FEW NON MINIMAL TYPES
AND NON-STRUCTURE
SH603

SAHARON SHELAH

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram Jerusalem 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

Abstract. We deal with abstract elementary classes $K$ which has amalgamation in
$\lambda$, categorical in $\lambda$ and in $\lambda^+$. Our main result is that if $(\beth_{\omega_1} \leq 2^\lambda < 2^{\lambda^+}$ and) the
minimal types on members of $K_\lambda$ are not dense (among non algebraic (complete)
types over models in $K_\lambda$, extending our given model), then the number of models in
$K$ of cardinality $\lambda^{++}$ is maximal. For this we deal with some claims in pcf. This
improves a result in [Sh 576], it is essentially self-contained.

Saharon:
(a) in 3.4 proof: $|\mathcal{S}_\kappa(M)| \geq \lim_\kappa(T)$, reference (but 3.4 contains a proof, 1.15
contains a statement)
(b) also in 1.12; there is fact in end of §1
(c) weaker coding: 3.4 refer to Definition ?

2000 Mathematics Subject Classification. - 03C45, 03C75, 03C95, 03E04.
Key words and phrases. model theory, abstract elementary classes, classification theory, cate-
goricity, nonstructure theory, pcf theory.

I thank Alice Leonhardt for the beautiful typing
Split from Sh600; work done Spring of 1995
Revised after proof reading for the Proc! and more
Latest version - 04/May/28

Typeset by AMSTeX
§0 Introduction

[We explain our aim and define our framework.]

§1 Non minimal types and nonstructure

[We define unique amalgamation, UQ, and try to use it for building many models in $\lambda^+$ when $2^\lambda < 2^{\lambda^+}$ (so the weak diamond holds). If this approach fails we still get the many models in $\lambda^{++}$ by the “easy” criterion of [Sh 576, §3] or [Sh:F603] but it works only if the weak diamond ideal on $\lambda^+$ is not $\lambda^{++}$-saturated.]

§2 Remarks on pcf

[We prove some pcf observations needed here.]

§3 Finishing the many models

[We prove the result of §1 without the extra assumption on the saturation of the weak diamond ideal.]

§4 A minor debt

[There was one point in [Sh 576] where we use $\lambda > \aleph_0$, though our aim there was to generalize theorem known for $\lambda = \aleph_0$. We eliminate this use.]
§0 Introductions

In [Sh 576] there was an important point where we used as assumption $I(\lambda^+, K) = 0$. This was fine for the purpose there, but is unsuitable in other frameworks, like [Sh 600]: we would like to analyze what occurs in higher cardinals, so our main aim here is to eliminate its use and add to our knowledge on non-structure.

The point was “the minimal triples in $K^3_\lambda$ are dense” ([Sh 576, 3.30, pg.88, 3.17t]). For this we assume we have a counterexample, and try to build many non-isomorphic models. Hence we get cases of amalgamation which are necessarily unique. Those “unique amalgamations” are normally too strong (even for first order superstable theories), but here they help us to prove positive theorems, controlling omitting types. So we try to build many models in $\lambda^+$ by omitting “types” over models of size $\lambda$, in a specific way where unique amalgamation holds. If this argument fails, we prove $\mathbf{C}^1_{\lambda, \lambda}$ has weak $\lambda^+$-coding (see [Sh 838]) and by it get $2^{\lambda^+}$ non-isomorphic models except when the weak diamond ideal on $\lambda^+$ is $\lambda^{++}$-saturated; this is done in §1. In §3 we work harder and by partition to cases relying on pcf theory we succeed to get the full result. We work also to get large IE (many models no one $\leq_{\mathbb{R}}$-embedding to another). The pcf lemmas (which are pure infinite combinatorics) are dealt with in §2.

Compared to its original [Sh 603] we:

(a) there was also another point left in [Sh 576, 4.2t], for the case $\lambda = \aleph_0$ only, this was filled in §4 but hence omitted as it was moved to citeSh:EF6

(b) we rely on [Sh 838] which correct, improves and simplify [Sh 576, §3]

(c) the main changes in this version is improving 1.8 to 1.11 and simplification of Definition 1.6.

The result of 3.3 is not meaningful outside the general aim here; move to [Sh 838, §1]?

An aim of the present revision is to weaken the assumption “$K$ is categorical in $\lambda^+$” to having an intermediate number of models in $\lambda^+$.

∗ ∗ ∗

0.1 Definition. We say $\mathcal{R} = (K, \leq_{\mathbb{R}})$ is an abstract elementary class, aec or a.e.c. in short, if $\tau = \tau_K$ is a fixed vocabulary, $K$ a class of $\tau$-models (and Ax0 holds and) AxI-VI hold where:

Ax0: The holding of $M \in K, N \leq_{\mathbb{R}} M$ depends on $N, M$ only up isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and $[\text{if } N \leq_{\mathbb{R}} M\text{ and } f \text{ is an isomorphism from } M \text{ onto the } \tau\text{-model } M' \text{ mapping } N \text{ onto } N' \text{ then } N' \leq_{\mathbb{R}} M']$. 
AxI: If $M \leq_{\mathcal{R}} N$ then $M \subseteq N$ (i.e. $M$ is a submodel of $N$).

AxII: $M_0 \leq_{\mathcal{R}} M_1 \leq_{\mathcal{R}} M_2$ implies $M_0 \leq_{\mathcal{R}} M_2$ and $M \leq_{\mathcal{R}} M$ for $M \in K$.

AxIII: If $\lambda$ is a regular cardinal, $M_i$ (for $i < \lambda$) is $\leq_{\mathcal{R}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathcal{R}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup_{i < \delta} M_i$) then $M_0 \leq_{\mathcal{R}} \bigcup_{i < \lambda} M_i$.

AxIV: If $\lambda$ is a regular cardinal, $M_i$ (for $i < \lambda$) is $\leq_{\mathcal{R}}$-increasing continuous, $M_i \leq_{\mathcal{R}} N$ for $i < \lambda$ then $\bigcup_{i < \lambda} M_i \leq_{\mathcal{R}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathcal{R}} N$ for $\ell = 0, 1$, then $M_0 \leq_{\mathcal{R}} M_1$.

AxVI: LS($\mathcal{R}$) exists\(^1\), where LS($\mathcal{R}$) is the minimal cardinal $\lambda$ such that: if $A \subseteq N$ and $|A| \leq \lambda$ then for some $M \leq_{\mathcal{R}} N$ we have $A \subseteq |M| \leq \lambda$ and we demand for simplicity $|\tau| \leq \lambda$.

0.2 Notation: 1) $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu$.

See more in [Sh 576, §0] or [Sh 600, §0].

0.3 Definition. 1) For $\mu \geq \text{LS}(\mathcal{R})$ and $M \in K_\mu$ we define $\mathcal{S}(M)$ as:

{$tp(a, M, N) : M \leq_{\mathcal{R}} N \in K_\mu$ and $a \in N$} where $tp(a, M, N) = (M, N, a)/E_M$ where $E_M$ is the transitive closure of $E^a_M$, and the two-place relation $E^a_M$ is defined by:

$$(M, N_1, a_1)E^a_M(M, N_2, a_2) \text{ iff there is } N \in K_\mu \text{ and } \leq_{\mathcal{R}}\text{-embeddings}$$

$$f_\ell : N_\ell \to N \text{ for } \ell = 1, 2 \text{ such that:}$$

$$f_1 \upharpoonright M = \text{id}_M = f_2 \upharpoonright M \text{ and } f_1(a_1) = f_2(a_2).$$

(of course $M \leq_{\mathcal{R}} N_1, M \leq_{\mathcal{R}} N_2$ and $a_1 \in N_1, a_2 \in N_2$)

2) We say “$a$ realizes $p$ in $N$” if $a \in N, p \in \mathcal{S}(M)$ and for some $N' \in K_\mu$ we have $M \leq_{\mathcal{R}} N' \leq_{\mathcal{R}} N$ and $a \in N'$ and $p = \text{tp}(a, M, N')$; so $M, N' \in K_\mu$ but possibly $N \notin K_\mu$.

3) We say “$a_2$ strongly realizes $(M, N^1, a^1)/E^a_M$ in $N$” if for some $N^2, a^2$ we have

\(^1\)We normally assume $M \in \mathcal{R} \Rightarrow \|M\| \geq \text{LS}(\mathcal{R})$, here there is no loss in it. It is also natural to assume $|\tau(\mathcal{R})| \leq \text{LS}(\mathcal{R})$ which means just increasing LS($\mathcal{R}$), but no real need.
$M \leq_{\mathcal{R}} N^2 \leq_{\mathcal{R}} N$ and $a_2 \in N^2$ and $(M, N^1, a^1) \in E_{M}^{a_1} (M, N^2, a^2)$.

(Note: if $M_0$ is an amalgamation base, see below, then the difference between realize and strongly realize disappears).

4) We say $M_0 \in \mathcal{R}_\lambda$ is an amalgamation base if: for every $M_1, M_2 \in \mathcal{R}_\lambda$ and $\leq_{\mathcal{R}}$-embeddings $f_\ell : M_0 \rightarrow M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathcal{R}_\lambda$ and $\leq_{\mathcal{R}}$-embeddings $g_\ell : M_\ell \rightarrow M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.

5) We say $\mathcal{R}$ is stable in $\lambda$ if $\text{LS}(\mathcal{R}) \leq \lambda$ and $M \in K_\lambda \Rightarrow \|S(M)\| \leq \lambda$.

6) We say $N$ is $\lambda$-universal over $M$ if for every $M', M \leq_{\mathcal{R}} M' \in K_\lambda$, there is a $\leq_{\mathcal{R}}$-embedding of $M'$ into $N$ over $M$. If we omit $\lambda$ we mean $\|N\|$.

7) We say $N$ is $(\lambda, \kappa)$-brimmed over $M$ if there is an $\leq_{\mathcal{R}}$-increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$, such that $M = M_0, M_\kappa = N, M_i \in K_\lambda, M_{i+1}$ is an amalgamation base, $M_{i+1}$ is universal over $M_i$. Replacing $\kappa$ by $*$ means “for some $\kappa = \text{cf}(\kappa) \leq \lambda$”. We omit “over $M$” to mean “for some $M \in K_\lambda$”.

8) $K_3^{\lambda} = \{(M, N, a) : M \leq_{\mathcal{R}} N, a \in N \setminus M \text{ and } M, N \in \mathcal{R}_\lambda\}$, with the partial order $\leq$ defined by $(M, N, a) \leq (M', N', a')$ iff $a = a', M \leq_{\mathcal{R}} M'$ and $N \leq_{\mathcal{R}} N'$.

9) We say $(M, N, a) \in K_3^{\lambda}$ is minimal if $(M, N, a) \leq (M', N_\ell, a) \in K_3^{\lambda}$ for $\ell = 1, 2$ implies $\text{tp}(a, M', N_1) = \text{tp}(a, M', N_2)$. We say $p \in \mathcal{S}(M)$ is minimal if $p = \text{tp}(a, M, N)$ for some minimal triple $(M, N, a)$ from $K_3^{\lambda}$, so $\lambda = \|M\|$.

10) We say $\mathcal{R}$ has amalgamation in $\lambda$ if every $M \in \mathcal{R}_\lambda$ is an amalgamation base.
§1 Non-minimal triples and non-structure

We shall quote here [Sh 838] but in a black box nature.

1.1 Context.

(a) $\mathcal{R}$ abstract elementary class with $\text{LS}(\mathcal{R}) \leq \lambda$ and $K_\lambda \neq \emptyset$

(b) $\mathcal{R}$ has amalgamation in $\lambda$.

Remark. Alternatively we can use a weaker context.

1.2 Definition. 1) For $x \in \{a, d\}$ we say $\text{UQ}_\lambda^x(M_0, M_1, M_2)$ if:

(a) $M_\ell \in K_\lambda$ for $\ell \leq 3$

(b) $M_0 \leq_{\mathcal{R}} M_\ell$ for $\ell = 1, 2$

(c) if for $i \in \{1, 2\}$ we have $M_i^0 \in K_\lambda$, for $\ell < 4$ and $M_i^0 \leq_{\mathcal{R}} M_i^0 \leq_{\mathcal{R}} M_3^3$ for $i = 1, 2$, $\ell = 1, 2$ and $x = d \Rightarrow M_i^1 \cap M_2^2 = M_0^0$ and $f_i^1$ is an isomorphism from $M_\ell$ onto $M_i^1$ for $\ell < 3$ and $f_0^1 \subseteq f_1^1, f_0^2 \subseteq f_2^2$ then there are $M_3', f_3$ such that $M_3^3 \leq_{\mathcal{R}} M_3'$ and $f_3$ is an $\leq_{\mathcal{R}}$-embedding of $M_3^1$ into $M_3'$ extending $(f_0^2 \circ (f_1^1)^{-1}) \cup (f_2^2 \circ (f_1^1)^{-1})$ i.e. $f_3 \circ f_1^1 = f_2^2$ & $f_3 \circ f_2^1 = f_3^2$

(d) $M_0 \leq_{\mathcal{R}} M_\ell \leq_{\mathcal{R}} M_3 \in K_\lambda$ for $\ell = 1, 2$

(e) $x = d \Rightarrow M_1 \cap M_2 = M_0$.

2) We say $\text{UQ}_\lambda^x(M_0, M_1, M_2)$ if $\text{UQ}_\lambda^x(M_0, M_1^1, M_2^2, M_3)$ for some $M_3$ and $M_1^1, M_2^2$ isomorphic to $M_1, M_2$ over $M_0$ respectively.

3) If we omit $x$, we mean $x = a$.

4) $K_\lambda^{3,*}$ is the family of triples $(M, N, a) \in K_\lambda^3$ such that there is no minimal triple above it.

5) $K_\lambda^{2,*}$ is the family $\{(M, N) : (M, N, a) \in K_\lambda^{3,*}\}$.

6) For $M \in K_\lambda$ let $\mathcal{S}(M) = \{p \in \mathcal{S}(M) : (M, N, a) \in K_\lambda^{3,*} \text{ we have } p = \text{tp}(a, M, N)\}$.

7) For $M \in K_\lambda^+$ let $\mathcal{S}(M) = \{\text{tp}(a, M, N) : (M, N, a) \in K_\lambda^+ \text{ and for some } M_0 \leq_{\mathcal{R}} N_0 \leq_{\mathcal{R}} N \text{ we have } M_0, N_0 \in K_\lambda, M_0 \leq_{\mathcal{R}} M \text{ and } (M_0, N_0, a) \in K_\lambda^{3,*}\}$.

The reader may find it helpful to look at the following example.

1.3 Example. Let $K$ be the class of $M = (|M|, E^M, |M| \geq \lambda$ and $E^M$ is an equivalence relation on $|M|$ and $\leq_{\mathcal{R}}$ be being a submodel. Then $\text{UQ}_\lambda(M_0, M_1, M_2)$ iff
Given

3) Chasing arrows, we should prove clause (c) of Definition 1.2(1). Assume we are trivial.
Proof.

UQ

and also

4) If θ, we are done.

5) Let λ1 Claim. 1) Symmetry: assuming x ∈ {a, d} we have UQ(1)(M0, M1, M2, M3) ⇒ UQ(2)(M0, M2, M1, M3); we can also omit M3.
2) UQ(1)(M0, M1, M2) ⇒ UQ(2)(M0, M1, M2) recalling M0 is an amalgamation base (in Rλ) by clause (b) of 1.1.
3) UQ(1)(M0, M1, M2, M3) iff clauses (a), (b), (d), (e) of Definition 1.2(1),(2) holds and also (c)-, i.e., clause (c) restricted to the case M1 = M2 for ℓ ≤ 3.
4) If UQ(1)(M0, M1, M2, M3), M3 ≤R M5 then UQ(1)(M0, M1, M2, M3); and also the inverse: if UQ(2)(M0, M1, M2, M3) and M1 ∪ M2 ⊆ M3 ≤R M5 then UQ(2)(M0, M1, M2, M3).
5) Assume (M, N, a) ∈ Kλ and it is not minimal (even less) then ¬UQ(M, N, N).

Proof. 1.2) Trivial.
3) Chasing arrows, we should prove clause (c) of Definition 1.2(1). Assume we are given ⟨M1 : ℓ < 4⟩, ⟨M2 : ℓ < 4⟩, ⟨f1i : ℓ < 3⟩ as there for i = 1, 2. First for i = 1, 2 apply clause (c)- to ⟨M1 : ℓ < 4⟩, ⟨f1i : ℓ < 3⟩. So there are N3, f3 such that: M3 ≤R M4 ∈ Kλ, and f3 a ≤R-embedding of M3 into N4 extending f1i ∪ f3.
As R has amalgamation in λ (by 1.1(b)) there are N ∈ Kλ and ≤R-embeddings g : N → N such that g1 ∘ f3 = g2 ∘ f3, so we are done.
4) Again by the amalgamation i.e., 1.1(b).
5) Let N0 = N, a0 = a.
We can find N1, a1 such that

⊕ N ≤R N1 ∈ Rλ, a* ∈ N1 \{a} and a* ≠ a
(this follows from non-minimality, and is all that we need).

Hence we can find N2, f1 such that: N1 ≤R N2, f1 is a ≤R-embedding of N into N2 and f(a) = a1. Clearly we have gotten two contradictory amalgamations, so we are done. □1.4

1.5 Claim. 1) transitivity: If UQ(1)(M0, N0, M0, N0) for ℓ = 0, 1 then UQ(1)(M0, N0, M1, N1).
2) If θ = cf(θ) < λ+, and ⟨Mi : i ≤ θ⟩ is ≤R-increasing continuous and ⟨Ni : i ≤ θ⟩ is ≤R-increasing and UQ(1)(M0, M0, N0) for each i < θ then UQ(1)(M0, N0, M0, N0).
3) Assume:
1.6 Definition. 1) $\alpha, \beta < \lambda^+$
2) $M_{i,j} \in K_\lambda$ for $i \leq \alpha, j \leq \alpha$
3) $i_1 \leq i_2 \leq \alpha$ & $j_1 \leq j_2 \leq \beta \Rightarrow M_{i_1,j_1} \leq M_{i_2,j_2}$
4) If $\langle M_{i,j} : i \leq \alpha \rangle$ is $\leq \mathcal{R}$-increasing continuous for each $j \leq \beta$
5) If $\langle M_{i,j} : j \leq \beta \rangle$ is $\leq \mathcal{R}$-increasing continuous for each $i \leq \alpha$
6) $UQ_\lambda(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ for every $i < \alpha, j < \beta$.

Then $UQ_\lambda(M_{0,0}, M_{\alpha,0}, M_{0,\beta}, M_{\alpha,\beta})$.
4) If $UQ_\lambda(M_0, M_1, M_2)$ and $M_0 \leq \mathcal{R} M_1' \leq \mathcal{R} M_1$ and $M_0 \leq \mathcal{R} M_2' \leq \mathcal{R} M_2$ then $UQ_\lambda(M_0, M_1', M_2')$.
5) If $M \leq \mathcal{R} N_\ell$ for $\ell = 1, 2$ and $N_1$ can be $\leq \mathcal{R}$-embedded into $N_2$ over $M$, then $UQ_\lambda(M, N_2, M')$ implies $UQ_\lambda(M, N_1, M')$.

Proof. Chasing arrows (note: $UQ = UQ^d$ is easier than $UQ^d$, for $UQ^d$ the parallel claim is not clear at this point, e.g. the straightforward proof of transitivity fails and we can construct a counterexample).

1.7 Observation. 1) $\leq^*$ is a partial order of $\mathcal{T}^\text{dis}_\lambda[\mathcal{K}]$.
2) $K_\lambda^\text{dis}[\mathcal{K}]$ is non-empty, e.g., $(M, \emptyset) \in \mathcal{T}^\text{dis}_\lambda[\mathcal{K}]$ when $M \in K_\lambda$.
3) If $\langle (M_i, \Gamma_i) : i < \delta \rangle$ is $\leq^*$-increasing then this sequence has a l.u.b. $(M_\delta, \Gamma_\delta) =: \cup\{(M_i, \Gamma_i) : i < \delta\}$ where $M_\delta = \cup\{M_i : i < \delta\}$ and $\Gamma = \{N_i : \text{for some } i < \delta\}$ and $N_i \in \Gamma_i$ letting $N_j \in \Gamma_j$ be the unique $\ell' \in \Gamma_j$ such that $N_i \leq \mathcal{R} \ell'$ (necessarily well defined we have $\ell = \cup\{N_j : j \in [i, \delta]\}$).
1.8 Claim. Assume $2^\lambda < 2^{\lambda^+}$ or at least the definitional weak diamond; i.e.,
$\text{DIW}^+(\lambda^+)$, see [Sh:35, 1.7]. If $(\ast)_\lambda$ or at least $(\ast)'_\lambda$ below holds (hence above some triple from $K^3_\lambda$ there is no minimal one), then $I(\lambda^+, K) \geq \mu_{\text{wd}}(\lambda^+)$ where

$(\ast)_\lambda$ for every $(M, \Gamma) \in K^{\text{dis}}_\lambda[\mathcal{R}]$ for some $\Gamma', \Gamma''$ we have $(M, \Gamma) \leq^* (M', \Gamma')$ or just

$(\ast)'_\lambda$ for some $(M_\mu, \Gamma_\mu) \in K^{\text{dis}}_\lambda[\mathcal{R}]$, if $(M_0, \Gamma_0) \leq^* (M, \Gamma)$ then for some $\Gamma', \Gamma''$ we have $(M, \Gamma) \leq^* (M', \Gamma')$.

Proof. Note that as $K^{\text{dis}}_\lambda[\mathcal{R}] \neq \emptyset$, see 1.7(2), clearly $(\ast)_\lambda \Rightarrow (\ast)'_\lambda$ hence we can assume $(\ast)'_\lambda$.

We choose by induction on $\alpha < \lambda, \langle (M_\eta, \Gamma_\eta, \Gamma^+_\eta) : \eta \in ^\alpha \lambda \rangle$ such that:

(a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$
(b) $(M_\eta, \Gamma_\eta) \in K^{\text{dis}}_\lambda[\mathcal{R}]
(c) N \in \Gamma_\eta \Rightarrow (N \setminus M_\eta) \cap \lambda^+ = \emptyset
(d) \nu < \eta \Rightarrow (M_\nu, \Gamma_\nu) \leq^* (M_\eta, \Gamma_\eta)
(e) $(M_\eta, \Gamma^+_\eta) \in K^{\text{dis}}_\lambda[\mathcal{R}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \setminus M_\eta) \cap \lambda^+ = \emptyset
(f) \Gamma_\eta \subseteq \Gamma^+_\eta
(g) (M_\eta, \Gamma^+_\eta) \leq^* (M_{\eta^{-}(0)}, \Gamma_{\eta^{-}(0)})
(h) for some $N \in \Gamma^+_\eta$ we have $N \cong_{M_\eta} M_{\eta^{-}(1)}$.

There is no serious problem to carry the induction with $\Gamma^+_\eta$ (for $\eta \in ^\alpha \lambda$) chosen in the $(\alpha+1)$-th step. For $\alpha = 0$ let $(M_<, \Gamma_<)$ be the $(M_\nu, \Gamma_\nu)$ from $(\ast)_\lambda$ except that we rename the elements to make the relevant parts of clauses (a), (c) true. For $\alpha$ limit use 1.7(3) (part on lub). For $\alpha = \beta + 1, \eta \in ^\beta \lambda$, by $(\ast)'_\lambda$ we can find $(M_{\eta^{-}(1)}, \Gamma_{\eta^{-}(1)})$ such that $(M_{\eta^{-}(1)}, \Gamma_{\eta^{-}(1)}) \in K^{\text{dis}}_\lambda[\mathcal{R}]$ and $(M_\eta, \Gamma_\eta) \leq^* (M_{\eta^{-}(1)}, \Gamma_{\eta^{-}(1)})$.

By renaming without loss of generality the universe of $M_{\eta^{-}(1)}$ is some $\gamma_{\eta^{-}(1)} \in (\gamma_\eta, \lambda^+)$ and clause (c) holds. Let $N_\eta$ be isomorphic to $M_{\eta^{-}(1)}$ over $M_\eta$ with $N_\eta \setminus M_\eta$ disjoint to $\lambda^+ \cup \{|N| : N \in \Gamma_\eta \}$ and let $\Gamma^+_\eta = \Gamma_\eta \cup \{N_\eta\}$, so $(M_\eta, \Gamma^+_\eta) \in K^{\text{dis}}_\lambda[\mathcal{R}]$ is disjoint, now apply to it $(\ast)'_\lambda$ to get $(M_{\eta^{-}(0)}, \Gamma_{\eta^{-}(0)})$. Why does clause (h) hold?

By the choice of $N_\eta$. So $M_\eta, \Gamma_\eta, \Gamma^+_\eta (\eta \in \lambda^+ \lambda)$ are defined.

Note: if $\eta^{-}(0) \not< \nu \in \lambda^+ \lambda$, then $M_{\eta^{-}(1)}$ is not $\leq_{\text{nd}}$-embeddable into $M_\nu$ over $M_\eta$ (by clause (g) + (h) because by 1.4(5) + 1.5(5) and clause (i) we have $\neg \UQ(M_\eta, M_{\eta^{-}(1)}, N_\eta)$).

By [Sh 383, 1.4] we get the desired conclusion (really, usually also on IE). \(\square_{1.8}\)

We in 1.8 - 1.11 we investigate the non-structure conclusion of “there is no maximal member of $(\mathcal{F}^{\text{dis}}_\lambda(\mathcal{R}), <^*)$".
1.9 Claim. 1) An equivalent condition for \((\ast)_{\lambda}\) from 1.8 is (respectively):
\[
(\ast_{\ast})_{\lambda}\text{ for every } M \leq_{R} N \text{ from } K_{\lambda}\text{ for some } M' \in K_{\lambda}\text{ and } UQ_{\lambda}(M, M', N).
\]
2) Also, the condition \((\ast')_{\lambda}\) from 1.8 is equivalent to:
\[
(\ast_{\ast'})_{\lambda}\text{ for some } M_0 \in K_{\lambda}\text{ if } M_0 \leq_{R} N \in K_{\lambda}\text{ then for some } M',
\]
\[
M <_{R} M' \in K_{\lambda}\text{ and } UQ_{\lambda}(M, M', N).
\]
3) We have \((\ast)_{\lambda} \Rightarrow (\ast')_{\lambda}\) if \(R\) is categorical in \(\lambda\) then \((\ast)_{\lambda} \iff (\ast')_{\lambda}\).

Proof. 1),2) For any \((M, \Gamma) \in K_{\lambda}^{\text{dis}}[R],\) by “\(R\) has amalgamation in \(\lambda\) and \(LS(R) \leq \lambda\)” (and properties of abstract elementary classes) there are \(N^*, \{f_N : N \in \Gamma\}\) such that:
\[
(a) M \leq_{R} N^* \in K_{\lambda}
\]
\[
(b) \text{ for } N \in \Gamma, f_N\text{ is a } \leq_{R}\text{-embedding of } N \text{ into } N^* \text{ over } M.
\]
This shows \((\ast_{\ast})_{\lambda} \Rightarrow (\ast)_{\lambda}\) and also \((\ast_{\ast'})_{\lambda} \Rightarrow (\ast')_{\lambda}\).

The other direction is deduced by applying \((\ast)_{\lambda} \text{ (or } (\ast')_{\lambda}\text{) to } (M, \{N\}).

3) Should be clear. \(\square_{1.9}\)

We continue 1.8

1.10 Claim. \(\check{I}(\lambda^+, R) = 2^{\lambda^+}\) \text{ if }\n\[\check{0} \quad 2^{\lambda} < 2^{\lambda^+} \text{ or at least the definitional weak diamond for } \lambda^+\]
\[\check{1} \quad (M_*, \Gamma_*) \in K_{\lambda}^{\text{dis}}[R]\]
\[\check{2} \quad \text{if } (M_*, \Gamma_*) \leq^* (M, \Gamma) \text{ then for some } M', \Gamma' \text{ we have } (M, \Gamma) <^* (M', \Gamma')\]
\[\check{3} \quad (M_*, \Gamma_*) \leq^* (M_1, \Gamma_1) \in K_{\lambda}^{\text{dis}}[R] \text{ then we can find } (M_2, \Gamma_2) \text{ such that}\]
\[
(a) (M_1, \Gamma_1) \leq^* (M_2, \Gamma_2) \in K_{\lambda}^{\text{dis}}[R]
\]
\[
(b) \text{ if } (M_2, \Gamma_2) \leq^* (M_\ell, \Gamma_\ell) \text{ for } \ell = 1, 2 \text{ then we can find } (M_3, \Gamma_3) \text{ such that } (M_\ell, \Gamma_\ell) \leq^* (M_\ell+2, \Gamma_\ell+2) \text{ for } \ell = 1, 2 \text{ and } M_3, M_4 \text{ are isomorphic over } M_1.
\]

Remark. 1) This corresponds to case A in the proof of [Sh:E45, 1.7](?).
2) The gain over 1.8 is not large.
Proof. By induction on $\alpha < \lambda$, we choose $\langle (M_\eta, \Gamma, \Gamma^+ : \eta \in \alpha \rangle$ such that:

(a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$
(b) $(M_\eta, \Gamma) \in K^\text{dis}_\lambda R$
(c) $N \in \Gamma_\eta \Rightarrow (N \setminus M_\eta) \cap \lambda^+ = \emptyset$
(d) $\nu < \eta \Rightarrow (M_\nu, \Gamma_\nu) <^* (M_\eta, \Gamma_\eta)$
(e) $(M_\eta^+, \Gamma_\eta^+) = (M_\eta, \Gamma_\eta)$ and $(M_\eta^+, \Gamma_\eta^+) \in K^\text{dis}_\lambda R$ and $N \in \Gamma_\eta^+ \Rightarrow (N \setminus M_\eta) \cap \lambda^+ = \emptyset$
(f) $\Gamma_\eta \subseteq \Gamma^+_\eta$
(g) $(M_\eta^+, \Gamma^+_\eta) \leq^* (M_\eta^{-}(0), \Gamma_\eta^{-}(0))$
(h) for some $N \in \Gamma^+_\eta$ we have $N \cong M_\eta^+$, $M_\eta^+ < 1$
(i) if $\eta \in \lambda^+ > 2$, $\nu \in \{ \eta^{-} < 0, \eta^{-} < 1 > \}$ and $(M_\nu, \Gamma_\nu) \leq^* (M^\ell, \Gamma^\ell) \in K^\text{dis}_\lambda R$
   for $\ell = 1, 2$ then we can find $(M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1, 2$ such that $(M^\ell, \Gamma^\ell) \leq (M^{\ell+2}, \Gamma^{\ell+2})$ and $M^3, M^4$ are isomorphic over $M_\eta$.

There is no problem to carry the induction, to guarantee (i) use assumption $\exists 3$.
Clearly, by clauses (g) + (h), as in the proof of 1.8

$\otimes_1$ if $\eta \in \lambda^+ > 2$ and $(M_\eta^+, \Gamma_\eta^+) \leq^* (M^\ell, \Gamma^\ell) \in K^\text{dis}_\lambda R$ for $\ell = 0, 1$ then we cannot find $(M, \Gamma) \in \mathcal{F}^\text{dis}_\lambda R$ and functions $f_0, f_1$ such that $(M, \Gamma) \leq^*_f (M^\ell, \Gamma^\ell)$ for $\ell = 0, 1$ and $f_0 \upharpoonright M = f_1 \upharpoonright M_1$.

Now we apply [Sh 838, 1.4K], we have to check it assumption.

The main point is proving (*) there, a stronger version of $\otimes_1$ above which says

$\otimes_2$ the following is impossible

(a) $\alpha < \beta < \lambda^+$
(b) $\eta_1, \eta_2 \in \alpha \beta$
(c) $\eta_1^{-} < 0 > \nu_1 \in \beta_2$ and $\eta_1^{-} < 1 > \nu_1 \in \beta_2$
(d) $\eta_2^{-} < 0 > \nu_2 \in \beta_2$ and $\eta_2^{-} < 0 > \nu_2 \in \beta_2$
(e) $f$ is an isomorphism from $M_{\nu_1}$ onto $M_{\nu_2}$ mapping $M_{\eta_1}$ onto $M_{\eta_2}$
(f) $g$ is an isomorphism from $M_{\nu_1}$ onto $M_{\nu_2}$ mapping $M_{\eta_1}$ onto $M_{\eta_2}$
(g) $g \upharpoonright M_{\eta_1} \upharpoonright f \downharpoonright M_{\eta_1}$

Proof of $\otimes_2$. We can find $\Gamma^*_\nu, \Gamma^*_\nu$ such that
\((i)\) \((M_{\nu_2}, \Gamma_{\nu_2}) \leq^* (M_{\nu_2}, \Gamma_{\nu_2}^*) \in K_{\lambda}^{\text{dis}}[\mathcal{F}]\)

\((ii)\) \((M_{\rho_2}, \Gamma_{\rho_2}) \leq^* (M_{\rho_2}, \Gamma_{\rho_2}^*) \in K_{\lambda}^{\text{dis}}[\mathcal{F}]\)

\((iii)\) if \(N \in \Gamma_{\nu_1}\) then \(f\) can be extended to an isomorphism from \(N\) onto some \(N' \in \Gamma_{\nu_2}^*\).

\((iv)\) if \(N \in \Gamma_{\rho_1}\) then \(g\) can be extended to an isomorphism from \(N\) onto some \(N' \in \Gamma_{\rho_2}^*\).

So \((M_{\eta_2}^{-1}, \Gamma_{\eta_2}^{-1}) \leq^* (M_{\nu_2}, \Gamma_{\nu_2}) \leq^* (M_{\eta_2}^{-1}, \Gamma_{\eta_2}^{-1}) \leq^* (M_{\rho_2}, \Gamma_{\rho_2}) \leq^* (M_{\rho_2}, \Gamma_{\rho_2}^*).\)

By applying clause (i) with \(\eta_2, \eta_2^{-1} < 0, (M_{\eta_2}, \Gamma_{\eta_2}^{-1}), (M_{\nu_2}, \Gamma_{\nu_2}), (M_{\rho_2}, \Gamma_{\rho_2})\) here standing for \(\eta, \nu, (M^1, \Gamma^1), (M^2, \Gamma^2)\) there we find \((\mathcal{N}, \Gamma^\xi)\) for \(\xi = 1, 2\) and \(h\) such that

\((v)\) \((M_{\nu_2}, \Gamma_{\nu_2}^*) \leq^* (\mathcal{N}^1, \Gamma_{\nu_2}^1)\)

\((vi)\) \((M_{\rho_2}, \Gamma_{\rho_2}^*) \leq^* (\mathcal{N}^2, \Gamma_{\rho_2}^1)\)

\((vii)\) \(h\) is an isomorphism from \(\mathcal{N}^1\) onto \(\mathcal{N}^2\) over \(M_{\eta_2}\).

But now the mapping \(h \circ f\) and \(g\) contradict the choice of \((M_{\eta_1}^{-1}, \Gamma_{\eta_1}^{-1}), (M_{\eta_1}^{-1}, \Gamma_{\eta_1}^{-1})\) that is \(\ominus_1\) above.

Having proved \(\ominus_2\) we are done. \(\square_{1.10}\)

1.11 Conclusion. If \(2^\lambda < 2^{\lambda^+}\) (or at least the definitional weak diamond for \(\lambda^+\)) and \((*)'_{\lambda}\) below then \(\mathcal{I}(\lambda^+, K) = 2^{\lambda^+}\), where

\((*)'_{\lambda}\) for some \(M_0 \in K_{\lambda}\) if \(M_0 \leq_R M \leq_R N \in K_{\lambda}\) then for some \(M'\) we have \(M <_R M' \in K_{\lambda}\) and UQ\(_{\lambda}(M, M', N)\).

Proof. Clearly \(\ominus_0 + \ominus_1\) from 1.10 holds, and also \(\ominus_2\) holds by our assumption \((*)'_{\lambda}\) and Claim 1.9(2). If \(\ominus_3\) holds too by 1.10 we are done so assume that it fails for \(M, (\mathcal{M}, \Gamma_1)\). Now we define \((M_{\eta}, \Gamma_{\eta})\) as the proof of 1.8 except that:

\((a) - (d)\) as there

\((e)'\) we choose \((M_{<}, \Gamma_{<}) = (M_0, \emptyset)\)

\((f)'\) for each \(\eta\), for no \((M^1, \Gamma^1), (M^2, \Gamma^2)\) do we have:

\((i)\) \((M_{\eta^{-1}} <_0, \eta_{\eta^{-1}} <_0) \leq^* (M^1, \Gamma^1)\)

\((ii)\) \((M_{\eta^{-1}} <_1, \eta_{\eta^{-1}} <_1) \leq^* (M^2, \Gamma^2)\)

\((iii)\) \(M^1, M^2\) are isomorphic over \(M\).
So we can carry the definition as there because we assume that $\mathfrak{p}_3$ of 1.8 fail. \langle M_\eta : \eta \in \lambda^+2 \rangle$ are pairwise non-isomorphic over $M$ (hence over $M_{<\lambda}$) and this suffices.

1.12 Claim. Assume

(a) $(**)_\lambda$ of 1.11 fails (equivalently 1.9)
(b) $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| > \lambda^+$ (follows from “above (M, N, a) $\in K^3_\lambda$ there is no minimal triple” $+2^\lambda > \lambda^+$ see [Sh:E46, xx?])
(c) $K$ is categorical in $\lambda$
(d) $K$ is categorical in $\lambda^+$
(e) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$.

Then $\dot{I}(\lambda^{++}, K) \geq \mu_wa(\lambda^{++}, 2^{\lambda^+})$, (which is $= 2^{\lambda^{++}}$ when $\lambda \geq \beth_\omega$) except possibly when

\(\otimes_\lambda \text{WDmId}(\lambda^+)\) is $\lambda^{++}$-saturated (normal ideal on $\lambda^+$).

Remark. (See [Sh 838, §4]) 1) We may omit the model theoretic assumption (d) in 1.12 if we strengthen the set theoretic assumptions, e.g.

\((*)_1\) for some stationary $S \subseteq S^{\lambda^{++}}_\alpha$ we have $S \in I[\lambda]$ but $S \notin \text{WDmId}(\lambda^{++})$.

2) Note that: if $\lambda = \lambda^{<\lambda}$ and $V = V^Q, Q$ is adding $\lambda^+$-Cohen set (1.1 and) the minimal types are not dense then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$.

3) If in part (2), if we omit 1.1(b), the amalgamation, but demand “no maximal model” [??]. However, the minimality may hold for uninteresting reasons.

Proof. This is done in [Sh 838, e.1].

1.13 Observation. [??] Under the assumptions of 1.12.

Assume

(a) $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is $\leq_{\mathfrak{p}_\lambda}$-increasing continuous, $M = \cup\{M_\alpha : \alpha < \lambda^+ \} \in K_{\lambda^+}$
(b) $M_{\alpha(0)} \leq_{\mathfrak{p}_\lambda} N_0, tp(a_0, M_{\alpha(0)}, N_0) \in \mathcal{S}(M_{\alpha(0)})$. 
Then we can find $\alpha(1), a_1, N_{\alpha(1)}, f$ such that

\begin{itemize}
  \item[(\alpha)] $\alpha(0) \leq \alpha(1) < \lambda^+, M_{\alpha(1)} \leq_{\forall \lambda} N_1,$
  \item[(\beta)] $f$ is a $\leq_{\forall \lambda}$-embedding of $N_0$ into $N_1$
  \item[(\gamma)] $f(a_0) = a_1$
  \item[(\delta)] $tp(a_1, M_{\alpha(1)}, N_1)$ is not realized in $M$.
\end{itemize}

**Question:** Can we find $\alpha(1), a_1, \varepsilon, N_{\alpha(1)}, f$ for $\varepsilon < 2^\lambda$ such that each $(a_1, \varepsilon, N_{\alpha(1)}, f_\varepsilon)$ is as above and $(tp(a_1, \varepsilon, M_{\alpha(1)}, N_{\alpha(1)}, \varepsilon) : \varepsilon < 2^\lambda)$ are pairwise distinct? [??]

1.14 **Remark.** 1) We can get more abstract results.
2) Note $\neg(\ast)_\lambda$ of 1.12 is a “light” assumption, in fact, e.g. its negation has high consistency strength.

1.15 **Fact.** Assume $\mathfrak{K}$ is an abstract elementary class with amalgamation in $\lambda$, and above $(M, N, a) \in K^3_\lambda$ there is no minimal pair and $K^3_\lambda$ has the weak extension property.

1) Assume $\mathcal{T}$ is a tree with $\delta < \lambda^+$ levels and $\leq \lambda$ nodes. Then we can find $(M^*, N_\eta, a) \in K^3_\lambda$ above $(M, N, a)$ for $\eta \in \lim_\delta(\mathcal{T})$ such that $tp(a, M^*, N_\eta)$ for $\eta \in \lim_\delta(T)$ are pairwise distinct so $|\mathcal{I}_*(M)| \geq |\lim_\delta(\mathcal{T})|$. We can add “$(M^*, N_\eta, a)$ is reduced”, (see [Sh 576]).

2) If $M \in K_\lambda$ is universal then $\mathcal{I}_*(M) \geq \sup\{|\lim_\delta(\mathcal{T})| : \mathcal{T} \text{ a tree with } \leq \lambda \text{ nodes and } \delta \text{ levels}\}$.

**Proof.** 1) Straight (or see the proof of 3.4(1)).
2) As for any $N \in K_\lambda$ there is a model $N' \leq_{\forall \lambda} M$ isomorphic to $N$, now $p \mapsto p \mid N'$ is a function from $\mathcal{I}(M)$ onto $\mathcal{I}(N')$ by [Sh 576, §2] hence $|\mathcal{I}_*(M)| + \lambda \geq |\mathcal{I}(N')| = |\mathcal{I}(N)|$. Now use part (1). \hfill $\Box_{1.15}$

---

\textsuperscript{2}If $2^\lambda > \lambda^+$ then for some such $T$ and $\delta, \lambda^+ < \lim_\delta(T)$; as in 3.4
§2 Remarks in pcf

The following will provide us a useful division into cases (it is from pcf theory; on \( \mu_{wd}(\lambda) \) see [Sh 576, 1.1t]), we can replace \( \lambda^+ \) by regular \( \lambda \) such that \( 2^\theta = 2^{<\lambda} < 2^\lambda \) for some \( \theta \).

2.1 Fact. Assume \( 2^\lambda < 2^{\lambda^+} \).

Then one of the following cases occurs:

\( (A) \lambda \) we can find \( \mu \) such that letting \( \chi^* = 2^{\lambda^+} \)

\( (\alpha) \lambda^+ < \mu \leq 2^\lambda \) and \( \text{cf}(\mu) = \lambda^+ \)

\( (\beta) \text{pp}(\mu) = \chi^* \), moreover \( \text{pp}(\mu) = + \chi^* \) and \( \chi^* > 2^\lambda \)

\( (\gamma) (\forall \mu')(\text{cf}(\mu') \leq \lambda^+ < \mu' < \mu \rightarrow \text{pp}(\mu') < \mu) \) hence \( \text{cf}(\mu') \leq \lambda^+ < \mu' < \mu \Rightarrow \text{pp}(\mu') < \mu \)

\( (\delta) \) for every regular cardinal \( \chi \) in the interval \( (\mu, \chi^*] \) there is an increasing sequence \( \langle \lambda_i : i < \lambda^+ \rangle \) of regular cardinals \( > \lambda^+ \) with limit \( \mu \) such that \( \chi = \text{tcf} \left( \prod_{i<\lambda^+} \lambda_i / J_{\lambda^+}^{bd} \right) \)

\( (\varepsilon) \) for some regular \( \kappa \leq \lambda \), for any \( \mu' < \mu \) there is a tree \( T \) with \( \leq \lambda \) nodes, \( \kappa \) levels and \( |\text{lim}_\kappa(T)| \geq \mu' \) (in fact e.g. \( \kappa = \text{Min}\{\kappa : 2^\kappa \geq \mu\} \) is appropriate; without loss of generality \( T \subseteq {\kappa^+}^\lambda \); we can get, of course, a tree \( T \) with \( \text{cf}(\kappa) \) levels, \( \leq \lambda \) nodes and \( |\text{lim}_{\text{cf}(\kappa)}(T)| \geq \mu' \)).

\( (B) \lambda \) for some \( \mu, \chi^* \) we have: clauses (\( \alpha \)) - (\( \varepsilon \)) from above (so \( 2^\lambda < \chi^* \)) and

\( (\zeta) \) there is \( \langle T_\zeta : \zeta < \chi^* \rangle \) such that: \( T_\zeta \subseteq \lambda^+ > 2 \) a tree, of cardinality \( \leq \lambda^+ \) and \( \lambda^+ 2 = \bigcup_{\zeta<\chi^*} \text{lim}_{\lambda^+}(T_\zeta) \) and \( \chi^* < 2^{\lambda^+} \)

\( (\eta) \) \( 2^\lambda < \chi^* < \mu_{wd}(\lambda^+, 2^\lambda) \) (but \( < \mu_{wd}(\lambda^+, 2^\lambda) \) is not used here, see [Sh 576, Definition 1.1t](5))

\( (\theta) \) for some \( \zeta < \chi^* \) we have \( \text{lim}_{\lambda^+}(T_\zeta) \notin \text{WDmTId}(\lambda^+), \) not used here

\( (\iota) \) if there is a normal \( \lambda^{++} \)-saturated ideal on \( \lambda^+ \), e.g. the ideal \( \text{WDmId}(\lambda^+) \) is, then \( 2^{\lambda^+} = \lambda^{++} \) (so as \( 2^\lambda < 2^{\lambda^+} \), necessarily \( 2^\lambda = \lambda^+ \) )

\( (\kappa) \) \( \text{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^* \), equivalently \( \chi^* = \sup\{\text{pp}(\chi) : \chi \leq 2^\lambda, \aleph_1 \leq \text{cf}(\chi) \leq \lambda^+ < \chi\} \) by [Sh:g, Ch.II,5.4]
(C)\(\lambda\) letting \(\chi^* = 2^\lambda\) we have \((\zeta), (\eta), (\theta), (\iota), (\kappa)\) of clause (B) and

(\lambda) for no \(\mu \in (\lambda^+, 2^\lambda)\) do we have \(\text{cf}(\mu) \leq \lambda^+, \text{pp}(\mu) > 2^\lambda\) equivalently

\[2^\lambda > \lambda^+ \Rightarrow \text{cf}([2^\lambda]^{\lambda^+}, \subseteq) = 2^\lambda\] hence (see the proof) \(\mu_{\text{wd}}(\lambda^+, 2^\lambda) = 2^\lambda\) except (maybe) when \(\lambda < \beth\) and there is \(\mathcal{A} \subseteq [\mu_{\text{wd}}(\lambda^+, 2^\lambda)]^\lambda\) such that \(A \neq B \in \mathcal{A} \Rightarrow |A \cap B| = \aleph_0\).

Remark. Remember that

\[\text{cov}(\chi, \mu, \theta, \sigma) = \chi + \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\mu, \text{and every member of} |\chi|^{<\theta} \text{is included in the union of} < \sigma \text{ members of} \mathcal{P}\} \]

Proof. This is related to [Sh:g, II,5.11]; we assume basic knowledge of pcf (or a readiness to believe). Note that if \(2^\lambda > \lambda^+\) then \(\text{cf}([2^\lambda]^{\leq \lambda^+}, \subseteq) = 2^\lambda \iff \text{cov}(2^\lambda, \lambda^{++}, \lambda^+, \aleph_\theta, \aleph_0) = 2^\lambda\) and \(\text{cov}(2^\lambda, \lambda^{++}, \lambda^+, \aleph_0, \aleph_0) \geq \text{cov}(2^\lambda, \lambda^{++}, \lambda^+, \aleph_\theta, \aleph_0) = 2^\lambda\) for \(\theta \in [\aleph_0, \lambda]\).

Possibility 1: \(\chi^* = \text{cov}(2^\lambda, \lambda^{++}, \lambda^+, \aleph_1) = 2^\lambda\)

We shall show that case (C) holds.

Now by the definition of \(\text{cov}\), clause \((\zeta)\) is obvious, as well as \((\kappa)\). As on the one hand by [Shf, AP.1.16 + 1.19] we have \((\mu_{\text{wd}}(\lambda^+, 2^\lambda))^{\aleph_0} = 2^\lambda > 2^\lambda = \chi^*\) and on the other hand \((\chi^*)^{\aleph_0} = (2^\lambda)^{\aleph_0} = 2^\lambda = \chi^*\) necessarily \(\chi^* < \mu_{\text{wd}}(\lambda^+, 2^\lambda)\) so clause \((\eta)\) follows; now clause \((\theta)\) follows from clause \((\zeta)\) as \(\text{WDMFdt}(\lambda^+)\) is \((2^\lambda)^\omega\) complete by [Sh 576, 1.21(\dagger)] and we have chosen \(\chi^* = 2^\lambda\). Now if \(2^\lambda > \lambda^{++}\), (so \(2^\lambda \geq \lambda^{++}\)), then for some \(\zeta < \chi^*, \mathcal{R}_\mathcal{Z}\) is a (tree with \(\leq \lambda^+\) nodes, \(\lambda^+\) levels and) at least \(\lambda^{++}\) \(\lambda^+\)-branches which is well known (see e.g. [J]) to imply “no normal ideal on \(\lambda^+\) is \(\lambda^{++}\)-saturated”; so we got clause \((\iota)\). As for \((\lambda)\) the definition of \(\chi^*\) and the assumption \(\chi^* = 2^\lambda\) we have the first two phrases, as for \(\mu_{\text{wd}}(\lambda^+, 2^\lambda) = 2^\lambda\) by [Shf, AP.14,p.956 + 1.16,p.958] there is \(\mathcal{A}\) as mentioned in \((\lambda)\) and by [Sh 460] we get \(\lambda < \beth\). The “equivalently” holds as \((2^\lambda)^{\aleph_0} = 2^\lambda\).

Possibility 2: \(\chi^* = \text{cov}(2^\lambda, \lambda^{++}, \lambda^+, \aleph_1) > 2^\lambda\)

Let \(\mu = \text{Min}\{\mu : \text{cf}(\mu) \leq \lambda^+, \lambda^+ < \mu \leq 2^\lambda\text{ and pp}(\mu) = \chi^*\}\); clearly we may replace \(\text{pp}(\mu) = \chi^*\) by \(\text{pp}(\mu) = \chi^*\). We know by [Sh:g, II.5.4] that \(\mu\) exists and (by [Sh:g, II.2.3] (2)) clause \((\gamma)\) holds, also \(2^\lambda < \text{pp}(\mu) \leq \mu^{\text{cf}(\mu)}\) hence \(\text{cf}(\mu) = \lambda^+\).
So clauses (α), (β), (γ) hold (of course, for clause (β) use [Sh:g, Ch.II,5.4](2)), and by (γ) + [Sh:g, VIII,§1] also clause (δ) holds.

For clause (ε) let $\Upsilon = \text{Min}\{\Upsilon : 2^\Upsilon \geq \mu\}$, clearly $\alpha < \Upsilon \Rightarrow 2^{[\alpha]} < \mu$ and $\Upsilon \leq \lambda$ (as $2^\lambda \geq \mu$) hence $\text{cf}(\Upsilon) \leq \lambda < \lambda^+ = \text{cf}(\mu)$ hence $2^{<\Upsilon} < \mu$. Now we shall first prove

(*) there is a tree with $\lambda^+$ nodes, $\text{cf}(\Upsilon)$ levels and $\geq \mu$ branches.

Why (*) holds? Otherwise we shall get contradiction to the claim 2.3 below with $\sigma, \kappa, \theta_0, \theta_1, \mu, \chi$ there standing for $\text{cf}(\Upsilon), \lambda^+, \lambda^+, 2^{<\Upsilon}, \mu, (2^\lambda)^+$ here and $T^*$ defined below; let us check the conditions there:

Clause (a): It says $\text{cf}(\Upsilon) < \lambda^+ = \text{cf}(\mu) \leq \lambda^+ \leq 2^{<\Upsilon} < \mu$ which is readily checked except the inequality $\lambda^+ \leq 2^{<\Upsilon}$ but if it fails we immediately get more than required.

Clause (b): This is clause (γ) of (A) which we have proved.

Clause (c): The tree $\mathcal{F}^*$ is $(T > 2, <)$ restricted to an unbounded set of levels of order type $\text{cf}(\Upsilon)$.

Clause (d): Let $\theta_2 =: \text{cov}(2^{<\Upsilon}, \lambda^+, \lambda^+, \text{cf}(\Upsilon)^+)$. So the statement we have to prove is $\text{pp}(\mu) \geq \chi = \text{cf}(\chi) > \theta_2^{\text{cf}(\Upsilon)}$. Now $\text{pp}(\mu) \geq \chi$ holds by the choice of $\mu$ and $\chi = \text{cf}(\chi)$ as $\chi = (2^\lambda)^+$. For the last inequality, by [Sh:g, Ch.II,5.4] and the choice of $\mu$, as we have shown $2^{<\Upsilon} < \mu$ we know $\theta_2 < \mu$, but $\mu \leq 2^\lambda$ so $\theta_2^{\text{cf}(\Upsilon)} \leq (2^\lambda)^{\text{cf}(\Upsilon)} \leq (2^\lambda)^Y \leq (2^\lambda)^\lambda = 2^\lambda < \chi$.

Clause (e): Trivial by the choice of $\theta_2$.

Clause (f): So $\kappa^*$ is $\text{cov}(\theta_0^+, \theta_1^+ + \kappa^+, \sigma^+)$ which means $\text{cov}((\lambda^+)^{\text{cf}(\Upsilon)}, \lambda^+, \lambda^+, \text{cf}(\Upsilon)^+)$. But $\text{cf}(\Upsilon) \leq \lambda$ so this number is $\leq (\lambda^+)^\lambda = 2^\lambda < (2^\lambda)^+ < (2^\lambda)^+ \leq (2^\lambda)^\lambda = 2^\lambda < \chi$.

So we have verified clauses (α) – (f) of 2.3 hence its conclusion holds, but this gives (*), i.e., the desired conclusion in clause (ε) of Case A in 2.1; well not exactly, it gives only $|\mathcal{F}^*| \leq \lambda^+$, so $\mathcal{F}^* = \bigcup_{i<\lambda^+} \mathcal{F}_i$, $\mathcal{F}_i$ increases continuously with each $\mathcal{F}_i$ of cardinality $\leq \lambda$, so for every $\mu' < \mu$ for some $i$ we have $|\lim_{i<\mu}(T_i)| \geq \mu'$, so we have finished proving clause (ε). Together we have, under possibility (2), clauses (α) – (ε) there.

Subpossibility 2a: $\chi^* < 2^{\lambda^+}$.

3 the less easy point is when $\text{cf}(\Upsilon) = \aleph_0$, otherwise we can get the conclusion differently (by [Sh:g, II,5.4]), so 2.1(A) suffice
We shall prove \((B)_\lambda\), so we are left with proving clauses \((\zeta) - (\kappa)\) when \(\chi^* < 2^{\lambda^+}\). By the choice of \(\chi^*\), easily clause \((\zeta)\) (in Case B of 2.1) holds. In clause \((\eta)\), “\(2^\lambda < \chi^*\)” holds as we are in possibility 2.

Also as \(\text{pp}(\mu) = \chi^*\) and \(\text{cf}(\mu) = \lambda^+\) by the choice of \(\mu\) necessarily (by transitivity of pcf, i.e., [Sh:g, Ch.II.2.3](2)) \(\text{cf}(\chi^*) > \lambda^+\) but \(\mu > \lambda^+\). Easily \(\chi \leq \chi^* \land \text{cf}(\chi) \leq \lambda^+ \Rightarrow \text{pp}(\chi) \leq \chi^*\) hence \(\text{cov}(\chi^*, \lambda^+, \lambda^+, \aleph_1) = \chi^*\) by [Sh:g, Ch.II,5.4], which gives clause \((\kappa)\). Now let \(\mathcal{A} \subseteq [\chi^*]^{\lambda^+}\) exemplify \(\text{cov}(\chi^*, \lambda^+, \lambda^+, \aleph_1) = \chi^*\) and let \(\mathcal{A}' = \{B : B\text{ is an infinite countable subset of some } A \in \mathcal{A}\}\). So \(\mathcal{A}' \subseteq [\chi^*]^{\aleph_0}\) and easily \(A \in [\chi^*]^{\lambda^+} \Rightarrow (\exists B \in \mathcal{A}')(B \subseteq A)\) and \(|\mathcal{A}'| \leq \chi^*\) as \((\lambda^+)^{\aleph_0} \leq 2^\lambda < \chi^*\) certainly there is no family of \(> \chi^*\) subsets of \(\chi^*\) each of cardinality \(\lambda^+\) with pairwise finite intersections, hence (by [Sh:b, Ch.XIV,§1] or see [Sh 576, 1.2](1) or [Sh:f, AP,1.16]) we have \(\chi^* < \mu_{\text{add}}(\lambda^+, 2^\lambda)\) thus completing the proof of \((\eta)\).

Now clause \((\theta)\) follows by \((\zeta) + (\eta)\) by [Sh 576, 1.2](5). Also if \(2^{\lambda^+} \neq \lambda^{++}\) then \(2^{\lambda^+} \geq \lambda^++3\) so by clause \((\zeta)\) (as \(\chi^* < 2^{\lambda^+}\)), we have \(\lim_{\lambda^+}(T_\chi) \geq \lambda^+3\) for some \(\zeta\) which is well known to imply no normal ideal on \(\lambda^+\) is \(\lambda^{++}\)-saturated; i.e., clause \((\iota)\). So we have proved that case \((B)_\lambda\) holds.

**Subpossibility 2b: \(\chi^* = 2^{\lambda^+}\).**

We have proved that case \((A)_\lambda\) holds, as we already defined \(\mu\) and \(\chi^*\) and proved \((\alpha), (\beta), (\gamma), (\delta), (\varepsilon)\) we are done.

Still we depend on 2.3 below but first we prove

**2.2 Claim.** Assume

\[
(a) \quad \sigma < \kappa = \text{cf}(\mu) \leq \theta_0 \leq \theta_1 < \mu \leq \theta_0^+
\]

\[
(b) \quad (\forall \mu')[(\theta_0 < \mu' < \mu \land \text{cf}(\mu') \leq \kappa) \Rightarrow \text{pp}(\mu') < \mu]
\]

\[
(c) \quad \theta_2 = \theta_1 + \text{cov}(\theta_1, \theta_0^+, \kappa^+, \sigma^+) \quad \text{(by clause \((b)\) and [Sh:g, Ch.II,5.4] we know that it is < } \mu)\]

\[
(d) \quad \text{pp}(\mu) \geq \chi = \text{cf}(\chi) > \theta_2^\sigma (\geq \theta_1^\sigma \geq \mu).
\]

Then \(\theta_0^\sigma \geq \mu\).

**Remark.** In fact \(\theta_2^\sigma \geq \text{cov}(\theta_1, \theta_0^+, \kappa^+, 2)\).

**Proof.** Assume toward contradiction \(\theta_0^\sigma < \mu\). By [Sh:g, Ch.II.2.3](2) and clause \((b)\) of the assumption we have \(\sup\{\text{pp}(\mu') : \theta_0^+ \leq \mu' \leq \theta_0^\sigma \text{ and } \text{cf}(\mu') \leq \kappa\} < \mu\) hence by [Sh:g, Ch.II,5.4] it follows that

\[
\Box \quad \kappa^* =: \text{cov}(\theta_0^\sigma, \theta_0^+, \kappa^+, \sigma^+) < \mu.
\]
We can by assumptions (b) + (d) and [Sh:g, Ch.II.3.5] + [Sh:g, Ch.VIII.8.1] find $T \subseteq \kappa \geq \mu$, a tree with $\leq \mu$ nodes, $|\lim_\kappa(T)| \geq \chi$, (if $\chi = \text{pp}(\mu)$, the supremum in the definition of $\text{pp}(\mu)$ is obtained by [Sh:g, II.5.4](2)). Moreover, by the construction there is $\Xi \subseteq \lim_\kappa(T), |\Xi| = \chi$ such that $\Xi' \subseteq \Xi$ & $|\Xi'| \geq \chi \Rightarrow |\{\eta | \alpha < \kappa, \eta \in \Xi'\}| = \mu$. By renaming (and also by the construction), without loss of generality

\[ \otimes \text{ if } \eta_0^\ast \langle \alpha_0 \rangle \neq \eta_1^\ast \langle \alpha_1 \rangle \text{ belongs to } T \text{ then } \alpha_0 \neq \alpha_1. \]

So let $\eta_i \in \lim_\kappa(T)$ for $i < \chi$ be pairwise distinct, listing $\Xi$.

As $\mu \leq \theta_1^\ast$ there is a sequence $F = \langle F_\varepsilon : \varepsilon < \sigma \rangle$ satisfying: $F_\varepsilon$ a function from $\mu$ to $\theta_1$ such that $\alpha < \beta < \mu \Rightarrow (\exists \varepsilon < \sigma) F_\varepsilon(\alpha) \neq F_\varepsilon(\beta).

Let $w_{i,\varepsilon} = \{F_\varepsilon(\eta_0(\alpha)) : \alpha < \kappa\}$, so $w_{i,\varepsilon} \in [\theta_1]^\kappa$. By assumption (c) we have $\theta_2 = \theta_1 + \text{cov}(\theta_1, \theta_0^\alpha, \kappa^+, \sigma^+)$ so there is $\mathcal{P} \subseteq [\theta_1]^\theta_0, \theta_2 = |\mathcal{P}|$ such that: any $w \in [\theta_1]^\kappa$ is included in a union of $\leq \sigma$ members of $\mathcal{P}$. So we can find $X_{i,\varepsilon,\zeta} \in \mathcal{P}$ for $\zeta < \sigma$ such that $w_{i,\varepsilon} \subseteq \bigcup \{X_{i,\varepsilon,\zeta} : \zeta < \sigma\}$. So $\bigcup w_{i,\varepsilon} \subseteq X_i = \bigcup \{X_{i,\varepsilon,\zeta} : \zeta < \sigma\}$.

\[ \mathcal{P}^* = \{\bigcup X_\varepsilon : X_\varepsilon \in \mathcal{P} \text{ for } \varepsilon < \sigma\}, \text{ so } \mathcal{P}^* \text{ is a family of } \leq |\mathcal{P}|^\sigma \leq \theta_2^\sigma \text{ sets and } i < \chi \Rightarrow X_i \in \mathcal{P}^*. \]

For each $Y \in \mathcal{P}^*$ let $Z_Y = \{\alpha < \mu : (\forall \varepsilon < \sigma)(F_\varepsilon(\alpha) \in Y)\}$

clearly $Y = X_i \Rightarrow \text{Rang}(\eta_i) \subseteq Z_Y$, also $|Y| \leq \theta_0$ hence $|Z_Y| \leq \theta_0^\sigma < \mu$ hence there is a family $\mathcal{D}_Y$ of cardinality $\kappa^* = (\text{cov}(\theta_0^\sigma, \theta_0^\kappa, \kappa^+, \sigma^+)) < \mu$ whose members are subsets of $Z_Y$ each of cardinality $\leq \theta_0$ such that any $X \in [Z_Y]^\kappa$ is included in the union of $\leq \sigma$ of them. For each $Y \in \mathcal{P}^*$ and $W \in \mathcal{D}_Y$ let $T_W' = \{\eta \in T : (\exists \nu)(\eta < \nu \in T_W)\}.$

So by $\otimes$ above we have: $T_W'$, hence $T_W$ is a set of $\leq |W| + \kappa \leq \theta_0$ nodes in $T$, $\leftarrow$-downward closed. Also

\[ (\ast) \quad |\bigcup_{Y \in \mathcal{P}^*} \mathcal{D}_Y| \leq |\mathcal{P}^*| \times \sup_{Y \in \mathcal{P}^*} |\mathcal{D}_Y| \leq \theta_2^\kappa \times \theta_2^\sigma + \text{cov}(\theta_0^\sigma, \theta_0^\kappa, \kappa^+, \sigma^+) < \mu. \]

However, for every $i < \chi, Y_i \in \mathcal{P}^*$ and $\text{Rang}(\eta_i) \in [Z_Y]^\kappa$ so for some $W \in \mathcal{D}_Y, (\exists \kappa < \kappa)[\eta_i(\alpha) \in W]$ hence $\eta_i \in \lim_\kappa(T_W)$.

By assumption (d) and $(\ast)$ above for some $W \in \bigcup_{Y \in \mathcal{P}^*} \mathcal{D}_Y$ we have

\[ |\{i < \chi : \eta_i \in \lim_\kappa(T_W)\}| = \chi. \]
$T_w$ is (essentially) a tree with $\leq \theta_0$ nodes and contradict the choice of $\Xi = \{\eta_i : i < \chi\}$.
(We could have instead using $\kappa^*, \mathcal{Q}$ to fix $Y = Y$ as $|\mathcal{P}^*| < \chi = \text{cf}(\chi)$.) \hfill $\Box_{2.2}$

2.3 Claim. Assume

(a) $\sigma < \kappa = \text{cf}(\mu) \leq \theta_0 < \theta_1 < \mu$

(b) $(\forall \mu^{'})[\theta_0 < \mu' < \mu \& \text{cf}(\mu') \leq \kappa \rightarrow \text{pp}(\mu') < \mu]$

(c) $\mathcal{T}^*$ is a tree with $\leq \theta_1$ nodes, $\sigma$ levels and $\geq \mu$ $\sigma$-branches

(d) $\text{pp}(\mu) > \chi = \text{cf}(\chi) > \theta_2$

(e) $\theta_2 = \text{cf}(\theta_1, \theta_0^+, \kappa^+, \sigma^+)$

(f) $\kappa^* = \text{cov}(\theta_0^+, \theta_0^+, \kappa^+, \sigma^+) < \chi$.

Then for some subtree $Y \subseteq \mathcal{T}^*, |Y| \leq \theta_0$ and $|\lim_\sigma(Y)| \geq \mu$ (for 2.1 it is enough to prove $\geq \mu'$ for any given $\mu' < \mu$).

Saharon: use of $F$??

Proof. Let $\mathcal{T}, \Xi = \{\eta_i : i < \chi\}$ be as in the proof of the previous claim. Let $\{\nu_\zeta : \zeta < \mu\}$ list $\mu$ distinct $\sigma$-branches of $\mathcal{T}^*$ (see clause (c)). Without loss of generality the set of nodes of $\mathcal{T}^*$ is $\theta_1$. Choose for each $\varepsilon < \sigma$ the function $F_\varepsilon : \mu \rightarrow \theta_1$ by $F_\varepsilon(\gamma) = \nu_\gamma(\varepsilon)$. Define $w_{i,\varepsilon}, \mathcal{P}, X_{i,\varepsilon,\zeta}, Y_i, \mathcal{P}^*$ as in the proof of 2.2. But for $Y \in \mathcal{P}^*$ we change the choice of $Z_Y$, first

$$Y' = \{\beta < \theta_1 : \text{for some } \alpha \in Y, \text{ we have } \beta <_{T^*} \alpha\}$$

So $|Y'| \leq \sigma + |Y|$ and let $Z_Y = \{\alpha < \mu : (\forall \varepsilon < \sigma)(F_\varepsilon(\alpha) \in Y')\}$.

We continue as in the proof of 2.2.

Let us present it in detail. We can by assumptions (b) + (d) and [Sh:g, Ch.II,3.5] + [Sh:g, Ch.VIII,§1] find $\mathcal{T} \subseteq \kappa \geq \mu$, a tree with $\leq \mu$ nodes, $|\lim_\kappa(\mathcal{T})| \geq \chi$, (if $\chi = \text{pp}(\mu)$, the supremum in the definition of $\text{pp}(\mu)$ is obtained by [Sh:g, II,5.4](2)). Moreover, by the construction there is $\Xi \subseteq \lim_\kappa(T), |\Xi| = \chi$ such that $\Xi' \subseteq \Xi \& |\Xi'| \geq \chi \Rightarrow |\{\eta \upharpoonright \alpha : \alpha < \kappa, \eta \in \Xi'\}| = \mu$. By renaming (and also by the construction), without loss of generality

\begin{itemize}
  \item $\otimes$ if $\eta_0 \uparrow (\alpha_0) \neq \eta_1 \uparrow (\alpha_1)$ belongs to $\mathcal{T}$ then $\alpha_0 \neq \alpha_1$.
\end{itemize}
So let $\eta_i \in \lim_\kappa(T)$ for $i < \chi$ be pairwise distinct, listing $\Xi$.

Let $\{\nu_{\zeta} : \zeta < \mu\}$ list $\mu$ distinct $\sigma$-branches of $\mathcal{F}^*$ (see clause (c)). Without loss of generality the set of nodes of $\mathcal{F}^*$ is $\theta_1$ and let $\nu_{\zeta}(\varepsilon)$ be the unique $\alpha < \theta_1$ which belong to the branch $\nu_{\zeta}$ and is of level $\varepsilon$. Define for each $\varepsilon < \sigma$ the function $F_{\varepsilon} : \mu \to \theta_1$ by $F_{\varepsilon}(\gamma) = \nu_{\zeta}(\varepsilon)$ hence $\alpha < \beta < \mu \Rightarrow (\exists \varepsilon < \sigma) F_{\varepsilon}(\alpha) \neq F_{\varepsilon}(\beta).$

For $\varepsilon < \sigma$ and $i < \chi$ let $w_{i,\varepsilon} = \{F_{\varepsilon}(\eta_i(\alpha)) : \alpha < \kappa\}$, so $w_{i,\varepsilon} \in [\theta_1]^\kappa$. By assumption (e) we have

$$\theta_2 = \theta_1 + \cov(\theta_1, \theta_0^+, \kappa^+, \sigma^+),$$

so there is $\mathcal{P} \subseteq [\theta_1]^{\theta_0}, \theta_2 \geq |\mathcal{P}|$ such that: any $w \in [\theta_1]^\kappa$ is included in a union of $\leq \sigma$ members of $\mathcal{P}$. So we can find $X_{i,\varepsilon,\zeta} \in \mathcal{P}$ for $\zeta < \sigma$ such that $w_{i,\varepsilon} \subseteq \bigcup_{\zeta < \sigma} X_{i,\varepsilon,\zeta}$. So $\bigcup_{\varepsilon < \sigma} w_{i,\varepsilon} \subseteq Y_i = \bigcup_{\zeta, \varepsilon < \sigma} X_{i,\varepsilon,\zeta} \in [\theta_1]^{\theta_0}$. Let

$$\mathcal{F}^* = \bigcup_{\varepsilon < \sigma} X_{\varepsilon} : X_{\varepsilon} \in \mathcal{P} \text{ for } \varepsilon < \sigma,$$

so $\mathcal{F}^*$ is a family of $\leq |\mathcal{P}|^\sigma \leq \theta_2^\sigma$ sets each of cardinality $\leq \kappa$ and $i < \chi \Rightarrow Y_i \in \mathcal{F}^*$.

For each $Y \in \mathcal{F}^*$ let

$$Z_Y = \{\beta < \theta_1 : \text{ for some } \alpha \in Y \text{ we have } \beta \leq \varpi_{\mathcal{F}^*} \alpha\}$$

Clearly $Y = Y_i \Rightarrow \text{Rang}(\eta_i) \subseteq Z_Y$, also $|Y| \leq \theta_0$ so $|X_Y| \leq \sigma + |Y| \leq \sigma + \theta_0 = \theta_2$ hence $|Z_Y| \leq \theta_0^\sigma \leq \theta_1^\sigma < \chi$ hence there is a family $\mathcal{P}_Y$ of cardinality $\kappa^* = \cov(\theta_0^\sigma, \theta_1^\sigma, \kappa^+, \sigma^+) < \chi$ by assumption (f) whose members are subsets of $Z_Y$ each of cardinality $\leq \theta_0$ such that any $X \in [Z_Y]^\kappa$ is included in the union of $\leq \sigma$ of them. For each $Y \in \mathcal{P}^*$ and $W \in \mathcal{P}_Y$ let $\mathcal{F}'_W = \{\eta \in \mathcal{F} : \text{ for some } \alpha < \kappa \text{ we have: } \alpha + 1 = \ell g(\eta) \text{ and } \eta(\alpha) \in W\}$ and $\mathcal{F}_W = \{\eta \in \mathcal{F} : (\exists \nu)(\eta < \nu \in \mathcal{F}'_W)\}.$

So by $\otimes$ above we have: $\mathcal{F}'_W$, hence $\mathcal{F}_W$ is a set of $\leq |W| + \kappa \leq \theta_0$ nodes in $T$ and is $\varpi$-downward closed. Also

$$\bigstar \bigcup_{Y \in \mathcal{P}^*} \mathcal{P}_Y \leq |\mathcal{P}^*| \times \sup_{Y \in \mathcal{P}^*} |\mathcal{P}_Y| \leq \theta_2^\sigma \times \kappa^* \leq \theta_2^\sigma + \cov(\theta_0^\sigma, \theta_1^\sigma, \kappa^+, \sigma^+) = \theta_2^\sigma + \kappa^* < \chi.$$

However, for every $i < \chi, Y_i \in \mathcal{P}^*$ and $\alpha < \kappa \Rightarrow \{F_{\varepsilon}(\eta_i(\alpha)) : \varepsilon < \sigma\} \subseteq \bigcup_{\varepsilon < \sigma} w_{i,\varepsilon} \subseteq Y_i \Rightarrow \eta_i(\alpha) \in Z_Y$ hence $\text{Rang}(\eta_i) \in [Z_Y]^\kappa$ so for some $W \in \mathcal{P}_Y, (\exists \nu < \kappa)[\eta_i(\alpha) \in W]$ hence $\eta_i \in \lim_\kappa(\mathcal{F}_W)$.

By assumption (d) and $\bigstar$ above for some $W \in \bigcup_{Y \in \mathcal{P}^*} \mathcal{P}_Y$ we have

$$|\{i < \chi : \eta_i \in \lim_\kappa(\mathcal{F}_W)\}| = \chi.$$
Now $\mathcal{T}_W$ is (essentially) a tree with $\leq \theta_0$ nodes and contradict the choice of $\Xi = \{\eta_i : i < \chi\}$.
(We could have instead using $\kappa^*, \mathcal{D}_Y$ to fix $Y_i = Y$ as $|\mathcal{D}_Y| < \chi = \text{cf}(\chi)$.)
$\square_{2.3}$, $\square_{2.1}$

2.4 Remark. 1) We could have used in 2.2, 2.3, $\theta_2 = \text{cov}(\theta_0^+, \theta_0^+, \kappa^+, \kappa^+)$ instead $\text{cov}(\theta_0^+, \theta_0^+, \kappa^+, \kappa^+)$ and similarly in the proof of 2.1.
2) We can also play with assumption (b) as 2.2, 2.3.

It may be useful to note (actually $\lambda^{<\kappa^{+\kappa}} = \lambda$ suffice)

2.5 Fact. If $\mathcal{T} \subseteq \lambda^{+\kappa} > 2$ is a tree, $|\mathcal{T}| \leq \lambda^+$ and $\lambda \geq \beth_\omega$ then for every regular $\kappa < \beth_\omega$ large enough, we can find $(Y_\delta : \delta < \lambda^+, \text{cf}(\delta) = \kappa), |Y_\delta| \leq \lambda$ such that:
for every $\eta \in \lim_{\lambda^+}(\mathcal{T})$ for a club of $\delta < \lambda^+$ we have $\text{cf}(\delta) = \kappa \Rightarrow \eta \restriction \delta \in Y_\delta$.

The following is needed when we like to get in the model theory not just many models but many models no one $\leq R$-embeddable into another.

2.6 Fact: Assume:

(a) $\text{cf}(\mu) \leq \kappa < \mu$, $\text{pp}_\kappa(\mu) = \lambda^+$, moreover $\text{pp}_\kappa(\mu) = \lambda^+$ and $\kappa^+ < \theta < \chi^+$
(b) $F$ is a function, with domain $[\mu]^\kappa$, such that: for $a \in [\mu]^\kappa$, $F(a)$ is a family of $< \theta$ members of $[\mu]^\kappa$
(c) $F$ is a function with domain $[\mu]^\kappa$ such that

$$a \in [\mu]^\kappa \Rightarrow a \subseteq F(a) \in F(a).$$

Then we can find pairwise distinct $a_i \in [\mu]^\kappa$ for $i < \chi^+$ such that $\mathcal{I} = \{a_i : i < \chi\}$ is $(F, F)$-independent which means

(*) $F, F_\mathcal{I}$ if $a \neq b \& a \in \mathcal{I} \& b \in \mathcal{I} \& c \in F(a) \Rightarrow \neg(F(b) \subseteq c)$.

2.7 Remark. 1) Clearly this is similar to Hajnal’s free subset theorem [Ha61].
2) Note that we can let $F(a) = a$.
3) Note that if $\lambda = \text{cf}([\mu]^\kappa, \subseteq)$ then for some $F, F$ as in the Fact

(*) if $a_i \in [\mu]^\kappa$ for $i < \lambda^+$ are pairwise distinct then not every pair $\{a_i, a_j\}$ is $(F, F)$-independent
why? let $\mathcal{D} \subseteq [\mu]^\kappa$ be cofinal (under $\subseteq$) of cardinality $\lambda$, and let $F$ be such
We can define \( F_i < \chi \ast \) is with no repetitions and \( \{ a : a \leq b \} \) has a \( \subseteq \)-maximal member \( F(a) \); clearly there is such \( F \). Now clearly

\[(*)_1 \text{ if } a \neq b \text{ are from } [\mu]^\kappa \text{ and } F(a) \cap F(b) \neq \emptyset \text{ then } \{ a, b \} \text{ is not } (F, F)-\text{independent.}
\]

Also if \( \mu_1 \leq \mu \), \( cf(\mu_1) \leq \kappa \leq \kappa + \theta < \mu_1 \) and \( pp(\xi) \geq \mu \) then by [Sh:g, Ch.II,2.3] the Fact for \( \mu_1 \) implies the one for \( \mu \).

**Proof.** We can define \( g : [\mu]^{\leq \kappa} \rightarrow [\mu]^\kappa \) and \( F', F' \) functions with domain \( [\mu]^{\leq \kappa} \) as follows:

\[
g(a) = \{ \kappa + \alpha : \alpha \in a \} \cup \{ \alpha : \alpha < \kappa \}
\]

\[
F'(a) = \{ \{ \alpha : \kappa + \alpha \in b \} : b \in F'(g(a)) \}
\]

\[
F'(a) = \{ \alpha : \kappa + \alpha \in F(g(a)) \}.
\]

Now \( F', F' \) are as above only replacing everywhere \( [\mu]^\kappa \) by \( [\mu]^{\leq \kappa} \), and if \( \mathcal{S} = \{ a_i : i < \chi \} \subseteq [\mu]^{\leq \kappa} \) with no repetitions satisfying \( (*)_{F', F', \mathcal{S}} \) then \( \mathcal{S}' = \{ g(a_i) : i < \chi \} \) is with no repetitions and \( (*)_{F, F, \mathcal{S}} \).

So we conclude that we can replace \( [\mu]^\kappa \) by \( [\mu]^{\leq \kappa} \). In fact we shall find the \( a_i \) in \( [\mu]^\alpha \) where \( \alpha \) chosen below.

Without loss of generality \( \kappa^{++} < \theta \).

Assume \( \theta < \chi = cf([\Pi\alpha]/J) \) where \( \alpha \subseteq \mu \cap Reg \setminus \kappa^+ \), \( |a| \leq \kappa, sup(a) = \mu, J^\alpha_a \subseteq J \) and for simplicity \( \chi = \max \text{ pcf}(a) \) and let \( f = \langle f_\alpha : \alpha < \chi \rangle \) be a sequence of members of \( [\Pi\alpha, < J \rangle \)-increasing, and cofinal in \( (\Pi\alpha, < J) \), so, of course, \( \chi \leq \chi^* \).

Without loss of generality \( f_\alpha(\tau) > \sup(\alpha \cap \tau) \) for \( \tau \in a \). Also for every \( a \in [\mu]^\kappa \), define \( f_\alpha \in [\Pi\alpha] \) by \( f_\alpha(\tau) = \sup(\alpha \cap \tau) \) for \( \tau \in a \) so for some \( \zeta(\alpha) < \chi \) we have \( f_\alpha < J f_\zeta(\alpha) \) (as \( \langle f_\alpha : \alpha < \chi \rangle \) is cofinal in \( (\Pi\alpha, < J) \)). So for each \( a \in [\mu]^\kappa \), as \( |F(a)| < \theta < \chi = cf(\chi) \) clearly \( \xi(\alpha) = \sup\{ \zeta(b) : b \in F(a) \} \) is \( \in \chi \), and clearly \( \forall b \in F(a), f_b < J f_\xi(\alpha) \).

So \( C = \{ \gamma < \chi : \text{for every } \beta < \gamma, \xi(\kappa \cup \text{Rang } f_\beta) < \gamma \} \) is a club of \( \chi \).

For each \( \alpha < \chi \), \( \text{Rang}(f_\alpha) \cup \kappa \in [\mu]^\kappa \), hence \( F(\text{Rang}(f_\alpha) \cup \kappa) \) has cardinality \( < \theta \), but \( \theta < \chi = cf(\chi) \) hence for some \( \theta_1 < \theta \) we have \( \theta_1 > \kappa^{++} \) and \( \chi = \sup\{ \alpha < \chi : |F(\text{Rang}(f_\alpha) \cup \kappa)| \leq \theta_1 \} \), so without loss of generality \( \alpha < \chi \Rightarrow \theta_1 \geq |F(\text{Rang}(f_\alpha) \cup \kappa)| \).
As $\kappa^+ < \theta_1$, there is a stationary $S \subseteq \{\delta < \theta_1^+ : \text{cf}(\delta) = \kappa^+\}$ which is in $I[\theta_1^+ \rightarrow 2]$, by [Sh 420, §1] and let $\langle d_i : i < \theta_1^+ \rangle$ witness it, so $\text{otp}(d_i) = \kappa^+$, $d_i \subseteq i$, $j \in d_i \Rightarrow d_j = d_i \cap i$ and $i \in S \Rightarrow i = \text{sup}(d_i)$, and for simplicity: for every club $E$ of $\theta_1^+$ for stationarily many $\delta \in S$ we have $(\forall \alpha \in d_\delta)[(\exists \beta \in E)(\sup(\alpha \cap d_\delta) < \beta < \alpha)]$, exists by [Sh 420, §1]. Now try to choose by induction on $i < \theta_1^+$, a triple $(g_i, \alpha_i, w_i)$ such that:

(a) $g_i \in \Pi a$
(b) $j < i \Rightarrow g_j <_J g_i$
(c) $(\forall \tau \in a)(\sup_{j \in d_i} g_j(\tau) < g_i(\tau))$
(d) $\alpha_i < \chi, \alpha_i > \sup(\bigcup_{j < i} w_j)$
(e) $j < i \Rightarrow \alpha_j < \alpha_i$
(f) $g_i <_J f_{\alpha_i}$
(g) $\beta \in \bigcup_{j < i} w_j \Rightarrow \xi(\beta) < \alpha_i$ & $f_\beta <_J g_i$
(h) $w_i$ is a maximal subset of $(\alpha_i, \chi)$ satisfying

\[
(\ast) \quad \beta \in w_i \& \gamma \in w_i \& \beta \neq \gamma \& a \in F(\text{Rang}(f_\beta)) \Rightarrow \neg(\text{Rang}(f_\gamma) \subseteq a)
\]

or just

\[
(\ast)^+ \quad \beta \in w_i \& \gamma \in w_i \& \beta \neq \gamma \& a \in F(\text{Rang}(f_\beta)) \Rightarrow \{\tau \in a : f_\gamma(\tau) \in a\} \notin J.
\]

[Note that really]

\[
\exists \text{ if } w \subseteq (\alpha_i, \chi) \text{ satisfies } (\ast)^+ \text{ then it satisfies } (\ast)
\]

why? let us check $(\ast)$, so let $\beta \in w, \gamma \in w, \beta \neq \gamma$ and $a \in F(\text{Rang}(f_\beta))$; by $(\ast)^+$ we know that $a' = \{\tau \in a : f_\gamma(\tau) \in a\} \notin J, J$ is a proper ideal on $a$ clearly for some $\tau \in a$ we have $\tau \notin a'$, hence $f_\gamma(\tau) \notin a$ but $f_\gamma(\tau) \in \text{Rang}(f_\gamma) \subseteq F(\text{Rang}(f_\gamma))$ hence $f_\gamma(\tau) \in F(\text{Rang}(f_\gamma)) \setminus a$ so $\neg(\text{Rang}(f_\gamma) \subseteq a)$, as required.]

We claim that we cannot carry the induction because if we succeed, then as $\text{cf}(\chi) = \chi > \theta \geq \theta_1^+$ there is $\alpha$ such that $\bigcup_{i < \theta_1^+} \alpha_i < \alpha < \chi$ and let $F(\text{Rang}(f_\alpha)) = \{a_\alpha^{\gamma} : \zeta < \theta_1^+\}$ (possible as $1 \leq |F(\text{Rang}(f_\alpha))| \leq \theta_1$). Now for each $i < \theta_1^+$, by the choice of $w_i$ clearly $w_i \cup \{\alpha\}$ does not satisfy the demand in clause $(h)$, so as $\beta \in w_i \Rightarrow \xi(\beta) < \alpha_{i+1} < \alpha$, necessarily for some $\beta_i \in w_i$ and $\zeta_i < \theta_1$ we have

\[
a_i = \{\tau \in a : f_{\beta_i}(\tau) \in a_\alpha^{\gamma}\} \notin J.\n\]
[why use the ideal? In order to show that $b_\varepsilon \neq \emptyset$. Now $\text{cf}(\theta_1^+) = \theta_1^+ > \theta_1$, for some $\zeta_1 < \theta_1^+$ we have $A = \{i : \zeta_i = \zeta_1\}$ is unbounded in $\theta_1^+$. Hence $E = \{\alpha < \theta_1^+ : \alpha$ a limit ordinal and $A \cap \alpha$ is unbounded in $\alpha\}$ is a club of $\theta_1^+$. So for some $\delta \in S$ we have $\delta = \sup(A \cap \delta)$, moreover if $d_\delta = \{\alpha_\varepsilon : \varepsilon < \kappa^+\}$ (increasing) then $(\forall \varepsilon)[E \cap (\sup \alpha_\zeta_\varepsilon, \alpha_\varepsilon) \neq \emptyset]$ hence we can find $i(\delta, \varepsilon) \in (\sup \alpha_\zeta_\varepsilon, \alpha_\varepsilon) \cap A$ for each $\varepsilon < \kappa^+$.

Clearly for each $\varepsilon < \kappa^+$

$$b_\varepsilon = \{\tau \in a : g_i(\delta, \varepsilon)(\tau) < f_{\alpha_i(\delta, \varepsilon)}(\tau) < f_{\beta_i(\delta, \varepsilon)}(\tau) \leq g_i(\delta, \varepsilon) + 1 < f_{\alpha_i(\delta, \varepsilon) + 1}(\tau) < f_\alpha(\tau)\} = a \mod J$$

hence $b_\varepsilon \cap a_{i(\delta, \varepsilon)} \neq \emptyset$. Moreover, $b_\varepsilon \cap a_{i(\delta, \varepsilon)} \neq J$. Now for each $\tau \in a$ let $\varepsilon(\tau)$ be sup$\{\varepsilon < \kappa^+ : \tau \in b_\varepsilon \cap a_{i(\delta, \varepsilon)}\}$ and let $\varepsilon(\tau) = \text{sup}\{\varepsilon(\tau) : \tau \in a$ and $\varepsilon(\tau) < \kappa^+\}$ so as $|a| \leq \kappa$ clearly $\varepsilon(\tau) < \kappa^+$. Let $\tau_* \in b_\varepsilon(\varepsilon) + 1 \cap a_{i(\delta, \varepsilon)(\varepsilon) + 1}$, so $B = \{\varepsilon < \kappa^+ : \tau_* \in b_\varepsilon \cap a_{i(\delta, \varepsilon)}\}$ is unbounded in $\kappa^+$ and $\{f_{\beta_i(\delta, \varepsilon)}(\tau_*), \varepsilon \in B\}$ is strictly increasing (see clause (c) above and the choice of $b_\varepsilon$) and $\varepsilon \in B \Rightarrow f_{\beta_i(\delta, \varepsilon)}(\tau_*) \in a_{\alpha(\varepsilon)}$ (by the definition of $a_{i(\delta, \varepsilon)}$ and $\zeta_\varepsilon$ as $\zeta_i(\delta, \varepsilon) = \zeta_\varepsilon(\varepsilon)$). We get contradiction to $a \in \mathbf{F}(\kappa \cup \text{Rang}(f_\alpha)) \Rightarrow |a| \leq \kappa$.

So really we cannot carry the induction so we are stuck at some $i$. If $i = 0$, or $i$ limit, or $i = j + 1$ & sup($w_j$) < $\chi$ we can find $g_i$ and then $\alpha_i$ and then $w_i$ as required. So necessarily $i = j + 1$, sup($w_j$) = $\chi$. Now if $\chi = \chi^*$, then this $w_j$ is as required in the fact. As $\text{pp}^+(\mu) = (\chi^*)^+$, the only case we cannot have is when $\chi^*$ is singular. Let $\chi^* = \sup \chi_\varepsilon$ and $\chi_\varepsilon \in (\mu, \chi^*) \cap \text{Reg}$ is (strictly) increasing with $\varepsilon < \text{cf}(\chi^*)$

$\varepsilon$. By [Sh:10, Ch.II,$\S1$] we can find, for each $\varepsilon < \text{cf}(\chi^*)$, $a_\varepsilon, J_\varepsilon, \bar{f}_\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \chi_\varepsilon\rangle$ as above, but in addition

$$(\ast) \bar{f}_\varepsilon \text{ is } \mu^+-\text{free i.e. for every } u \in [\chi_\varepsilon]^\mu, \text{ there is } \langle b_\alpha : \alpha \in u\rangle \text{ such that } b_\alpha \in J_\varepsilon \text{ and for each } \tau \in a_\varepsilon, \langle f^\varepsilon_\alpha(\tau) : \alpha \text{ satisfies } \tau \notin b_\alpha\rangle \text{ is strictly increasing.}$$

So for every $a \in [\mu]^\kappa$ and $\varepsilon < \text{cf}(\chi^*)$ we have

$$\{\alpha < \chi_\varepsilon : \{\tau \in a_\varepsilon : f_\alpha(\tau) \in a\} \notin J_\varepsilon\} \text{ has cardinality } \leq \kappa.$$  

Hence for each $a \in [\mu]^\kappa$

$$\{(\varepsilon, \alpha) : \varepsilon < \text{cf}(\chi^*) \text{ and } \alpha < \chi_\varepsilon \text{ and } \{\tau \in a_\varepsilon : f_\alpha(\tau) \in a\} \notin J_\varepsilon\}$$

has cardinality $\leq \kappa + \text{cf}(\chi^*) = \text{cf}(\chi^*)$ as for singular $\mu > \kappa \geq \text{cf}(\mu)$ we have $\text{cf}(\text{pp}_\kappa(\mu)) > \kappa$.  


Define: $X = \{(\varepsilon, \alpha) : \varepsilon < \text{cf}(\chi^*), \alpha < \chi_\varepsilon\}$

$F'(\varepsilon, \alpha) = \{(\varepsilon', \alpha') : (\varepsilon', \alpha') \in X \setminus \{(\varepsilon, \alpha)\} \text{ and for some } d \in F(\text{Rang}(f^\varepsilon_\alpha)) \text{ we have } \{\tau \in a_\varepsilon : f^\varepsilon_\alpha(\tau) \in d\} \notin J_{\varepsilon'}\}$

so $F'(\varepsilon, \alpha)$ is a subset of $X$ of cardinality $< \text{cf}(\chi^*)^+ + \theta < \chi^*$.

So by Hajnal’s free subset theorem [Ha61] we finish (we could alternatively, for $\chi^*$ singular, have imitated his proof). □
§3 Finishing the many models

Recall from [Sh 576, 3.13t](3) (see [Sh 838]??).

3.1 Claim. 1) Assume

(a) $2^\lambda < 2^{\lambda^+}$ and Case A or B of Fact 2.1 holds for $\mu, \chi^*$ (or just the conclusion there)
(b) $\mathfrak{R}$ is an abstract elementary class with $\text{LS}(\mathfrak{R}) \leq \lambda$
(c) $K_{\lambda^+} \neq 0$
(d) $\mathfrak{R}$ has amalgamation in $\lambda$
(e) in $K^3_\lambda$, the minimal triples are not dense.

Then

(*) for any regular $\chi < \mu$ we have:

($\lambda$) there is $M \in K_\lambda, |\mathcal{P}(M)| > \chi$.

2) If in part (1) we strengthen clause (d) to (d)$^+$, then we get ($\ast$)$_1^+$ where:

(d)$^+$ $\mathfrak{R}$ has amalgamation in $\lambda$ and has a universal member in $\lambda$

($\ast$)$_1^+$ for some $M \in K_\lambda$ we have $|\mathcal{P}(M)| \geq \mu$.

3) Assume (a), (b), (c), (e) of part (1) and (d)$^+$ of part (2) then:

($\ast$)$_2$ $I(\lambda^+, K) \geq \chi^*$ and if $(2^\lambda)^+ < \chi^*$ then $I^E(\lambda^+, \mathfrak{R}) \geq \chi^*$.

4) If in clause (a) of part (1) we restrict ourselves to Case A of 2.1, then $\chi^* = 2^{\lambda^+}$ so in part (3) we get

($\ast$)$_2^+$ $I(\lambda^+, K) = 2^{\lambda^+}$ and $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow I^E(\lambda^+, K) \geq 2^{\lambda^+}$.

3.2 Remark. 1) We can restrict clause (b) to $\mathfrak{R}_\lambda$, interpreting in (c) + (e), $K_{\lambda^+}$ as

$\{ \bigcup M_i : M_i \in K_\lambda \text{ is } <_R\text{-increasing (strictly and) continuous} \}$, but see [Sh 576, §0], mainly 0.31t.

2) Part (3) of 3.1 (and 3.3) below are like [Sh:g, Ch.II,4.10E], Kojman Shelah [KjSh 409, §2].

3) We can apply this to $\lambda^+$ standing for $\lambda$ here.
4) We can state the part of (A) of 2.1 used (and can replace $2^{\lambda^+}$ by smaller cardinals).
5) We can replace $\lambda^+$ by a weakly inaccessible cardinal with suitable changes.

**Proof.**

1) Note that $\mu$ is singular (as by clause (α) of (A) of 2.1 (so also (B)), $\text{cf}(\mu) = \lambda^+ < \mu$). By 1.15(1) it suffices for each $\mu' < \mu$ to have $\delta < \lambda^+$ and a tree with $\leq \lambda$ nodes and $\geq \mu'$ $\delta$-branches. They exist by clause (ε) of (A) of 2.1 (so also of (B)).

2) Similarly using 1.15(2).

3) By part (2) we can find $M^* \in K_\lambda$ satisfying $S(M^*)$ has cardinality $\geq \mu$, and apply 3.3 elow.

4) Should be clear from the proof of part (3). □

3.3 Fact. Assume

(a) $\chi^* = \text{cf}([\mu]^{\lambda^+}, \subseteq) > 2^\lambda$
(b) $\mathfrak{A}$ is an a.e.c. with $\text{LS}(\mathfrak{A}) \leq \lambda$
(c) $M^* \in K_\lambda$ is an amalgamation base
(d) $M^* \in K_\lambda$ satisfies $|\mathcal{S}(M^*)| \geq \mu$.

Then $\dot{I}(\lambda^+, K) \geq \chi^*$ and if $(2^\lambda)^+ < \chi^*$ then IE($\lambda^+, K) \geq \chi^*$.

**Proof.** Let $p_\eta \in \mathcal{S}(M^*)$ for $\eta \in Z$ be pairwise distinct, $|Z| \geq \mu$ and let $M^* \leq_{\mathfrak{A}} N_\eta \in K_\lambda$, $p_\eta = \text{tp}(a_\eta, M^*, N_\eta)$.

Now for every $X \in [Z]^{\lambda^+}$, as $\mathfrak{A}$ has amalgamation in $\lambda$ there is $M_X \in K_{\lambda^+}$ such that $M^* \leq_{\mathfrak{A}} M_X$ and $\eta \in X \Rightarrow N_\eta^*$ is embeddable into $M_X$ over $M^*$ (hence $p_\eta$ is realized in $M_X$). Let $Y[X] = \{\eta \in Z : p_\eta$ is realized in $M_X\}$. So $X \subseteq Y[X] \in [Z]^{\lambda^+}$, so $\{Y[X] : X \in [Z]^{\lambda^+}\}$ is a cofinal subset of $[Z]^{\lambda^+}$, hence (see clause (β) of case (A) of Fact 2.1)

$$|\{(M_X, c)_{c \in M^*} \models X \in [Z]^{\lambda^+}\}| \geq$$

$$|\{Y[X] : X \in [Z]^{\lambda^+}\}| \geq \text{cf}([Z]^{\lambda^+}, \subseteq) \geq \text{cf}([\mu]^{\lambda^+}, \subseteq) \geq \text{pp}(\mu) = \chi^*.$$

As $2^\lambda < \chi^*$ also $|\{M_X/ \models X \in [Z]^{\lambda^+}\}| \geq \chi^*$ (clear or see [Sh:a, Ch.VIII,1.2] because $\|M_X\| = \lambda^+, \|M^*\| = \lambda$ and $(\lambda^+)^\lambda < \mu$) but $\dot{I}E(\lambda^+, K)$ is $\geq$ than the former.

---

**Revision note:**

- Page dimensions: 595.0x842.0
- Modified: 2004-05-28
- Revised: 2004-05-30

---

**Revision:**

- Text adjustments for formatting and clarity.
Lastly we shall prove \((2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) \geq \chi^*(\text{so the reader may skip this, sufficing himself with the estimate on } \dot{I}\dot{E}(\lambda^+, K))\).

For each \(X \in [\mu]^{\lambda^+}\), let \(F_X = \{f : f\text{ a }\leq R\text{-embedding of } M^* \text{ into } M^*_X\}\), and for \(f \in F_X\) let

\[\mathcal{X}_{X,f} = \{X_1 \in [Z]^{\lambda^+} : \text{there is a }\leq_R\text{-embedding of } M_{X_1} \text{ into } M_X \text{ extending } f\},\]

and let \(\mathcal{I}_{X,f} = \{p \in \mathcal{I}(M^*) : f(p) \text{ is realized in } M_X\}\), so \(\mathcal{X}_{X,f} \subseteq \{X_1 : X_1 \subseteq \mathcal{I}_{X,f}\}\) and \(|\mathcal{X}_{X,f}| \leq \lambda^+\).

Now the result follows from the the fact 2.6 above. \(\square_{3.3}\)

**3.4 Claim.** 1) Assume

\[(a) 2^\lambda < 2^{\lambda^+} < 2^{\lambda^+^2} \text{ and case B or C of Fact 2.1 for } \lambda\text{ occurs (so } \chi^*, \mathcal{T}_e\text{ are determined)}\]

\[(b) \mathcal{R} \text{ is an abstract elementary class } LS(\mathcal{R}) \leq \lambda\]

\[(c) K_{\lambda^+} \neq 0,\]

\[(d) \mathcal{R} \text{ has amalgamation in } \lambda \text{ [was: and in } \lambda^+, \text{ seem irrelevant]}\]

\[(e) \text{in } K^3_{\lambda^+}, \text{ the minimal triples are not dense}.\]

Then

\[(*) \text{ for each } \zeta < \chi^* \text{ for some } M \in K_{\lambda^+} \text{ we have } |\mathcal{I}_*(M)| \geq |\operatorname{lim}_{\lambda^+}(\mathcal{T}_e)|\]

(see 1.2(7) the tree from clause (\(\zeta\)) of 2.1); on \(\mathcal{I}_*\) see 1.2(7).

2) If \(K\) satisfies (a)-(e) and is categorical in \(\lambda^+\) or just has a universal member in \(\lambda^+\) and amalgamation in \(\lambda^+\), \text{then for some } M \in K_{\lambda^+} \text{ we have } |\mathcal{I}_*(M)| = 2^{\lambda^+}\).

3) If clauses (a)–(e) from above and clause (f)\(^+\), \text{then } I(\lambda^{++}, K) \geq \mu_{\text{wd}}(\lambda^{++}, 2^{\lambda^+})\]

where

\[(d)^+ \mathcal{R} \text{ has amalgamation also in } \lambda^+\]

\[(f)^+ \mathcal{R} \text{ is categorical in } \lambda \text{ and } \lambda^+.\]

**Remark.** 1) Assume (a)-(e) of part (1) then \(C^0_{R,\lambda^+}\) has weaker \(\lambda\)-coding if we have restricted to \((M, N, a)\) above which there is not minimal triple in Definition of \(C\). But not used.

2) Note that for 3.6 below we do not use 3.4.

3) We would like to weaken in [Sh:E46] the assumption “\(\mathcal{R}\) categorical in \(\lambda^+\)” to
“no maximal model in $R_\lambda$”. So by 3.3 for amalgamation bases $M^* \in K_{\lambda^+}$, $\mathcal{S}(M^*)$ cannot be too large (used in the proof of 3.6 and as $I(\lambda^{++}, K) < \mu_{\omega \lambda}(\lambda^{++}, 2\lambda^+)$, there are many amalgamation basis and by 3.4(1)-pf there are many $M \in K_{\lambda^+}$ with $\mathcal{S}(M)$ large. But we have to put them together (20045/4).

**Proof.** 1) Let $\zeta < \chi^*$. Recall the $\mathcal{T}_\zeta$ is a subtree of $\lambda^+ > 2$ of cardinality $\leq \lambda^+$, hence let $\mathcal{T}_\zeta = \bigcup_{\alpha < \lambda^+} \mathcal{T}_\zeta^\alpha$ where $\mathcal{T}_\zeta^\alpha$ are pairwise disjoint for $\alpha < \lambda^+$, each $\mathcal{T}_\zeta^\alpha$ has cardinality $\leq \lambda$, $\mathcal{T}_0^\alpha = \{<>\}$ and $\eta \in T_\alpha^\zeta$ & $\beta < \ell g(\eta) \Rightarrow \eta \upharpoonright \beta \in \bigcup_{\gamma < \alpha} \mathcal{T}_\zeta^\gamma$, and $\eta \in \mathcal{T}_\zeta^\alpha \Rightarrow \bigwedge_{\ell < 2} \eta^\ell (\ell) \in \mathcal{T}_{\zeta, \alpha+1}$ so $\mathcal{T}_{\zeta, \alpha+1} = \{\eta^\ell (\ell) : \eta \in T_\alpha^\zeta \text{ and } \ell < 2\}$. For $\eta \in \mathcal{T}_\delta^\zeta$, $\delta$ a limit ordinal, necessarily both $\ell g(\eta)$ and $\alpha(\eta) = \sup\{\gamma : \text{for some } \varepsilon < \ell g(\eta), \eta \upharpoonright \varepsilon \in \mathcal{T}_\delta^\zeta\}$ are limit ordinals $\leq \delta$.

Let $(M, N, a) \in K_\lambda^3$ be such that there is no minimal triple above it. We now by induction on $\alpha < \lambda^+$ choose $\langle M_\alpha^\zeta, M_\eta^\zeta, N_\eta^\zeta : \eta \in \mathcal{T}_\alpha^\zeta \rangle$ such that:

(a) $(M_\alpha^\zeta, N_\eta^\zeta, a) \in K_\lambda^3$ and is reduced (see [Sh 576]) if $\eta \in T_\alpha^\zeta$, $\alpha$ non-limit

(b) $(M_0^\zeta, N_{>\alpha}^\zeta, a) = (M, N, a)$

(c) if $\nu \in T_\beta^\zeta$, $\eta \in T_\alpha^\zeta, \nu < \eta, \beta < \alpha$ and $\alpha, \beta$ are non-limit then $\Rightarrow (M_\beta^\zeta, N_\eta^\zeta, a) \leq (M_\alpha^\zeta, N_\eta^\zeta, a)$ in the order of $K_\lambda^3$

(d) if $\delta$ is a limit ordinal then: $M_\delta^\zeta = \bigcup_{\beta < \delta} M_\beta^\zeta$

(e) if $\delta$ is a limit ordinal and $\eta \in T_\delta^\zeta$ then $N_{\ell g(\eta)}^\zeta = \bigcup_{\beta < \delta} N_{\eta, \beta}^\zeta$ hence $(M_{\alpha(\eta)}, N_\eta^\zeta, a) \in K_\lambda^3$

(f) if $\eta \in \mathcal{T}_\alpha^\zeta$ then $\text{tp}(a, M_{\alpha+1}, N_{\eta^\zeta < 0}) \neq \text{tp}(a, M_{\alpha+1}, N_{\eta^\zeta < 1})$

(g) $M_\zeta^\zeta \neq M_\zeta^\zeta+1$.

There is no problem to carry the definition. Let $M_\zeta = \bigcup_{\alpha < \lambda^+} M_\alpha^\zeta \in K_{\lambda^+}$, and for each $\nu \in \lim_{\lambda^+}(\mathcal{T}_\zeta)$ let $N_\nu^\zeta = \bigcup_{\alpha < \lambda^+} N_{\nu, \alpha}^\zeta$ clearly $M_\zeta \leq R N_\zeta^\zeta$ and $a \in N_\nu^*$ and $\langle \text{tp}(a, M_\zeta, N_\nu^\zeta) : \nu \in \lim_{\lambda^+}(\mathcal{T}_\zeta) \rangle$ is pairwise distinct members of $\mathcal{S}(M_\zeta)$ (if $R_\lambda$ fails the amalgamation property, we should add: pairwise contradiction). This proves clause $(*)$ of part (1).

2) This part follows by 1.15(2).
3) Assume that the conclusion fails. Now if $(A)\lambda \lor (B)\lambda$ of 2.1 let $\chi^*$ be as there then by 3.1(3) we get $I(\lambda^+, K) \geq \chi^* > 1$ contradicting assumption $(f)^+$. Hence in 2.1, case $(C)$ holds, so let $\langle T, \zeta : \zeta < \chi^* \rangle$ be as there, so by part (2) for some $M \in K_{\lambda^+}$ we have $|S_M(M)| = 2^{\lambda^+}$. By 3.3 with $\lambda^+$ here standing for $\lambda$ there (!) and our assumption toward contradiction we deduce $\text{cf}([2^{\lambda^+}]^{\lambda^+}, \subseteq) < 2^{\lambda^+}$. By 2.1 with $\lambda^+$ here standing for $\lambda$ there; case $(A)\lambda^+$ cannot hold (recall that in this case $\chi^* = 2^{\lambda^+}$), so we can assume $(C)\lambda^+ \lor (B)\lambda^+$ occurs.

Now if $(2^\lambda > \lambda^+) + (\text{WDmId}(\lambda^+) \text{ is not } \lambda^{++}-saturated)$ we get the desired result as follows.

**Case 1:** $\neg(\ast\ast)'_\lambda$ of 1.9 holds.
The result follows by 1.12.

**Case 2:** $(\ast\ast)'_\lambda$ of 1.11 holds.
By 1.11 we get a contradiction to $I(\lambda^+, K) = 1$.
So one of the assumptions of the previous paragraph fails. If the second fails (i.e. $\text{WDmId}(\lambda^+)$ is $\lambda^{++}$-saturated as we are in case $(B)\lambda$ or $(C)\lambda$ (see (a) of 3.4(1)) so by clause (ii) of 2.1 we have $2^\lambda = \lambda^+$, $2^{\lambda^+} = \lambda^{++}$. So in both cases $2^\lambda = \lambda^+$. However, once we know $2^\lambda = \lambda^+$ we deduce that there is a model in $\lambda^+$ saturated over $\lambda$ and we apply the claim below. \(\square_{3.4}\)

**3.5 Claim.** Assume $(a)-(e), (f)'$ of 3.4 and $(g)$ below then $I(\mathfrak{A}, \lambda^{++}) \geq \mu_{wd}(\lambda^{++}, 2^\lambda)$ where

$(g)$ there is $M \in K_{\lambda^+}$ saturated over $\lambda$.

**Proof.** Claim [Sh 576, 3.16](3) possibility $(\ast)_2$ applies. Alternatively see [Sh 838]. \(\square_{3.4}\)

**3.6 Claim.** Assume

$(a)$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
$(b)$ $\mathfrak{A}$ an a.e.c. is categorical in $\lambda, \lambda^+$ and $\text{LS}(\mathfrak{A}) \leq \lambda$
$(c)$ $1 \leq I(\lambda^{++}, K) < \mu_{wd}(\lambda^{++}, 2^\lambda)$
$(d)$ $K^3_\lambda$ has the weak extension property above $(M^*, N^*, a)$.

Then above $(M^*, N^*, a) \in K^3_\lambda$ the minimal triples are dense.
Proof. Assume toward contradiction that above \((M^*, N^*, a) \in K_\lambda^3\) there is no minimal type. If \(2^\lambda = \lambda^+\), then there is a \(M \in K_{\lambda^+}\) saturated over \(\lambda\) hence we finish by 3.5 above. So we can assume \(2^\lambda > \lambda^+\), hence [Sh 430, 6.3] (with \(\lambda^+\) here standing for \(\lambda\) there so \(\mu\) there is \(\leq \lambda\) so \(\delta < \lambda^+\) hence \(|\mathcal{T}| \leq |\delta| \leq \lambda\) and let \(\kappa = \text{cf}(\delta)\)) there are \(\kappa \leq \lambda\) and\(^4\) tree \(\mathcal{T}\) with \(\leq \lambda\) nodes and \(\kappa\) levels with \(|\lim_{\kappa}(\mathcal{T})| > \lambda^+\) hence for some \(M \in K_{\lambda}, |\mathcal{T}_*(M)| > \lambda^+\) (e.g. by the proof of 3.4(1)). If WDmId(\(\lambda^+\)) is not \(\lambda^{++}\)-saturated then in 1.12 assumption (b) holds, and assumptions (c) + (d) + (e) holds by the assumptions of the present claim but not the conclusion, so (a) fails, that is \((**)_\lambda\) of 1.9 holds hence by 1.9, \((*)_\lambda\) of 1.8 holds. But now 1.8 contradicts clause (b) of the assumption, so we have to assume that WDmId(\(\lambda^+\)) is \(\lambda^{++}\)-saturated. Hence clause (i) of 2.1, Case B does not occur, hence Cases B,C of 2.1 do not occur and hence Case A occurs. So by 3.1(3) we get a contradiction to categoricity in \(\lambda^+\).

\[\square_{3.6}\]
4.1 Claim. We can prove [Sh 576, 4.2t] also for $\lambda = \aleph_0$.

Proof. We ask:

**Question 1:** are there $M <_\mathcal{K} N$ in $K_\lambda$ such that for no $a \in N \setminus M$ is $\text{tp}(a, M, N)$ minimal?

If the answer is yes, we can find $\langle M^1_1 : i < \lambda^+ \rangle$ a representation of a model $M^1 \in K_{\lambda^+}$ such that: $a \in M^1_{i+1} \setminus M^1_i \Rightarrow \text{tp}(a, M^1_i, M^1_{i+1})$ is not minimal. This implies $a \in M^1_i \setminus M^1_i \Rightarrow \text{tp}(a, M^1_i, M^1)$ is not minimal (as for some $j \in [i, \lambda^+]$ we have $a \in M^1_j \setminus M^1_j$ so $(M^1_i, M^1_j, a)$ and the latter is not minimal). But we can build another representation $\langle M^2_i : i < \lambda^+ \rangle$ of $M^2 \in K_{\lambda^+}$ such that for each $i < \lambda^+$ for some $a \in M^1_{i+1} \setminus M^1_i$, $\text{tp}(a, M^1_i, M^1_{i+1})$ is minimal (as there is a minimal triple). So $M^1 \not\leq M^2$.

So we assume the answer is no.

**Question 2:** If $M \in K_\lambda, \Gamma \subseteq \Gamma^*_M =: \{p \in \mathcal{S}(M) : p \text{ minimal}\}$ and $|\Gamma| \leq \lambda$, is there $N$ such that: $M <_\mathcal{K} N \in K_\lambda$ and $N$ omit every $p \in \Gamma$?

If the answer to question 2 is yes, we can build $\langle M_\eta : \eta \in \lambda^+ > 2 \rangle$ as in the proof of 1.8 (more exactly $\eta \triangleleft \nu \Rightarrow M_\eta \leq_\mathcal{K} M_\nu, M_\eta \in K_\lambda$) and we also have $\Gamma_\eta \subseteq \{p : \text{for some } N \leq_\mathcal{K} M_\eta, N \in K_\lambda \text{ and } p \in \mathcal{S}(N) \text{ is minimal not realized in } M_\eta\}$ have cardinality $\leq \lambda, \eta \triangleleft \nu \Rightarrow \Gamma_\eta \subseteq \Gamma_\nu$ and there is $p \in \Gamma_\eta^{<\lambda^+} > 0$ realized in $M_\eta^{<\lambda^+}$ (and if you like also $p' \in \Gamma_\eta^{<\lambda^+} > 0$ realized in $M_\eta^{<\lambda^+}$). So by [Sh 576, 1.6t] we get $I(\lambda^+, K) = 2^\lambda$. So assume the answer is no and for every $M \in K_\lambda$ let $\Gamma_M$ be a counterexample. Let $\langle M^1_i : i < \lambda^+ \rangle$, representing a model $M^1 \in K_{\lambda^+}$ be such that $i < \lambda^+ \land p \in \Gamma_M \Rightarrow p$ realizes in $M$. Now as in the proof of saturated = model homogeneous (see [Sh 576, 0.21t]) we can prove $M^1$ is saturated. But this proves more than required: $|\mathcal{S}(M^1_i)| \leq \lambda^+$.
REFERENCES.


