ON T_3-TOPOLOGICAL SPACE OMITTING MANY CARDINALS

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Abstract. We prove that for every (infinite cardinal) λ there is a T_3-space X with clopen basis, 2^μ points where $\mu = 2^\lambda$, such that every closed subspace of cardinality $< |X|$ has cardinality $< \lambda$.

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§0 Introduction

Juhasz has asked on the spectrums $c-sp(X)=\{|Y|:Y$ an infinite closed subspace of $X\}$ and $w-sp(X)=\{w(Y):Y$ a closed subspace of $X\}$. He proved [Ju93] that if X is a compact Hausdorff space, then $|X|>\kappa\Rightarrow c-sp(X)\cap[\kappa,\sum\limits_{\lambda<\kappa}2^{2^\lambda}]\neq\emptyset$ and $w(X)>\kappa\Rightarrow w-sp(X)\cap[\kappa,2^{<\kappa}]\neq\emptyset$. So under GCH the cardinality spectrum of a compact Hausdorff space does not omit two successive regular cardinals, and omit no inaccessible. Of course, the space $\beta(\omega)\backslash\omega$, the space of nonprincipal ultrafilters on ω, satisfies $c-sp(X)=\{\beth_2\}$. Now Juhasz Shelah [JuSh 612] shows that we can omit many singular cardinals, e.g. under GCH for every regular $\lambda>\kappa$, there is a compact Hausdorff space X with $c-sp(X)=\{\mu:\mu\leq\lambda,\mathrm{cf}(\mu)\geq\kappa\}$; see more there and in [Sh 652]. In fact [JuSh 612] constructs a Boolean Algebra, so relevant to the parallel problems of Monk [M]. Here we deal with the noncompact case and get a strong existence theorem. Note that trivially for a Hausdorff space $X,|X|\geq\kappa\Rightarrow c-sp(X)\cap[\kappa,2^{2^\kappa}]\neq\emptyset$, using the closure of any set with κ points, so our result is in this respect best possible.

We prove

0.1 Theorem. *For every infinite cardinal λ there is a T_3 topological space X, even with clopen basis, with 2^{2^λ} points such that every closed subset with $\geq\lambda$ points has $|X|$ points.*

In §1 we prove a somewhat weaker theorem but with the main points of the proof present, in §2 we complete the proof of the full theorem.

§1

1.1 Theorem. *Assume* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. *Let* $\mu = 2^\lambda, \kappa = \mathrm{Min}\{\kappa : 2^\kappa > \mu\}$. *There is a Hausdorff space* X *with a clopen basis with* $|X| = 2^\kappa$ *such that: if for* $Y \subseteq \lambda$ *is closed and* $|Y| < |X|$ *then* $|Y| < \lambda$.

Proof. Let $S \subseteq \{\delta < \kappa : \delta \text{ limit}\}$ be stationary. Let $T_\alpha = {}^\alpha\mu$ for $\alpha \leq \kappa$ and let $T = \bigcup\limits_{\alpha \leq \kappa} T_\alpha$. Let $\zeta_\alpha = \cup\{\mu\delta + \mu : \delta \in S \cap (\alpha + 1)\}$ and let $\zeta_{<\alpha} = \cup\{\zeta_\beta : \beta < \alpha\}$.

Stage A: We shall choose sets $u_\zeta \subseteq T_\kappa$ (for $\zeta < \mu \times \kappa$). Those will be clopen sets generating the topology. For each ζ we choose (I_ζ, J_ζ) such that: I_ζ is a $\lhd$-antichain of $({}^{\kappa>}\mu, \lhd)$ such that for every $\rho \in T_\kappa$, $(\exists!\alpha)(\rho \restriction \alpha \in I_\zeta)$ and $J_\zeta \subseteq I_\zeta$ and we shall let $u_\zeta = \bigcup\limits_{\nu \in J_\zeta} (T_\kappa)^{[\nu]}$ where $(T_\kappa)^{[\nu]} = \{\rho \in T_\kappa : \nu \lhd \rho\}$. Let $I_{\alpha,\zeta} = T_\alpha \cap I_\zeta$, $J_{\alpha,\zeta} = T_\alpha \cap J_\zeta$ but we shall have $\alpha \notin S \Rightarrow I_{\alpha,\zeta} = \emptyset = J_{\alpha,\zeta}$.

Stage B: Let $Cd : \mu \to {}^{\lambda^+>}(T_{<\kappa})$ be onto such that for every $x \in$ Rang(Cd) we have $\mathrm{otp}\{\alpha < \mu : \mathrm{Cd}(\alpha) = x\} = \mu$.
We say α codes x (by Cd) if $\mathrm{Cd}(\alpha) = x$.

Stage C:Definition: For $\delta \leq \kappa$ we call $\bar\eta$ a δ-candidate if

- (a) $\bar\eta = \langle \eta_i : i \leq \lambda \rangle$
- (b) $\eta_i \in T_\delta$
- (c) $(\exists \gamma < \delta)(\bigwedge\limits_{i<j<\lambda} \eta_i \restriction \gamma \neq \eta_j \restriction \gamma)$
- (d) for every odd $\beta < \delta$, we have
  $Cd(\eta_\lambda(\beta)) = \langle \eta_i \restriction \beta : i \leq \lambda \rangle$
- (e) $\eta_\lambda(0)$ codes $\langle \eta_i \restriction \gamma : i < \lambda \rangle$, where $\gamma = \gamma(\eta \restriction \lambda) =$ Min$\{\gamma < \delta : i < j < \lambda \Rightarrow \eta_i \restriction \gamma \neq \eta_j \restriction \gamma\}$, it is well defined by clause (c) and
- (f) $\eta_\lambda(0) > \sup\{\eta_i(0) : i < \lambda\}$.

Stage D:Choice: Choose $A_{\xi,\varepsilon} \subseteq \lambda$ for $\xi < \mu \times \kappa, \varepsilon < \lambda$ such that:

$$\xi < \mu \times \kappa \ \& \ \varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow |A_{\xi,\varepsilon_1} \cap A_{\xi,\varepsilon_2}| < \lambda \text{ and even } = \emptyset$$

and

$$\xi_1 < \ldots < \xi_n < \mu \times \kappa, \varepsilon_1 \ldots \varepsilon_{n_1} < \lambda \Rightarrow \bigcap_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell} \text{ is a stationary subset of } \lambda.$$

Let $\Xi = \big\{\{(\xi_1,\varepsilon_1),\ldots,(\xi_n,\varepsilon_n)\} : \xi_1,\ldots,\xi_n < \mu \times \kappa$ is with no repetitions and $\varepsilon_1,\ldots,\varepsilon_n < \lambda\big\}$ and for $x \in \Xi$ let $A_x = \bigcap_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell}$. Let D_0 be a maximal filter on λ extending the club filter such that $x \in \Xi \Rightarrow A_x \neq \emptyset \mod D_0$.

For $A \subseteq \lambda$ let

$$\mathscr{B}^+(A) = \{x \in \Xi : A \cap A_x = \emptyset \mod D_0 \text{ but } y \subsetneqq x \Rightarrow A \cap A_y \neq \emptyset \mod D_0\}$$

$$\mathscr{B}(A) =: \mathscr{B}^+(A) \cup \mathscr{B}^+(\lambda \backslash A).$$

<u>Fact</u>: $\mathscr{B}(A) =: \mathscr{B}^+(A) \cup \mathscr{B}^+(\lambda\backslash A)$ is predense in Ξ i.e.

$$(\forall x \subseteq \Xi)(\exists y \in \mathscr{B}(A))(x \cup y \in \Xi).$$

Proof. If $x \in \Xi$ contradict it then we can add to D_0 the set $\lambda\backslash(A_x \cap A)$ getting D_0'. Now D_0' thus properly extends D_0 otherwise $A_x \cap A = \emptyset \mod D_0$ hence, let $x' \subseteq x$ be minimal with this property so $x' \in \mathscr{B}^+(A)$ and x by assumption satisfies: $\neg(\exists y \in \Xi)(x \cup y \in \mathscr{B}(A))$ so try $y = x$. For every $z \in \Xi$ we have $A_z \neq \emptyset \mod D_0$.

<u>Fact</u>: $|\mathscr{B}(A)| \leq \lambda$ for $A \subseteq \lambda$.

Proof. Let $\mathbf{B}_0$ be the Boolean Algebra freely generated by $\{x_{\xi,\varepsilon} : \xi < \mu \times \kappa, \varepsilon < \lambda\}$, by Δ-system argument, except $x_{\xi,\varepsilon_1} \cap x_{\xi,\varepsilon_2} = 0$ if $\varepsilon_1 \neq \varepsilon_2$; clearly $\mathbf{B}_0$ satisfies λ^+-c.c.

Let $\mathbf{B}^*$ be the completion of $\mathbf{B}_0$. Let f^* be a homomorphism from $\mathscr{P}(\lambda)$ into $\mathbf{B}^*$ such that $C \in D_0 \Rightarrow f^*(C) = 1_{\mathbf{B}^*}$ and

$$f(A_{\xi,\varepsilon}) = x_{\xi,\varepsilon}.$$

[Why exists? Look at the Boolean Algebra $\mathscr{P}(\lambda)$ let $I_\lambda = \{A \subseteq \lambda : \lambda\backslash A \in D_0\}$ and $\mathfrak{A}_0 = I_\lambda \cup \{\lambda\backslash A : A \in I_\lambda\}$ is a subalgebra of $\mathscr{P}(\lambda)$, and let $I_\lambda \cup \{A_{\xi,\varepsilon} : \xi \leq$

$\mu \times \kappa, \varepsilon = \lambda\}$ generate a subalgebra $\mathfrak{A}$ of $\mathscr{P}(\lambda)$; it extends $\mathfrak{A}_0$. Let $f_0^* : \mathfrak{A}_0 \to \mathbf{B}_0$ be the homomorphism with kernel I_λ. Let f_1^* be the homomorphism from $\mathfrak{A}$ into $\mathbf{B}_0$ extending f_0 such that $f_1^*(A_{\xi,\varepsilon}) = x_{\xi,\varepsilon}$, clearly exists and is onto. Now as $\mathbf{B}^*$ is a complete Boolean Algebra, f_1^* can be extended to a homomorphism f_2^* from $\mathscr{P}(\lambda)$ into $\mathbf{B}^*$. Clearly $\mathrm{Ker}(f_2^*) = \mathrm{Ker}(f_2^*) = \mathrm{Ker}(f_0^*) = I_\lambda$ so f_1^* induces an isomorphism from $\mathscr{P}(\lambda)/D_0$ onto $\mathrm{Rang}(f_1^*) \subseteq \mathbf{B}^*$, so the problem translates to $\mathbf{B}^*$. So $\mathbf{B}_0$ satisfies the λ^+-c.c and is a dense subalgebra of $\mathbf{B}^*$ hence of range(f_2^*), so this range is a λ^+-c.c. Boolean Algebra hence $\mathscr{P}(\lambda)/D_0$ satisfies the fact.]
Let $\mathbf{B}^*_\gamma$ be the complete Boolean subalgebra of $\mathbf{B}^*$ generated (as a complete subalgebra) by $\{x_{\xi,\varepsilon} : \xi < \gamma, \varepsilon < \lambda\}$. Clearly $\mathbf{B}^* = \bigcup\limits_{\gamma<\kappa} \mathbf{B}^*_\gamma$ and $\mathbf{B}^*_\gamma$ is increasing with γ.

Stage E: We choose by induction on $\delta \in S$ the following

- (A) $w_{\delta,\zeta} \subseteq T_\delta$ (for $\zeta < \mu\delta + \mu$) and $J_{\delta,\zeta} \subseteq I_{\delta,\zeta} \subseteq w_{\delta,\zeta}$
- (B) for each δ-candidate $\bar\eta = \langle \eta_i : i \leq \lambda\rangle$, a uniform filter $D_{\bar\eta}$ on λ extending the filter D_0
- (C) for each $\nu_1 \neq \nu_2$ in T_δ for some $\zeta < \mu \times \delta + \mu$ we have $\{\nu_1, \nu_2\} \subseteq w_{\delta,\zeta}$ and: $(\exists \delta' \in S \cap (\delta+1))(\nu_1 \in J_{\delta',\zeta}) \equiv (\exists \delta' \in S \cap (\delta+1))(\nu_2 \in J_{\delta',\zeta})$
- (D) if $n < \omega, \mu \times \delta + \mu \leq \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\bigcap\limits_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset \mod D_{\bar\eta}$
- (E) if $\delta_1 \in S \cap \delta, \bar\eta$ is a δ-candidate and $\bar\eta \restriction \delta_1 = \langle \eta_i \restriction \delta_1 : i \leq \lambda\rangle$ is a δ_1-candidate then $D_{\bar\eta\restriction\delta_1} \subseteq D_{\bar\eta}$
- $(F)_1$ $\eta \in w_{\delta,\zeta}$ iff $(\exists \delta')(\delta' \in S \cap (\delta+1)\ \&\ \eta \restriction \delta \in I_{\delta',\zeta})$
- $(F)_2$ if $\bar\eta = \langle \eta_i : i \leq \lambda\rangle$ is a δ-candidate and $\eta_\lambda \in w_{\delta,\zeta}$ then $\{i < \lambda : \eta_i \in w_{\delta,\zeta}\} \in D_{\bar\eta}$ and
  $\langle (\exists \delta' \in S \cap (\delta+1))(\eta_\lambda \restriction \delta' \in J_{\delta',\zeta})\rangle =$
  $\mathrm{LIM}_{D_{\bar\eta}} \langle (\exists \delta' \in S \cap (\delta+1))(\eta_i \restriction \delta' \in J_{\delta',\zeta}) : i < \lambda\rangle$
- $(F)_3$ $w_{\delta,\zeta}$ satisfies the following
  - (a) it is empty if $\zeta < \zeta_{<\delta}$
  - (b) has $\leq \lambda$ members if $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$
  - (c) otherwise $w_{\delta,\zeta}$ is the disjoint union $w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$ where
    $w^0_{\delta,\zeta} = \{\eta \in T_\delta : (\exists \delta' \in S \cap \delta)(\eta \restriction \delta' \in w_{\delta',\zeta})\}$
    $w^1_{\delta,\zeta} = \{\eta \in T_\delta : \eta \notin w^0_{\delta,\zeta}$ and for no κ-candidate $\bar\eta$ is $\eta \lhd \eta_\lambda\}$
    $w^2_{\delta,\zeta} = \{\eta \in T_\delta : \eta \notin w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta}$ and for some δ-candidate $\bar\eta, \eta_\lambda = \eta$ and $(\forall i < \lambda)(\exists \delta' \in S \cap \delta)(\eta_i \restriction \delta' \in w_{\delta',\zeta})$

and the set $\{i < \lambda : (\exists \delta' \in S \cap \delta)(\eta_i \restriction \delta' \in J_{\delta,\zeta})\}$
or its compliment belongs to $D_{\bar\eta \restriction \delta^*}$ for some $\delta^* < \delta\}$

$(F)_4$ $I_{\delta,\zeta} = w^2_{\delta,\zeta} \cup w^1_{\delta,\zeta}$

(G) if $\bar\eta$ is a δ-candidate and $B \subseteq \lambda$, $f^*(B) \in \mathbf{B}^*_{\mu \times (\delta+1)}$, then $B \in D_{\bar\eta} \vee (\lambda \backslash B) \in D_{\bar\eta}$.

We can ask more explicitly: there is an ultrafilter $D'_{\bar\eta}$ on the Boolean Algebra $\mathbf{B}^*_{\mu \times (\delta+1)}$ such that $D_{\bar\eta} = \{B \subseteq \lambda : f^*(B) \in D'_{\bar\eta}\}$.

The rest of the proof is split into carrying the construction and proving it is enough.

Stage F:This is Enough: First for every κ-candidate $\bar\eta$ lets $D_{\bar\eta} = \cup\{D_{\bar\nu,\delta} : \delta \in S, \bar\nu$ is a δ-candidate and $i \leq \lambda \Rightarrow \nu_i \triangleleft \eta_i\}$. Easily $D_{\bar\eta}$ is a uniform ultrafilter on λ. Let us define the space. The set of points of the space is $T_\kappa = {}^\kappa\mu$ and a subbase of clopen sets will be u_ζ: for $\zeta < \mu \times \kappa$ where u_ζ is defined as $u_\zeta =: \cup\{(T_\kappa)^{[\nu]} : \nu \in J_\zeta\}$ and $J_\zeta =: \bigcup_{\delta \in S} J_{\delta,\zeta}$. Now note that

(α) $I_\zeta = \cup\{I_{\delta,\zeta} : \delta \in S\}$ is an antichain and $\forall \rho \in T_\kappa \exists! \delta(\rho \restriction \delta \in I_{\delta,\zeta})$
[Why? We prove this by induction on $\rho(0)$ and is straight. In details, it is an antichain by the choice $I_{\delta,\zeta} = w^2_{\delta,\zeta}, w^2_{\delta,\zeta} \subseteq T_\delta \backslash w^0_{\delta,\zeta}$. As for the second phrase by the first there is at most one such δ; let $\rho \in T_\kappa$ and assume we have proved it for every $\rho' \in T_\kappa$ such that $\rho'(0) < \rho(0)$. By the definition of κ-candidate, if there is no κ-candidate $\bar\eta$ with $\eta_\lambda = \rho$, then for every large enough $\delta \in S$, there is no δ-candidate $\bar\eta$ with $\eta_\lambda = \rho \restriction \delta$, hence for any such $\delta, \rho \restriction \delta$ belongs to $w^0_{\delta,\zeta}$ or to $w^1_{\delta,\zeta}$, in the first case for some $\delta' \in \delta \cap S$ we have $(\rho \restriction \delta) \restriction \delta' \in I_{\delta',\zeta}$ so $\rho \restriction \delta' \in I_{\delta',\zeta}$ and we are done, in the second case $\rho \restriction \delta \in w^1_{\delta,\zeta} \subseteq I_{\delta,\zeta}$ and we are done. So assume that there is a κ-candidate $\bar\eta$ with $\eta_\lambda = \rho$, by the definition of a candidate it is unique and $i < \lambda \Rightarrow \eta_i(0) < \rho(0)$, so for each $i < \lambda$ there is $\delta_i \in S$ such that $\eta_i \restriction \delta_i \in I_{\delta_i,\zeta}$ and let $\gamma =$ Min$\{\gamma < \mu : \langle \eta_i \restriction \gamma : i < \lambda\rangle$ is with no repetition$\}$. Let $A = \{i < \lambda : \eta_i \restriction \delta_i \in J_{\delta,\zeta}\}$ so for some $\beta < \mu$ we have $f^*_2(A) \in \mathbf{B}^*_\beta$. For $\delta \in S$, which is $> \sup[\{\gamma, \delta_i : i < \lambda\}]$ we get $\rho \restriction \delta \in w_{\delta,\zeta}$ and we can finish as before.]

(β) X is a T_3 space
[why? as we use a clopen basis we really need just to separate points which holds by clause (C), i.e. if $\nu_1 \neq \nu_2 \in X$ then for some $\delta \in S$ we have $\nu_1 \restriction \delta \neq \nu_2 \restriction \delta$ and apply clause (C) to $\nu_1 \restriction \delta, \nu_2 \restriction \delta$]

(γ) $|X| = \mu^\kappa = 2^\kappa$
[why? as T_κ is the set of points of X]

(δ) suppose $Y = \{\eta_i : i < \lambda\} \subseteq X = T_\kappa$ and $\bigwedge_{i<j} \eta_i \neq \eta_j$. We need to show that $|c\ell(Y)|$ large, i.e. has cardinality 2^κ.

Choose γ such that $\langle \eta_i \restriction \gamma : i < \lambda \rangle$ is with no repetitions.

Let

$$W_{\bar\eta} = \{<>\} \cup \big\{\rho : \text{for some } \alpha \leq \kappa, \rho \in T_\alpha, \rho(0) \text{ code } \langle \eta_i \restriction \gamma : i < \lambda \rangle, \rho(0) > \sup\{\eta_i(0) : i < \lambda\} \text{ and } (\forall \beta < \ell g(\rho))(\beta \text{ odd } \Rightarrow \rho(\beta) \text{ code } \langle \eta_i \restriction \beta : i < \lambda \rangle \hat{} \langle \rho \restriction \beta \rangle)\big\}.$$

So clearly:

(i) $W_{\bar\eta} \cap T_1 \neq \emptyset$

(ii) $W_{\bar\eta}$ is a subtree of $(\bigcup_{\alpha \leq \kappa} T_\alpha, \triangleleft)$ (i.e. closed under initial segments, closed under limits),

(iii) every $\rho \in W_{\bar\eta} \cap T_\alpha$ where $\alpha < \kappa$ has a successor and if α is even has μ successors.

So $|W_{\bar\eta} \cap T_\kappa| = \mu^\kappa$.

So enough to prove

($*$) if $\rho \in W_{\bar\eta} \cap T_\kappa$ then $\rho \in c\ell\{\eta_i : i < \lambda\}$.

Let $\bar\eta = \langle \eta_i : i < \lambda \rangle, \eta_\lambda = \rho, \bar\eta' = \bar\eta \hat{} \langle \rho \rangle$ and the filter $D_{\bar\eta'} = \cup\{D_{\langle \bar\eta'_i \restriction \delta : i \leq \lambda \rangle} : \delta \in S \text{ and } \delta \geq \gamma\}$ is a filter by clause (E) and even ultrafilter by clause (G).

Now for every ζ, by clause $(\mathrm{F})_2$ for δ large enough

$$\text{Truth Value}(\rho \in u_\zeta) = \lim_{D_{\langle \bar\eta'_i \restriction \delta : i \leq \delta \rangle}} \langle \text{Truth Value}(\eta_i \in u_\zeta) : i < \lambda \rangle.$$

As $\{u_\zeta : \zeta < \mu \times \kappa\}$ is a clopen basis of the topology, we are done.

<u>Stage G: The construction</u>:

We arrive to stage $\delta \in S$. So for every δ-candidate $\bar\eta = \langle \eta_i : i \leq \lambda \rangle$, let

$$D'_{\bar\eta} = \cup\{D_{\langle \eta_i \restriction \delta_1 : i \leq \lambda \rangle} : \delta_1 \in \delta \cap S \text{ and } \langle \eta_i \restriction \delta_1 : i \leq \lambda \rangle \text{ a } \delta_1\text{-candidate}\} \cup D_0.$$

Note: $|T_\delta| = \mu$ by the choice of κ.

Let $<^*_\delta$ be a well ordering of T_δ such that: $\nu_1(0) < \nu_2(0) \Rightarrow \nu_1 <^*_\delta \nu_2$.

Hence

(∗) $\langle \eta_i : i \le \lambda \rangle$ a δ-candidate $\Rightarrow \bigwedge_{i<\lambda} \eta_i <^*_\delta \eta_\lambda$.

So let $\{\langle \nu_{1,\zeta}, \nu_{2,\zeta} \rangle : \zeta_{<\delta} \le \zeta < \zeta_\delta\}$ list $\{(\nu_1, \nu_2) : \nu_1 <^*_\delta \nu_2\}$; such a list exists as $\zeta_\delta \ge \zeta_{<\delta} + \mu$ and $|T_\delta| = \mu$. Now we choose by induction on $\zeta < \zeta_\delta$ the following

(α) $D^\zeta_{\bar\eta}$ for $\bar\eta$ a δ-candidate when $\zeta \ge \zeta_{<\delta}$

(β) $w^*_{\delta,\zeta}, I_{\delta,\zeta}, J_{\delta,\zeta}$

(γ) $D^{\zeta_{<\delta}}_{\bar\eta}$ is $D'_{\bar\eta}$ which was defined above

such that

(δ) $D^\zeta_{\bar\eta}$ for ζ in $[\zeta_{<\delta}, \zeta_\delta]$ is increasing continuous

(ε) if $n < \omega, \zeta_{<\delta} \le \zeta \le \xi_1 < \xi_2 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda^+$ then $\bigcap_{\ell=1}^{n} A_{\xi_\ell, \varepsilon_\ell} \ne \emptyset \mod D^\zeta_{\bar\eta}$

(ζ) $D^{\zeta+1}_{\bar\eta}, I_{\delta,\zeta}, J_{\delta,\zeta}$ satisfies the requirement $(\mathrm{F})_2$

(η) $\nu_{1,\zeta} \in J_{\delta,\zeta} \Leftrightarrow \nu_{2,\zeta} \notin J_{\delta,\zeta}$ or $\nu_{1,\zeta}, \nu_{2,\zeta} \in w^0_{\delta,\zeta}$

(θ) $D^\zeta_{\bar\eta}$ is $D'_{\bar\eta} + \{A_{\zeta_1, \varepsilon_{\bar\eta}(\zeta_0)} : \zeta_1 < \zeta\}$ for some function $\varepsilon_{\bar\eta} : [\zeta_{<\delta}, \zeta) \Rightarrow \lambda$.

Note: For $\zeta = 0$, condition (ε) holds by the induction hypothesis (i.e. clause (D)) and choice of D'_η (and choice of the $A_{\xi,\varepsilon}$'s if for no $\delta_1, \bar\eta \restriction \delta_1$ is a δ_1-candidate).

(ι) if $\zeta < \zeta_{<\delta}$ then:

$$w_{\delta,\zeta} = w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta} \text{ are defined as in } (F)_2$$

$$I^\zeta_{\delta,\zeta} = w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$$

$$\begin{aligned} J^\zeta_{\delta,\zeta} = \{\eta \in T_\delta : & \delta \in w^2_{\delta,\zeta} \text{ and for some } \delta\text{-candidate } \bar\eta \text{ we have } \eta_\lambda = \eta \\ & \text{hence } (\forall i < \lambda)(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in w_{\delta',\zeta}] \\ & \text{and } \{i < \lambda : (\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in J_{\delta',\zeta}]\} \text{ belongs to } D'_{\bar\eta}\}. \end{aligned}$$

[Note in the context above, by the induction hypothesis $(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in w_{\delta',\zeta}]$ is equivalent to $(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in I_{\delta',\zeta}]$ and thus δ' is unique. Of course, they have to satisfy the relevant requirements from (A)-(G)].

The cases $\zeta \leq \zeta_{<\delta}, \zeta$ limit are easy.
The crucial point is: we have $\langle D^\zeta_{\bar\eta} : \bar\eta$ a δ-candidate$\rangle$ and $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$ and we should define $w_{\delta,\zeta}, I_{\delta,\zeta}$ and $D^{\zeta+1}_{\bar\eta}$ to which the last stage is dedicated.

<u>Stage H</u>: Define by induction on $n < \omega$,

$$w^\zeta_0 = \{\nu_{1,\zeta}, \nu_{2,\zeta}\}$$

$$w^\zeta_{n+1} = \{\eta^\rho_i : i < \lambda, \rho \in w_n \text{ and } \bar\eta^\rho \text{ is a } \delta\text{-candidate with } \eta^\rho_\lambda = \rho\}.$$

Note that $\eta^\rho_i <^*_\delta \rho$.
Let $w = w_{\delta,\zeta} = I_{\delta,\zeta} = \bigcup\limits_{n<\omega} w^\zeta_n$, so $|w_{\delta,\zeta}| \leq \lambda$ (note that this is the first "time" we deal with ζ).

We need: to choose $J_{\alpha,\zeta} \cap w_{\delta,\zeta}$ so that the cases of condition (ζ) (i.e. $(\mathrm{F})_2$) for $\bar\eta^\rho, \rho \in w$ hold and condition (η) (i.e. (C) for $\nu_{1,\zeta}, \nu_{2,\zeta}$) holds.
Let $w'_{\delta,\zeta} = \{\rho \in w_{\delta,\zeta} : \bar\eta^\rho$ is well defined$\}$, (so $w'_{\delta,\zeta} \subseteq w_{\delta,\zeta}$). Let $w'_{\delta,\zeta} = \{\rho[\zeta,\varepsilon] : \varepsilon < \varepsilon^* \leq \lambda\}$. Now we define $D^{\zeta+1}_{\bar\eta^{\rho[\zeta,\varepsilon]}}$ as $D^\zeta_{\eta^{\rho[\zeta,\varepsilon]}} + A_{\zeta,\varepsilon}$, clearly "legal".

Let $A'_{\zeta,\varepsilon} = \{i < \lambda : i \in A_{\zeta,\varepsilon}$ and $i > \varepsilon$ and $\eta^{\rho[\zeta,\varepsilon]}_i \notin \{\eta^{\rho[\zeta,\varepsilon_1]}_{i_1} : \varepsilon_1 < i$ and $i_1 < i\}$ and $\eta^{\rho[\zeta,\varepsilon]}_i \neq \nu_{1,\zeta}, \nu_{2,\zeta}\}$.
Observe

- $(*)_1$ $A_{\zeta,\varepsilon} \backslash A'_\varepsilon$ is not stationary by Fodor's lemma as $\langle \eta^{\rho[\varepsilon]}_i : i < \lambda\rangle$ is with no repetition.

Now we shall prove that

- $(*)_2$ the sets $\{\eta^{\rho[\varepsilon]}_i : i \in A'_\varepsilon\}$ for $\varepsilon > \varepsilon^*$ are pairwise disjoint.

So toward contradiction suppose $i_1 \in A'_{\varepsilon_1}, i_2 \in A'_{\varepsilon_2}, \varepsilon_1 < \varepsilon_2 < \varepsilon^*$ and $\eta^{\rho^{[\zeta,\varepsilon_1]}}_{i_1} = \eta^{\rho^{[\zeta,\varepsilon_2]}}_{i_2}$ and try to get a contradiction.

<u>Case 1</u>: $i_2 > i_1$.
As $i_1 \in A'_{\varepsilon_1}$ we have $i_1 > \varepsilon_1$ similarly $i_2 > \varepsilon_2$ but $\varepsilon_1 < \varepsilon_2$ so $i_2 > \varepsilon_2 > \varepsilon_1$, and by the assumption $i_2 > i_1$. So $\eta^{\rho^{[\zeta,\varepsilon_1]}}_{i_1}$ belongs to the set $\{\eta^{\rho^{[\zeta,\varepsilon]}}_i : \varepsilon < i_2 \ \& \ i < i_2\}$ so $\eta^{\rho[\zeta,\varepsilon_2]}_{i_2} \neq \eta^{\rho[\zeta,\varepsilon_1]}_{i_1}$ as $\eta^{\rho[\zeta,\varepsilon_2]}_{i_2}$ does not belong to this set as $i_2 \in A'_{\varepsilon_2}$.

Case 2: $i_2 < i_1$.

As $i_2 \in A'_{\zeta,\varepsilon_2}$ necessarily $\varepsilon_2 < i_2$. So $\varepsilon_2 < i_2 < i_1$ so $\eta^{\rho^{[\zeta,\varepsilon_2]}}_{i_2} \in \{\eta^{\rho^{[\varepsilon]}}_i : \varepsilon < i_1 \ \& \ \ell^i < i_1\}$ but $\eta^{\rho^{[\zeta,\varepsilon_1]}}_{i_2}$ does not belong to this set as $i_1 \in A'_{\varepsilon_1}$ hence $\eta^{[\zeta,\varepsilon_1]}_{i_1}, \eta^{[\zeta,\varepsilon_2]}_{i_2}$ cannot be equal.

Case 3: $i_1 = i_2$.

As $i_1 \in A'_{\varepsilon_1}$ we have $i_1 \in A_{\zeta,\varepsilon_1}$ similarly $i_2 \in A_{\zeta,\varepsilon_2}$ but those sets are disjoint; a contradiction.
So $(*)_2$ holds.

Now define $w^{\zeta,\ell}_n$ for $\ell = 1,2, n < \omega$ by induction on

$$n : w^{\zeta,\ell}_0 = \{\nu_{\ell,\zeta}\}$$

$$w^{\zeta,\ell}_{n+1} = \{\eta^{\rho^{[\zeta,\varepsilon]}}_i : \rho[\zeta,\varepsilon] \in w^{\zeta,\ell}_n \text{ and } i \in A'_\varepsilon \text{ and } \varepsilon < \varepsilon^*\}.$$

Let $w^{\zeta,\ell} = \bigcup\limits_{n<\omega} w^{\zeta,\ell}_n$, now by $(*)_2$, $w^{\zeta,1} \cap w^{\zeta,2} = \emptyset$ (note the clause $\eta^{\rho^{[\zeta,\varepsilon]}}_i \neq \nu_{1,\zeta}$ in the definition of A'_ε).
So we define

$$J_{\delta,\zeta} = w^{\zeta,2}.$$

Now it is easy to check clause (F), i.e. (ζ) and we have finished the induction on $\zeta < \zeta_\delta$. Now choose $D_{\bar\eta}$ to satisfy clause (G) and to extend $\bigcup\limits_{\zeta<\zeta_\delta} D^\zeta_{\bar\eta}$, so we are done. $\square_{1.1}$

* * *

§2 The singular case and the full result

2.1 Theorem. *Assume $\lambda > \mathrm{cf}(\lambda)$. Let $\mu = 2^\lambda, \kappa = \mathrm{Min}\{\kappa : 2^\kappa > \mu\}$. There is a Hausdorff space X with a clopen basis with $|X| = 2^\kappa$ such that for $Y \subseteq \lambda$ closed $|Y| < |X| \Rightarrow |Y| < \lambda$.*

Proof. For λ singular we should replace the filter D_0 on λ. So let $\lambda = \sum\limits_{j<\mathrm{cf}(\lambda)} \lambda_j, \lambda_j$ strictly increasing $\bar\lambda = \langle \lambda_j : j < \mathrm{cf}(\lambda)\rangle$. Let $D^*_{\bar\lambda} = \{A \subseteq \lambda :$ for every $j < \mathrm{cf}(\lambda)$ large enough, the set $A \cap \lambda^+_j$ contains a club of $\lambda^+_j\}$.

We can find a partition $\langle A^j_\alpha : \alpha < \lambda^+_j\rangle$ of $\lambda^+_j \backslash \lambda_j$ to stationary sets; let us stipulate $A^j_\alpha = \emptyset$ when $\lambda^+_j \le \alpha < \lambda$ and let $\bar A^* = \langle A_\alpha = \bigcup\limits_{j<\mathrm{cf}(\lambda)} A^j_\alpha : \alpha < \lambda\rangle$ (so $A_\alpha \ne \emptyset \mod D^*_{\bar\lambda}$ and $\alpha < \beta < \lambda \Rightarrow A_\alpha \cap A_\lambda = \emptyset$). Let $\{f_\xi : \xi < \mu \times \kappa\}$ be a family of functions from λ to λ such that if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\{\alpha < \lambda : f_{\varepsilon_\ell}(\alpha) = \varepsilon_\ell$ for $\ell = 1, \ldots, n\}$ is not empty (exists by [EK]). Now for $\xi < \mu \times \kappa$ and $\varepsilon < \lambda$ we let $A_{\xi,\varepsilon} = \cup\{A_\alpha : f_\xi(\alpha) = \varepsilon\}$. Clearly $\xi < \mu \times \kappa$ & $\varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow A_{\xi,\varepsilon_1} \cap A_{\xi,\varepsilon_2} = \emptyset$, and also: if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\bigcap\limits_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \ne \emptyset \mod D^*_{\bar\lambda}$. Let D_0 be a maximal filter on λ extending $D^*_{\bar\lambda}$ and still satisfying $\bigcap\limits_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \ne \emptyset \mod D_0$ for $n, \xi_\ell, \varepsilon_\ell (\ell < n)$ as above.

Now the proof proceeds as before. All is the same except in stage H where we use λ regular, D_0 contains all clubs of λ.

The point is that we define A'_ε as before, the main question is: why $A'_\varepsilon = A_\varepsilon \mod D^*_{\bar\lambda}$.

Choose $j^* < \mathrm{cf}(\lambda)$ such that:

$$\varepsilon < \lambda_{j^*}.$$

So it is enough to show

(∗) if $j^* \le j < \mathrm{cf}(\lambda)$ then
$A'_\varepsilon \cap [\lambda_j, \lambda^+_j) = A_\varepsilon \cap [\lambda_j, \lambda^+_j) \mod D_{\lambda^+_j}$

(where $D_{\lambda_j^+}$-the club filter on λ_j^+).
Looking at the definition of $A'_{\zeta,\varepsilon}$,

$$A'_{\zeta,\varepsilon} \cap [\lambda_j, \lambda_j^+) = \big\{ i \in [\lambda_j, \lambda_j^+) : i \in A_{\zeta,\varepsilon} \cap [\lambda_j, \lambda_j^+) \text{ and } \eta_{i_1}^{\rho[\zeta,\varepsilon]} \notin \{\eta_{i_1}^{\rho[\zeta,\varepsilon_1]} : \varepsilon_1 < i \text{ and } i_1 < i\} \text{ and } \eta_i^{\rho[\varepsilon]} \neq \nu_{1,\zeta} \big\}$$

as $\langle \eta_i^{\rho^{[\zeta,\varepsilon]}} : \lambda_j \leq i < \lambda_j^+ \rangle$ is with no repetition and Fodor's theorem holds (can formulate the demand on D). Just check that the use of $A'_{\zeta,\varepsilon}$ in §1 still works.

2.2 Conclusion: If $\lambda \geq \aleph_0, \kappa = \text{ Min}\{\kappa : 2^\kappa > 2^\lambda\}$, then there is a T_3-space $\lambda, |X| = 2^\kappa$ with no closed subspace of cardinality $\in [\lambda, 2^\kappa)$. $\square_{2.1}$

* * *

We still would like to replace 2^κ by 2^{2^λ}.

2.3 Theorem. *For $\lambda \geq \aleph_0$ there is a T_3 space X with clopen basis such that: no closed subspace has cardinality in $[\lambda, 2^{2^\lambda}]$.*

Proof. For $\lambda = \aleph_0$ it is known so let $\lambda > \aleph_0$. Like the proof of 1.1 with $\kappa = 2^\mu$. The only problem is that $T_\delta = {}^\delta\mu$ may have cardinality $> 2^\mu$ so we have to redefine a δ-candidate (as there are too many $\eta_i \restriction \gamma$ to code) and in the crucial Stages G and H we have the list $\{(\nu^\delta_{1,\varepsilon}, \nu^\delta_{2,\varepsilon}) : \varepsilon < |T_\delta|\}$ but possibly $|T_\delta| > 2^\mu$. Still $|T_\delta| \leq \mu^{|\delta|} \leq 2^\mu$; so instead dedicating one $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$ to deal with any such pair we just do it for each "kind" of pairs such that the number of kinds is $\leq \mu$, (but we can deal with all of them at once).

Stage B':
Let $Cd : \mu \to \mathscr{H}_{<\lambda^+}(\mu)$ be such that for every $x \in \mathscr{H}_{<\lambda^+}(\mu)$ for μ ordinals $\alpha < \mu$ we have $Cd(\alpha) = x$.

Stage C':
For limit $\delta \leq \kappa$ we call $\bar\eta$ a δ-candidate if:

(a) $\bar\eta = \langle \eta_i : i \leq \lambda \rangle$

(b) $\eta_i \in T_\delta$
(c) for some $\gamma, \langle \eta_i \restriction \gamma : i < \lambda \rangle$ is with no repetition
(d) for odd $\beta < \delta$ we have
$Cd(\eta_\lambda(\beta)) = \langle (\eta_i(\beta - 1), \eta_i(\beta)) : i < \lambda \rangle$
(e) $Cd(\eta_\lambda(0)) = \{(i, j, \gamma, \eta_i(\gamma), \eta_j(\gamma)) : i < j < \lambda$ and for some $i_1 < j_1 < \lambda, \gamma$ minimal such that $\eta_{i_1}(\gamma) \neq \eta_{j_1}(\gamma)\}$
(f) $\eta_\lambda(0) > \sup\{\eta_i(0) : i < \lambda\}$.

So

$(*)_1$ if $\langle \eta_i : i \leq \lambda \rangle$ is a δ_1-candidate, $\delta_0 < \delta_1$ limit and $(\exists \gamma < \delta_0)(\langle \eta_i \restriction \gamma : i \leq \lambda \rangle$ with no repetitions then $\langle \eta_i \restriction \delta_0 : i \leq \lambda \rangle$ is a δ_0-candidate
$(*)_2$ if $\eta_i \in T_\kappa$ for $i < \kappa$ are pairwise distinct then for 2^μ sequences $\eta_\lambda \in T_\kappa$ we have $\langle \eta_i : i \leq \lambda \rangle$ is a κ-candidate.

Stage H':
For each $\varepsilon < |T_\delta|$ we can choose $v_{\delta,\varepsilon} = \cup\{v_{\delta,\varepsilon,n} : n < \omega\}$ where we define $v_{\delta,\varepsilon,n}$ by induction on n as follows:
$v_{\delta,\varepsilon,0} = \{\nu^\delta_{1,\varepsilon}, \nu^\delta_{2,\varepsilon}\}, v_{\delta,\varepsilon,n+1} = v_{\delta,\varepsilon,n} \cup \{\eta^\rho_i : \rho \in v_{\delta,\varepsilon,n}$ and $\bar{\eta}^\rho$ is a δ-candidate such that $\eta^\rho_\lambda = \rho\}$. We choose $u_\varepsilon = u_{\delta,\varepsilon} \in [\delta]^{\leq\lambda}$ such that: if $\bar{\eta}$ is a δ-candidate satisfying $\eta_\lambda \in v_{\delta,\varepsilon}$ (so $\eta_i \in v_{\delta,\varepsilon}$ for $i < \lambda$) then $0 \in u_\varepsilon$ & $i < j < \lambda \Rightarrow \text{Min}\{\gamma : \eta_i(\gamma) \neq \eta_j\gamma)\} \in u_\varepsilon$.
As $|T_\delta| \leq 2^\mu$ and $\mu^\lambda = \mu$ by Engelking Karlowic [EK] there are functions $H^\delta_\Upsilon : T_\delta \to \mathscr{H}_{<\lambda^+}(\mu)$ for $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ such that for every $w \in [T_\delta]^\lambda$ and $h : w \to \mathscr{H}_{<\lambda^+}(\mu)$ there is $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ such that $h \subseteq H^\delta$.

As $\mu = \mu^\lambda = |\mathscr{H}_{<\lambda^+}(\mu)|$, without loss of generality $|\text{Rang}(H^\delta_\Upsilon)| \leq \lambda$ (divide H^δ_Υ to $\leq 2^\lambda = \mu$ functions).

For each $\varepsilon < |T_\delta|$ let $h^\varepsilon_\delta : v_{\delta,\varepsilon} \to \mathscr{H}_{<\lambda^+}(\mu)$ be $h^\varepsilon_\delta(\eta) = (h^{\varepsilon,0}_\delta(\eta), h^{\varepsilon,1}_\delta(\eta), h^{\varepsilon,2}_\delta(\eta))$ where

$$h^{\varepsilon,0}_\delta(\eta) = \text{ otp}(\{\nu \in w^\varepsilon_\delta : \nu <^*_\delta \eta\}, <^*_\delta)$$

$$h^{\varepsilon,1}_\delta(\eta) = \{\langle \gamma, \eta(\gamma) \rangle : \gamma \in u_{\delta,\varepsilon}\}$$

$$h^{\varepsilon,2}_\delta(\eta) = \text{ truth value of } \eta \in v_{\delta,\varepsilon,0}$$

(the function h^ε_δ belongs to $\mathscr{H}_{<\lambda^+}(\mu)$ as $|v_{\delta,\varepsilon}| \leq \lambda$); let

$$\Upsilon_\varepsilon = \text{ Min}\{\Upsilon \in [\zeta_{<\delta}, \zeta_\delta) : h^\varepsilon_\delta \subseteq H^\delta_\Upsilon\}$$

(well defined). Let $\gamma^\delta_\Upsilon =: \sup\{\gamma < \lambda^+ : \gamma$ is the first cardinal in some sequence $\bar\lambda$ from $(\text{Rang}(H^\delta_\Upsilon)\}$, let g^δ_Υ be a one-to-one function from γ^δ_Υ into λ.

Next we can define the $D^\Upsilon_{\bar\eta}$ for $\bar\eta$ a δ-candidate; for $\Upsilon < \mu$:

$$D^{\Upsilon+1}_{\bar\eta} = D^\Upsilon_{\bar\eta} + A_{\Upsilon,\gamma^\delta_\Upsilon}.$$

In Stage $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ we deal with all $\varepsilon < |T_\delta|$ such that $\Upsilon_\varepsilon = \Upsilon$. Now we treat the choice of $I_{\delta,\zeta}, J_{\delta,\zeta}, w_{\delta,\zeta}$. We can finish as before (but dealing with many cases at once). $\square_{2.3}$

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