FEW NON MINIMAL TYPES AND NON-STRUCTURE SH603

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Abstract. We deal with abstract elementary classes $\mathfrak{K}$ which has amalgamation in λ, categorical in λ and in λ^+. Our main result is that if ($\beth_\omega \le 2^\lambda < 2^{\lambda^+}$ and) the minimal types on members of K_λ are not dense (among non algebraic (complete) types over models in K_λ, extending our given model), <u>then</u> the number of models in K of cardinality λ^{++} is maximal. For this we deal with some claims in pcf. This improves a result in [Sh 576], it is essentially self-contained.

Saharon:
(a) in 3.4 proof: $|\mathscr{S}_*(M)| \ge \lim_\kappa(T)$, reference (but 3.4 contains a proof, 1.15 contains a statement)
(b) also in 1.12; there is fact in end of §1
(c) weaker coding: 3.4 refer to Definition ?

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Annotated Content

§0 Introductions

In [Sh 576] there was an important point where we used as assumption $\dot{I}(\lambda^{+3}, K) = 0$. This was fine for the purpose there, but is unsuitable in other frameworks, like [Sh 600]: we would like to analyze what occurs in higher cardinals, so our main aim here is to eliminate its use and add to our knowledge on non-structure.

The point was "the minimal triples in K^3_λ are dense" ([Sh 576, 3.30,pg.88,3.17t]). For this we assume we have a counterexample, and try to build many non-isomorphic models. Hence we get cases of amalgamation which are necessarily unique. Those "unique amalgamations" are normally too strong (even for first order superstable theories), but here they help us to prove positive theorems, controlling omitting types. So we try to build many models in λ^+ by omitting "types" over models of size λ, in a specific way where unique amalgamation holds. If this argument fails, we prove $\mathbf{C}^1_{\mathfrak{K},\lambda}$ has weak λ^+-coding (see [Sh 838]) and by it get $2^{\lambda^{++}}$ non-isomorphic models except when the weak diamond ideal on λ^+ is λ^{++}-saturated; this is done in §1. In §3 we work harder and by partition to cases relying on pcf theory we succeed to get the full result. We work also to get large IE (many models no one $\le_{\mathfrak{K}}$-embedding to another). The pcf lemmas (which are pure infinite combinatorics) are dealt with in §2.

Compared to its original [Sh 603] we:

(a) there was also another point left in [Sh 576, 4.2t], for the case $\lambda = \aleph_0$ only, this was filled in §4 but hence omitted as it was moved to citeSh:EF6

(b) we rely on [Sh 838] which correct, improves and simplify [Sh 576, §3]

(c) the main changes in this version is improving 1.8 to 1.11 and simplification of Definition 1.6.

The result of 3.3 is not meaningful outisde the general aim here; move to [Sh 838, §1]?
An aim of the present revision is to weaken the assumption "K is categorical in λ^+" to having an intermediate number of models in λ^+.

* * *

0.1 Definition. We say $\mathfrak{K} = (K, \le_{\mathfrak{K}})$ is an abstract elementary class, aec or a.e.c. in short, if ($\tau = \tau_K$ is a fixed vocabulary, K a class of τ-models (and Ax0 holds and) AxI-VI hold where:
*Ax*0: The holding of $M \in K, N \le_{\mathfrak{K}} M$ depends on N, M only up isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \le_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' then $N' \le_{\mathfrak{K}} M'$].

AxI: If $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

AxII: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ implies $M_0 \leq_{\mathfrak{K}} M_2$ and $M \leq_{\mathfrak{K}} M$ for $M \in K$.

AxIII: If λ is a regular cardinal, M_i (for $i < \lambda$) is $\leq_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup\limits_{i<\delta} M_i$) <u>then</u> $M_0 \leq_{\mathfrak{K}} \bigcup\limits_{i<\lambda} M_i$.

AxIV: If λ is a regular cardinal, M_i (for $i < \lambda$) is $\leq_{\mathfrak{K}}$-increasing continuous, $M_i \leq_{\mathfrak{K}} N$ for $i < \lambda$ <u>then</u> $\bigcup\limits_{i<\lambda} M_i \leq_{\mathfrak{K}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 0, 1$, <u>then</u> $M_0 \leq_{\mathfrak{K}} M_1$.

AxVI: $\mathrm{LS}(\mathfrak{K})$ exists[1], where $\mathrm{LS}(\mathfrak{K})$ is the minimal cardinal λ such that: if $A \subseteq N$ and $|A| \leq \lambda$ <u>then</u> for some $M \leq_{\mathfrak{K}} N$ we have $A \subseteq |M| \leq \lambda$ and we demand for simplicity $|\tau| \leq \lambda$.

<u>0.2 Notation</u>: 1) $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup\limits_{\mu<\lambda} K_\mu$.
See more in [Sh 576, §0] or [Sh 600, §0].

0.3 Definition. 1) For $\mu \geq \mathrm{LS}(\mathfrak{K})$ and $M \in K_\mu$ we define $\mathscr{S}(M)$ as $\{\mathrm{tp}(a, M, N) : M \leq_{\mathfrak{K}} N \in K_\mu \text{ and } a \in N\}$ where $\mathrm{tp}(a, M, N) = (M, N, a)/E_M$ where E_M is the transitive closure of E^{at}_M, and the two-place relation E^{at}_M is defined by:

$$\begin{aligned}(M, N_1, a_1) E^{\mathrm{at}}_M (M, N_2, a_2) \ \underline{\text{iff}} \text{ there is } N \in K_\mu \text{ and } \leq_{\mathfrak{K}}\text{-embeddings}\\ f_\ell : N_\ell \to N \text{ for } \ell = 1, 2 \text{ such that:}\\ f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).\end{aligned}$$

$$(\text{of course } M \leq_{\mathfrak{K}} N_1, M \leq_{\mathfrak{K}} N_2 \text{ and } a_1 \in N_1, a_2 \in N_2)$$

2) We say "a <u>realizes</u> p in N" if $a \in N, p \in \mathscr{S}(M)$ and for some $N' \in K_\mu$ we have $M \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N$ and $a \in N'$ and $p = \mathrm{tp}(a, M, N')$; so $M, N' \in K_\mu$ but possibly $N \notin K_\mu$.
3) We say "a_2 <u>strongly realizes</u> $(M, N^1, a^1)/E^{\mathrm{at}}_M$ in N" if for some N^2, a^2 we have

[1]We normally assume $M \in \mathfrak{K} \Rightarrow \|M\| \geq \mathrm{LS}(\mathfrak{K})$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{K})| \leq \mathrm{LS}(\mathfrak{K})$ which means just increasing $\mathrm{LS}(\mathfrak{K})$, but no real need.

$M \leq_{\mathfrak{K}} N^2 \leq_{\mathfrak{K}} N$ and $a_2 \in N^2$ and $(M, N^1, a^1)\, E^{\text{at}}_M\, (M, N^2, a^2)$.
(Note: if M_0 is an amalgamation base, see below, then the difference between realize and strongly realize disappears).
4) We say $M_0 \in \mathfrak{K}_\lambda$ is an amalgamation base if: for every $M_1, M_2 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.
5) We say $\mathfrak{K}$ is stable in λ if $\text{LS}(\mathfrak{K}) \leq \lambda$ and $M \in K_\lambda \Rightarrow |\mathscr{S}(M)| \leq \lambda$.
6) We say N is λ-universal over M if for every $M', M \leq_{\mathfrak{K}} M' \in K_\lambda$, there is a $\leq_{\mathfrak{K}}$-embedding of M' into N over M. If we omit λ we mean $\|N\|$.
7) We say N is (λ, κ)-brimmed over M if there is as $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$, such that $M = M_0, M_\kappa = N, M_i \in K_\lambda, M_{i+1}$ is an amalgamation base, M_{i+1} is universal over M_i. Replacing κ by $*$ means "for some $\kappa = \text{cf}(\kappa) \leq \lambda$". We omit "over M" to mean "for some $M \in K_\lambda$".
8) $K^3_\lambda = \{(M, N, a) : M \leq_{\mathfrak{K}} N, a \in N \backslash M \text{ and } M, N \in \mathfrak{K}_\lambda\}$, with the partial order $\leq$ defined by $(M, N, a) \leq (M', N', a')$ iff $a = a', M \leq_{\mathfrak{K}} M'$ and $N \leq_{\mathfrak{K}} N'$.
9) We say $(M, N, a) \in K^3_\lambda$ is minimal if $(M, N, a) \leq (M', N_\ell, a) \in K^3_\lambda$ for $\ell = 1, 2$ implies $\text{tp}(a, M', N_1) = \text{tp}(a, M', N_2)$. We say $p \in \mathscr{S}(M)$ is minimal if $p = \text{tp}(a, M, N)$ for some minimal triple (M, N, a) from K^3_λ, so $\lambda = \|M\|$.
10) We say $\mathfrak{K}$ has amalgamation in λ if every $M \in \mathfrak{K}_\lambda$ is an amalgamation base.

§1 Non-minimal triples and non-structure

We shall quote here [Sh 838] but in a black box nature.

1.1 Context.

(a) $\mathfrak{K}$ abstract elementary class with $\mathrm{LS}(\mathfrak{K}) \le \lambda$ and $K_\lambda \ne \emptyset$

(b) $\mathfrak{K}$ has amalgamation in λ.

Remark. Alternatively we can use a weaker context.

1.2 Definition. 1) For $x \in \{a, d\}$ we say $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$ <u>if</u>:

(a) $M_\ell \in K_\lambda$ for $\ell \le 3$

(b) $M_0 \le_{\mathfrak{K}} M_\ell$ for $\ell = 1, 2$

(c) if for $i \in \{1, 2\}$ we have $M^i_\ell \in K_\lambda$, for $\ell < 4$ and $M^i_0 \le_{\mathfrak{K}} M^i_\ell \le_{\mathfrak{K}} M^i_3$ for $i = 1, 2, \ell = 1, 2$ and $[x = d \Rightarrow M^i_1 \cap M^i_2 = M^i_0]$ and f^i_ℓ is an isomorphism from M_ℓ onto M^i_ℓ for $\ell < 3$ and $f^i_0 \subseteq f^i_1, f^i_0 \subseteq f^i_2$ <u>then</u> there are M'_3, f_3 such that $M^2_3 \le_{\mathfrak{K}} M'_3$ and f_3 is a $\le_{\mathfrak{K}}$-embedding of M^1_3 into M'_3 extending $(f^2_1 \circ (f^1_1)^{-1}) \cup (f^2_2 \circ (f^1_2)^{-1})$ i.e. $f_3 \circ f^1_1 = f^2_1$ & $f_3 \circ f^1_2 = f^2_2$

(d) $M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3 \in K_\lambda$ for $\ell = 1, 2$

(e) $x = d \Rightarrow M_1 \cap M_2 = M_0$.

2) We say $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2)$ if $\mathrm{UQ}^x_\lambda(M_0, M'_1, M'_2, M_3)$ for some M_3 and M'_1, M'_2 isomorphic to M_1, M_2 over M_0 respectively.
3) If we omit x, we mean $x = a$.
4) $K^{3,*}_\lambda$ is the family of triples $(M, N, a) \in K^3_\lambda$ such that there is no minimal triple above it.
5) $K^{2,*}_\lambda$ is the family $\{(M, N) : \text{for some } a, (M, N, a) \in K^{3,*}_\lambda\}$.
6) For $M \in K_\lambda$ let $\mathscr{S}_*(M) = \{p \in \mathscr{S}(M)\colon$ for some $(M, N, a) \in K^{3,*}_\lambda$ we have $p = \mathrm{tp}(a, M, N)\}$.
7) For $M \in K_{\lambda^+}$ let $\mathscr{S}_*(M) = \{\mathrm{tp}(a, M, N) : (M, N, a) \in K_{\lambda^+}$ and for some $M_0 \le_{\mathfrak{K}} N_0 \le_{\mathfrak{K}} N$ we have $M_0, N_0 \in K_\lambda, M_0 \le_{\mathfrak{K}} M$ and $(M_0, N_0, a) \in K^{3,*}_\lambda\}$.

The reader may find it helpful to look at the following example.

1.3 Example. Let K be the class of $M = (|M|, E^M), \|M\| \ge \lambda$ and E^M is an equivalence relation on $|M|$ and $\le_{\mathfrak{K}}$ be being a submodel. <u>Then</u> $\mathrm{UQ}_\lambda(M_0, M_1, M_2)$ <u>iff</u>

(a) $M_\ell \in K_\lambda$ for $\ell = 0, 1, 2$

(b) $M_0 \subseteq M_\ell$ for $\ell = 1, 2$

(c) if $aE^{M_1}c, bE^{M_2}d, b \in M_1 \backslash M_0, d \in M_2 \backslash M_0$ then $\neg(aE^{M_0}b)$

(d) for some $\ell \in \{1, 2\}$ for every $c \in M_\ell$ we have $(c/E^{M_\ell}) \cap M_0 \neq 0$.

1.4 Claim. *1) Symmetry: assuming $x \in \{a, d\}$ we have $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3) \Rightarrow \mathrm{UQ}^x_\lambda(M_0, M_2, M_1, M_3)$; we can also omit M_3.*
2) $\mathrm{UQ}^a_\lambda(M_0, M_1, M_2) \Rightarrow \mathrm{UQ}^d_\lambda(M_0, M_1, M_2)$ recalling M_0 is an amalgamation base (in $\mathfrak{K}_\lambda$) by clause (b) of 1.1.
3) $\mathrm{UQ}^a_\lambda(M_0, M_1, M_2, M_3)$ iff clauses (a), (b), (d), (e) of Definition 1.2(1),(2) holds and also $(c)^-$, i.e., clause (c) restricted to the case $M^1_\ell = M_\ell$ for $\ell \leq 3$.
4) If $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3), M_3 \leq_{\mathfrak{K}} M'_3 \in K_\lambda$ then $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M'_3)$; and also the inverse: if $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M'_3)$ and $M_1 \cup M_2 \subseteq M_3 \leq_{\mathfrak{K}} M'_3$ then $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$.
5) Assume $(M, N, a) \in K^3_\lambda$ and it is not minimal (even less) then $\neg\mathrm{UQ}(M, N, N)$.

Proof. 1),2) Trivial.
3) Chasing arrows, we should prove clause (c) of Definition 1.2(1). Assume we are given $\langle M^1_\ell : \ell < 4\rangle, \langle M^2_\ell : \ell < 4\rangle, \langle f^i_\ell : \ell < 3\rangle$ as there for $i = 1, 2$. First for $i = 1, 2$ apply clause $(c)^-$ to $\langle M^i_\ell : \ell < 4\rangle, \langle f^i_\ell : \ell < 3\rangle$. So there are N^i_3, f^i_3 such that: $M^i_3 \leq_{\mathfrak{K}} N^i_3 \in K_\lambda$, and f^i_3 a $\leq_{\mathfrak{K}}$-embedding of M_3 into N^i_3 extending $f^i_1 \cup f^i_2$. As $\mathfrak{K}$ has amalgamation in λ (by 1.1(b)) there are $N \in K_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $g^i : N^i \to N$ such that $g^1 \circ f^1_3 = g^2 \circ f^2_3$, so we are done.
4) Again by the amalgamation i.e., 1.1(b).
5) Let $N_0 = N, a_0 = a$.
We can find N_1, a_1 such that

⊛ $N \leq_{\mathfrak{K}} N_1 \in \mathfrak{K}_\lambda, a^* \in N_1 \backslash \{a\}$ and $a^* \neq a$
(this follows from non-minimality, and is all that we need).

Hence we can find N_2, f_1 such that: $N_1 \leq_{\mathfrak{K}_\lambda} N_2, f_1$ is a $\leq_{\mathfrak{K}}$-embedding of N into N_2 and $f_a(a) = a_1$. Clearly we have gotten two contradictory amalgamations, so we are done. $\square_{1.4}$

1.5 Claim. *1) transitivity: If $\mathrm{UQ}_\lambda(M_\ell, N_\ell, M_{\ell+1}, N_{\ell+1})$ for $\ell = 0, 1$ then $\mathrm{UQ}_\lambda(M_0, N_0, M_2, N_2)$.*
2) If $\theta = \mathrm{cf}(\theta) < \lambda^+$, and $\langle M_i : i \leq \theta\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $\langle N_i : i \leq \theta\rangle$ is $\leq_{\mathfrak{K}}$-increasing and $\mathrm{UQ}_\lambda(M_i, N_i, M_{i+1}, N_{i+1})$ for each $i < \theta$ then $\mathrm{UQ}_\lambda(M_0, N_0, M_\theta, N_\theta)$.
3) Assume:

(a) $\alpha, \beta < \lambda^+$
(b) $M_{i,j} \in K_\lambda$ *for* $i \leq \alpha, j \leq \alpha$
(c) $i_1 \leq i_2 \leq \alpha \ \& \ j_1 \leq j_2 \leq \beta \Rightarrow M_{i_1,j_1} \leq_{\mathfrak{K}} M_{i_2,j_2}$
(d) $\langle M_{i,j} : i \leq \alpha \rangle$ *is* $\leq_{\mathfrak{K}}$*- increasing continuous for each* $j \leq \beta$
(e) $\langle M_{i,j} : j \leq \beta \rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing continuous for each* $i \leq \alpha$
(f) $\mathrm{UQ}_\lambda(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ *for every* $i < \alpha, j < \beta$.

Then $\mathrm{UQ}_\lambda(M_{0,0}, M_{\alpha,0}, M_{0,\beta}, M_{\alpha,\beta})$.
4) If $\mathrm{UQ}_\lambda(M_0, M_1, M_2)$ *and* $M_0 \leq_{\mathfrak{K}} M'_1 \leq_{\mathfrak{K}} M_1$ *and* $M_0 \leq_{\mathfrak{K}} M'_2 \leq_{\mathfrak{K}} M_2$ *then* $\mathrm{UQ}_\lambda(M_0, M'_1, M'_2)$.
5) If $M \leq_{\mathfrak{K}} N_\ell$ *for* $\ell = 1, 2$ *and* N_1 *can be* $\leq_{\mathfrak{K}}$*-embedded into* N_2 *over* M*, then* $\mathrm{UQ}_\lambda(M, N_2, M')$ *implies* $\mathrm{UQ}_\lambda(M, N_1, M')$.

Proof. Chasing arrows (note: $\mathrm{UQ} = \mathrm{UQ}^a$ is easier than UQ^d, for UQ^d the parallel claim is not clear at this point, e.g. the straightforward proof of transitivity fails and we can construct a counterexample). $\square_{1.5}$

1.6 Definition. 1) $K^{\mathrm{dis}}_\lambda[\mathfrak{K}] = \{(M, \Gamma) : M \in K_\lambda, \Gamma \subseteq K_\lambda, |\Gamma| \leq \lambda$ satisfies $N \in \Gamma \Rightarrow M <_{\mathfrak{K}} N$ and $N_1 \neq N_2 \in \Gamma \Rightarrow N_1 \cap N_2 = M\}$.
The last demand $N_1 \cap N_2 = M$ is for technical reasons.
2) Let $(M_1, \Gamma_1) \leq^*_f (M_2, \Gamma_2)$ mean that

(a) $(M_\ell, \Gamma_\ell) \in \mathscr{K}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ for $\ell = 1, 2$
(b) f is a function with domain $\cup\{N : N \in \Gamma\}$
(c) $f \restriction M_1$ is a $\leq_{\mathfrak{K}}$-embedding of M_1 into M_2
(d) for every $N_1 \in \Gamma_2$ for some (unique) $N_2 \in \Gamma_2$ the function $f \restriction N_1$ is a $\leq_{\mathfrak{K}}$-embedding of N_1 into N_2.

3) $(M_1, \Gamma_1) \leq^* (M_2, \Gamma_2)$ means that $(M_1, \Gamma_1) \leq^*_f (M_2, \Gamma_2)$ for $f = \cup\{\mathrm{id}_N : N \in \Gamma_1\} \cup \{\mathrm{id}_M\}$.

1.7 Observation. 1) $\leq^*$ is a partial order of $\mathscr{K}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$.
2) $K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ is non-empty, e.g., $(M, \emptyset) \in \mathscr{K}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ when $M \in K_\lambda$.
3) If $\langle (M_i, \Gamma_i) : i < \delta \rangle$ is $\leq^*$-increasing then this sequence has a l.u.b. $(M_\delta, \Gamma_\delta) =: \cup\{(M_i, \Gamma_i) : i < \delta\}$ where $M_\delta = \cup\{M_i : i < \delta\}$ and $\Gamma = \{N$: for some $i < \delta$ and $N_i \in \Gamma_i$ letting $N_j \in \Gamma_j$ be the unique $N' \in \Gamma_j$ such that $N_i \leq_{\mathfrak{K}} N'$ (necessarily well defined we have $N = \cup\{N_j : j \in [i, \delta)\}$.

1.8 Claim. *Assume $2^\lambda < 2^{\lambda^+}$ or at least the definitional weak diamond; i.e., $\mathrm{DfWD}^+(\lambda^+)$, see [Sh:E35, 1.7]. If $(*)_\lambda$ or at least $(*)'_\lambda$ below holds (hence above some triple from K^3_λ there is no minimal one), then $\dot I(\lambda^+,K) \ge \mu_{\mathrm{wd}}(\lambda^+)$ where*

- $(*)_\lambda$ *for every $(M,\Gamma) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ for some M',Γ' we have $(M,\Gamma) <^* (M',\Gamma')$ or just*
- $(*)'_\lambda$ *for some $(M_*,\Gamma_*) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$, if $(M_0,\Gamma_0) \le^* (M,\Gamma)$ then for some M',Γ' we have $(M,\Gamma) <^* (M',\Gamma')$.*

Proof. Note that as $K^{\mathrm{dis}}_\lambda[\mathfrak{K}] \ne \emptyset$, see 1.7(2), clearly $(*)_\lambda \Rightarrow (*)'_\lambda$ hence we can assume $(*)'_\lambda$.
We choose by induction on $\alpha < \lambda$, $\langle (M_\eta,\Gamma_\eta,\Gamma^+_\eta) : \eta \in {}^\alpha 2\rangle$ such that:

- (a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$
- (b) $(M_\eta,\Gamma_\eta) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$
- (c) $N \in \Gamma_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$
- (d) $\nu \triangleleft \eta \Rightarrow (M_\nu,\Gamma_\nu) <^* (M_\eta,\Gamma_\eta)$
- (e) $(M_\eta,\Gamma^+_\eta) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$
- (f) $\Gamma_\eta \subseteq \Gamma^+_\eta$
- (g) $(M_\eta,\Gamma^+_\eta) \le^* (M_{\eta\char`^\langle 0\rangle},\Gamma_{\eta\char`^\langle 0\rangle})$
- (h) for some $N \in \Gamma^+_\eta$ we have $N \cong_{M_\eta} M_{\eta\char`^<1>}$.

There is no serious problem to carry the induction with Γ^+_η (for $\eta \in {}^\alpha 2$) chosen in the $(\alpha+1)$-th step. For $\alpha = 0$ let $(M_{<>},\Gamma_{<>})$ be the (M_*,Γ_*) from $(*)'_\lambda$ except that we rename the elements to make the relevant parts of clauses (a), (c) true. For α limit use 1.7(3) (part on lub). For $\alpha = \beta+1, \eta \in {}^\beta 2$, by $(*)'_\lambda$ we can find $(M_{\eta\char`^\langle 1\rangle},\Gamma_{\eta\char`^\langle 1\rangle})$ such that $(M_{\eta\char`^\langle 1\rangle},\Gamma_{\eta\char`^\langle 1\rangle}) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ and $(M_\eta,\Gamma_\eta) <^* (M_{\eta\char`^\langle 1\rangle},\Gamma_{\eta\char`^\langle 1\rangle})$.

By renaming without loss of generality the universe of $M_{\eta\char`^\langle 1\rangle}$ is some $\gamma_{\eta\char`^\langle 1\rangle} \in (\gamma_\eta,\lambda^+)$ and clause (c) holds. Let N_η be isomorphic to $M_{\eta\char`^\langle 1\rangle}$ over M_η with $N_\eta \backslash M_\eta$ disjoint to $\lambda^+ \cup \bigcup\{|N| : N \in \Gamma_\eta\}$ and let $\Gamma^+_\eta = \Gamma_\eta \cup \{N_\eta\}$, so $(M_\eta,\Gamma^+_\eta) \in \mathscr{T}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ is disjoint, now apply to it $(*)'_\lambda$ to get $(M_{\eta\char`^\langle 0\rangle},\Gamma_{\eta\char`^\langle 0\rangle})$. Why does clause (h) hold? By the choice of N_η. So $M_\eta,\Gamma_\eta,\Gamma^+_\eta$ $(\eta \in {}^{\lambda^+}2)$ are defined.
Note: if $\eta\char`^\langle 0\rangle \triangleleft \nu \in {}^{\lambda^+>}2$, then $M_{\eta\char`^<1>}$ is not $\le_{\mathfrak{K}}$-embeddable into M_ν over M_η (by clause (g) + (h) because by 1.4(5) + 1.5(5) and clause (i) we have $\neg\, \mathrm{UQ}(M_\eta,M_{\eta\char`^\langle 1\rangle},N_\eta))$.
By [Sh 838, 1.4] we get the desired conclusion (really, usually also on IE). $\square_{1.8}$

We in 1.8 - 1.11 we investigate the non-structure conclusion of "there is no maximal member of $(\mathscr{T}^{\mathrm{dis}}_\lambda(\mathfrak{K}),<^*)$".

1.9 Claim. *1) An equivalent condition for $(*)_\lambda$ from 1.8 is (respectively):*

$(**)_\lambda$ *for every $M \le_{\mathfrak{K}} N$ from K_λ for some M' we have $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$.*

2) Also, the condition $(*)'_\lambda$ from 1.8 is equivalent to:

$(**)'_\lambda$ *for some $M_0 \in K_\lambda$ if $M_0 \le_{\mathfrak{K}} M \le_{\mathfrak{K}} N \in K_\lambda$ then for some M', $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$.*

3) We have $(*)_\lambda \Rightarrow (*)'_\lambda$ if $\mathfrak{K}$ is categorical in λ __then__ $(*)_\lambda \Leftrightarrow (*)'_\lambda$.

Proof. 1),2) For any $(M, \Gamma) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$, by "$\mathfrak{K}$ has amalgamation in λ and $\mathrm{LS}(\mathfrak{K}) \le \lambda$" (and properties of abstract elementary classes) there are $N^*, \langle f_N : N \in \Gamma\rangle$ such that:

(a) $M \le_{\mathfrak{K}} N^* \in K_\lambda$
(b) for $N \in \Gamma, f_N$ is a $\le_{\mathfrak{K}}$-embedding of N into N^* over M.

This shows $(**)_\lambda \Rightarrow (*)_\lambda$ and also $(**)'_\lambda \Rightarrow (*)'_\lambda$.
The other direction is deduced by applying $(*)_\lambda$ (or $(*)'_\lambda$) to $(M, \{N\})$.
3) Should be clear. $\square_{1.9}$

We continue 1.8

1.10 Claim. $\dot{I}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ *__if__:*

$\boxtimes_0$ $2^\lambda < 2^{\lambda^+}$ *or at least the definitional weak diamond for λ^+*
$\boxtimes_1$ $(M_*, \Gamma_*) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$
$\boxtimes_2$ *if $(M_*, \Gamma_*) \le^* (M, \Gamma)$ then for some M', Γ' we have $(M, \Gamma) <^* (M', \Gamma')$*
$\boxtimes_3$ $(M_*, \Gamma_*) \le^* (M_1, \Gamma_1) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ *__then__ we can find (M_2, Γ_2) such that*
(a) $(M_1, \Gamma_1) \le^* (M_2, \Gamma_2) \in K^{\mathrm{dis}}_\lambda[\mathfrak{K}]$
(b) *if $(M_2, \Gamma_2) \le^* (M^\ell, \Gamma^\ell)$ for $\ell = 1, 2$ then we can find (M^3, Γ^ℓ) for $\ell = 3, 4$ such that $(M^\ell, \Gamma^\ell) \le^* (M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1, 2$ and M^3, M^4 are isomorphic over M_1.*

Remark. 1) This corresponds to case A in the proof of [Sh:E45, 1.7](?).
2) The gain over 1.8 is not large.

Proof. By induction on $\alpha < \lambda$, we choose $\langle (M_\eta, \Gamma_\eta, \Gamma^+_\eta) : \eta \in {}^\alpha 2\rangle$ such that:

- (a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$
- (b) $(M_\eta, \Gamma_\eta) \in K^{\rm dis}_\lambda[\mathfrak{K}]$
- (c) $N \in \Gamma_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$
- (d) $\nu \triangleleft \eta \Rightarrow (M_\nu, \Gamma_\nu) <^* (M_\eta, \Gamma_\eta)$
- (e) $(M_{<>}, \Gamma_{<>}) = (M_*, \Gamma_*)$ and $(M_\eta, \Gamma^+_\eta) \in K^{\rm dis}_\lambda[\mathfrak{K}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$
- (f) $\Gamma_\eta \subseteq \Gamma^+_\eta$
- (g) $(M_\eta, \Gamma^+_\eta) \le^* (M_{\eta\hat{}\langle 0\rangle}, \Gamma_{\eta\hat{}\langle 0\rangle})$
- (h) for some $N \in \Gamma^+_\eta$ we have $N \cong_{M_\eta} M_{\eta\hat{}<1>}$
- (i) if $\eta \in {}^{\lambda^+>}2, \nu \in \{\eta\hat{}<0>, \eta\hat{}<1>\}$ and $(M_\nu, \Gamma_\nu) \le^* (M^\ell, \Gamma^\ell) \in K^{\rm dis}_\lambda[\mathfrak{K}]$ for $\ell = 1, 2$ then we can find $(M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1, 2$ such that $(M^\ell, \Gamma^\ell) \le (M^{\ell+2}, \Gamma^{\ell+2})$ and M^3, M^4 are isomorphic over M_η.

There is no problem to carry the induction, to guarantee (i) use assumption $\boxtimes_3$. Clearly, by clauses (g) + (h), as in the proof of 1.8

- $\otimes_1$ if $\eta \in {}^{\lambda^+>}2$ and $(M_{\eta\hat{}<\ell>}, \Gamma_{\eta\hat{}<\ell>}) \le^* (M_\ell, \Gamma_\ell) \in K^{\rm dis}_\lambda[\mathfrak{K}]$ for $\ell = 0, 1$ then we cannot find $(M, \Gamma) \in \mathscr{T}^{\rm dis}_\lambda[\mathfrak{K}]$ and functions f_0, f_1 such that $(M, \Gamma) \le^*_{f_\ell} (M_\ell, \Gamma_\ell)$ for $\ell = 0, 1$ and $f_0 \restriction M = f_1 \restriction M_1$.

Now we apply [Sh 838, 1.4K], we have to check it assumption.

The main point is proving $(*)$ there, a stronger version of $\otimes_1$ above which says

- $\circledast_2$ the following is impossible
  - (α) $\alpha < \beta < \lambda^+$
  - (β) $\eta_1, \eta_2 \in {}^\alpha 2$
  - (γ) $\eta_1\hat{}<0> \triangleleft \nu_1 \in {}^\beta 2$ and $\eta_1\hat{}<1> \triangleleft \rho_1 \in {}^\beta 2$
  - (δ) $\eta_2\hat{}<0> \triangleleft \nu_2 \in {}^\beta 2$ and $\eta_2\hat{}<0> \triangleleft \rho_2 \in {}^\beta 2$
  - (ε) f is an isomorphism from M_{ν_1} onto M_{ν_2} mapping M_{η_1} onto M_{η_2}
  - (ζ) g is an isomorphism from M_{ρ_1} onto M_{ρ_2} mapping M_{η_1} onto M_{η_1}
  - (η) $g \restriction M_{\eta_1} = f \restriction M_{\eta_1}$.

Proof of $\circledast_2$. We can find $\Gamma^*_{\nu_2}, \Gamma^*_{\rho_2}$ such that

(i) $(M_{\nu_2}, \Gamma_{\nu_2}) \le^* (M_{\nu_2}, \Gamma^*_{\nu_2}) \in K^{\rm dis}_\lambda[\mathfrak{K}]$

(ii) $(M_{\rho_2}, \Gamma_{\rho_2}) \le^* (M_{\rho_2}, \Gamma^*_{\rho_2}) \in K^{\rm dis}_\lambda[\mathfrak{K}]$

(iii) if $N \in \Gamma_{\nu_1}$ then f can be extended to an isomorphism from N onto some $N' \in \Gamma^*_{\nu_2}$

(iv) if $N \in \Gamma_{\rho_1}$ then g can be extended to an isomorphism from N onto some $N' \in \Gamma^*_{\rho_2}$.

So $(M_{\eta_2\hat{\ }<1>}, \Gamma_{\eta_2\hat{\ }<1>}) \le^* (M_{\nu_2}, \Gamma_{\nu_2}) \le^* (M_{\nu_2}, \Gamma^*_{\nu_2})$ and $(M_{\eta_2 \frown <0>}, \Gamma_{\eta_2 \frown 94 <1>}) \le^* (M_{\rho_2}, \Gamma_{\rho_2}) \le^* (M_{\rho_2}, \Gamma^*_{\rho_2})$.
By applying clause (i) with $\eta_2, \eta_2\hat{\ } < 0 >, (M_{\eta_2\hat{\ }<0>}, \Gamma_{\eta_2\hat{\ }<0>}), (M_{\nu_2}, \Gamma^*_{\nu_2}), (M_{\rho_2}, \Gamma^*_{\rho_2})$ here standing for $\eta, \nu, (M^1, \Gamma^1), (M^2, \Gamma^2)$ there we can find (N^ℓ, Γ^ℓ_*) for $\ell = 1, 2$ and h such that

(v) $(M_{\nu_2}, \Gamma^*_{\nu_2}) \le^* (N^1, \Gamma^1_*)$

(vi) $(M_{\rho_2}, \Gamma^*_{\rho_2}) \le^* (N^2, \Gamma^2_*)$

(vii) h is an isomorphism from N^1 onto N^2 over M_{η_2}.

But now the mapping $h \circ f$ and g contradict the choice of $(M_{\eta_1\hat{\ }<0>}, \Gamma_{\eta_1\hat{\ }<0>}), (M_{\eta_1\hat{\ }<1>}, \Gamma_{\eta_1\hat{\ }<1>})$ that is $\circledast_1$ above.

Having proved $\circledast_2$ we are done. $\square_{1.10}$

1.11 Conclusion. If $2^\lambda < 2^{\lambda^+}$ (or at least the definitional weak diamond for λ^+) and $(**)'_\lambda$ below <u>then</u> $\dot I(\lambda^+, K) = 2^{\lambda^+}$, where

$(**)'_\lambda$ for some $M_0 \in K_\lambda$ if $M_0 \le_{\mathfrak{K}} M \le_{\mathfrak{K}} N \in K_\lambda$ then for some M' we have $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$.

Proof. Clearly $\boxtimes_0 + \boxtimes_1$ from 1.10 holds, and also $\boxtimes_2$ holds by our assumption $(**)'_\lambda$ and Claim 1.9(2). If $\boxtimes_3$ holds too by 1.10 we are done so assume that it fails for $M, (M_1, \Gamma_1)$. Now we define (M_η, Γ_η) as the proof of 1.8 except that:

$(a) - (d)$ as there

$(e)'$ we choose $(M_{<>}, \Gamma_{<>}) = (M_0, \emptyset)$

$(f)'$ for each η, for no $(M^1, \Gamma^1), (M^2, \Gamma^2)$ do we have:

(i) $(M_{\eta\hat{\ }<0>}, \Gamma_{\eta\hat{\ }<0>}) \le^* (M^1, \Gamma^1)$

(ii) $(M_{\eta\hat{\ }<1>}, \Gamma_{\eta\hat{\ }<1>}) \le^* (M^2, \Gamma^2)$

(iii) M^1, M^2 are isomorphic over M.

So we can carry the definition as there because we assume that $\boxtimes_3$ of 1.8 fail. $\langle M_\eta : \eta \in {}^{\lambda^+}2\rangle$ are pairwise non-isomorphic over M (hence over $M_{<>}$) and this suffices. $\square_{1.11}$

1.12 Claim. *Assume*

(a) $(**)'_\lambda$ *of 1.11 fails (equivalently 1.9)*

(b) $M \in K_\lambda \Rightarrow |\mathscr{S}_*(M)| > \lambda^+$ *(follows from "above $(M,N,a) \in K^3_\lambda$ there is no minimal triple" $+2^\lambda > \lambda^+$ see [Sh:E46, xx?])*

(c) K *is categorical in* λ

(d) K *is categorical in* λ^+

(e) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.

Then $\dot I(\lambda^{++},K) \ge \mu_{\text{wd}}(\lambda^{++},2^{\lambda^+})$, *(which is* $= 2^{\lambda^{++}}$ *when* $\lambda \ge \beth_\omega$*) except possibly when*

$\circledast_\lambda$ $\text{WDmId}(\lambda^+)$ *is* λ^{++}*-saturated (normal ideal on* λ^+*).*

Remark. (See [Sh 838, §4]) 1) We may omit the model theoretic assumption (d) in 1.12 if we strengthen the set theoretic assumptions, e.g.

$(*)_1$ for some stationary $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ we have $S \in \check I[\lambda]$ but $S \notin \text{WDmId}(\lambda^{++})$.

2) Note that: if $\lambda = \lambda^{<\lambda}$ and $\mathbf{V} = \mathbf{V}^{\mathbb{Q}}, \mathbb{Q}$ is adding λ^+-Cohen set (1.1 and) the minimal types are not dense then $\dot I(\lambda^+,K) = 2^{\lambda^+}$.
3) If in part (2), if we omit 1.1(b), the amalgamation, but demand "no maximal model" [??]. However, the minimality may hold for uninteresting reasons.

Proof. This is done in [Sh 838, e.1].

1.13 Observation. [??] Under the assumptions of 1.12.
Assume

(a) $\langle M_\alpha : \alpha < \lambda^+\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous, $M = \cup\{M_\alpha : \alpha < \lambda^+\} \in K_{\lambda^+}$

(b) $M_{\alpha(0)} \le_{\mathfrak{K}_\lambda} N_0$, $\text{tp}(a_0, M_{\alpha(0)}, N_0) \in \mathscr{S}_*(M_{\alpha(0)})$.

<u>Then</u> we can find $\alpha(1), a_1, N_{\alpha(1)}, f$ such that

- (α) $\alpha(0) \le \alpha(1) < \lambda^+, M_{\alpha(1)} \le_{\mathfrak{K}_\lambda} N_1$,
- (β) f is a $\le_{\mathfrak{K}}$-embedding of N_0 into N_1
- (γ) $f(a_0) = a_1$
- (δ) $\mathrm{tp}(a_1, M_{\alpha(1)}, N_1)$ is not realized in M.

<u>Question</u>: Can we find $\alpha(1), a_{1,\varepsilon}, N_{\alpha(1),\varepsilon}, f_\varepsilon$ for $\varepsilon < 2^\lambda$ such that each $(a_{1,\varepsilon}, N_{\alpha(1),\varepsilon}, f_\varepsilon)$ is as above and $\langle \mathrm{tp}(a_{1,\varepsilon}, M_{\alpha(1)}, N_{\alpha(1),\varepsilon}) : \varepsilon < 2^\lambda \rangle$ are pairwise distinct? [??]

1.14 Remark. 1) We can get more abstract results.
2) Note $\neg(*)_\lambda$ of 1.12 is a "light" assumption, in fact, e.g. its negation has high consistency strength.

1.15 Fact. Assume $\mathfrak{K}$ is an abstract elementary class with amalgamation in λ, and above $(M, N, a) \in K^3_\lambda$ there is no minimal pair and K^3_λ has the weak extension property.
1) Assume $\mathscr{T}$ is a tree with $\delta < \lambda^+$ levels and $\le \lambda$ nodes[2]. <u>Then</u> we can find $(M^*, N_\eta, a) \in K^3_\lambda$ above (M, N, a) for $\eta \in \lim_\delta(\mathscr{T})$ such that $\mathrm{tp}(a, M^*, N_\eta)$ for $\eta \in \lim_\delta(T)$ are pairwise distinct so $|\mathscr{S}_*(M)| \ge |\lim_\delta(\mathscr{T})|$. We can add "$(M^*, N_\eta, a)$ is reduced", (see [Sh 576]).
2) If $M \in K_\lambda$ is universal <u>then</u> $\mathscr{S}_*(M) \ge \sup\{\lim_\delta(\mathscr{T}) : \mathscr{T}$ a tree with $\le \lambda$ nodes and δ levels$\}$.

Proof. 1) Straight (or see the proof of 3.4(1)).
2) As for any $N \in K_\lambda$ there is a model $N' \le_{\mathfrak{K}} M$ isomorphic to N, now $p \mapsto p \restriction N'$ is a function from $\mathscr{S}(M)$ onto $\mathscr{S}(N')$ by [Sh 576, §2] hence $|\mathscr{S}_*(M)| + \lambda \ge |\mathscr{S}(N')| = |\mathscr{S}(N)|$. Now use part (1). $\square_{1.15}$

[2]If $2^\lambda > \lambda^+$ then for some such T and $\delta, \lambda^+ < \lim_\delta(T)$; as in 3.4

§2 Remarks in pcf

The following will provide us a useful division into cases (it is from pcf theory; on $\mu_{\rm wd}(\lambda)$ see [Sh 576, 1.1t]), we can replace λ^+ by regular λ such that $2^\theta = 2^{<\lambda} < 2^\lambda$ for some θ).

2.1 Fact. Assume $2^\lambda < 2^{\lambda^+}$.

Then one of the following cases occurs:

$(A)_\lambda$ we can find μ such that letting $\chi^* = 2^{\lambda^+}$

- (α) $\lambda^+ < \mu \le 2^\lambda$ and $\mathrm{cf}(\mu) = \lambda^+$
- (β) $\mathrm{pp}(\mu) = \chi^*$, moreover $\mathrm{pp}(\mu) =^+ \chi^*$ and $\chi^* > 2^\lambda$
- (γ) $(\forall \mu')(\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \rightarrow \ \mathrm{pp}(\mu') < \mu)$ hence $\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \Rightarrow \ \mathrm{pp}_{\lambda^+}(\mu') < \mu$
- (δ) for every regular cardinal χ in the interval $(\mu, \chi^*]$ there is an increasing sequence $\langle \lambda_i : i < \lambda^+ \rangle$ of regular cardinals $> \lambda^+$ with limit μ such that $\chi = \ \mathrm{tcf}\left(\prod\limits_{i<\lambda^+} \lambda_i / J^{\rm bd}_{\lambda^+}\right)$
- (ε) for some regular $\kappa \le \lambda$, for any $\mu' < \mu$ there is a tree T with $\le \lambda$ nodes, κ levels and $|\lim_\kappa(T)| \ge \mu'$ (in fact e.g. $\kappa = \ \mathrm{Min}\{\kappa : 2^\kappa \ge \mu\}$ is appropriate; without loss of generality $T \subseteq {}^{\kappa >}\lambda$; we can get, of course, a tree T with $\mathrm{cf}(\kappa)$ levels, $\le \lambda$ nodes and $|\lim_{\mathrm{cf}(\kappa)}(T)| \ge \mu'$).

$(B)_\lambda$ for some μ, χ^* we have: clauses $(\alpha) - (\varepsilon)$ from above (so $2^\lambda < \chi^*$) and

- (ζ) there is $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ such that: $T_\zeta \subseteq {}^{\lambda^+ >}2$ a tree, of cardinality $\le \lambda^+$ and ${}^{\lambda^+}2 = \bigcup\limits_{\zeta<\chi^*} \lim_{\lambda^+}(T_\zeta)$ and $\chi^* < 2^{\lambda^+}$
- (η) $2^\lambda < \chi^* < \mu_{\rm wd}(\lambda^+, 2^\lambda)$ (but $< \mu_{\rm wd}(\lambda^+, 2^\lambda)$ is not used here, see [Sh 576, Definition 1.1t](5))
- (θ) for some $\zeta < \chi^*$ we have $\lim_{\lambda^+}(T_\zeta) \notin \mathrm{WDmTId}(\lambda^+)$, not used here
- (ι) if there is a normal λ^{++}-saturated ideal on λ^+, e.g. the ideal $\mathrm{WDmId}(\lambda^+)$ is, then $2^{\lambda^+} = \lambda^{++}$ (so as $2^\lambda < 2^{\lambda^+}$, necessarily $2^\lambda = \lambda^+$)
- (κ) $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$, equivalently $\chi^* = \sup\{\mathrm{pp}(\chi) : \chi \le 2^\lambda, \aleph_1 \le \mathrm{cf}(\chi) \le \lambda^+ < \chi\}$ by [Sh:g, Ch.II,5.4]

$(C)_\lambda$ letting $\chi^* = 2^\lambda$ we have $(\zeta), (\eta), (\theta), (\iota), (\kappa)$ of clause (B) and

(λ) for no $\mu \in (\lambda^+, 2^\lambda]$ do we have $\mathrm{cf}(\mu) \le \lambda^+, \mathrm{pp}(\mu) > 2^\lambda$ equivalently $2^\lambda > \lambda^+ \Rightarrow \ \mathrm{cf}([2^\lambda]^{\lambda^+}, \subseteq) = 2^\lambda$ hence (see the proof) $\mu_{\mathrm{wd}}(\lambda^+, 2^\lambda) = 2^{\lambda^+}$ except (maybe) when $\lambda < \beth_\omega$ and there is $\mathscr{A} \subseteq [\mu_{\mathrm{wd}}(\lambda^+, 2^\lambda)]^\lambda$ such that $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| = \aleph_0$.

Remark. Remember that

$$\mathrm{cov}(\chi, \mu, \theta, \sigma) = \chi + \ \mathrm{Min}\big\{|\mathscr{P}| : \mathscr{P} \subseteq [\chi]^{<\mu} \text{ and every member of } [\chi]^{<\theta} \text{ is included in the union of } < \sigma \text{ members of } \mathscr{P}\big\}.$$

Proof. This is related to [Sh:g, II,5.11]; we assume basic knowledge of pcf (or a readiness to believe). Note that if $2^\lambda > \lambda^+$ then $\mathrm{cf}([2^\lambda]^{\le \lambda^+}, \subseteq) = 2^\lambda \Leftrightarrow \ \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_0) = 2^\lambda$ and $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_0) \ge \ \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \theta) = 2^\lambda$ for $\theta \in [\aleph_0, \lambda]$.

<u>Possibility 1</u>: $\chi^* =: \ \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) = 2^\lambda$.

We shall show that case (C) holds.
Now by the definition of cov, clause (ζ) is obvious, as well as (κ). As on the one hand by [Sh:f, AP,1.16 + 1.19] we have $\big(\mu_{\mathrm{wd}}(\lambda^+, 2^\lambda)\big)^{\aleph_0} = 2^{\lambda^+} > 2^\lambda = \chi^*$ and on the other hand $(\chi^*)^{\aleph_0} = (2^\lambda)^{\aleph_0} = 2^\lambda = \chi^*$ necessarily $\chi^* < \mu_{\mathrm{wd}}(\lambda^+, 2^\lambda)$ so clause (η) follows; now clause (θ) follows from clause (ζ) as $\mathrm{WDmTId}(\lambda^+)$ is $(2^\lambda)^+$-complete by [Sh 576, 1.2t](5) and we have chosen $\chi^* = 2^\lambda$. Now if $2^{\lambda^+} > \lambda^{++}$, (so $2^{\lambda^+} \ge \lambda^{+3}$), then for some $\zeta < \chi^*$, $\mathscr{T}_\zeta$ is (a tree with $\le \lambda^+$ nodes, λ^+ levels and) at least λ^{+3} λ^+-branches which is well known (see e.g. [J]) to imply "no normal ideal on λ^+ is λ^{++}-saturated"; so we got clause (ι). As for (λ) the definition of χ^* and the assumption $\chi^* = 2^\lambda$ we have the first two phrases, as for $\mu_{\mathrm{wd}}(\lambda^+, 2^\lambda) = 2^{\lambda^+}$ by [Sh:f, AP1.14,pg.956 + 1.16,pg.958] there is $\mathscr{A}$ as mentioned in (λ) and by [Sh 460] we get $\lambda < \beth_\omega$. The "equivalently" holds as $(2^\lambda)^{\aleph_0} = 2^\lambda$.

<u>Possibility 2</u>: $\chi^* =: \ \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) > 2^\lambda$.

Let $\mu = \ \mathrm{Min}\{\mu : \mathrm{cf}(\mu) \le \lambda^+, \lambda^+ < \mu \le 2^\lambda$ and $\mathrm{pp}(\mu) = \chi^*\}$; clearly we may replace $\mathrm{pp}(\mu) = \chi^*$ by $\mathrm{pp}(\mu) =^+ \chi^*$. We know by [Sh:g, II,5.4] that μ exists and (by [Sh:g, II,2.3](2)) clause (γ) holds, also $2^\lambda < \ \mathrm{pp}(\mu) \le \mu^{\mathrm{cf}(\mu)}$ hence $\mathrm{cf}(\mu) = \lambda^+$.

So clauses $(\alpha), (\beta), (\gamma)$ hold (of course, for clause (β) use [Sh:g, Ch.II,5.4](2)), and by (γ) + [Sh:g, VIII,§1] also clause (δ) holds.

For clause (ε) let $\Upsilon = \mathrm{Min}\{\Upsilon : 2^\Upsilon \geq \mu\}$, clearly $\alpha < \Upsilon \Rightarrow 2^{|\alpha|} < \mu$ and $\Upsilon \leq \lambda$ (as $2^\lambda \geq \mu$) hence $\mathrm{cf}(\Upsilon) \leq \Upsilon \leq \lambda < \lambda^+ = \mathrm{cf}(\mu)$ hence $2^{<\Upsilon} < \mu$. Now we shall first prove

$(*)$ there[3] is a tree with λ^+ nodes, $\mathrm{cf}(\Upsilon)$ levels and $\geq \mu\, \Upsilon$-branches.

Why $(*)$ holds? Otherwise we shall get contradiction to the claim 2.3 below with $\sigma, \kappa, \theta_0, \theta_1, \mu, \chi$ there standing for $\mathrm{cf}(\Upsilon), \lambda^+, \lambda^+, 2^{<\Upsilon}, \mu, (2^\lambda)^+$ here and T^* defined below; let us check the conditions there:

Clause (a): It says $\mathrm{cf}(\Upsilon) < \lambda^+ = \mathrm{cf}(\mu) \leq \lambda^+ \leq 2^{<\Upsilon} < \mu$ which is readily checked except the inequality $\lambda^+ \leq 2^{<\Upsilon}$ but if it fails we immediately get more than required.

Clause (b): This is clause (γ) of (A) which we have proved.

Clause (c): The tree $\mathscr{T}^*$ is $({}^{\Upsilon>}2, \triangleleft)$ restricted to an unbounded set of levels of order type $\mathrm{cf}(\Upsilon)$.

Clause (d): Let $\theta_2 =: \mathrm{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \mathrm{cf}(\Upsilon)^+)$.
So the statement we have to prove is $\mathrm{pp}(\mu) \geq \chi = \mathrm{cf}(\chi) > \theta_2^{\mathrm{cf}(\Upsilon)}$. Now $\mathrm{pp}(\mu) \geq \chi$ holds by the choice of μ and $\chi = \mathrm{cf}(\chi)$ as $\chi = (2^\lambda)^+$. For the last inequality, by [Sh:g, Ch.II,5.4] and the choice of μ, as we have shown $2^{<\Upsilon} < \mu$ we know $\theta_2 < \mu$, but $\mu \leq 2^\lambda$ so $\theta_2^{\mathrm{cf}(\Upsilon)} \leq (2^\lambda)^{\mathrm{cf}(\Upsilon)} \leq (2^\lambda)^\Upsilon \leq (2^\lambda)^\lambda = 2^\lambda < \chi$.

Clause (e): Trivial by the choice of θ_2.

Clause (f): So κ^* is $\mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+)$ which means $\mathrm{cov}((\lambda^+)^{\mathrm{cf}(\Upsilon)}, \lambda^+, \lambda^+, \mathrm{cf}(\Upsilon)^+)$ but $\mathrm{cf}(\Upsilon) \leq \lambda$ so this number is $\leq (\lambda^+)^\lambda = 2^\lambda < (2^\lambda)^+$ which means $\kappa^* < \chi$.

So we have verified clauses $(a) - (f)$ of 2.3 hence its conclusion holds, but this gives $(*)$, i.e., the desired conclusion in clause (ε) of Case A in 2.1; well not exactly, it gives only $|\mathscr{T}^*| \leq \lambda^+$, so $\mathscr{T}^* = \bigcup\limits_{i<\lambda^+} \mathscr{T}_i, \mathscr{T}_i$ increases continuously with each $\mathscr{T}_i$ of cardinality $\leq \lambda$, so for every $\mu' < \mu$ for some i we have $|\lim_{\mathrm{cf}(\Upsilon)}(T_i)| \geq \mu'$, so we have finished proving clause (ε). Together we have, under possibility (2), clauses $(\alpha) - (\varepsilon)$ there.

Subpossibility 2a: $\chi^* < 2^{\lambda^+}$.

[3]the less easy point is when $\mathrm{cf}(\Upsilon) = \aleph_0$, otherwise we can get the conclusion differently (by [Sh:g, II,5.4]), so 2.1(A) suffice

We shall prove $(B)_\lambda$, so we are left with proving clauses $(\zeta)-(\kappa)$ when $\chi^* < 2^{\lambda^+}$. By the choice of χ^*, easily clause (ζ) (in Case B of 2.1) holds. In clause (η), "$2^\lambda < \chi^*$" holds as we are in possibility 2.

Also as $\mathrm{pp}(\mu) = \chi^*$ and $\mathrm{cf}(\mu) = \lambda^+$ by the choice of μ necessarily (by transitivity of pcf, i.e., [Sh:g, Ch.II,2.3](2)) $\mathrm{cf}(\chi^*) > \lambda^+$ but $\mu > \lambda^+$. Easily $\chi \le \chi^* \wedge \mathrm{cf}(\chi) \le \lambda^+ \Rightarrow \mathrm{pp}(\chi) \le \chi^*$ hence $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ by [Sh:g, Ch.II,5.4], which gives clause (κ). Now let $\mathscr{A} \subseteq [\chi^*]^{\lambda^+}$ exemplify $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ and let $\mathscr{A}' = \{B : B$ is an infinite countable subset of some $A \in \mathscr{A}\}$. So $\mathscr{A}' \subseteq [\chi^*]^{\aleph_0}$ and easily $A \in [\chi^*]^{\lambda^+} \Rightarrow (\exists B \in \mathscr{A}')(B \subseteq A)$ and $|\mathscr{A}'| \le \chi^*$ as $(\lambda^+)^{\aleph_0} \le 2^\lambda < \chi^*$ certainly there is no family of $> \chi^*$ subsets of χ^* each of cardinality λ^+ with pairwise finite intersections, hence (by [Sh:b, Ch.XIV,§1] or see [Sh 576, 1.2](1) or [Sh:f, AP,1.16]) we have $\chi^* < \mu_{\mathrm{wd}}(\lambda^+, 2^\lambda)$ thus completing the proof of (η).

Now clause (θ) follows by $(\zeta) + (\eta)$ by [Sh 576, 1.2t](5). Also if $2^{\lambda^+} \neq \lambda^{++}$ then $2^{\lambda^+} \ge \lambda^{+3}$ so by clause (ζ) (as $\chi^* < 2^{\lambda^+}$), we have $|\lim_{\lambda^+}(\mathscr{T}_\zeta)| \ge \lambda^{+3}$ for some ζ which is well known to imply no normal ideal on λ^+ is λ^{++}-saturated; i.e., clause (ι). So we have proved that case $(B)_\lambda$ holds.

<u>Subpossibility 2b</u>: $\chi^* = 2^{\lambda^+}$.

We have proved that case $(A)_\lambda$ holds, as we already defined μ and χ^* and proved $(\alpha), (\beta), (\gamma), (\delta), (\varepsilon)$ we are done.

Still we depend on 2.3 below but first we prove

2.2 Claim. *Assume*

(a) $\sigma < \kappa = \mathrm{cf}(\mu) \le \theta_0 \le \theta_1 < \mu \le \theta_1^\sigma$

(b) $(\forall \mu')[\theta_0 < \mu' < \mu \ \& \ \mathrm{cf}(\mu') \le \kappa \Rightarrow \mathrm{pp}(\mu') < \mu]$

(c) $\theta_2 = \theta_1 + \mathrm{cov}(\theta_1, \theta_0^+, \kappa^+, \sigma^+)$ *(by clause (b) and [Sh:g, Ch.II,5.4] we know that it is* $< \mu$*)*

(d) $\mathrm{pp}(\mu) \ge \chi = \mathrm{cf}(\chi) > \theta_2^\sigma$ $(\ge \theta_1^\sigma \ge \mu)$.

<u>*Then*</u> $\theta_0^\sigma \ge \mu$.

Remark. In fact $\theta_2^\sigma \ge \mathrm{cov}(\theta_1, \theta_0^+, \kappa^+, 2)$.

Proof. Assume toward contradiction $\theta_0^\sigma < \mu$. By [Sh:g, Ch.II,2.3](2) and clause (b) of the assumption we have $\sup\{\mathrm{pp}(\mu') : \theta_0^+ \le \mu' \le \theta_0^\sigma$ and $\mathrm{cf}(\mu') \le \kappa\} < \mu$ hence by [Sh:g, Ch.II,5.4] it follows that

$\boxdot$ $\kappa^* =: \mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \mu$.

We can by assumptions (b) + (d) and [Sh:g, Ch.II,3.5] + [Sh:g, Ch.VIII,§1] find $T \subseteq {}^{\kappa \geq}\mu$, a tree with $\leq \mu$ nodes, $|\lim_\kappa(T)| \geq \chi$, (if $\chi = \mathrm{pp}(\mu)$, the supremum in the definition of $\mathrm{pp}(\mu)$ is obtained by [Sh:g, II,5.4](2)). Moreover, by the construction there is $\Xi \subseteq \lim_\kappa(T), |\Xi| = \chi$ such that $\Xi' \subseteq \Xi \ \& \ |\Xi'| \geq \chi \Rightarrow |\{\eta \restriction \alpha : \alpha < \kappa, \eta \in \Xi'\}| = \mu$. By renaming (and also by the construction), without loss of generality

$\otimes$ if $\eta_0 \hat{}\langle\alpha_0\rangle \neq \eta_1 \hat{}\langle\alpha_1\rangle$ belongs to T then $\alpha_0 \neq \alpha_1$.

So let $\eta_i \in \lim_\kappa(T)$ for $i < \chi$ be pairwise distinct, listing Ξ.
As $\mu \leq \theta_1^\sigma$ there is a sequence $\bar{F} = \langle F_\varepsilon : \varepsilon < \sigma\rangle$ satisfying: F_ε a function from μ to θ_1 such that $\alpha < \beta < \mu \Rightarrow (\exists \varepsilon < \sigma) F_\varepsilon(\alpha) \neq F_\varepsilon(\beta)$.

Let $w_{i,\varepsilon} = \{F_\varepsilon(\eta_i(\alpha)) : \alpha < \kappa\}$, so $w_{i,\varepsilon} \in [\theta_1]^\kappa$. By assumption (c) we have $\theta_2 = \theta_1 + \mathrm{cov}(\theta_1, \theta_0^+, \kappa^+, \sigma^+)$ so there is $\mathscr{P} \subseteq [\theta_1]^{\theta_0}, \theta_2 = |\mathscr{P}|$ such that: any $w \in [\theta_1]^\kappa$ is included in a union of $\leq \sigma$ members of $\mathscr{P}$. So we can find $X_{i,\varepsilon,\zeta} \in \mathscr{P}$ for $\zeta < \sigma$ such that $w_{i,\varepsilon} \subseteq \bigcup\limits_{\zeta<\sigma} X_{i,\varepsilon,\zeta}$. So $\bigcup\limits_{\varepsilon<\sigma} w_{i,\varepsilon} \subseteq Y_i =: \bigcup\limits_{\zeta,\varepsilon<\sigma} X_{i,\varepsilon,\zeta}$. Let $\mathscr{P}^* = \{\bigcup\limits_{\varepsilon<\sigma} X_\varepsilon : X_\varepsilon \in \mathscr{P}$ for $\varepsilon < \sigma\}$, so $\mathscr{P}^*$ is a family of $\leq |\mathscr{P}|^\sigma \leq \theta_2^\sigma$ sets and $i < \chi \Rightarrow Y_i \in \mathscr{P}^*$.

For each $Y \in \mathscr{P}^*$ let

$$Z_Y = \{\alpha < \mu : (\forall \varepsilon < \sigma)(F_\varepsilon(\alpha) \in Y)\}$$

clearly $Y = Y_i \Rightarrow \mathrm{Rang}(\eta_i) \subseteq Z_Y$, also $|Y| \leq \theta_0$ hence $|Z_Y| \leq \theta_0^\sigma < \mu$ hence there is a family $\mathscr{Q}_Y$ of cardinality $\kappa^* =: \mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \mu$ whose members are subsets of Z_Y each of cardinality $\leq \theta_0$ such that any $X \in [Z_Y]^{\leq\kappa}$ is included in the union of $\leq \sigma$ of them. For each $Y \in \mathscr{P}^*$ and $W \in \mathscr{Q}_Y$ let $T'_W = \{\eta \in T :$ for some $\alpha < \kappa$ we have: $\alpha + 1 = \ell g(\eta)$ and $\eta(\alpha) \in W\}$ and $T_W = \{\eta \in T : (\exists \nu)(\eta \triangleleft \nu \in T_W)\}$.

So by $\otimes$ above we have: T'_W, hence T_W is a set of $\leq |W| + \kappa \leq \theta_0$ nodes in T, $\triangleleft$-downward closed. Also

$$(*) \quad |\bigcup_{Y \in \mathscr{P}^*} \mathscr{Q}_Y| \leq |\mathscr{P}^*| \times \sup_{Y \in \mathscr{P}^*} |\mathscr{Q}_Y| \leq \theta_2^\sigma \times \kappa^* \leq \theta_2^\sigma + \mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \mu.$$

However, for every $i < \chi, Y_i \in \mathscr{P}^*$ and $\mathrm{Rang}(\eta_i) \in [Z_{Y_i}]^{\leq\kappa}$ so for some $W \in \mathscr{Q}_{Y_i}, (\exists^\kappa \alpha < \kappa)[\eta_i(\alpha) \in W]$ hence $\eta_i \in \lim_\kappa(T_W)$.

By assumption (d) and $(*)$ above for some $W \in \bigcup\limits_{Y \in \mathscr{P}^*} \mathscr{Q}_Y$ we have

$$|\{i < \chi : \eta_i \in \lim_\kappa(T_W)\}| = \chi.$$

T_W is (essentially) a tree with $\leq \theta_0$ nodes and contradict the choice of $\Xi = \{\eta_i : i < \chi\}$.
(We could have instead using $\kappa^*, \mathscr{Q}_Y$ to fix $Y_i = Y$ as $|\mathscr{P}^*| < \chi = \text{cf}(\chi)$.) $\square_{2.2}$

2.3 Claim. *Assume*

(a) $\sigma < \kappa = \text{cf}(\mu) \leq \theta_0 \leq \theta_1 < \mu$
(b) $(\forall \mu')[\theta_0 < \mu' < \mu \ \& \ \text{cf}(\mu') \leq \kappa \rightarrow \text{pp}(\mu') < \mu]$
(c) $\mathscr{T}^*$ *is a tree with* $\leq \theta_1$ *nodes,* σ *levels and* $\geq \mu$ σ*-branches*
(d) $\text{pp}(\mu) \geq \chi = \text{cf}(\chi) > \theta_2^\sigma$ *where*
(e) $\theta_2 = \theta_1 + \text{cov}(\theta_1, \theta_0^+, \kappa^+, \sigma^+)$
(f) $\kappa^* = \text{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \chi$.

Then for some subtree $Y \subseteq \mathscr{T}^*, |Y| \leq \theta_0$ *and* $|\text{lim}_\sigma(Y)| \geq \mu$ *(for 2.1 it is enough to prove* $\geq \mu'$ *for any given* $\mu' < \mu$*).*
Saharon: use of F*??*

Proof. Let $\mathscr{T}, \Xi = \{\eta_i : i < \chi\}$ be as in the proof of the previous claim. Let $\{\nu_\zeta : \zeta < \mu\}$ list μ distinct σ-branches of $\mathscr{T}^*$ (see clause (c)). Without loss of generality the set of nodes of T^* is θ_1. Choose for each $\varepsilon < \sigma$ the function $F_\varepsilon : \mu \rightarrow \theta_1$ by $F_\varepsilon(\gamma) = \nu_\gamma(\varepsilon)$. Define $w_{i,\varepsilon}, \mathscr{P}, X_{i,\varepsilon,\zeta}, Y_i, \mathscr{P}^*$ as in the proof of 2.2. But for $Y \in \mathscr{P}^*$ we change the choice of Z_Y, first

$$Y' = \{\beta < \theta_1 : \text{for some } \alpha \in Y, \text{ we have } \beta <_{T^*} \alpha\}$$

So $|Y'| \leq \sigma + |Y|$ and let

$$Z_Y = \{\alpha < \mu : (\forall \varepsilon < \sigma)(F_\varepsilon(\alpha) \in Y')\}.$$

We continue as in the proof of 2.2.

Let us present it in detail. We can by assumptions (b) + (d) and [Sh:g, Ch.II,3.5] + [Sh:g, Ch.VIII,§1] find $\mathscr{T} \subseteq {}^{\kappa \geq}\mu$, a tree with $\leq \mu$ nodes, $|\text{lim}_\kappa(\mathscr{T})| \geq \chi$, (if $\chi = \text{pp}(\mu)$, the supremum in the definition of $\text{pp}(\mu)$ is obtained by [Sh:g, II,5.4](2)). Moreover, by the construction there is $\Xi \subseteq \text{lim}_\kappa(T), |\Xi| = \chi$ such that $\Xi' \subseteq \Xi \ \& \ |\Xi'| \geq \chi \Rightarrow |\{\eta \restriction \alpha : \alpha < \kappa, \eta \in \Xi'\}| = \mu$. By renaming (and also by the construction), without loss of generality

$\otimes$ if $\eta_0{}^\frown\langle\alpha_0\rangle \neq \eta_1{}^\frown\langle\alpha_1\rangle$ belongs to $\mathscr{T}$ then $\alpha_0 \neq \alpha_1$.

So let $\eta_i \in \lim_\kappa(T)$ for $i < \chi$ be pairwise distinct, listing Ξ.
Let $\{\nu_\zeta : \zeta < \mu\}$ list μ distinct σ-branches of $\mathscr{T}^*$ (see clause (c)). Without loss of generality the set of nodes of $\mathscr{T}^*$ is θ_1 and let $\nu_\zeta(\varepsilon)$ be the unique $\alpha < \theta_1$ which belong to the branch ν_ζ and is of level ε. Define for each $\varepsilon < \sigma$ the function $F_\varepsilon : \mu \to \theta_1$ by $F_\varepsilon(\gamma) = \nu_\gamma(\varepsilon)$ hence $\alpha < \beta < \mu \Rightarrow (\exists \varepsilon < \sigma) F_\varepsilon(\alpha) \neq F_\varepsilon(\beta)$.]

For $\varepsilon < \sigma$ and $i < \chi$ let $w_{i,\varepsilon} = \{F_\varepsilon(\eta_i(\alpha)) : \alpha < \kappa\}$, so $w_{i,\varepsilon} \in [\theta_1]^\kappa$. By assumption (e) we have
$\theta_2 = \theta_1 + \text{cov}(\theta_1, \theta_0^+, \kappa^+, \sigma^+)$ so there is $\mathscr{P} \subseteq [\theta_1]^{\theta_0}, \theta_2 \geq |\mathscr{P}|$ such that: any $w \in [\theta_1]^\kappa$ is included in a union of $\leq \sigma$ members of $\mathscr{P}$. So we can find $X_{i,\varepsilon,\zeta} \in \mathscr{P}$ for $\zeta < \sigma$ such that $w_{i,\varepsilon} \subseteq \bigcup\limits_{\zeta<\sigma} X_{i,\varepsilon,\zeta}$. So $\bigcup\limits_{\varepsilon<\sigma} w_{i,\varepsilon} \subseteq Y_i =: \bigcup\limits_{\zeta,\varepsilon<\sigma} X_{i,\varepsilon,\zeta} \in [\theta_1]^{\theta_0}$. Let $\mathscr{P}^* = \{\bigcup\limits_{\varepsilon<\sigma} X_\varepsilon : X_\varepsilon \in \mathscr{P} \text{ for } \varepsilon < \sigma\}$, so $\mathscr{P}^*$ is a family of $\leq |\mathscr{P}|^\sigma \leq \theta_2^\sigma$ sets each of cardinality $\leq \kappa$ and $i < \chi \Rightarrow Y_i \in \mathscr{P}^*$.

For each $Y \in \mathscr{P}^*$ let

$$X_Y = \{\beta < \theta_1 : \text{ for some } \alpha \in Y \text{ we have } \beta \leq_{\mathscr{T}^*} \alpha\}$$

$$Z_Y = \{\alpha < \mu : (\forall \varepsilon < \sigma)(F_\varepsilon(\alpha) \in X_Y)\}$$

clearly $Y = Y_i \Rightarrow \text{Rang}(\eta_i) \subseteq Z_Y$, also $|Y| \leq \theta_0$ so $|X_Y| \leq \sigma + |Y| \leq \sigma + \theta_0 = \theta_0$ hence $|Z_Y| \leq \theta_0^\sigma \leq \theta_1^\sigma \leq \theta_2^\sigma < \chi$ hence there is a family $\mathscr{Q}_Y$ of cardinality $\kappa^* =: \text{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \chi$ by assumption (f) whose members are subsets of Z_Y each of cardinality $\leq \theta_0$ such that any $X \in [Z_Y]^{\leq\kappa}$ is included in the union of $\leq \sigma$ of them. For each $Y \in \mathscr{P}^*$ and $W \in \mathscr{Q}_Y$ let $\mathscr{T}'_W = \{\eta \in \mathscr{T} : \text{for some } \alpha < \kappa \text{ we have: } \alpha + 1 = \ell g(\eta) \text{ and } \eta(\alpha) \in W\}$ and $\mathscr{T}_W = \{\eta \in \mathscr{T} : (\exists \nu)(\eta \triangleleft \nu \in \mathscr{T}_W)\}$.

So by $\otimes$ above we have: $\mathscr{T}'_W$, hence $\mathscr{T}_W$ is a set of $\leq |W| + \kappa \leq \theta_0$ nodes in T and is $\triangleleft$-downward closed. Also

$$\begin{aligned}(*)\ |\bigcup_{Y\in\mathscr{P}^*} \mathscr{Q}_Y| &\leq |\mathscr{P}^*| \times \sup_{Y\in\mathscr{P}^*} |\mathscr{Q}_Y| \\ &\leq \theta_2^\sigma \times \kappa^* \leq \theta_2^\sigma + \text{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) = \theta_2^\sigma + \kappa^* < \chi.\end{aligned}$$

However, for every $i < \chi, Y_i \in \mathscr{P}^*$ and $\alpha < \kappa \Rightarrow \{F_\varepsilon(\eta_i(\alpha)) : \varepsilon < \sigma\} \subseteq \bigcup\limits_{\varepsilon<\sigma} w_{i,\varepsilon} \subseteq Y_i \Rightarrow \eta_i(\alpha) \in Z_Y$ hence $\text{Rang}(\eta_i) \in [Z_{Y_i}]^{\leq\kappa}$ so for some $W \in \mathscr{Q}_{Y_i}, (\exists^\kappa \alpha < \kappa)[\eta_i(\alpha) \in W]$ hence $\eta_i \in \lim_\kappa(\mathscr{T}_W)$.

By assumption (d) and $(*)$ above for some $W \in \bigcup\limits_{Y\in\mathscr{P}^*} \mathscr{Q}_Y$ we have

$$|\{i < \chi : \eta_i \in \lim_\kappa(\mathscr{T}_W)\}| = \chi.$$

Now $\mathscr{T}_W$ is (essentially) a tree with $\leq \theta_0$ nodes and contradict the choice of $\Xi = \{\eta_i : i < \chi\}$.
(We could have instead using $\kappa^*, \mathscr{Q}_Y$ to fix $Y_i = Y$ as $|\mathscr{P}^*| < \chi = \mathrm{cf}(\chi)$.)

$\square_{2.3}$, $\square_{2.1}$

2.4 Remark. 1) We could have used in 2.2, 2.3,
$\theta_2 = \mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \kappa^+)$ instead $\mathrm{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+)$ and similarly in the proof of 2.1.
2) We can also play with assumption (b) as 2.2, 2.3.

It may be useful to note (actually $\lambda^{<\kappa>_{\mathrm{tr}}} = \lambda$ suffice)

2.5 Fact. If $\mathscr{T} \subseteq {}^{\lambda^+ >}2$ is a tree, $|\mathscr{T}| \leq \lambda^+$ and $\lambda \geq \beth_\omega$ <u>then</u> for every regular $\kappa < \beth_\omega$ large enough, we can find $\langle Y_\delta : \delta < \lambda^+, \mathrm{cf}(\delta) = \kappa \rangle, |Y_\delta| \leq \lambda$ such that:
for every $\eta \in \lim_{\lambda^+}(\mathscr{T})$ for a club of $\delta < \lambda^+$ we have
$\mathrm{cf}(\delta) = \kappa \Rightarrow \eta \restriction \delta \in Y_\delta$.

The following is needed when we like to get in the model theory not just many models but many models no one $\leq_{\mathfrak{K}}$-embeddable into another.
<u>2.6 Fact</u>: Assume:

- (a) $\mathrm{cf}(\mu) \leq \kappa < \mu$, $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$, moreover $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$ and $\kappa^+ < \theta < \chi^*$
- (b) $\mathbf{F}$ is a function, with domain $[\mu]^\kappa$, such that: for $a \in [\mu]^\kappa, \mathbf{F}(a)$ is a family of $< \theta$ members of $[\mu]^\kappa$
- (c) F is a function with domain $[\mu]^\kappa$ such that

$$a \in [\mu]^\kappa \Rightarrow a \subseteq F(a) \in \mathbf{F}(a).$$

<u>Then</u> we can find pairwise distinct $a_i \in [\mu]^\kappa$ for $i < \chi^*$ such that $\mathscr{I} = \{a_i : i < \chi\}$ is $(F, \mathbf{F})$-independent which means

$$(*)_{F,\mathbf{F},\mathscr{I}} \qquad \text{if } a \neq b \ \& \ a \in \mathscr{I} \ \& \ b \in \mathscr{I} \ \& \ c \in \mathbf{F}(a) \Rightarrow \neg(F(b) \subseteq c).$$

2.7 Remark. 1) Clearly this is similar to Hajnal's free subset theorem [Ha61].
2) Note that we can let $F(a) = a$.
3) Note that if $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$ then for some $\mathbf{F}, F$ as in the Fact

- $(*)$ if $a_i \in [\mu]^\kappa$ for $i < \lambda^+$ are pairwise distinct then not every pair $\{a_i, a_j\}$ is $(\mathbf{F}, F)$-independent
  [why? let $\mathscr{P} \subseteq [\mu]^\kappa$ be cofinal (under $\subseteq$) of cardinality λ, and let $\mathbf{F}$ be such

that
$\mathbf{F}(a) \subseteq \{b \in [\mu]^\kappa : a \subseteq b \text{ and } b \in \mathscr{P}\}$ has a $\subseteq$-maximal member $F(a)$; clearly there is such $\mathbf{F}$. Now clearly

$(*)_1$ if $a \neq b$ are from $[\mu]^\kappa$ and $\mathbf{F}(a) \cap \mathbf{F}(b) \neq \emptyset$ then $\{a, b\}$ is not $(\mathbf{F}, F)$-independent.
Also if $\mu_1 \leq \mu$, $\mathrm{cf}(\mu_1) \leq \kappa \leq \kappa + \theta < \mu_1$ and $\mathrm{pp}_\kappa(\mu_1) \geq \mu$ then by [Sh:g, Ch.II,2.3] the Fact for μ_1 implies the one for μ.]

Proof. We can define $g : [\mu]^{\leq\kappa} \to [\mu]^\kappa$ and $F', \mathbf{F}'$ functions with domain $[\mu]^{\leq\kappa}$ as follows:

$$g(a) = \{\kappa + \alpha : \alpha \in a\} \cup \{\alpha : \alpha < \kappa\}$$

$$\mathbf{F}'(a) = \{\{\alpha : \kappa + \alpha \in b\} : b \in \mathbf{F}'(g(a))\}$$

$$F'(a) = \{\alpha : \kappa + \alpha \in F(g(a))\}.$$

Now $\mathbf{F}', F'$ are as above only replacing everywhere $[\mu]^\kappa$ by $[\mu]^{\leq\kappa}$, and if $\mathscr{I} = \{a_i : i < \chi\} \subseteq [\mu]^{\leq\kappa}$ with no repetitions satisfying $(*)_{F',\mathbf{F}',\mathscr{I}}$ then $\mathscr{I}' = \{g(a_i) : i < \chi\}$ is with no repetitions and $(*)_{F,\mathbf{F},\mathscr{I}'}$.

So we conclude that we can replace $[\mu]^\kappa$ by $[\mu]^{\leq\kappa}$. In fact we shall find the a_i in $[\mu]^{\mathfrak{a}}$ where $\mathfrak{a}$ chosen below.
Without loss of generality $\kappa^{++} < \theta$.

Assume $\theta < \chi = \mathrm{cf}(\Pi\mathfrak{a}/J)$ where $\mathfrak{a} \subseteq \mu \cap \mathrm{Reg}\backslash\kappa^+, |\mathfrak{a}| \leq \kappa, \sup(\mathfrak{a}) = \mu, J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq J$ and for simplicity $\chi = \max \mathrm{pcf}(\mathfrak{a})$ and let $f = \langle f_\alpha : \alpha < \chi\rangle$ be a sequence of members of $\Pi\mathfrak{a}, <_J$-increasing, and cofinal in $(\Pi\mathfrak{a}, <_J)$, so, of course, $\chi \leq \chi^*$. Without loss of generality $f_\alpha(\tau) > \sup(\mathfrak{a} \cap \tau)$ for $\tau \in \mathfrak{a}$. Also for every $a \in [\mu]^\kappa$, define $f_a \in \Pi\mathfrak{a}$ by $f_a(\tau) = \sup(a \cap \tau)$ for $\tau \in \mathfrak{a}$ so for some $\zeta(a) < \chi$ we have $f_a <_J f_{\zeta(a)}$ (as $\langle f_\alpha : \alpha < \chi\rangle$ is cofinal in $(\Pi\mathfrak{a}, <_J)$). So for each $a \in [\mu]^\kappa$, as $|\mathbf{F}(a)| < \theta < \chi = \mathrm{cf}(\chi)$ clearly $\xi(a) = \sup\{\zeta(b) : b \in \mathbf{F}(a)\}$ is $< \chi$, and clearly $(\forall b \in \mathbf{F}(a))[f_b <_J f_{\xi(a)}]$. So
$C = \{\gamma < \chi : \text{for every } \beta < \gamma, \xi(\kappa \cup \mathrm{Rang}\ f_\beta) < \gamma\}$ is a club of χ.

For each $\alpha < \chi$, $\mathrm{Rang}(f_\alpha) \cup \kappa \in [\mu]^\kappa$, hence $\mathbf{F}(\mathrm{Rang}(f_\alpha) \cup \kappa)$ has cardinality $< \theta$, but $\theta < \chi = \mathrm{cf}(\chi)$ hence for some $\theta_1 < \theta$; we have $\theta_1 > \kappa^{++}$ and $\chi = \sup\{\alpha < \chi : |\mathbf{F}(\mathrm{Rang}(f_\alpha) \cup \kappa)| \leq \theta_1\}$, so without loss of generality $\alpha < \chi \Rightarrow \theta_1 \geq |\mathbf{F}(\mathrm{Rang}(f_\alpha) \cup \kappa)|$.

As $\kappa^+ < \theta_1$, there is a stationary $S \subseteq \{\delta < \theta_1^+ : \mathrm{cf}(\delta) = \kappa^+\}$ which is in $I[\theta_1^+]$, by [Sh 420, §1] and let $\langle d_i : i < \theta_1^+\rangle$ witness it, so $\mathrm{otp}(d_i) \le \kappa^+, d_i \subseteq i, [j \in d_i \Rightarrow d_j = d_i \cap i]$ and $i \in S \Rightarrow i = \sup(d_i)$, and for simplicity: for every club E of θ_1^+ for stationarily many $\delta \in S$ we have $(\forall \alpha \in d_\delta)[(\exists \beta \in E)(\sup(\alpha \cap d_\delta) < \beta < \alpha)]$, exists by [Sh 420, §1]. Now try to choose by induction on $i < \theta_1^+$, a triple (g_i, α_i, w_i) such that:

- (a) $g_i \in \Pi\mathfrak{a}$
- (b) $j < i \Rightarrow g_j <_J g_i$
- (c) $(\forall \tau \in \mathfrak{a})(\sup\limits_{j \in d_i} g_j(\tau) < g_i(\tau))$
- (d) $\alpha_i < \chi, \alpha_i > \sup(\bigcup\limits_{j<i} w_j)$
- (e) $j < i \Rightarrow \alpha_j < \alpha_i$
- (f) $g_i <_J f_{\alpha_i}$
- (g) $\beta \in \bigcup\limits_{j<i} w_j \Rightarrow \xi(\beta) < \alpha_i \ \& \ f_\beta <_J g_i$
- (h) w_i is a maximal subset of (α_i, χ) satisfying
  - $(*)$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \ne \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \neg(F(\mathrm{Rang}\ f_\gamma) \subseteq a)$

  or just
  - $(*)^+$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \ne \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \{\tau \in \mathfrak{a} : f_\gamma(\tau) \in a\} \in J$.

  [note that really

  $\otimes$ if $w \subseteq (\alpha_i, \chi)$ satisfies $(*)^+$ then it satisfies $(*)$

  why? let us check $(*)$, so let $\beta \in w, \gamma \in w, \beta \ne \gamma$ and $a \in \mathbf{F}(\mathrm{Rang}(f_\beta))$; by $(*)^+$ we know that $\mathfrak{a}' = \{\tau \in \mathfrak{a} : f_\gamma(\tau) \in a\} \in J, J$ is a proper ideal on $\mathfrak{a}$ clearly for some $\tau \in \mathfrak{a}$ we have $\tau \notin \mathfrak{a}'$, hence $f_\gamma(\tau) \notin a$ but $f_\gamma(\tau) \in \mathrm{Rang}(f_\gamma) \subseteq F(\mathrm{Rang}(f_\gamma))$ hence $f_\gamma(\tau) \in F(\mathrm{Rang}(f_\gamma)) \backslash a$ so $\neg(F(\mathrm{Rang}(f_\gamma) \subseteq a)$, as required.]

We claim that we cannot carry the induction because if we succeed, then as $\mathrm{cf}(\chi) = \chi > \theta \ge \theta_1^+$ there is α such that $\bigcup\limits_{i<\theta_1^+} \alpha_i < \alpha < \chi$ and let $\mathbf{F}(\mathrm{Rang}(f_\alpha)) = \{a^\alpha_\zeta : \zeta < \theta_1\}$ (possible as $1 \le |\mathbf{F}(\mathrm{Rang}(f_\alpha))| \le \theta_1$). Now for each $i < \theta_1^+$, by the choice of w_i clearly $w_i \cup \{\alpha\}$ does not satisfy the demand in clause (h), so as $\beta \in w_i \Rightarrow \xi(\beta) < \alpha_{i+1} < \alpha$, necessarily for some $\beta_i \in w_i$ and $\zeta_i < \theta_1$ we have

$$\mathfrak{a}_i = \{\tau \in \mathfrak{a} : f_{\beta_i}(\tau) \in a^\alpha_{\zeta_i}\} \notin J.$$

[why use the ideal? In order to show that $\mathfrak{b}_\varepsilon \neq \emptyset$.] Now $\mathrm{cf}(\theta_1^+) = \theta_1^+ > \theta_1$, for some $\zeta(*) < \theta_1^+$ we have $A = \{i : \zeta_i = \zeta(*)\}$ is unbounded in θ_1^+. Hence $E = \{\alpha < \theta_1^+ : \alpha$ a limit ordinal and $A \cap \alpha$ is unbounded in $\alpha\}$ is a club of θ_1^+. So for some $\delta \in S$ we have $\delta = \sup(A \cap \delta)$, moreover if $d_\delta = \{\alpha_\varepsilon : \varepsilon < \kappa^+\}$ (increasing) then $(\forall \varepsilon)[E \cap (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \neq \emptyset]$ hence we can find $i(\delta,\varepsilon) \in (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \cap A$ for each $\varepsilon < \kappa^+$.

Clearly for each $\varepsilon < \kappa^+$

$$\begin{aligned}\mathfrak{b}_\varepsilon = \big\{\tau \in \mathfrak{a} : g_{i(\delta,\varepsilon)}(\tau) < f_{\alpha_{i(\delta,\varepsilon)}}(\tau) < f_{\beta_{i(\delta,\varepsilon)}}(\tau) \\ < g_{i(\delta,\varepsilon)+1} < f_{\alpha_{i(\delta,\varepsilon)+1}}(\tau) < f_\alpha(\tau)\big\} = \mathfrak{a} \bmod J\end{aligned}$$

hence $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin \emptyset$. Moreover, $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin J$. Now for each $\tau \in \mathfrak{a}$ let $\varepsilon(\tau)$ be $\sup\{\varepsilon < \kappa^+ : \tau \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ and let $\varepsilon(*) = \sup\{\varepsilon(\tau) : \tau \in \mathfrak{a}$ and $\varepsilon(\tau) < \kappa^+\}$ so as $|\mathfrak{a}| \leq \kappa$ clearly $\varepsilon(*) < \kappa^+$. Let $\tau^* \in \mathfrak{b}_{\varepsilon(*)+1} \cap \mathfrak{a}_{i(\delta,\varepsilon(*)+1)}$, so $B = \{\varepsilon < \kappa^+ : \tau^* \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ is unbounded in $\kappa^+, \langle f_{\beta_{i(\delta,\varepsilon)}}(\tau^*) : \varepsilon \in B\rangle$ is strictly increasing (see clause (c) above and the choice of $\mathfrak{b}_\varepsilon$) and $\varepsilon \in B \Rightarrow f_{\beta_{i(\delta,\varepsilon)}}(\tau^*) \in a_{\zeta(*)}$ (by the definition of $\mathfrak{a}_{i(\delta,\varepsilon)}$, and $\zeta(*)$ as $\zeta_{i(\delta,\varepsilon)} = \zeta(*)$). We get contradiction to $a \in \mathbf{F}(\kappa \cup \mathrm{Rang}(f_\alpha)) \Rightarrow |a| \leq \kappa$.

So really we cannot carry the induction so we are stuck at some i. If $i = 0$, or i limit, or $i = j+1$ & $\sup(w_j) < \chi$ we can find g_i and then α_i and then w_i as required. So necessarily $i = j+1, \sup(w_j) = \chi$. Now if $\chi = \chi^*$, then this w_j is as required in the fact. As $\mathrm{pp}^+(\mu) = (\chi^*)^+$, the only case we cannot have is when χ^* is singular. Let $\chi^* = \sup\limits_{\varepsilon < \mathrm{cf}(\chi^*)} \chi_\varepsilon$ and $\chi_\varepsilon \in (\mu, \chi^*) \cap \mathrm{Reg}$ is (strictly) increasing with ε. By [Sh:g, Ch.II,§1] we can find, for each $\varepsilon < \mathrm{cf}(\chi^*), \mathfrak{a}_\varepsilon, J_\varepsilon, \bar{f}^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \chi_\varepsilon\rangle$ as above, but in addition

- $(*)$ $\bar{f}^\varepsilon$ is μ^+-free i.e. for every $u \in [\chi_\varepsilon]^\mu$ there is $\langle \mathfrak{b}_\alpha : \alpha \in u\rangle$ such that $\mathfrak{b}_\alpha \in J_\varepsilon$ and for each $\tau \in \mathfrak{a}_\varepsilon, \langle f^\varepsilon_\alpha(\tau) : \alpha$ satisfies $: \tau \notin \mathfrak{b}_\alpha\rangle$ is strictly increasing.

So for every $a \in [\mu]^\kappa$ and $\varepsilon < \mathrm{cf}(\chi^*)$ we have

$$\big\{\alpha < \chi_\varepsilon : \{\tau \in \mathfrak{a}_\varepsilon : f_\alpha(\tau) \in a\} \notin J_\varepsilon\big\} \text{ has cardinality } \leq \kappa.$$

Hence for each $a \in [\mu]^\kappa$

$$\big\{(\varepsilon,\alpha) : \varepsilon < \mathrm{cf}(\chi^*) \text{ and } \alpha < \chi_\varepsilon \text{ and } \{\tau \in \mathfrak{a}_\varepsilon : f_\alpha(\tau) \in a\} \notin J_\varepsilon\big\}$$

has cardinality $\leq \kappa + \mathrm{cf}(\chi^*) = \mathrm{cf}(\chi^*)$ as for singular $\mu > \kappa \geq \mathrm{cf}(\mu)$ we have $\mathrm{cf}(\mathrm{pp}_\kappa(\mu) > \kappa$.

Define: $X = \{(\varepsilon, \alpha) : \varepsilon < \text{cf}(\chi^*), \alpha < \chi_\varepsilon\}$

$$F'\big((\varepsilon,\alpha)\big) = \big\{(\varepsilon',\alpha') : (\varepsilon',\alpha') \in X \backslash \{(\varepsilon,\alpha)\} \text{ and for some } \\ d \in \mathbf{F}(\text{Rang}(f^\varepsilon_\alpha)) \text{ we have } \{\tau \in \mathfrak{a}_\varepsilon : f^{\varepsilon'}_{\alpha'}(\tau) \in d\} \notin J_{\varepsilon'}\big\}$$

so $F'\big((\varepsilon,\alpha)\big)$ is a subset of X of cardinality $< \text{cf}(\chi^*)^+ + \theta < \chi^*$.

So by Hajnal's free subset theorem [Ha61] we finish (we could alternatively, for χ^* singular, have imitated his proof). $\square_{2.6}$

§3 Finishing the many models

Recall from [Sh 576, 3.13t](3) (see [Sh 838]??).

3.1 Claim. *1) Assume*

- (a) $2^\lambda < 2^{\lambda^+}$ *and Case A or B of Fact 2.1 holds for* μ, χ^* *(or just the conclusion there)*
- (b) $\mathfrak{K}$ *is an abstract elementary class with* $\mathrm{LS}(\mathfrak{K}) \le \lambda$
- (c) $K_{\lambda^+} \neq 0$
- (d) $\mathfrak{K}$ *has amalgamation in* λ
- (e) *in* K^3_λ, *the minimal triples are not dense.*

Then

- $(*)_1$ *for any regular* $\chi < \mu$ *we have:*
- $(*)^1_\chi$ *there is* $M \in K_\lambda, |\mathscr{S}(M)| > \chi$.

2) If in part (1) we strengthen clause (d) *to* $(d)^+$, *then we get* $(*)^+_1$ *where:*

- $(d)^+$ $\mathfrak{K}$ *has amalgamation in* λ *and has a universal member in* λ
  - $(*)^+_1$ *for some* $M \in K_\lambda$ *we have* $|\mathscr{S}(M)| \ge \mu$.

3) Assume (a), (b), (c), (e) of part (1) and $(d)^+$ *of part (2) then:*

- $(*)_2$ $\dot{I}(\lambda^+, K) \ge \chi^*$ *and if* $(2^\lambda)^+ < \chi^*$ *then* $\dot{I}\dot{E}(\lambda^+, \mathfrak{K}) \ge \chi^*$.

4) If in clause (a) *of part (1) we restrict ourselves to Case A of 2.1, then* $\chi^* = 2^{\lambda^+}$ *so in part (3) we get*

- $(*)^+_2$ $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ *and* $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) \ge 2^{\lambda^+}$.

3.2 Remark. 1) We can restrict clause (b) to $\mathfrak{K}_\lambda$, interpreting in (c) + (e), K_{λ^+} as $\{\bigcup\limits_{i<\lambda^+} M_i : M_i \in K_\lambda$ is $<_{\mathfrak{K}}$-increasing (strictly and) continuous$\}$, but see [Sh 576, §0], mainly 0.31t.
2) Part (3) of 3.1 (and 3.3) below are like [Sh:g, Ch.II,4.10E], Kojman Shelah [KjSh 409, §2].
3) We can apply this to λ^+ standing for λ here.

4) We can state the part of (A) of 2.1 used (and can replace 2^{λ^+} by smaller cardinals).
5) We can replace λ^+ by a weakly inaccessible cardinal with suitable changes.

Proof. 1) Note that μ is singular (as by clause (α) of (A) of 2.1 (so also (B)), $\mathrm{cf}(\mu) = \lambda^+ < \mu$). By 1.15(1) it suffices for each $\mu' < \mu$ to have $\delta < \lambda^+$ and a tree with $\leq \lambda$ nodes and $\geq \mu'$ δ-branches. They exist by clause (ε) of (A) of 2.1 (so also of (B)).
2) Similarly using 1.15(2).
3) By part (2) we can find $M^* \in K_\lambda$ satisfying $\mathscr{S}(M^*)$ has cardinality $\geq \mu$, and apply 3.3 elow.
4) Should be clear from the proof of part (3). $\square_{3.1}$

3.3 Fact. Assume

(a) $\chi^* = \mathrm{cf}([\mu]^{\lambda^+}, \subseteq) > 2^\lambda$
(b) $\mathfrak{K}$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) \leq \lambda$
(c) $M^* \in K_\lambda$ is an amalgamation base
(d) $M^* \in K_\lambda$ satisfies $|\mathscr{S}(M^*)| \geq \mu$.

Then $\dot{I}(\lambda^+, K) \geq \chi^*$ and if $(2^\lambda)^+ < \chi^*$ then $\mathrm{IE}(\lambda^+, K) \geq \chi^*$.

Proof. Let $p_\eta \in \mathscr{S}(M^*)$ for $\eta \in Z$ be pairwise distinct, $|Z| \geq \mu$ and let $M^* \leq_{\mathfrak{K}} N_\eta \in K_\lambda, p_\eta = \mathrm{tp}(a_\eta, M^*, N_\eta)$.
Now for every $X \in [Z]^{\lambda^+}$, as $\mathfrak{K}$ has amalgamation in λ there is $M_X \in K_{\lambda^+}$ such that $M^* \leq_{\mathfrak{K}} M_X$ and $\eta \in X \Rightarrow N^*_\eta$ is embeddable into M_X over M^* (hence p_η is realized in M_X). Let $Y[X] = \{\eta \in Z : p_\eta$ is realized in $M_X\}$. So $X \subseteq Y[X] \in [Z]^{\lambda^+}$, so $\{Y[X] : X \in [Z]^{\lambda^+}\}$ is a cofinal subset of $[Z]^{\lambda^+}$, hence (see clause (β) of case (A) of Fact 2.1)

$$\begin{aligned}|\{(M_X, c)_{c \in M^*}/\cong : X \in [Z]^{\lambda^+}\}| \geq \\ |\{Y[X] : X \in [Z]^{\lambda^+}\}| \geq \mathrm{cf}([Z]^{\lambda^+}, \subseteq) \geq \\ \mathrm{cf}([\mu]^{\lambda^+}, \subseteq) \geq \mathrm{pp}(\mu) = \chi^*.\end{aligned}$$

As $2^\lambda < \chi^*$ also $|\{M_X/\cong : X \in [Z]^{\lambda^+}\}| \geq \chi^*$ (clear or see [Sh:a, Ch.VIII,1.2] because $\|M_X\| = \lambda^+, \|M^*\| = \lambda$ and $(\lambda^+)^\lambda < \mu$) but $\dot{I}\dot{E}(\lambda^+, K)$ is $\geq$ than the former.

Lastly we shall prove $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) \geq \chi^*$ (so the reader may skip this, sufficing himself with the estimate on $\dot{I}\dot{E}(\lambda^+, K)$).
For each $X \in [\mu]^{\lambda^+}$, let $\mathbf{F}_X = \{f : f$ a $\leq_{\mathfrak{K}}$ -embedding of M^* into $M_X\}$, and for $f \in \mathbf{F}_X$ let

$$\mathscr{X}_{X,f} = \{X_1 \in [Z]^{\lambda^+} : \text{there is a } \leq_{\mathfrak{K}} \text{-embedding of } M_{X_1} \text{ into } M_X \text{ extending } f\},$$

and let $\mathscr{S}_{X,f} = \{p \in \mathscr{S}(M^*) : f(p)$ is realized in $M_X\}$, so $\mathscr{X}_{X,f} \subseteq \{X_1 : X_1 \subseteq \mathscr{S}_{X,f}\}$ and $|\mathscr{S}_{X,f}| \leq \lambda^+$.

Now the result follows from the the fact 2.6 above. $\square_{3.3}$

3.4 Claim. *1) Assume*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ *and case B or C of Fact 2.1 for λ occurs (so $\chi^*, \mathscr{T}_\zeta$ are determined)*
- (b) *$\mathfrak{K}$ is an abstract elementary class* $\mathrm{LS}(\mathfrak{K}) \leq \lambda$
- (c) *$K_{\lambda^{++}} \neq \emptyset$,*
- (d) *$\mathfrak{K}$ has amalgamation in λ [was: and in λ^+, seem irrelevant]*
- (e) *in K^3_λ, the minimal triples are not dense.*

<u>*Then*</u>

- $(*)$ *for each $\zeta < \chi^*$ for some $M \in K_{\lambda^+}$ we have $|\mathscr{S}_*(M)| \geq |\lim_{\lambda^+}(\mathscr{T}_\zeta)|$ (see 1.2(7) the tree from clause (ζ) of 2.1); on $\mathscr{S}_*$ see 1.2(7).*

2) If K satisfies (a)-(e) and is categorical in λ^+ or just has a universal member in λ^+ and amalgamation in λ^+, <u>then</u> for some $M \in K_{\lambda^+}$ we have $|\mathscr{S}_*(M)| = 2^{\lambda^+}$.
3) If clauses $(a)-(e)$ from above and clause $(f)^+$, <u>then</u> $\dot{I}(\lambda^{++}, K) \geq \mu_{\rm wd}(\lambda^{++}, 2^{\lambda^+})$ where

- $(d)^+$ *$\mathfrak{K}$ has amalgamation also in λ^+*
- $(f)^+$ *$\mathfrak{K}$ is categorical in λ and λ^+.*

Remark. 1) Assume (a)-(e) of part (1) <u>then</u> $\mathbf{C}^0_{\mathfrak{K},\lambda}$ has weaker λ-coding if we have restricted to (M, N, a) above which there is not minimal triple in Definition of $\mathbf{C}$). But not used.
2) Note that for 3.6 below we do not use 3.4.
3) We would like to weaken in [Sh:E46] the assumption "$\mathfrak{K}$ categorical in λ^+" to

"no maximal model in $\mathfrak{K}_\lambda$". So by 3.3 for amalgamation bases $M^* \in K_{\lambda^+}, \mathscr{S}(M^*)$ cannot be too large (used in the proof of 3.6 and as $\dot{I}(\lambda^{++}, K) < \mu_{\rm wd}(\lambda^{++}, 2^{\lambda^+})$, there are many amalgamation basis and by 3.4(1)-pf there are many $M \in K_{\lambda^+}$ with $\mathscr{S}(M)$ large. But we have to put them together (20045/4).

Proof. 1) Let $\zeta < \chi^*$. Recall the $\mathscr{T}_\zeta$ is a subtree of ${}^{\lambda^+ >}2$ of cardinality $\leq \lambda^+$, hence let $\mathscr{T}_\zeta = \bigcup\limits_{\alpha<\lambda^+} \mathscr{T}^\zeta_\alpha$ where $\mathscr{T}^\zeta_\alpha$ are pairwise disjoint for $\alpha < \lambda^+$, each $\mathscr{T}^\zeta_\alpha$ has cardinality $\leq \lambda, \mathscr{T}^\zeta_0 = \{<>\}$ and $\eta \in T^\zeta_\alpha$ & $\beta < \ell g(\eta) \Rightarrow \eta \restriction \beta \in \bigcup\limits_{\gamma<\alpha} \mathscr{T}^\zeta_\gamma$, and $\eta \in \mathscr{T}^\zeta_\alpha \Rightarrow \bigwedge\limits_{\ell<2} \eta\hat{\ }\langle \ell \rangle \in \mathscr{T}^\zeta_{\zeta,\alpha+1}$ so $\mathscr{T}_{\zeta,\alpha+1} = \{\eta\hat{\ }\langle\ell\rangle : \eta \in T^\zeta_\alpha$ and $\ell < 2\}$. For $\eta \in \mathscr{T}^\zeta_\delta, \delta$ a limit ordinal, necessarily both $\ell g(\eta)$ and $\alpha(\eta) = \sup\{\gamma :$ for some $\varepsilon < \ell g(\eta), \eta \restriction \varepsilon \in \mathscr{T}^\zeta_\gamma\}$ are limit ordinals $\leq \delta$.

Let $(M, N, a) \in K^3_\lambda$ be such that there is no minimal triple above it.
We now by induction on $\alpha < \lambda^+$ choose $\langle M^\zeta_\alpha, M^\zeta_\eta, N^\zeta_\eta : \eta \in \mathscr{T}^\zeta_\alpha \rangle$ such that:

- (a) $(M^\zeta_\alpha, N^\zeta_\eta, a) \in K^3_\lambda$ and is reduced (see [Sh 576]) <u>if</u> $\eta \in T^\zeta_\alpha, \alpha$ non-limit
- (b) $(M^\zeta_0, N^\zeta_{<>}, a) = (M, N, a)$
- (c) if $\nu \in T^\zeta_\beta, \eta \in T^\zeta_\alpha, \nu \triangleleft \eta, \beta < \alpha$ and α, β are non-limit then $\Rightarrow (M^\zeta_\beta, N^\zeta_\nu, a) \leq (M^\zeta_\alpha, N^\zeta_\eta, a)$ in the order of K^3_λ
- (d) if δ is a limit ordinal then: $M^\zeta_\delta = \bigcup\limits_{\beta<\delta} M^\zeta_\beta$
- (e) if δ is a limit ordinal and $\eta \in T^\zeta_\delta$ then
  $N^\zeta_{\ell g(\eta)} = \bigcup\limits_{\beta<\delta} N^\zeta_{\eta\restriction\beta}$ hence $(M_{\alpha(\eta)}, N^\zeta_\eta, a) \in K^3_\lambda$
- (f) if $\eta \in \mathscr{T}^\zeta_\alpha$ then ${\rm tp}(a, M_{\alpha+1}, N_{\eta\hat{\ }<0>}) \neq {\rm tp}(a, M_{\alpha+1}, N_{\eta\hat{\ }<1>})$
- (g) $M^\zeta_\alpha \neq M^\zeta_{\alpha+1}$.

There is no problem to carry the definition. Let $M_\zeta = \bigcup\limits_{\alpha<\lambda^+} M^\zeta_\alpha \in K_{\lambda^+}$, and for each $\nu \in \lim_{\lambda^+}(\mathscr{T}_\zeta)$ let $N^\zeta_\nu = \bigcup\limits_{\alpha<\lambda^+} N^\zeta_{\nu\restriction\alpha}$, clearly $M_\zeta \leq_{\mathfrak{K}} N^\zeta_\nu$ and $a \in N^*_\nu$ and $\langle {\rm tp}(a, M_\zeta, N^\zeta_\nu) : \nu \in \lim_{\lambda^+}(\mathscr{T}_\zeta)\rangle$ are pairwise distinct members of $\mathscr{S}(M_\zeta)$ (if $\mathfrak{K}_\lambda$ fails the amalgamation property, we should add: pairwise contradiction). This proves clause $(*)$ of part (1).
2) This part follows by 1.15(2).

3) Assume that the conclusion fails. Now if $(A)_\lambda \vee (B)_\lambda$ of 2.1 let χ^* be as there then by 3.1(3) we get $\dot I(\lambda^+, K) \geq \chi^* > 1$ contradicting assumption $(f)^+$. Hence in 2.1, case (C) holds, so let $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ be as there, so by part (2) for some $M \in K_{\lambda^+}$ we have $|\mathscr{S}_*(M)| = 2^{\lambda^+}$. By 3.3 with λ^+ here standing for λ there (!) by our assumption toward contradiction we deduce $\mathrm{cf}([2^{\lambda^+}]^{\lambda^{++}}, \subseteq) < 2^{\lambda^{++}}$. By 2.1 with λ^+ here standing for λ there; case $(A)_{\lambda^+}$ cannot hold (recall that in this case $\chi^* = 2^{\lambda^{++}}$), so we can assume $(C)_{\lambda^+} \vee (B)_{\lambda^+}$ occurs.

Now if $(2^\lambda > \lambda^+)+$ (WDmId(λ^+) is not λ^{++}-saturated) we get the desired result as follows.

<u>Case 1</u>: $\neg(**)'_\lambda$ of 1.9 holds.

The result follows by 1.12.

<u>Case 2</u>: $(**)'_\lambda$ of 1.11 holds.

By 1.11 we get a contradiction to $\dot I(\lambda^+, K) = 1$.
So one of the assumptions of the previous paragraph fails. If the second fails (i.e. WdmId(λ^+) is λ^{++}-saturated as we are in case $(B)_\lambda$ or $(C)_\lambda$ (see (a) of 3.4(1)) so by clause (ι) of 2.1 we have $2^\lambda = \lambda^+, 2^{\lambda^+} = \lambda^{++}$. So in both cases $2^\lambda = \lambda^+$. However, once we know $2^\lambda = \lambda^+$ we deduce that there is a model in λ^+ saturated over λ and we apply the claim below. $\square_{3.4}$

3.5 Claim. *Assume* $(a)-(e), (f)'$ *of 3.4 and* (g) *below* <u>then</u> $\dot I(\mathfrak{K}, \lambda^{++}) \geq \mu_{\mathrm{wd}}(\lambda^{++}, 2^\lambda)$ *where*

(g) *there is* $M \in K_{\lambda^+}$ *saturated over* λ.

Proof. Claim [Sh 576, 3.16](3) possibility $(*)_2$ applies.
Alternatively see [Sh 838]. $\square_{3.4}$

3.6 Claim. *Assume*

(a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$

(b) $\mathfrak{K}$ *an* a.e.c. *is categorical in* λ, λ^+ *and* $\mathrm{LS}(\mathfrak{K}) \leq \lambda$

(c) $1 \leq \dot I(\lambda^{++}, K) < \mu_{\mathrm{wd}}(\lambda^{++}, 2^{\lambda^+})$

(d) K^3_λ *has the weak extension property above* (M^*, N^*, a).

<u>Then</u> *above* $(M^*, N^*, a) \in K^3_\lambda$ *the minimal triples are dense.*

Proof. Assume toward contradiction that above $(M^*, N^*, a) \in K^3_\lambda$ there is no minimal type. If $2^\lambda = \lambda^+$, then there is a $M \in K_{\lambda^+}$ saturated over λ hence we finish by 3.5 above. So we can assume $2^\lambda > \lambda^+$, hence [Sh 430, 6.3] (with λ^+ here standing for λ there so μ there is $\le \lambda$ so $\delta < \lambda^+$ hence $|\mathscr{T}| \le |\delta| \le \lambda$ and let $\kappa =$ cf(δ)) there are $\kappa \le \lambda$ and[4] tree $\mathscr{T}$ with $\le \lambda$ nodes and κ levels with $|\lim_\kappa(\mathscr{T})| > \lambda^+$ hence for some $M \in K_\lambda, |\mathscr{S}_*(M)| > \lambda^+$ (e.g. by the proof of 3.4(1)). If WDmId(λ^+) is not λ^{++}-saturated then in 1.12 assumption (b) holds, and assumptions (c) + (d) + (e) holds by the assumptions of the present claim but not the conclusion, so (a) fails, that is $(**)_\lambda$ of 1.9 holds hence by 1.9, $(*)_\lambda$ of 1.8 holds. But now 1.8 contradicts clause (b) of the assumption, so we have to assume that WDmId(λ^+) is λ^{++}-saturated. Hence clause (ι) of 2.1, Case B does not occur, hence Cases B,C of 2.1 do not occur and hence Case A occurs. So by 3.1(3) we get a contradiction to categoricity in λ^+.

$\square_{3.6}$

[4]let $\kappa = \text{Min}\{\sigma : 2^\sigma > \lambda^+\}$, so if $2^{<\sigma} \le \lambda$ then $({}^{\sigma>}2, \triangleleft)$ is okay, otherwise ${}^{\sigma>}2 = \bigcup\limits_{i<\lambda^+} \mathscr{T}_i, |\mathscr{T}_i| \le \lambda, \mathscr{T}_i$ increasing with i so for some $i, |\{\eta \in {}^\sigma 2 : \bigwedge\limits_{\alpha<\sigma} \eta \restriction \alpha \in \mathscr{T}_i\}| > \lambda^+$

§4 A minor debt

4.1 Claim. *We can prove [Sh 576, 4.2t] also for* $\lambda = \aleph_0$.

Proof. We ask:
<u>Question 1</u>: are there $M <_{\mathfrak{K}} N$ in K_λ such that for no $a \in N \backslash M$ is $\mathrm{tp}(a, M, N)$ minimal?

If the answer is yes, we can find $\langle M^1_i : i < \lambda^+ \rangle$ a representation of a model $M^1 \in K_{\lambda^+}$ such that: $a \in M^1_{i+1} \backslash M^1_i \Rightarrow \mathrm{tp}(a, M^1_i, M^1_{i+1})$ is not minimal. This implies $a \in M^1 \backslash M^1_i \Rightarrow \mathrm{tp}(a, M^1_i, M^1)$ is not minimal (as for some $j \in [i, \lambda^+)$ we have $a \in M^1_{j+1} \backslash M^1_j$ so $(M^1_i, M^1_{j+1}, a) \le (M^1_j, M^1_{j+1}, a)$ and the latter is not minimal). But we can build another representation $\langle M^2_i : i < \lambda^+ \rangle$ of $M^2 \in K_{\lambda^+}$ such that for each $i < \lambda^+$ for some $a \in M^1_{i+1} \backslash M^1_i$, $\mathrm{tp}(a, M^1_i, M^1_{i+1})$ is minimal (as there is a minimal triple). So $M^1 \ncong M^2$.

So we assume the answer is no.

<u>Question 2</u>: If $M \in K_\lambda, \Gamma \subseteq \Gamma^*_M =: \{p \in \mathscr{S}(M) : p \text{ minimal}\}$ and $|\Gamma| \le \lambda$, is there N such that: $M <_{\mathfrak{K}} N \in K_\lambda$ and N omit every $p \in \Gamma$?

If the answer to question 2 is yes, we can build $\langle M_\eta : \eta \in {}^{\lambda^+ >}2 \rangle$ as in the proof of 1.8 (more exactly $\eta \triangleleft \nu \Rightarrow M_\eta \le_{\mathfrak{K}} M_\nu, M_\eta \in K_\lambda$) and we also have $\Gamma_\eta \subseteq \{p :$ for some $N \le_{\mathfrak{K}} M_\eta, N \in K_\lambda$ and $p \in \mathscr{S}(N)$ is minimal not realized in $M_\eta\}$ have cardinality $\le \lambda, \eta \triangleleft \nu \Rightarrow \Gamma_\eta \subseteq \Gamma_\nu$ and there is $p \in \Gamma_{\eta \hat{} <1>}$ realized in $M_{\eta \hat{} <0>}$ (and if you like also $p' \in \Gamma_{\eta \hat{} <0>}$ realized in $M_{\eta \hat{} <1>}$). So by [Sh 576, 1.6t] we get $I(\lambda^+, K) = 2^\lambda$. So assume the answer is no and for every $M \in K_\lambda$ let Γ_M be a counterexample. Let $\langle M^1_i : i < \lambda^+ \rangle$, representing a model $M^1 \in K_{\lambda^+}$ be such that $i < \lambda^+ \ \& \ p \in \Gamma_{M_i} \Rightarrow p$ realizes in M. Now as in the proof of saturated = model homogeneous (see [Sh 576, 0.21t]) we can prove M^1 is saturated. But this proves more than required: $|\mathscr{S}(M^1_\ell)| \le \lambda^+$. $\square_{4.1}$

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