CONSTRUCTING STRONGLY EQUIVALENT NONISOMORPHIC MODELS FOR UNSUPERSTABLE THEORIES, PART C

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Abstract

In this paper we prove a strong nonstructure theorem for $\kappa(T)$-saturated models of a stable theory T with dop. This paper continues the work started in [HT].

1. Introduction and basic definitions

By a strong nonstructure theorem we mean a theorem, which claims that in a given class of structures, there are very equivalent nonisomorphic models. The equivalence is usually measured by the length of Ehrenfeucht-Fraisse games in which $\exists$ has a winning strategy. The idea behind this is, that if models are very equivalent but still nonisomorphic, they must be very complicated, i.e. there is a lot nonstructure in the class.

For more background for the theorems of this kind, see [HT].

In this paper we prove the following strong nonstructure theorem (see Definitions 1.2 and 1.3).

1.1 Theorem. *Let T be a stable theory with dop and $\kappa = cf(\kappa) = \lambda(T) + \kappa^{<\kappa(T)} \geq \omega_1$, $\lambda = \lambda^{<\lambda} > \kappa^+$ and for all $\xi < \lambda$, $\xi^\kappa < \lambda$. Then there is F^a_κ-saturated model $M_0 \models T$ of power λ such that the following is true: for all λ^+, λ-trees t there is a F^a_κ-saturated model M_1 of power λ such that $M_0 \equiv^\lambda_t M_1$ and $M_o \not\cong M_1$.*

In [HT] Theorem 1.1 was proved for F^a_ω-saturated models of a countable superstable theory with dop. There we used Ehrenfeucht-Mostowski models to construct

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the required models. To prove that the models are not isomorphic, it was essential that the sequences in the skeletons of the models were of finite length. In the case of unsuperstable theories we cannot quarantee this. Another problem was, of course, that with Ehrenfeucht-Fraisse models we cannot construct more than F^a_ω-saturated models.

In this paper we overcome these problems by using F^a_κ-prime models instead of Ehrenfeucht-Mostowski models.

1.2 Definition.

(i) Let λ be a cardinal and α an ordinal. Let t be a tree (i.e. for all $x \in t$, the set $\{y \in t|\ y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t|\ z < x\} = \{z \in t|\ z < y\}$, then we denote $x \sim y$, and the equivalence class of x for $\sim$ we denote $[x]$. By a λ, α-tree t we mean a tree which satisfies:

(a) $|[x]| < \lambda$ for every $x \in t$;

(b) there are no branches of length $\geq \alpha$ in t;

(c) t has a unique root;

(d) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

(ii) If η is a tree and α is an ordinal then we define the tree $\alpha \times \eta = (\alpha \times \eta, <)$ so that $(x, y) < (v, w)$ iff $y < w$ or $y = w$ and $x < v$.

1.3 Definition. *Let t be a tree and κ a cardinal. The Ehrenfeucht-Fraisse game of length t between models $\mathcal{A}$ and $\mathcal{B}$, $G^\kappa_t(\mathcal{A}, \mathcal{B})$, is the following. At each move α:*

(i) player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$ and either $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$ or $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$, we will denote this sequence by X_α;

(ii) if $\forall$ chose from $\mathcal{A}$ then $\exists$ chooses $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$, we will denote this sequence by Y_α.

$\forall$ must move so that $(x_\beta)_{\beta \leq \alpha}$ form a strictly increasing sequence in t. $\exists$ must move so that $\{(a^\beta_\gamma, b^\beta_\gamma)|\gamma \leq \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$. The player who first has to break the rules loses.

We write $\mathcal{A} \equiv^\kappa_t \mathcal{B}$ if $\exists$ has a winning strategy for $G^\kappa_t(\mathcal{A}, \mathcal{B})$.

The following theorem is frequently used in this paper.

1.4 Theorem. *([Sh]) Let T be a stable theory. Assume I is an infinite indiscernible sequence over A, $I \subseteq B$ and $J \subseteq I$ is countable.*

(i) $Av(I, B)$ does not fork over J and $Av(I, J)$ is stationary.

(ii) $I \cup \{a\}$ is indiscernible over A iff $t(a, A \cup I) = Av(I, A \cup I)$.

Proof. See [Sh] Lemma III 4.17. □

1.5 Corolary. *Let T be a stable theory. Assume I is an infinite indiscernible sequence over A and $J \subseteq I$ is infinite. Then $I - J$ is independent over $A \cup J$.*

Proof. Follows immediately from Theorem 1.4. □

2. Construction

Through out this paper we assume that T is a stable theory with dop, $\kappa = cf(\kappa) = \lambda(T) + \kappa^{<\kappa(T)} \geq \omega_1$, $\lambda = \lambda^{<\lambda} > \kappa^+$ and for all $\xi < \lambda$, $\xi^\kappa < \lambda$.

2.1 Theorem. *([Sh]) There are models $\mathcal{A}_i$, $i < 3$, of cardinality $< \kappa$ and infinite indiscernible sequence I over $\mathcal{A}_1 \cup \mathcal{A}_2$ such that*

(i) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,

(ii) $Av(I, I \cup \mathcal{A}_1 \cup \mathcal{A}_2) \perp \mathcal{A}_1$, $Av(I, I \cup \mathcal{A}_1 \cup \mathcal{A}_2) \perp \mathcal{A}_2$,

(iii) $t(I, \mathcal{A}_1 \cup \mathcal{A}_2)$ *is almost orthogonal to* $\mathcal{A}_1$ *and to* $\mathcal{A}_2$,

(iv) if B_i, $i < 3$ *are such that* $B_0 \downarrow_{\mathcal{A}_0} \mathcal{A}_1 \cup \mathcal{A}_2$, $B_1 \downarrow_{\mathcal{A}_1 \cup B_0} \mathcal{A}_2 \cup B_2$ *and* $B_2 \downarrow_{\mathcal{A}_3 \cup B_0} \mathcal{A}_1 \cup B_1$ *then*

$$t(I, \mathcal{A}_1 \cup \mathcal{A}_2) \vdash t(I, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \bigcup_{i<3} B_3).$$

Proof. This is [Sh] X Lemma 2.4, except that in (iv), only

$$(*) \qquad stp(I, \mathcal{A}_1 \cup \mathcal{A}_2) \vdash t(I, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \bigcup_{i<3} B_3)$$

is proved. But since $\kappa \geq \kappa_r(T)$, by [Sh] XI Lemma 3.1 $\mathcal{A}_1 \cup \mathcal{A}_2$ is a good set. It is easy to see that this together with (*) implies

$$t(I, \mathcal{A}_1 \cup \mathcal{A}_2) \vdash t(I, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \bigcup_{i<3} B_3).$$

□

In [HT] the following theorem is proved.

2.2 Theorem. *([HT] Theorem 3.4) There is a* $\lambda^+, \lambda + 1$*-tree* η *such that it has a branch of length* λ *and for every* λ^+, λ*-tree* t *there is a* λ^+, λ*-tree* ξ *such that* $\eta \equiv_t^\lambda \xi$.

Let η be a tree. We define a model $M(\eta)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and I be as $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ and I in Theorem 2.1. We may assume that $|I| = \lambda$.

For all $t \in \eta$ we choose $\mathcal{A}_t, \mathcal{B}_t$ and $\mathcal{C}_t$ so that

(i) there is an automorphism f_t (of the monster model) such that $f_t(\mathcal{B}_t) = \mathcal{B}$, $f_t(\mathcal{C}_t) = \mathcal{C}$ and $f_t^{-1} \restriction \mathcal{A} = id_{\mathcal{A}}$,

(ii) $\mathcal{B}_t \cup \mathcal{C}_t \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_s \cup \mathcal{C}_s |\ s \in \eta,\ s \neq t\}$.

For all $s, t \in \eta$, $s < t$, we choose I_{st} so that

(i) there is an automorphism g_{st} such that $g_{st} \restriction \mathcal{B}_s = f_s \restriction \mathcal{B}_s$, $g_{st} \restriction \mathcal{C}_t = f_t \restriction \mathcal{C}_t$ and $g_{st}(I_{st}) = I$,

(ii) $I_{st} \downarrow_{\mathcal{B}_s \cup \mathcal{C}_t} \bigcup\{\mathcal{B}_p \cup \mathcal{C}_p |\ p \in \eta\} \cup \bigcup\{I_{pr} |\ p, r \in \eta,\ p < r,\ p \neq s \text{ or } r \neq t\}$.

We define $M(\eta)$ to be the F^a_κ-primary model over $S(\eta) = \bigcup\{\mathcal{B}_t \cup \mathcal{C}_t |\ t \in \eta\} \cup \bigcup\{I_{st} |\ s, t \in \eta,\ s < t\}$.

By Theorem 2.2, Theorem 1.1 follows immediately from the theorem below.

2.3 Theorem. *Let* η *be as in Theorem 2.2 and* $M_0 = M(\eta)$. *Assume* t *is a* λ^+, λ*-tree. Let* ξ *be a* λ^+, λ*-tree such that* $\eta \equiv^\lambda_{\kappa \times t} \xi$. *If* $M_1 = M(\xi)$, *then* $M_0 \equiv_t^\lambda M_1$, $M_o \not\cong M_1$ *and the cardinality of the models is* λ.

The claim on the cardinality of the models follows immediately from the assumptions on λ. The other two claims are proved in the next two chapters.

Notice that in ξ there are no brances of length λ. Since in η there is such a branch, this enables us to prove the nonisomorphism of the models.

3. Equivalence

In this chapter we prove the first part of Theorem 2.3. We want to remind the reader of the assumptions made in the beginning of Chapter 2.

Let $(S(\eta), \{d_i|\ i<\alpha\}, (D_i|\ i<\alpha))$ and $(S(\xi), \{e_i|\ i<\alpha\}, (E_i|\ i<\beta))$ be F^a_κ-constructions of $M(\eta)$ and $M(\xi)$, respectively, see [Sh] IV Definition 1.2. If we choose the constructions carefully we can assume $\alpha=\beta=\lambda$.

We enumerate η and ξ: $\eta=\{t^\eta_i|\ i<\lambda\}$ and $\xi=\{t^\xi_i|\ i<\lambda\}$. Furthermore we do this so that if $t^*_i<t^*_j$ then $i<j$, $*\in\{\eta,\xi\}$. If $\gamma\le\lambda$, we write $\eta(\gamma)=\{t^\eta_i|\ i<\gamma\}$ and similarly for $\xi(\gamma)$.

We also enumerate all I_{st}: $I_{st}=\{a^i_{st}|\ i<\lambda\}$.

We write $S(\eta,\gamma)$ for

$$\bigcup\{\mathcal{B}_t|\ t\in\eta(\gamma)\}\cup\bigcup\{\mathcal{C}_t|\ t\in\eta(\gamma)\}\cup$$

$$\bigcup\{a^i_{st}|\ s<t,\ s,t\in\eta(\gamma),\ i<\gamma\}$$

and similarly for $S(\xi,\gamma)$.

If $\gamma<\lambda$ and $g:\eta(\gamma)\to\xi(\gamma)$ is a partial isomorphism then by g^* we mean the function from $S(\eta,\gamma)$ onto $S(\xi,\gamma)$ which satisfies:

(i) if $g(t)=t'$ then for all $a\in B_t$ and $b\in C_t$, $g^*(a)=f^{-1}_{t'}(f_t(a))$ and $g^*(b)=f^{-1}_{t'}(f_t(b))$,

(ii) if $g(t)=t'$, $g(s)=s'$, $t<s$ and $a\in I_{ts}$ then $g^*(a)=g^{-1}_{t's'}(g_{ts}(a))$.

3.1 Lemma. *If $\gamma<\lambda$ and $g:\eta(\gamma)\to\xi(\gamma)$ is a partial isomorphism then g^* is a partial isomorphism.*

Proof. Immediate by the definitions. □

We write

$$M(\eta,\gamma)=S(\eta,\gamma)\cup\{d_i|\ i<\gamma\}$$

and similarly for $M(\xi,\gamma)$. We say that $\gamma<\lambda$ is good if for all $i<\gamma$, $D_i\subseteq M(\eta,\gamma)$ and $E_i\subseteq M(\xi,\gamma)$. Notice that the set of all good ordinals is cub in λ. Notice also that the set of those ordinals $\gamma<\lambda$ for which $M(\eta,\gamma)$ is F^a_κ-saturated, is $\ge\kappa$-cub, i.e. it is unbounded in λ and closed under increasing sequences of cofinality $\ge\kappa$.

3.2 Lemma. *Assume $A\subseteq B$, a_i and C_i, $i<\alpha$, are such that*

(i) $C_i\subseteq A\cup\{a_j|\ j<i\}$ is of power $<\kappa$,

(ii) $t(a_i,C_i)\vdash t(a_i,B\cup\{a_j|\ j<i\})$.

Then for all sequences $\overline{d}\in\{a_i|\ i<\alpha\}$, there is $D\subseteq A$ of power $<\kappa$ such that $t(\overline{d},D)\vdash t(\overline{d},B)$. Especially, $\overline{d}\downarrow_A B$.

Proof. See the proof of [Sh] Theorem IV 3.2. □

3.3 Lemma. *Let $\gamma<\lambda$ be good, $\gamma<\delta<\lambda$, $g:\eta(\delta)\to\xi(\delta)$ is a partial isomorphism, $f:M(\eta,\gamma)\to M(\xi,\gamma)$ is a partial isomorphism and $g^*\restriction S(\eta,\gamma)\subseteq f$. Then $f\cup g^*$ is a partial isomorphism from $M(\eta,\gamma)\cup S(\eta,\delta)$ onto $M(\xi,\gamma)\cup S(\xi,\delta)$.*

Proof. Follows immediately from Lemmas 3.1, 3.2 and the definition of a good ordinal. □

3.4 Lemma. *Assume $\gamma < \lambda$ is good, $g : \eta(\gamma) \to \xi(\gamma)$ and $f : M(\eta,\gamma) \to M(\xi,\gamma)$ are partial isomorphism, $g^* \subseteq f$ and*

$$(\eta, a)_{a\in\eta(\gamma)} \equiv^\lambda_\kappa (\xi, f(a))_{a\in\eta(\gamma)}.$$

If $A \subseteq M_0$ is of power $< \lambda$ then there are good $\gamma' < \lambda$, partial isomorphisms $g' : \eta(\gamma') \to \xi(\gamma')$ and $f' : M(\eta,\gamma') \to M(\xi,\gamma')$ such that $(g')^* \subseteq f'$, $f \subseteq f'$, $g \subseteq g'$ and $A \subseteq M(\eta,\gamma')$.

Proof. By playing the Ehrenfeucht-Fraisse game we can find a good $\gamma' < \lambda$ such that

(i) there is a partial isomorphism $g' : \eta(\gamma') \to \xi(\gamma')$ such that $g \subseteq g'$,

(ii) $M(\eta,\gamma')$ is F^a_κ-primary over $S(\eta,\gamma')$ and $M(\xi,\gamma')$ is F^a_κ-primary over $S(\xi,\gamma')$,

(iii) $A \subseteq M(\eta,\gamma')$.

By (i) above and Lemma 3.3, $f \cup (g')^*$ is a partial isomorphism from $M(\eta,\gamma) \cup S(\eta,\gamma')$ onto $M(\xi,\gamma) \cup S(\xi,\gamma')$. From (ii) it follows that $M(\eta,\gamma')$ is F^a_κ-primary over $M(\eta,\gamma) \cup S(\eta,\gamma')$ and $M(\xi,\gamma')$ is F^a_κ-primary over $M(\xi,\gamma) \cup S(\xi,\gamma')$. So the existence of the required f' follows from the uniqueness of the F^a_κ-primary models ([Sh] Conclusion IV 3.9). □

3.5 Theorem. $M_0 \equiv^\lambda_t M_1$.

Proof. By Lemma 3.4, it is easy to translate the winning strategy of $\exists$ in $G^\lambda_{\kappa\times t}(\eta,\xi)$ to her winning strategy in $G^\lambda_t(M_0, M_1)$. □

4. Nonisomorphism

In this chapter we prove the second part of Theorem 2.3, i.e. $M_0 \not\cong M_1$. Again we want to remind the reader of the assumptions made in the beginning of Chapter 2.

For a contradiction we assume that $f : M_0 \to M_1$ is an isomorphism.

If $a \in M_0$ then we write α_a for the least α such that $a \in M(\eta,\alpha)$ and similarly for $a \in M_1$. By α_A we mean $\bigcup\{\alpha_a |\ a \in A\}$.

Let $X \subseteq \eta$ be such that $|X| = \lambda$ and for all $x, y \in X$ if $x \neq y$ then either $x < y$ or $y < x$. For every $x \in X$ we choose u^i_x, S^i_x and N^i_x, $i \in \{0, 1\}$, so that

(i) $x \in u^0_x \subseteq \eta$ and $u^1_x \subseteq \xi$,

(ii) $S^i_x = \bigcup\{\mathcal{B}_t |\ t \in u^i_x\} \cup \bigcup\{\mathcal{C}_t |\ t \in u^i_x\} \cup \bigcup\{I^x_{st} |\ s, t \in u^i_x,\ s < t\}$, where $I^x_{st} \subseteq I_{st}$ is of infinite power at most κ,

(iii) $N^i_x \subseteq M_i$ is F^a_κ-primary over S^i_x and furthermore if $a \in N^0_x - S(\eta)$ and $a = d_i$ in the construction of M_0 then $D_i \subseteq N^0_x$ and similarly for N^1_x,

(iv) $f \restriction N^0_x$ is onto N^1_x,

(v) $|N^i_x| \leq \kappa$,

(vi) if $M(\eta,\alpha)$ is F^a_κ-saturated, then so is $M(\eta,\alpha) \cap N^0_x$.

It is easy to see that these sets exist.

4.1 Lemma. *Assume A_i, $i < \lambda$, are sets of power $\leq \kappa$. Then there are $X \subseteq \lambda$ and B such that $|X| = \lambda$ and for all $i, j \in X$, $A_i \cap A_j = B$.*

Proof. Without loss of generality we may assume that for all $i < \lambda$, $A_i \subseteq \lambda$. We define $f(\alpha) = sup(A_i \cap (\cup_{j<i} A_j))$. Since $\lambda > \kappa^+$ is regular, this function is regressive on a stationary set. So by Fodor's lemma, it is constant on some set X' of power λ. Since for all $\theta < \lambda$, $\theta^\kappa < \lambda$, the claim follows by the pigeon hole principle. □

By Lemma 4.1 and the pigeon hole principle we may assume that X is chosen so that it satisfies the following:

(i) There are u^i, S^i and N^i, $i \in \{1,2\}$, such that for all $x, y \in X$, if $x \neq y$ then $u^i_x \cap u^i_y = u^i$, $S^i_x \cap S^i_y = S^i$ and $N^i_x \cap N^i_y = N^i$.

(ii) For all $x \in X$, $M(\eta, \alpha_{N^0}) \cap N^o_x = N^0$ and if $x < y$ then $M(\eta, \alpha_{N^0_x}) \cap N^o_y = N^0$ and similarly for 1 instead of 0.

(iii) For all $x, y \in X$, there are elementary maps $f^i_{xy} : N^i_x \to N^i_y$ and an order isomorphisms $g^i_{xy} : u^i_x \to u^i_y$ such that

(a) $f^i_{xy} \restriction N^i = id_{N^i}$, $g^i_{xy} \restriction u^i = id_{u^i}$ and $g^0_{xy}(x) = y$,

(b) for all $t \in u^i_x$ and $a \in \mathcal{B}_t \cup \mathcal{C}_t$, $f^i_{xy}(a) = f^{-1}_{g^i_{xy}(t)}(f_t(a))$,

(c) for all $s, t \in u^i_x$, $s < t$, $f^i_{xy} \restriction I^x_{st}$ is onto $I^y_{g^i_{xy}(s) g^i_{xy}(t)}$

(d) for all $a \in N^0_x$, $f(f^0_{xy}(a)) = f^1_{xy}(f(a))$.

4.2 Lemma. *Let $x, y \in X$, $x < y$.*

(i) N^i is F^a_κ-primary over S^i.

(ii) N^i_x is F^a_κ-primary over $N^i \cup S^i_x$.

(iii) $N^i \downarrow_{S^i} S^i_x \cup S^i_y$.

(iv) $N^i_x \downarrow_{N^i} N^i_y$.

(v) $I_{xy} \downarrow_{\mathcal{B}_x \cup \mathcal{C}_y} N^o_x \cup N^o_y$.

Proof. Immediate by (ii) in the choice of X and Lemma 3.2. □

4.3 Corollary. *Let $x, y \in X$, $x < y$.*

(i) If A, B and C are such that $A \downarrow_{N^0} N^0_x \cup N^0_y$, and $B \cup N^0_x \downarrow_{N^0 \cup A} N^0_y \cup C$ then

$$t(I_{xy}, \mathcal{B}_x \cup \mathcal{C}_y) \vdash t(I_{xy}, I_{xy} \cup N^o_x \cup N^0_y \cup A \cup B \cup C)$$

(ii) $t(I_{xy} \cup N^0_x \cup N^0_y, \emptyset)$ does not depend on x and y.

Proof. (i) By the first assumption on A and Lemma 4.2 (iii)

$$A \cup N^0 \downarrow_{S^0} \mathcal{B}_x \cup \mathcal{C}_y.$$

By the construction of $\mathbf{M}_0$, this implies

$$(A) \qquad A \cup N^0 \downarrow_{\mathcal{A}} \mathcal{B}_x \cup \mathcal{C}_y.$$

From the second assumption it follows easily that

$$(B) \qquad B \cup N^0_x \downarrow_{\mathcal{B}_x \cup N^0 \cup A} N^0_y \cup C$$

and

$$(C) \qquad C \cup N^0_y \downarrow_{\mathcal{C}_y \cup N^0 \cup A} N^0_x \cup B.$$

By Theorem 2.1 (iv), (A),(B) and (C) imply the claim.

(ii) By (iii) in the choice of X and Lemma 4.2 (iv), for all $x' < y'$, $f^0_{xx'} \cup f^0_{yy'}$ is an elementary map. So the claim follows from (A), (B) and (C) above and Theorem 2.1 (iv). □

For $x, y \in X$, $x < y$, let I^c_{xy} be some countable subset of I_{xy}.

4.4 Lemma. *Assume $x, y \in X$, $x < y$. Then there are $s \in u^1_x - u^1$ and $t \in u^1_y - u^1$ such that either*

(i) $s < t$ and $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is not orthogonal to $Av(I^c_{st}, I^c_{st} \cup \mathcal{B}_s \cup \mathcal{C}_t)$,

or

(ii) $t < s$ and $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is not orthogonal to $Av(I^c_{ts}, I^c_{ts} \cup \mathcal{B}_t \cup \mathcal{C}_s)$.

Proof. For a contradiction, we assume that such s and t do not exist.

Let

$\xi^0(x, y) = \{(s, t) |\ s < t$ and $s \in u^1_x - u^1,\ t \notin u^1_y - u^1$ or $t \in u^1_x - u^1,\ s \notin u^1_y - u^1\}$

$\xi^1(x, y) = \{(s, t) |\ s < t$ and $s \notin u^1_x - u^1,\ t \in u^1_y - u^1$ or $t \notin u^1_x - u^1,\ s \in u^1_y - u^1\}$

and

$\xi^2(x, y) = \{(s, t) |\ s < t$ and $s \in u^1_x - u^1,\ t \in u^1_y - u^1$ or $t \in u^1_x - u^1,\ s \in u^1_y - u^1\}$.

For $i \in \{0, 1, 2\}$, let

$S^i(x, y) = S(\xi) - (S^1_x \cup S^1_y \cup \bigcup_{j \geq i}\{I_{st} |\ (s, t) \in \xi^j(x, y)\})$

and

$R^i(x, y) = \{I_{st} |\ (s, t) \in \xi^i(x, y)\}$.

Now it is easy to see that $S^0(x, y) \downarrow_{S^1} S^1_x \cup S^1_y$. By Lemma 3.2 $N^1 \downarrow_{S^1} S^0(x, y) \cup S^1_x \cup S^1_y$. So

$$S^0(x, y) \downarrow_{N^1} S^1_x \cup S^1_y.$$

By Lemma 4.2 this implies

$$(A) \quad S^0(x, y) \downarrow_{N^1} N^1_x \cup N^1_y.$$

By the construction

$$(B) \quad R^0(x, y) \cup S^1_x \downarrow_{S^1 \cup S^0(x,y)} R^1(x, y) \cup S^1_y.$$

By Lemma 3.2

$$N^1_x \downarrow_{S^1_x} S^0(x, y) \cup R^0(x, y) \cup R^1(x, y) \cup S^1_y$$

and so

$$R^0(x, y) \cup N^1_x \downarrow_{S^1_x \cup S^0(x,y) \cup R^0(x,y)} R^1(x, y) \cup S^1_y.$$

By (B) this implies

$$(C) \quad R^0(x, y) \cup N^1_x \downarrow_{N^1 \cup S^0(x,y)} R^1(x, y) \cup S^1_y.$$

By Lemma 3.2 and (ii) in the choice of X,

$$N^1_y \downarrow_{S^1_y} S^0(x, y) \cup R^0(x, y) \cup R^1(x, y) \cup N^1_x$$

and so

$$R^1(x, y) \cup N^1_y \downarrow_{S^1_y \cup S^0(x,y) \cup R^1(x,y)} R^0(x, y) \cup N^1_x.$$

By (C) this implies

$$(D) \quad R^1(x, y) \cup N^1_y \downarrow_{N^1 \cup S^0(x,y)} R^0(x, y) \cup N^1_x.$$

Then by (A), (D) and Corollary 4.3 (i), $f(I_{xy})$ is indiscernible over $N_x^1 \cup N_y^1 \cup S^2(x,y)$.

By Lemma 3.2 and (ii) in the choice of X, we see that for all $(s,t) \in \xi^2(x,y)$, I_{st} is indiscernible over $N_x^1 \cup N_y^1 \cup S^2(x,y)$ and $(I_{st})_{(s,t)\in\xi^2(x,y)}$ is independent over $N_x^1 \cup N_y^1 \cup S^2(x,y)$.

For all $(u,v) \in \xi^2(x,y) \cup \{(x,y)\}$ we choose infinite $I^*_{uv} \subseteq I_{uv}$ of power $< \lambda$ such that

(i) for all $(u,v) \in \xi^2(x,y)$, if we write $B(u,v) = N_x^1 \cup N_y^1 \cup S^2(x,y) \cup I^*_{uv}$, then

$$I_{uv} - I^*_{uv} \downarrow_{B(u,v)} f(I^*_{xy}) \cup \bigcup\{I^*_{st} |\ (s,t) \in \xi^2(x,y),\ (s,t) \neq (u,v)\}.$$

(ii) $I^c_{xy} \subseteq I^*_{xy}$ and if we write $B(x,y) = N_x^1 \cup N_y^1 \cup S^2(x,y) \cup f(I^*_{xy})$, then

$$f(I_{xy} - I^*_{xy}) \downarrow_{B(x,y)} \bigcup\{I^*_{st} |\ (s,t) \in \xi^2(x,y)\}.$$

Because $|\xi^2(x,y)| < \lambda$, it is easy to see that such I^*_{uv} exist.

Since $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is orthogonal to $Av(I^c_{st}, I^c_{st} \cup \mathcal{B}_s \cup \mathcal{C}_s)$ for all $(s,t) \in \xi^2(x,y)$ we see that $I_{xy} - I^*_{xy}$ is indiscernible over $S(\xi)$. Because $|I_{xy} - I^*_{xy}| = \lambda$, this contradicts [Sh] Theorem IV 4.9 (2). □

If $s,t \in \xi$, then we write Θ_{st} for the set of all infinite J such that for some J', $J \subseteq J'$ and there is an automorphism g for which $g \restriction \mathcal{B}_s = f_s \restriction \mathcal{B}_s$, $g \restriction \mathcal{C}_t = f_t \restriction \mathcal{C}_t$ and $g(J') = I$.

4.5 Lemma. *Assume $x, y \in X$, $x < y$, $s \in u^1_x - u^1$, $t \in u^1_y - u^1$ and s and t are incomparable in ξ. If $J \in \Theta_{st}$, then $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is orthogonal to $Av(J, J \cup \mathcal{B}_s \cup \mathcal{C}_t)$. Also if $J \in \Theta_{ts}$, then $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is orthogonal to $Av(J, J \cup \mathcal{B}_t \cup \mathcal{C}_s)$.*

Proof. For a contradiction assume that $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is not orthogonal to $Av(J, J \cup \mathcal{B}_s \cup \mathcal{C}_t)$, the other case is similar. Then we can choose J so that in addition, $|J| = \omega$ and $J \subseteq M_1$.

By Theorem 2.1 (iv), J is indiscernible over $S(\xi)$. By [Sh] Theorem IV 4.14, $Av(J, M_1)$ is $F^a_{\kappa^+}$-isolated. Then we can find a model $D \subseteq M_1$ of power $\leq \kappa$ such that

(a) $f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y) \cup J \cup \mathcal{B}_s \cup \mathcal{C}_t \subseteq D$,

(b) $Av(f(I^c_{xy}), D)$ is not almost orthogonal to $Av(J, D)$,

(c) $Av(J, D) \vdash Av(J, M_1)$.

(For (c), notice that because D is a model, $t(a, D) \vdash stp(a, D)$.) But since $|D| < \lambda$ and $|f(I_{xy})| = \lambda$, it is easy to see that $Av(f(I^c_{xy}), D)$ is satisfied in M_1, a contradiction. □

Let $x, y \in X$ be such that $x < y$. By Lemma 4.4 we can find s_{xy} and t_{xy} such that there is $J \in \Theta_{s_{xy}t_{xy}} \cup \Theta_{t_{xy}s_{xy}}$ for which $Av(f(I^c_{xy}), f(I^c_{xy} \cup \mathcal{B}_x \cup \mathcal{C}_y))$ is not orthogonal to $Av(J, J \cup \mathcal{B}_{s_{xy}} \cup \mathcal{C}_{t_{xy}})$ or to $Av(J, J \cup \mathcal{B}_{t_{xy}} \cup \mathcal{C}_{s_{xy}})$. By Lemma 4.3 (ii) we can choose these so that for all y and y' from X, if $x < y$ and $x < y'$ then $s_{xy} = s_{xy'}$. We call this element just s_x. Similarly we can choose t_{xy} so that it does not depend on x $(x < y)$. We call this element t_y.

4.6 Lemma. *For all x and x' from X, s_x and $s_{x'}$ are comparable in ξ.*

Proof. By Lemma 4.5, for all $y \in X$, if $y > x$ and $y > x'$ then t_y is comparable to s_x and to $s_{x'}$. Since $|\{z \in \xi | \ z \leq s_x \ \vee \ z \leq s_{x'}\}| < \lambda$ and if $y \neq y'$ then $t_y \neq t_{y'}$, we can find $y \in X$ such that $s_x < t_y$ and $s_{x'} < t_y$, which implies the claim. □

4.7 Theorem. $M_0 \not\cong M_1$.

Proof. If $M_0 \cong M_1$ then by Lemma 4.6 we can find $Y \subseteq \xi$ of power λ such that for all $s, t \in Y$ if $s \neq t$ then either $s < t$ or $t < s$. Clearly this contradicts the fact that ξ is a λ^+, λ-tree. □

Together with Theorem 3.5, Theorem 4.7 implies Theorem 2.3, and so Theorem 1.1 is proved.

4.8 Remark. *As in [HT], we can see that Theorem 1.1 implies the following: Under the assumptions of Theorem 1.1, for every λ^+, λ-tree t there are models $M_i \models T$, $i < \lambda^+$, such that for all $i < j < \lambda^+$, $M_i \equiv_t^\lambda M_j$ and $M_i \not\cong M_j$.*

References.

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