(n))}2$ and $w \subseteq [i_\ell(n), j_\ell(n)),\ |w| > i_\ell(n)\}$ for $\ell \leq k_n$, $n < \omega$,

(e) $|V^n_\ell| \leq i_\ell(n)$,

(f) for every $n < \omega$, for some $\ell \leq k_n$ and $(\eta, w) \in V^n_\ell$ we have $w = w_{i_\ell(n)}$, $\eta_{i_\ell(n)} \subseteq \eta$.

[Note that $i_\ell(n)$ corresponds to $i(\ell)+m(\ell)$ in definition 1.17(1), so we do not have $m_\ell(n)$ here.] Working in $\mathbf{V}$, by 2.2, for each $n<\omega$, $\ell\leq k_n$ there is $\rho^n_\ell\in {}^{[i_\ell(n),j_\ell(n))}2$ such that:

$$(\forall(\eta,w)\in V^n_\ell)(\exists m_1,m_2\in w)(m_2=\min(w\setminus(m_1+1))\ \&\ \eta\restriction[m_1,m_2)\subseteq\rho^n_\ell).$$

Let $\rho_n\in {}^{[i_0(n),i_0(n+1))}2$ be such that $\ell\leq k_n \quad\Rightarrow\quad \rho^n_\ell\subseteq\rho_n$. As we have worked in $\mathbf{V}$, $\langle\rho_n:n<\omega\rangle\in\mathbf{V}$. Let

$$A^*=\{\rho\in {}^{<\omega}2:\neg(\exists n\in\omega)(\rho_n\subseteq\rho)\}.$$

Clearly $A^*\in\mathbf{V}$ is as required. ■

Let us recall definition 0.1 reformulating it slightly for technical purposes. (Of course, the two definitions are equivalent; see the discussion at the beginning of this section.)

Definition 2.4 *We say that a non-principal ultrafilter $\mathcal{D}$ on ω is an NWD-ultrafilter* if *for any sequence $\langle\eta_n:n<\omega\rangle\subseteq {}^{<\omega}2$ for some $A\in\mathcal{D}$ the set $\{\eta_n:n\in A\}$ is nowhere dense in ${}^{<\omega}2$.*

Lemma 2.5 *Let $\mathcal{D}$ be a non-principal ultrafilter on ω and I be the dual ideal (and $h:\omega\longrightarrow\omega$ non-decreasing $\lim\limits_{n\to\infty}h(n)=\infty$). Then:*

1. *in $\mathbf{V}^{\mathbb{Q}^1_{I,h}}$ we cannot extend $\mathcal{D}$ to an NWD-ultrafilter.*
2. *If $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{Q}^1_{I,h}$-name of a proper forcing notion with the PP–property,* then *also in $\mathbf{V}^{\mathbb{Q}^1_{I,h}*\underset{\sim}{\mathbb{Q}}}$ we cannot extend $\mathcal{D}$ to an NWD-ultrafilter.*

Proof 1) Let $\bar{\underset{\sim}{\eta}}=\langle\underset{\sim}{\eta}_n:n<\omega\rangle$ be the name defined in 1.6, but now we interpret the value -1 as 0. So $\Vdash$"$\underset{\sim}{\eta}_n\in {}^{h(n)}2$" (for each $n<\omega$). Clearly it is enough to show that

$$(*)\qquad \Vdash_{\mathbb{Q}^1_{I,h}} \text{" if } X\subseteq\omega \text{ and the set } \{\underset{\sim}{\eta}_n:n\in X\} \text{ is nowhere dense then there is } Y\in\mathcal{D} \text{ disjoint from } X\text{"}.$$

So suppose that $\underset{\sim}{\tau}$ is a $\mathbb{Q}^1_{I,h}$-name for a subset of ω and a condition $p^*\in\mathbb{Q}^1_{I,h}$ forces that $\{\underset{\sim}{\eta}_n:n\in\underset{\sim}{\tau}\}$ is nowhere dense. By 2.1, for some $\mathbb{Q}^1_{I,h}$-names $\bar{\underset{\sim}{\nu}}=\langle\underset{\sim}{\nu}_m:m<\omega\rangle$ we have

$$p^*\Vdash \text{"}\underset{\sim}{\nu}_m\in\bigcup_{\ell\geq m}{}^{[m,\ell)}2 \text{ and for every } m<\omega \text{ for no } n\in\underset{\sim}{\tau} \text{ we have } \underset{\sim}{\nu}_m\subseteq\underset{\sim}{\eta}_n\text{"}.$$

By 1.14 (or actually by its proof) without loss of generality:

> for every $n\in A^{p^*}$, for some $k_n\in(n,\min(A^{p^*}\setminus(n+1)))$, for every $f:\{x^m_j:m\in A^{p^*}\cap(n+1) \text{ and } j<h(m)\}\longrightarrow\{-1,1\}$, the condition $p^{*[f]}$ forces a value to $\underset{\sim}{\tau}\cap k_n$, and $\underset{\sim}{\tau}\cap k_n\setminus n\neq\emptyset$.

[Why? Give a strategy to Player I in the game there for p^* trying to force the needed information, so for some such play Player II wins and replaces p^* by q from there.]
Again by 1.14 we may assume that

> for every $f : \{x^m_j : j < h(m) \text{ and } m \in A^{p^*} \cap (n+1)\} \longrightarrow \{-1,1\}$, $n \in A^{p^*}$, for some $\bar{\nu}^f$ we have
>
> $$p^{*[f]} \Vdash \text{“}\bar{\nu}^f \text{ is an initial segment of } \underset{\sim}{\bar{\nu}} \text{ and } \ell g(\bar{\nu}^f) = n+1\text{ ”}.$$

For $n \in A^{p^*}$ and $f : \{x^m_j : j < h(m) \text{ and } m \in A^{p^*} \cap (n+1)\} \longrightarrow \{-1,1\}$ and $k \in A^{p^*} \setminus (n+1)$ let:

(a) $f^{[k,p^*]}$ be the function with domain $\{x^m_j : j < h(m) \text{ and } m \in A^{p^*} \cap (k+1)\}$ extending f that is constantly 1 on $\mathrm{dom}(f^{[k,p^*]}) \setminus \mathrm{dom}(f)$,

(b) $\bar{\rho}^f$ be an ω-sequence $\langle \rho^f_\ell : \ell < \omega\rangle$ such that for each $k \in A^{p^*} \setminus (n+1)$ we have $\bar{\rho}^f \restriction (k+1) = \bar{\nu}^{f^{[k,p^*]}} \restriction (k+1)$.

Now, for every $n \in A^{p^*}$, we can find $\rho^*_n \in {}^{<\omega}2$ such that for every function

$$f : \{x^m_j : j < h(m) \text{ and } m \in A^{p^*} \cap (n+1)\} \longrightarrow \{-1,1\}$$

for some $\ell(f) \in (h(n),\omega)$ we have $\rho^f_{\ell(f)} \subseteq \rho^*_n$ (so $\ell(f) < \ell g(\rho^*_n)$).
[Why? Let $\{f_j : j < j(*)\}$ list the possible f's, and we chose by induction on $j \leq j(*)$, $\rho^j \in {}^{<\omega}2$ such that $\rho^j \vartriangleleft \rho^{j+1}$, and ρ^{j+1} satisfies the requirement on f_j, e.g. $\rho_0 = \langle \underbrace{0,\ldots,0}_{h(n)}\rangle$, $\rho^{j+1} = \rho^j{}^\frown \rho^{f_j}_{\ell g(\rho^j)}$].

Now choose by induction on $\zeta < \omega$, $n_\zeta \in A^{p^*}$ such that $n_\zeta < n_{\zeta+1}$, and $\ell g(\rho^*_{n_\zeta}) < h(n_{\zeta+1})$. Without loss of generality $\bigcup\limits_{\zeta<\omega} (n_\zeta/E^{p^*}) \in I$. Then

> either $\bigcup\{n/E^{p^*} : n \in A^{p^*} \text{ and } (\exists \zeta < \omega)(n_{2\zeta} < n < n_{2\zeta+1})\} \in \mathcal{D}$
> or $\bigcup\{n/E^{p^*} : n \in A^{p^*} \text{ and } (\exists \zeta < \omega)(n_{2\zeta+1} < n < n_{2\zeta+2})\} \in \mathcal{D}$,

so by renaming the latter holds. (Again, it suffices that the ideal I is such that the quotient algebra $\mathcal{P}(\omega)/I$ satisfies the c.c.c.) Lastly we define a condition $r \in \mathbb{Q}^1_{I,h}$:

$$\mathrm{dom}(E^r) = \bigcup_{\zeta<\omega} n_{2\zeta}/E^{p^*} \cup \bigcup\{n/E^{p^*} : n \in A^{p^*} \text{ and } (\exists \zeta < \omega)(n_{2\zeta+1} < n < n_{2\zeta+2})\},$$

$$n_{2\zeta}/E^r = (n_{2\zeta}/E^{p^*}) \cup \bigcup\{m/E^{p^*} : m \in A^{p^*} \cap (n_{2\zeta+1}, n_{2\zeta+2})\}$$

(note that this defines correctly an I–equivalence relation E^r), $A^r = \{n_{2\zeta} : \zeta < \omega\}$. The function H^r is defined by cases (interpreting the value 0 as -1, where appears):

$$\begin{array}{lll} H^r(x^m_j)=H^{p^*}(x^m_j) & \text{if} & m\in(\omega\setminus\mathrm{dom}(E^{p^*}))\text{ and } j<h(m),\\ H^r(x^m_j)=H^{p^*}(x^m_j) & \text{if} & m\in\mathrm{dom}(E^{p^*})\text{ and } j\in[h(\min(m/E^{p^*})),h(m))\\ H^r(x^m_j)=1 & \text{if} & m\in\mathrm{dom}(E^{p^*})\text{ and } \min(m/E^{p^*})\in(n_{2\zeta},n_{2\zeta+1}]\\ & & \text{and } j<h(\min(m/E^{p^*}))\\ H^r(x^m_j)=\rho^*_{n_{2\zeta}}(j) & \text{if} & m\in\mathrm{dom}(E^{p^*})\text{ and } \min(m/E^{p^*})\in(n_{2\zeta+1},n_{2\zeta+2})\\ & & \text{and } j\in\mathrm{dom}(\rho^*_{n_{2\zeta}})\text{ and } j\geq h(n_{2\zeta})\\ H^r(x^m_j)=1 & & \text{otherwise (but } x^m_j\in\mathrm{dom}(H^r)). \end{array}$$

Now check that $p^*\leq r\in\mathbb{Q}^1_{I,h}$ and for each $n\in\mathrm{dom}(E^r)\setminus\bigcup\limits_{\zeta<\omega}n_{2\zeta}/E^{p^*}$:

$$r\Vdash\text{“ }\underset{\sim}{\eta}_n\text{ violates the property of }\underset{\sim}{\bar\nu}\text{ and hence }n\notin\underset{\sim}{\tau}\text{”}.$$

As $\mathrm{dom}(E^r)\setminus\bigcup\limits_{\zeta<\omega}n_{2\zeta}/E^{p^*}\in\mathcal{D}$ we have finished.

2) Should be clear by (*) of the proof of 2.5(1) and 2.3.
However we will give an alternative proof of 2.5(2). We start as in the proof of 2.5(1): suppose some $(p^*,\underset{\sim}{r}^*)\in\mathbb{Q}^1_{I,h}*\underset{\sim}{\mathbb{Q}}$ forces “$\underset{\sim}{F}$ is an NWD-ultrafilter on ω extending $\mathcal{D}$”. As $\Vdash$“$\underset{\sim}{\eta}_n[\underset{\sim}{G}_{\mathbb{Q}^1_{I,h}}]\in{}^{h(n)}2$”, for some $(\mathbb{Q}^1_{I,h}*\underset{\sim}{\mathbb{Q}})$-name $\underset{\sim}{\tau}$ for a subset of ω

$$(p^*,\underset{\sim}{r}^*)\Vdash\text{“ }\underset{\sim}{\tau}\in\underset{\sim}{F}\text{ and }(\forall\eta\in{}^{<\omega}2)(\exists\nu\in{}^{<\omega}2)(\eta\trianglelefteq\nu\ \&\ (\forall n\in\underset{\sim}{\tau})(\neg\nu\trianglelefteq\underset{\sim}{\eta}_n))\text{ ”}.$$

So for some $\mathbb{Q}^1_{I,h}*\underset{\sim}{\mathbb{Q}}$-name $\underset{\sim}{\bar\nu}=\langle\underset{\sim}{\nu}_n:n<\omega\rangle$

$$(p^*,\underset{\sim}{r}^*)\Vdash\text{“ }\underset{\sim}{\nu}_\ell\in\bigcup_{j\in[\ell,\omega)}{}^{[\ell,j)}2\text{ and for no }n\in\underset{\sim}{\tau}\text{ we have }\underset{\sim}{\nu}_\ell\subseteq\underset{\sim}{\eta}_n\text{”}.$$

So for some $\mathbb{Q}^1_{I,h}*\underset{\sim}{\mathbb{Q}}$–names $\underset{\sim}{d}_\ell$, $\underset{\sim}{w}_\ell$

$$\begin{array}{ll}(p^*,\underset{\sim}{r}^*)\Vdash & \text{“ }\omega>\underset{\sim}{d}_\ell>\ell,\ \underset{\sim}{w}_\ell\subseteq[\ell,\underset{\sim}{d}_\ell),\ |\underset{\sim}{w}_\ell|>(4\cdot\prod_{s\leq n}h(s))!\text{ and}\\ & [m_1<m_2\text{ in }\underset{\sim}{w}_\ell\quad\Rightarrow\quad\max\mathrm{dom}(\underset{\sim}{\nu}_{m_1})<m_2]\text{”}.\end{array}$$

Let $p^*\in G_{\mathbb{Q}^1_{I,h}}\subseteq\mathbb{Q}^1_{I,h}$ and $G_{\mathbb{Q}^1_{I,h}}$ generic over $\mathbf{V}$. Now in $\mathbf{V}[G_{\mathbb{Q}^1_{I,h}}]$, the forcing notion $\underset{\sim}{\mathbb{Q}}[G_{\mathbb{Q}^1_{I,h}}]$ is ${}^\omega\omega$-bounding (this follows from the PP-property) and also $\mathbb{Q}^1_{I,h}$ is ${}^\omega\omega$-bounding. Hence for some $r'\in\underset{\sim}{\mathbb{Q}}[G_{\mathbb{Q}^1_{I,h}}]$ and strictly increasing $x\in{}^\omega\omega\cap\mathbf{V}$ we have:

$$r'\Vdash_{\underset{\sim}{\mathbb{Q}}[G_{\mathbb{Q}^1_{I,h}}]}\text{“ }\underset{\sim}{d}_n<x(n)\text{ and }m\in\underset{\sim}{w}_n\ \Rightarrow\ \mathrm{dom}(\underset{\sim}{\nu}_m)\subseteq[0,x(n))\text{”}.$$

In $\mathbf{V}[G_{\mathbb{Q}^1_{I,h}}]$, by the property of $\underset{\sim}{\mathbb{Q}}$, there are r^{**}, $r'\leq r^{**}\in\underset{\sim}{\mathbb{Q}}[G_{\mathbb{Q}^1_{I,h}}]$ and a sequence $\langle\langle i_\ell(n),j_\ell(n)):\ell\leq k_n\rangle:n<\omega\rangle$ such that

$$i_0(n)<j_0(n)<i_1(n)<j_1(n)<\ldots<j_{k_n}(n)<i_\ell(n+1),j_\ell(n)>x(i_\ell(n))$$

and there are $\bar{\nu}^*_{n,\ell,t} = \langle \nu^*_{n,\ell,t,j} : j \in [i_\ell(n), j_\ell(n))\rangle$ for $t < i_\ell(n), \ell \leq k_n$ and $\bar{w}^*_{n,\ell,t} = \langle w^*_{n,\ell,t,j} : j \in [i_\ell(n), i_{\ell+1}(n))$ for $t < i_\ell(n), \ell \leq k_n\rangle$ such that

$$r^{**} \Vdash_{\mathbb{Q}} \text{“}\langle \underset{\sim}{\nu}_{i_\ell(n)+j} : j \in [i_\ell(n), j_\ell(n))\rangle \text{ is } \bar{\nu}^*_{n,\ell,t} \text{ and } \\ \langle \underset{\sim}{w}_{i_\ell(n)+j} : j \in [i_\ell(n) : j_\ell(n))\rangle \text{ is } \bar{w}^*_{n,\ell,t} \text{ for some } t < i_\ell(n)\text{”}.$$

Back in $\mathbf{V}$ we have a $\mathbb{Q}^1_{I,h}$-name $\underset{\sim}{r}^{**}$ and $\langle\langle(\underset{\sim}{i}_\ell(n), \underset{\sim}{j}_\ell(n)) : \ell \leq \underset{\sim}{k}_n\rangle : n < \omega\rangle$ and $\langle\langle \underset{\sim}{\bar{\nu}}^*_{n,\ell,t} : t < i_\ell(n)\rangle : \ell < \underset{\sim}{k}_n, n < \omega\rangle$ and $\langle\langle \underset{\sim}{\bar{w}}^*_{n,\ell,t} : t < i_\ell(n)\rangle : \ell < \underset{\sim}{k}_n, n < \omega\rangle$ are forced (by p^*) to be as above.

By 1.14, increasing p^*, we get

for every $f : \{x^n_i : i < h(m), m \in A^{p^*} \cap (n+1)\} \longrightarrow \{-1, 1\}, n \in A^{p^*}$, the condition $p^{*[f]}$ forces a value to

$$\langle\langle(\underset{\sim}{i}_\ell(m), \underset{\sim}{j}_\ell(m)) : \ell \leq \underset{\sim}{k}_m\rangle : m \leq n\rangle, \\ \langle \underset{\sim}{\bar{\nu}}^*_{n,\ell,t} : t < \underset{\sim}{i}_\ell(n), \ell \leq \underset{\sim}{k}_n\rangle, \\ \langle \underset{\sim}{\bar{w}}^*_{m,\ell,t} : t < \underset{\sim}{i}_\ell(n), \ell < \underset{\sim}{k}_n\rangle$$

moreover, without loss of generality

$$n \in A^{p^*} \quad \Rightarrow j_{\underset{\sim}{k}_n}(n) < \min(A^{p^*} \setminus (n+1)).$$

Now by 2.2, without loss of generality for each $n \in A^{p^*}$ we can find a function ρ_n from $[n, \min(A^{p^*} \setminus (n+1))]$ to $\{-1, 1\}$ such that:

if $f : \{x^m_i : i < h(m), m \in A^{p^*} \cap (n+1)\} \longrightarrow \{-1, 1\}, n \in A^{p^*}$ then $(p^{*[f]}, \underset{\sim}{r}^{**})$ forces that ρ_n extends some $\underset{\sim}{\nu}_\ell$.

Now we continue as in the proof of 2.5(1). ■

3 The consistency proof

Theorem 3.1 *Assume CH and $\diamondsuit_{\{\gamma<\omega_2 : \mathrm{cf}(\gamma)=\omega_1\}}$. Then there is an $\aleph_2$–cc proper forcing notion $\mathbb{P}$ of cardinality $\aleph_2$ such that*

$$\Vdash_{\mathbb{P}} \text{“ there are no NWD–ultrafilters on } \omega \text{ ”}.$$

Proof Define a countable support iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \omega_2, j < \omega_2\rangle$ of proper forcing notions and sequences $\langle \underset{\sim}{\mathcal{D}}_i : i < \omega_2\rangle$ and $\langle \underset{\sim}{\bar{\eta}}^i : i < \omega_2\rangle$ such that for each $i < \omega_2$:

1. $\underset{\sim}{\mathcal{D}}_i$ is a $\mathbb{P}_i$–name for a non–principal ultrafilter on ω,
2. $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}_i$–name for a proper forcing notion of size $\aleph_1$ with the PP–property,
3. $\underset{\sim}{\bar{\eta}}^i$ is a $\mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i$–name for a function from ω to ${}^{<\omega}2$,

4. $\Vdash_{\mathbb{P}_i*\underset{\sim}{\mathbb{Q}}_i}$ "if $X\subseteq\omega$ and the set $\{\underset{\sim}{\eta}^i_n: n\in X\}\subseteq{}^{<\omega}2$ is nowhere dense then there is $Y\in\underset{\sim}{\mathcal{D}}_i$ disjoint from X",

5. if $\underset{\sim}{\mathcal{D}}$ is a $\mathbb{P}_{\omega_2}$–name for an ultrafilter on ω then the set

$$\{i<\omega_2:\ \mathrm{cf}(i)=\omega_1\quad\&\quad \underset{\sim}{\mathcal{D}}_i=\underset{\sim}{\mathcal{D}}{\restriction}\mathcal{P}(\omega)^{\mathbf{V}^{\mathbb{P}_i}}\}$$

is stationary.

Let us first argue that if we succeed with the construction then, in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, we will have

$$2^{\aleph_0}=\aleph_2\quad+\quad\text{"there is no NWD-ultrafilter on }\omega\text{"}.$$

Why? As each $\underset{\sim}{\mathbb{Q}}_i$ is (a name) for a proper forcing notion of size $\aleph_1$, the limit $\mathbb{P}_{\omega_2}$ is a proper forcing notion with a dense subset of size $\aleph_2$ and satisfying the $\aleph_2$–cc. Since $\mathbb{P}_{\omega_2}$ is proper, each subset of ω (in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$) has a canonical countable name (i.e. a name which is a sequence of countable antichains; every condition in the $n^{\rm th}$ antichain decides if the integer n is in the set or not; of course we do not require that the antichains are maximal). Hence $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_0}\leq\aleph_2$ (remember that we have assumed $\mathbf{V}\models$CH). Moreover, by 1.20 + 2.3 we know that $\mathbb{P}_{\omega_2}$ satisfies $(\oplus^{\rm nwd})$ of 2.3, i.e.

$$\Vdash_{\mathbb{P}_{\omega_2}}\quad\begin{array}{l}\text{"each nowhere dense subset of }{}^{<\omega}2\text{ can be covered}\\ \text{by a nowhere dense subset of }{}^{<\omega}2\text{ from }\mathbf{V}\text{".}\end{array}$$

Now suppose that $\underset{\sim}{\mathcal{D}}$ is a $\mathbb{P}_{\omega_2}$–name for an ultrafilter on ω. By the fifth requirement, we find $i<\omega_2$ such that $\underset{\sim}{\mathcal{D}}_i=\underset{\sim}{\mathcal{D}}{\restriction}\mathcal{P}(\omega)^{\mathbf{V}^{\mathbb{P}_i}}$ (and $\mathrm{cf}(i)=\omega_1$). Since $\mathbb{P}_{\omega_2}$ satisfies $(\oplus^{\rm nwd})$, we have

$\Vdash_{\mathbb{P}_{\omega_2}}$ "if $X\subseteq\omega$ and the set $\{\underset{\sim}{\eta}^i_n: n\in X\}\subseteq{}^{<\omega}2$ is nowhere dense then there is an element of $\underset{\sim}{\mathcal{D}}{\restriction}\mathcal{P}(\omega)^{\mathbf{V}^{\mathbb{P}_i}}$ disjoint from X"

[Why? Cover $\{\underset{\sim}{\eta}^i_n: n\in X\}$ by a nowhere dense set $A\subseteq{}^{<\omega}2$ from $\mathbf{V}$ and look at the set $X_0=\{n\in\omega:\underset{\sim}{\eta}^i_n\in A\}$. Clearly $X_0\in\mathbf{V}^{\mathbb{P}_i*\underset{\sim}{\mathbb{Q}}_i}$ and $X\subseteq X_0$. Applying the fourth clause to X_0 we find $Y\in\underset{\sim}{\mathcal{D}}_i=\underset{\sim}{\mathcal{D}}{\restriction}\mathcal{P}(\omega)^{\mathbf{V}^{\mathbb{P}_i}}$ such that $Y\cap X_0=\emptyset$. Then $Y\cap X=\emptyset$ too.]
But this means that, in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, the function $\underset{\sim}{\bar\eta}^i$ exemplifies that $\underset{\sim}{\mathcal{D}}$ is not an NWD ultrafilter (remember $\underset{\sim}{\mathcal{D}}{\restriction}\mathcal{P}(\omega)^{\mathbf{V}^{\mathbb{P}_i}}\subseteq\underset{\sim}{\mathcal{D}}$). Moreover, as CH implies the existence of NWD-ultrafilters, we conclude that actually $\Vdash_{\mathbb{P}_{\omega_2}} 2^{\aleph_0}=\aleph_2$.

Let us describe how one can carry out the construction. Each $\underset{\sim}{\mathbb{Q}}_i$ will be $\mathbb{Q}^1_{\underset{\sim}{I}_i,h}$ for some increasing function $h\in{}^{\omega}\omega$ (e.g. $h(n)=n$) and a ($\mathbb{P}_i$–name for a) maximal non–principal ideal $\underset{\sim}{I}_i$ on ω. By 2.4, 1.19 we know that $\mathbb{Q}^1_{\underset{\sim}{I}_i,h}$ satisfies the demands (2)–(4) for the ultrafilter $\underset{\sim}{\mathcal{D}}_i$ dual to $\underset{\sim}{I}_i$ and the function $\underset{\sim}{\bar\eta}^i$ as in the proof of 2.4. Thus, what we have to do is to say what are the names $\underset{\sim}{\mathcal{D}}_i$. To choose them we will use the assumption of $\diamondsuit_{\{\gamma<\omega_2:\mathrm{cf}(\gamma)=\omega_1\}}$. In the process of building the iteration we choose an enumeration $\langle(p_i,\underset{\sim}{\tau}_i): i<\omega_2\rangle$ of all pairs

$(p, \underset{\sim}{\tau})$ such that p is a condition in $\mathbb{P}_{\omega_2}$ (in its standard dense subset of size $\aleph_2$) and $\underset{\sim}{\tau}$ is a canonical (countable) $\mathbb{P}_{\omega_2}$–name for a subset of ω. We require that $p_i \in \mathbb{P}_i$ and $\underset{\sim}{\tau}_i$ is a $\mathbb{P}_i$–name (of course, it is done by a classical bookkeeping argument). Note that each subset of $\mathcal{P}(\omega)$ from $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ has a name which may be interpreted as a subset X of ω_2: if $i \in X$ then p_i forces that $\underset{\sim}{\tau}_i$ is in our set. Now we may describe how we choose the names $\underset{\sim}{\mathcal{D}}_i$. By $\diamondsuit_{\{\gamma<\omega_2 : \mathrm{cf}(\gamma)=\omega_1\}}$ we have a sequence $\langle X_i : i < \omega_2 \ \& \ \mathrm{cf}(i) = \omega_1 \rangle$ such that

(i) $X_i \subseteq i$ for each $i \in \omega_2$, $\mathrm{cf}(i) = \omega_1$,

(ii) if $X \subseteq \omega_2$ then the set

$$\{i \in \omega_2 : \mathrm{cf}(i) = \omega_1 \ \& \ X_i = X \cap i\}$$

is stationary.

Arriving at stage $i < \omega_2$, $\mathrm{cf}(i) = \omega_1$ we look at the set X_i. We ask if it codes a $\mathbb{P}_i$–name for an ultrafilter on ω (i.e. we look at $\{(p_\alpha, \underset{\sim}{\tau}_\alpha) : \alpha \in X_i\}$ which may be interpreted as a $\mathbb{P}_i$–name for a subset of $\mathcal{P}(\omega)$). If yes, then we take this name as $\underset{\sim}{\mathcal{D}}_i$. In all remaining cases we take whatever we wish, we may even not define the name $\underset{\sim}{\bar{\eta}}^i$ (note: this leaves us a lot of freedom and one may use this to get some additional properties of the final model). So why we may be sure that the fifth requirement is satisfied? Suppose that we have a $\mathbb{P}_{\omega_2}$–name for an ultrafilter on ω. This name can be thought of as a subset X of ω_2. If $i < \omega_2$ is sufficiently closed then $X \cap i$ is a $\mathbb{P}_i$–name for an ultrafilter on ω which is the restriction of $\underset{\sim}{\mathcal{D}}$ to $\mathbf{V}^{\mathbb{P}_i}$. So we have a club $C \subseteq \omega_2$ such that for each $i \in C$, if $\mathrm{cf}(i) = \omega_1$ the $X \cap i$ is of this type. By **(ii)** the set

$$S \stackrel{\mathrm{def}}{=} \{i < \omega_2 : i \in C \ \& \ \mathrm{cf}(i) = \omega_1 \ \& \ X_i = X \cap i\}$$

is stationary. But easily, for each $i \in S$, the name $\underset{\sim}{\mathcal{D}}_i$ has been chosen in such a way that $\underset{\sim}{\mathcal{D}}_i = \underset{\sim}{\mathcal{D}} \restriction \mathcal{P}(\omega)$, so we are done. ■

We note that this implies that there is also no ultrafilter with property M. This was asked by Benedikt in [Bn].

Definition 3.2 *A non-principal ultrafilter $\mathcal{D}$ on ω has the M-property (or property M) if:*

> *if for some real $\varepsilon > 0$, for $n < \omega$ we have a tree $T_n \subseteq {}^{<\omega}2$ such that $\mu(\lim(T_n)) \geq \varepsilon$*
> *then $(\exists A \in \mathcal{D})(\bigcap_{n \in A} \lim(T_n) \neq \emptyset)$*

(where μ stands for the Lebesgue measure on ${}^{\omega}2$).

Proposition 3.3 *If a non-principal ultrafilter $\mathcal{D}$ on ω is not NWD, then $\mathcal{D}$ does not have the property M.*

Proof Let

$$S^\varepsilon_\ell = \left\{ T \cap {}^{\ell \geq} 2 : T \subseteq {}^{<\omega}2,\ T \text{ a tree not containing a cone, } \mu(\lim(T)) > \varepsilon \right\}$$

(note that S^ε_ℓ is a set of trees not a set of nodes) and let $S^\varepsilon = \bigcup\limits_\ell S^\varepsilon_\ell$.

Now let $t_1 \prec t_2$ if: $t_1 \in S^\varepsilon_{\ell_1}$, $t_2 \in S^\varepsilon_{\ell_2}$, $\ell_1 < \ell_2$ and $t_1 = t_2 \cap {}^{\ell_1 \geq}2$. So S^ε is a tree with ω levels, each level is finite. As $\mathcal{D}$ is not NWD, we can find $\eta^\varepsilon_n \in \lim(S^\varepsilon)$ for $n < \omega$ such that:

if $A \in \mathcal{D}$ then $\{\eta^\varepsilon_n : n \in A\}$ is somewhere dense.

Now let $T^\varepsilon_n \subseteq {}^{<\omega}2$ be a tree such that $\langle T^\varepsilon_n \cap {}^{\ell \geq}2 : \ell < \omega \rangle = \eta^\varepsilon_n$. We claim that:

$\langle T^\varepsilon_n : n < \omega \rangle$ exemplifies $\mathcal{D}$ does not have the M-property.

Clearly T^ε_n is a tree of the right type, in particular

$$\mu(\lim(T^\varepsilon_n)) = \inf\{|T^\varepsilon_n \cap {}^\ell 2|/2^\ell : \ell < \omega\} \geq \varepsilon.$$

So assume $A \in \mathcal{D}$ and we are going to prove that $\bigcap\limits_{n \in A} \lim(T^\varepsilon_n)$ is empty. We know that $\{\eta^\varepsilon_n : n \in A\}$ is somewhere dense, so there is $\ell^* < \omega$ and $t^* \in S^\varepsilon_{\ell^*}$ such that:

$$\ell^* < \ell < \omega \ \&\ t^* \prec t \in S^\varepsilon_\ell \quad \Rightarrow \quad (\exists n \in A)(t \lhd \eta^\varepsilon_n).$$

Now $\frac{|t^* \cap {}^{\ell^*}2|}{2^{\ell^*}}$ is $> \varepsilon$ (so S^ε_ℓ was defined). So we choose $\ell > \ell^*$, such that:

if $\nu \in {}^\ell 2$, $\nu \restriction \ell^* \in t^*$
then $t'_\nu = \{\rho \in {}^\ell 2 : \rho \restriction \ell^* \in t^* \text{ and } \rho \neq \nu\} \in S^\varepsilon_\ell$,

hence there is $n = n_\nu \in A$ such that t'_ν appears in η^ε_n. Now clearly

$$\begin{aligned} \bigcap_{n \in A} \lim(T^\varepsilon_n) \quad &\supseteq \bigcap_{\substack{\nu \in {}^\ell 2 \\ \nu \restriction \ell^* \in t^*}} \lim(T^\varepsilon_{n_\nu}) \\ &\supseteq \{\eta \in {}^{<\omega}2 : \eta \restriction \ell \in \bigcap \{t'_\nu : \nu \in {}^\rho 2, \nu \restriction \ell \in t^*\}\} = \emptyset, \end{aligned}$$

finishing the proof. ■

Conclusion 3.4 *In the universe $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ from 3.1, there is no (non-principal) ultrafilter (on ω) with property M.* ■

Concluding Remarks 3.5 We may consider some variants of $\mathbb{Q}^2_{I,h}$.

In definition 1.2 we have $\mathrm{dom}(H^p)$ is as in 1.2(1) but: $H^p \restriction B^p_1$ gives constants (not functions) and for $x^m_i \in B^p_3 \setminus B^p_1$, letting $n = \min(m/E^p)$ the function $H^p(x^m_i)$ depends just on $\{x^n_j : j \leq i\}$. Moreover, it is such that changing the value of x^n_i changes the value, so $H^p(x^m_i) = x^n_i \times f^p_{x^m_i}(x^n_0, \ldots, x^n_{i-1})$. Call this $\mathbb{Q}^3_{I,h}$.

A second variant is when we demand the functions $f^p_{x^m_i}(x^n_0,\dots,x^n_{i-1})$ to be constant, call it $\mathbb{Q}^4_{I,h}$.

Both have the properties proved $\mathbb{Q}^2_{I,h}$. In particular, in the end of the proof of 1.9(5), we should change: $H^r(x^m_i)$ is defined exactly as in the proof of 1.9(4) except that when $i < h(n^*)$, $k = \min(m/E^p)$, $k \notin \mathrm{dom}(E^q)$, $k \notin \mathbf{u}$ (so m, k, n^* are E^r–equivalent) we let $H^r(x^k_i) = H^q(x^m_i) \times f(x^{n^*}_i) \times x^{n^*}_i$ (the first two are constant), so $H^r(x^m_i)$ is computed as before using this value.

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