ON $T_3$-TOPOLOGICAL SPACE OMITTING MANY CARDINALS

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Abstract. We prove that for every (infinite cardinal) $\lambda$ there is a $T_3$-space $X$ with clopen basis, $2^\mu$ points where $\mu = 2^\lambda$, such that every closed subspace of cardinality $< |X|$ has cardinality $< \lambda$.

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§0 Introduction

Juhasz has asked on the spectrums $c - sp(X) = \{ |Y| : Y \text{ an infinite closed subspace of } X \}$ and $w - sp(X) = \{ w(Y) : Y \text{ a closed subspace of } X \}$. He proved [Ju93] that if $X$ is a compact Hausdorff space, then $|X| > \kappa \Rightarrow c - sp(X) \cap [\kappa, \sum_{\lambda < \kappa} 2^{2^{\lambda}}] \neq \emptyset$ and $w(X) > \kappa \Rightarrow w - sp(X) \cap [\kappa, 2^{<\kappa}] \neq \emptyset$. So under GCH the cardinality spectrum of a compact Hausdorff space does not omit two successive regular cardinals, and omit no inaccessible. Of course, the space $\beta(\omega) \setminus \omega$, the space of nonprincipal ultrafilters on $\omega$, satisfies $c - sp(X) = \{ \beth_2 \}$. Now Juhasz Shelah [JuSh 612] shows that we can omit many singular cardinals, e.g. under GCH for every regular $\lambda > \kappa$, there is a compact Hausdorff space $X$ with $c - sp(X) = \{ \mu : \mu \leq \lambda, \text{cf}(\mu) \geq \kappa \}$; see more there and in [Sh 652]. In fact [JuSh 612] constructs a Boolean Algebra, so relevant to the parallel problems of Monk [M]. Here we deal with the noncompact case and get a strong existence theorem. Note that trivially for a Hausdorff space $X$, $|X| \geq \kappa \Rightarrow c - sp(X) \cap [\kappa, 2^{\geq \kappa}] \neq \emptyset$, using the closure of any set with $\kappa$ points, so our result is in this respect best possible.

We prove

0.1 Theorem. For every infinite cardinal $\lambda$ there is a $T_3$ topological space $X$, even with clopen basis, with $2^{2^{\lambda}}$ points such that every closed subset with $\geq \lambda$ points has $|X|$ points.

In §1 we prove a somewhat weaker theorem but with the main points of the proof present, in §2 we complete the proof of the full theorem.
\section*{1.1 Theorem.}
Assume $\lambda = \text{cf}(\lambda) > \aleph_0$. Let $\mu = 2^\lambda$, $\kappa = \text{Min}\{\kappa : 2^\kappa > \mu\}$. There is a Hausdorff space $X$ with a clopen basis with $|X| = 2^\kappa$ such that: if for $Y \subseteq \lambda$ is closed and $|Y| < |X|$ then $|Y| < \lambda$.

\textit{Proof.} Let $S \subseteq \{\delta < \kappa : \delta \text{ limit}\}$ be stationary. Let $T_\alpha = \alpha^\mu$ for $\alpha \leq \kappa$ and let $T = \bigcup_{\alpha \leq \kappa} T_\alpha$. Let $\zeta_\alpha = \bigcup\{\mu \delta + \mu : \delta \in S \cap (\alpha + 1)\}$. Let $\xi_{<\alpha} = \bigcup\{\zeta_\beta : \beta < \alpha\}$.

\textbf{Stage A:} We shall choose sets $u_\zeta \subseteq T_\kappa$ (for $\zeta < \mu \times \kappa$). Those will be clopen sets generating the topology. For each $\zeta$ we choose $(I_\zeta, J_\zeta)$ such that: $I_\zeta$ is a $\prec$-antichain of $(\kappa^\delta, \prec)$ such that for every $\rho \in T_\kappa$, $\exists! \alpha (\rho \upharpoonright \alpha \in I_\zeta)$ and $J_\zeta \subseteq I_\zeta$ and we shall let $u_\zeta = \bigcup_{\nu \in J_\zeta} (T_\kappa)^{[\nu]}$ where $(T_\kappa)^{[\nu]} = \{\rho \in T_\kappa : \nu \prec \rho\}$. Let $I_{\alpha, \zeta} = T_\alpha \cap I_\zeta, J_{\alpha, \zeta} = T_\alpha \cap J_\zeta$ but we shall have $\alpha \notin S \Rightarrow I_{\alpha, \zeta} = \emptyset = J_{\alpha, \zeta}$.

\textbf{Stage B:} Let $Cd : \mu \to \lambda^+(T_{<\kappa})$ be onto such that for every $x \in \text{Rang}(Cd)$ we have $\text{otp}\{\alpha < \mu : Cd(\alpha) = x\} = \mu$.
We say $\alpha$ codes $x$ (by $Cd$) if $Cd(\alpha) = x$.

\textbf{Stage C: Definition:} For $\delta \leq \kappa$ we call $\bar{\eta}$ a $\delta$-candidate if

(a) $\bar{\eta} = \langle \eta_i : i \leq \lambda \rangle$
(b) $\eta_i \in T_\delta$
(c) $\exists! \gamma \prec \delta (\bigwedge_{i < j < \lambda} \eta_i \upharpoonright \gamma \neq \eta_j \upharpoonright \gamma)$
(d) for every odd $\beta < \delta$, we have $Cd(\eta_\lambda(\beta)) = \langle \eta_i \upharpoonright \beta : i \leq \lambda \rangle$
(e) $\eta_\lambda(0)$ codes $\langle \eta_i \upharpoonright \gamma : i < \lambda\rangle$, where $\gamma = \gamma(\eta_\lambda \upharpoonright \lambda) = \text{Min}\{\gamma < \delta : i < j < \lambda \Rightarrow \eta_i \upharpoonright \gamma \neq \eta_j \upharpoonright \gamma\}$, it is well defined by clause (c) and
(f) $\eta_\lambda(0) > \sup\{\eta_i(0) : i < \lambda\}$.

\textbf{Stage D: Choice:} Choose $A_{\xi, \varepsilon} \subseteq \lambda$ for $\xi < \mu \times \kappa, \varepsilon < \lambda$ such that:

$\xi < \mu \times \kappa \& \varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow |A_{\xi, \varepsilon_1} \cap A_{\xi, \varepsilon_2}| < \lambda$ and even $= \emptyset$.
and

$$\xi_1 < \ldots < \xi_n < \mu \times \kappa, \varepsilon_1 \ldots \varepsilon_n < \lambda \Rightarrow \bigcap_{\ell=1}^{n} A_{\xi_\ell, \varepsilon_\ell} \text{ is a stationary subset of } \lambda.$$

Let \( \Xi = \{(\xi_1, \varepsilon_1), \ldots, (\xi_n, \varepsilon_n)\} : \xi_1, \ldots, \xi_n < \mu \times \kappa \) is with no repetitions and \( \varepsilon_1, \ldots, \varepsilon_n < \lambda \) and for \( x \in \Xi \) let \( A_x = \bigcap_{\ell=1}^{n} A_{\xi_\ell, \varepsilon_\ell} \). Let \( D_0 \) be a maximal filter on \( \lambda \) extending the club filter such that \( x \in \Xi \Rightarrow A_x \neq \emptyset \text{ mod } D_0 \).

For \( A \subseteq \lambda \) let

$$\mathcal{B}^+(A) = \{x \in \Xi : A \cap A_x = \emptyset \text{ mod } D_0 \text{ but } y \nsubseteq x \Rightarrow A \cap A_y = \emptyset \text{ mod } D_0\}$$

$$\mathcal{B}(A) =: \mathcal{B}^+(A) \cup \mathcal{B}^+(\lambda \setminus A).$$

**Fact:** \( \mathcal{B}(A) =: \mathcal{B}^+(A) \cup \mathcal{B}^+(\lambda \setminus A) \) is predense in \( \Xi \) i.e.

$$\forall x \subseteq \Xi \exists y \in \mathcal{B}(A) (x \cup y \in \Xi).$$

**Proof.** If \( x \in \Xi \) contradict it then we can add to \( D_0 \) the set \( \lambda \setminus (A_x \cap A) \) getting \( D'_0 \). Now \( D'_0 \) thus properly extends \( D_0 \) otherwise \( A_x \cap A = \emptyset \text{ mod } D_0 \) hence, let \( x' \subseteq x \) be minimal with this property so \( x' \in \mathcal{B}^+(A) \) and \( x \) by assumption satisfies:

$$\neg(\exists y \in \Xi)(x \cup y \in \mathcal{B}(A))$$

so try \( y = x \). For every \( z \in \Xi \) we have \( A_z \neq \emptyset \text{ mod } D_0 \).

**Fact:** \( |\mathcal{B}(A)| \leq \lambda \) for \( A \subseteq \lambda \).

**Proof.** Let \( \mathcal{B}_0 \) be the Boolean Algebra freely generated by \( \{x_{\xi, \varepsilon} : \xi < \mu \times \kappa, \varepsilon < \lambda\} \), by \( \Delta \)-system argument, except \( x_{\xi_1, \varepsilon_1} \cap x_{\xi_2, \varepsilon_2} = 0 \) if \( \varepsilon_1 \neq \varepsilon_2 \); clearly \( \mathcal{B}_0 \) satisfies \( \lambda^+ \)-c.c.

Let \( \mathcal{B}^* \) be the completion of \( \mathcal{B}_0 \). Let \( f^* \) be a homomorphism from \( \mathcal{P}(\lambda) \) into \( \mathcal{B}^* \) such that \( C \in D_0 \Rightarrow f^*(C) = 1_{\mathcal{B}^*} \) and

$$f(A_{\xi, \varepsilon}) = x_{\xi, \varepsilon}.$$

[Why exists? Look at the Boolean Algebra \( \mathcal{P}(\lambda) \) let \( I_\lambda = \{A \subseteq \lambda : \lambda \setminus A \in D_0\} \) and \( \mathcal{A}_0 = I_\lambda \cup \{\lambda \setminus A : A \in I_\lambda\} \) is a subalgebra of \( \mathcal{P}(\lambda) \), and let \( I_\lambda \cup \{A_{\xi, \varepsilon} : \xi \leq \lambda, \varepsilon < \lambda\} \) is a subalgebra of \( \mathcal{P}(\lambda) \).]
Let $\mathfrak{A}_0$ be the homomorphism with kernel $I_\lambda$. Let $f_0^* : \mathfrak{A}_0 \to B_0$ be the homomorphism from $\mathfrak{A}$ into $B_0$ extending $f_0$ such that $f_0^*(A_{\xi,\varepsilon}) = x_{\xi,\varepsilon}$, clearly exists and is onto. Now as $B^*$ is a complete Boolean Algebra, $f_1^*$ can be extended to a homomorphism $f_2^*$ from $\mathscr{P}(\lambda)$ into $B^*$. Clearly $\text{Ker}(f_2^*) = \text{Ker}(f_1^*) = \text{Ker}(f_0^*) = I_\lambda$ so $f_1^*$ induces an isomorphism from $\mathscr{P}(\lambda)/D_0$ onto $\text{Rang}(f_1^*) \subseteq B^*$, so the problem translates to $B^*$. So $B_0$ satisfies the $\lambda^+$-c.c and is a dense subalgebra of $B^*$ hence of range($f_2^*$), so this range is a $\lambda^+$-c.c. Boolean Algebra hence $\mathscr{P}(\lambda)/D_0$ satisfies the fact.

Let $B_\gamma^*$ be the complete Boolean subalgebra of $B^*$ generated (as a complete subalgebra) by $\{x_{\xi,\varepsilon} : \xi < \gamma, \varepsilon < \lambda\}$. Clearly $B^* = \bigcup_{\gamma < \kappa} B_\gamma^*$ and $B_\gamma^*$ is increasing with $\gamma$.

Stage E: We choose by induction on $\delta \in S$ the following

\begin{enumerate}
  \item[(A)] $w_{\delta,\zeta} \subseteq T_\delta$ (for $\zeta < \mu \delta + \mu$) and $J_{\delta,\zeta} \subseteq I_{\delta,\zeta} \subseteq w_{\delta,\zeta}$
  \item[(B)] for each $\delta$-candidate $\bar{\eta} = \langle \eta_i : i \leq \lambda \rangle$, a uniform filter $D_{\bar{\eta}}$ on $\lambda$ extending the filter $D_0$
  \item[(C)] for each $\nu_1 \neq \nu_2$ in $T_\delta$ for some $\zeta < \mu \times \delta + \mu$ we have $\{\nu_1, \nu_2\} \subseteq w_{\delta,\zeta}$ and:
  \begin{enumerate}
    \item[$(\exists \delta') \in S \cap (\delta + 1)$](\nu_1 \in J_{\delta',\zeta}) \equiv (\exists \delta') \in S \cap (\delta + 1) (\nu_2 \in J_{\delta',\zeta})$
  \end{enumerate}
  \item[(D)] if $n < \omega, \mu \times \delta + \mu \leq \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then
  \[ \cap_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset \mod D_{\bar{\eta}} \]
  \item[(E)] if $\delta_1 \in S \cap \delta, \eta$ is a $\delta$-candidate and $\bar{\eta} \upharpoonright \delta_1 = \langle \eta_i \upharpoonright \delta_1 : i \leq \lambda \rangle$ is a $\delta_1$-candidate
    \begin{enumerate}
      \item[$(F)_1$] $\eta \in w_{\delta,\zeta}$ \iff $(\exists \delta') \in S \cap (\delta + 1)$ & $\bar{\eta} \upharpoonright \delta \in I_{\delta',\zeta}$
      \item[$(F)_2$] if $\bar{\eta} = \langle \eta_i : i \leq \lambda \rangle$ is a $\delta$-candidate and $\eta_\lambda \in w_{\delta,\zeta}$ then $\{i < \lambda : \eta_i \in w_{\delta,\zeta} \in D_{\bar{\eta}}$ and
        \[ (\exists \delta') \in S \cap (\delta + 1))(\eta_i \upharpoonright \delta' \in J_{\delta',\zeta}) \]
      \item[$(F)_3$] $w_{\delta,\zeta}$ satisfies the following
        \begin{enumerate}
          \item[(a)] it is empty if $\zeta < \zeta_{\delta}$
          \item[(b)] has $\lambda$ members if $\zeta \in [\zeta_{\delta}, \zeta_{\delta})$
          \item[(c)] otherwise $w_{\delta,\zeta}$ is the disjoint union $w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$ where
            \[ w^0_{\delta,\zeta} = \{ \eta \in T_\delta : (\exists \delta' \in S \cap \delta)(\eta \upharpoonright \delta' \in w_{\delta',\zeta}) \} \]
            \[ w^1_{\delta,\zeta} = \{ \eta \in T_\delta : \eta \notin w^0_{\delta,\zeta} \text{ and for no } \kappa \text{-candidate } \bar{\eta} \text{ is } \eta < \eta_\lambda \} \]
            \[ w^2_{\delta,\zeta} = \{ \eta \in T_\delta : \eta \notin w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \text{ and for some } \delta \text{-candidate } \bar{\eta}, \eta_\lambda = \eta \text{ and } (\forall i < \lambda)(\exists \delta' \in S \cap \delta)(\eta_i \upharpoonright \delta' \in w_{\delta,\zeta}) \}
        \end{enumerate}
  \end{enumerate}
\end{enumerate}
Ju-sets will be defined the space. The set of points of the space is \( J = (\forall \eta =: \zeta) \).

\[
\delta \text{ is a } T \text{-candidate on the Boolean Algebra } B_{\mu \times (\delta + 1)}^* \text{ such that } D_{\eta} = \{ B \subseteq \lambda : f^*(B) \in D_{\eta}' \}.
\]

We can ask more explicitly: there is an ultrafilter \( D_{\eta}' \) on the Boolean Algebra \( B_{\mu \times (\delta + 1)}^* \) such that \( D_{\eta} = \{ B \subseteq \lambda : f^*(B) \in D_{\eta}' \} \).

The rest of the proof is split into carrying the construction and proving it is enough.

Stage F: This is Enough: First for every \( \kappa \)-candidate \( \bar{\eta} \) lets \( D_{\eta} = \cup\{ D_{\rho, \delta} : \delta \in S, \bar{\nu} \text{ is a } \delta \text{-candidate and } i \leq \lambda \} \). Easily \( D_{\eta} \) is a uniform ultrafilter on \( \lambda \). Let us define the space. The set of points of the space is \( T_{\kappa} = \kappa \mu \) and a subbase of clopen sets will be \( u_\zeta \): for \( \zeta < \mu \times \kappa \) where \( u_\zeta \) is defined as \( u_\zeta =: \cup\{ (T_{\kappa})^{[\nu]} : \nu \in J_{\zeta} \} \) and \( J_{\zeta} =: \bigcup_{\delta \in S} J_{\delta, \zeta} \). Note that

\[
\begin{align*}
(\alpha) & \quad I_{\delta, \zeta} = \cup\{ I_{\delta, \zeta} : \delta \in S \} \text{ is an antichain and } \forall \rho \in T_{\kappa} \exists ! \delta (\rho \upharpoonright \delta \in I_{\delta, \zeta}) \\
& \text{[Why?] We prove this by induction on } \rho(0) \text{ and is straight. In details, it is an antichain by the choice } I_{\delta, \zeta} = w_{\delta, \zeta}^2 \cup w_{\delta, \zeta}^1 \subseteq T_{\delta} \setminus w_{\delta, \zeta}^0. \text{ As for the second phrase by the first there is at most one such } \delta; \text{ let } \rho \in T_{\kappa} \text{ and assume we have proved it for every } \rho' \in T_{\kappa} \text{ such that } \rho'(0) < \rho(0). \text{ By the definition of } \kappa \text{-candidate, if there is no } \kappa \text{-candidate } \bar{\eta} \text{ with } \eta_\lambda = \rho, \text{ then for every large enough } \delta \in S, \text{ there is no } \delta \text{-candidate } \bar{\eta} \text{ with } \eta_\lambda = \rho \upharpoonright \delta, \text{ hence for any such } \delta, \rho \upharpoonright \delta \text{ belongs to } w_{\delta, \zeta}^0 \text{ or to } w_{\delta, \zeta}^1, \text{ in the first case for some } \delta' \in \delta \cap S \text{ we have } (\rho \upharpoonright \delta') \upharpoonright \delta' \in I_{\delta', \zeta} \text{ so } \rho \upharpoonright \delta' \in I_{\delta', \zeta} \text{ and we are done, in the second case } \rho \upharpoonright \delta \in w_{\delta, \zeta}^1 \subseteq I_{\delta, \zeta} \text{ and we are done. So assume that there is a } \kappa \text{-candidate } \bar{\eta} \text{ with } \eta_\lambda = \rho, \text{ by the definition of a candidate it is unique and } i < \lambda \Rightarrow \eta_i(0) < \rho(0), \text{ so for each } i < \lambda \text{ there is } \delta_i \in S \text{ such that } \eta_i \upharpoonright \delta_i \in I_{\delta_i, \zeta} \text{ and } \gamma = \text{Min}\{ \gamma < \mu : (\eta_i \upharpoonright \gamma : i < \lambda) \} \text{ is with no repetition}. \text{ Let } A = \{ i < \lambda : \eta_i \upharpoonright \delta_i \in J_{\delta, \zeta} \} \text{ so for some } \beta < \mu \text{ we have } f_2^\beta(A) \in B_{\beta}^\kappa. \text{ For } \delta \in S, \text{ which is } > \sup\{ (\eta_i, \delta_i : i < \lambda) \} \text{ we get } \rho \upharpoonright \delta \in w_{\delta, \zeta} \text{ and we can finish as before.} \\
(\beta) & \quad X \text{ is a } T_\kappa \text{ space} \\
& \text{[why? as we use a clopen basis we really need just to separate points which holds by clause (C), i.e. if } \nu_1 \neq \nu_2 \in X \text{ then for some } \delta \in S \text{ we have } \nu_1 \upharpoonright \delta \neq \nu_2 \upharpoonright \delta \text{ and apply clause (C) to } \nu_1 \upharpoonright \delta, \nu_2 \upharpoonright \delta] \\
(\gamma) & \quad |X| = \mu^\kappa = 2^\kappa \\
& \text{[why? as } T_\kappa \text{ is the set of points of } X]
suppose \( Y = \{ \eta_i : i < \lambda \} \subseteq X = T_\kappa \) and \( \bigwedge_{i < j} \eta_i \neq \eta_j \). We need to show that \( |\text{cl}(Y)| \) large, i.e. has cardinality \( 2^\kappa \).

Choose \( \gamma \) such that \( \langle \eta_i \upharpoonright \gamma : i < \lambda \rangle \) is with no repetitions.

Let

\[
W_{\bar{\eta}} = \{<>\} \cup \{ \rho : \text{for some } \alpha \leq \kappa, \rho \in T_\alpha, \rho(0) \text{ code } \langle \eta_i \upharpoonright \gamma : i < \lambda \rangle, \\
\rho(0) > \sup \{ \eta_i(0) : i < \lambda \} \text{ and } \\
(\forall \beta < \ell(g(\rho)) (\beta \text{ odd } \Rightarrow \rho(\beta) \text{ code } \langle \eta_i \upharpoonright \beta : i < \lambda \rangle \} \}.
\]

So clearly:

(i) \( W_{\bar{\eta}} \cap T_1 \neq \emptyset \)

(ii) \( W_{\bar{\eta}} \) is a subtree of \( \bigcup_{\alpha \leq \kappa} T_\alpha \) (i.e. closed under initial segments, closed under limits),

(iii) every \( \rho \in W_{\bar{\eta}} \cap T_\alpha \) where \( \alpha < \kappa \) has a successor and if \( \alpha \) is even has \( \mu \) successors.

So \( |W_{\bar{\eta}} \cap T_\kappa| = \mu^\kappa \).

So enough to prove

\[
(\ast) \text{ if } \rho \in W_{\bar{\eta}} \cap T_\kappa \text{ then } \rho \in \text{cl}\{\eta_i : i < \lambda \}.
\]

Let \( \bar{\eta} = \langle \eta_i : i < \lambda \rangle, \eta_\lambda = \rho, \bar{\eta}' = \bar{\eta} \upharpoonright \langle \rho \rangle \text{ and the filter } D_{\bar{\eta}'} = \bigcup \{ D_{\langle \eta_i \upharpoonright \delta : i \leq \lambda \rangle} : \delta \in S \text{ and } \delta \geq \gamma \} \text{ is a filter by clause (E) and even ultrafilter by clause (G).}

Now for every \( \zeta \), by clause (F) for \( \delta \) large enough

\[
\text{Truth Value}(\rho \in u_\zeta) = \lim_{D_{\langle \eta_i \upharpoonright \delta : i \leq \delta \rangle}} \langle \text{Truth Value}(\eta_i \in u_\zeta) \rangle : i < \lambda \rangle.
\]

As \( \{ u_\zeta : \zeta < \mu \times \kappa \} \) is a clopen basis of the topology, we are done.

Stage G: The construction:

We arrive to stage \( \delta \in S \). So for every \( \delta \)-candidate \( \bar{\eta} = \langle \eta_i : i \leq \lambda \rangle \), let

\[
D_{\bar{\eta}'} = \bigcup \{ D_{\langle \eta_i \upharpoonright \delta : i \leq \lambda \rangle} : \delta \in \delta \cap S \text{ and } \langle \eta_i \upharpoonright \delta : i \leq \lambda \rangle \text{ a } \delta \text{-candidate} \} \cup D_0.
\]
Note: $|T_\delta| = \mu$ by the choice of $\kappa$. Let $<_\delta$ be a well ordering of $T_\delta$ such that: $\nu_1(0) < \nu_2(0) \Rightarrow \nu_1 <_\delta \nu_2$. Hence

$$(\ast) \langle \eta_i : i \leq \lambda \rangle \text{ a } \delta\text{-candidate } \Rightarrow \bigwedge_{i<\lambda} \eta_i <^*_\delta \eta_\lambda.$$ 

So let $\{\langle \nu_{1,\zeta}, \nu_{2,\zeta} \rangle : \zeta < \zeta_\delta \leq \zeta < \zeta_\delta \}$ list $\{\langle \nu_1, \nu_2 \rangle : \nu_1 <^*_\delta \nu_2 \}$; such a list exists as $\zeta_\delta \geq \zeta < \zeta_\delta + \mu$ and $|T_\delta| = \mu$. Now we choose by induction on $\zeta < \zeta_\delta$ the following

$(\alpha) \ D^\zeta_\eta$ for $\vec{\eta}$ a $\delta$-candidate when $\zeta \geq \zeta < \zeta_\delta$

$(\beta) \ w^*_{\delta,\zeta}, I_{\delta,\zeta}, J_{\delta,\zeta}$

$(\gamma) \ D^c_{\eta < \delta}$ is $D'_{\eta}$ which was defined above

such that

$(\delta) \ D^\zeta_\eta$ for $\zeta$ in $[\zeta < \delta, \zeta_\delta]$ is increasing continuous

$(\varepsilon)$ if $n < \omega, \zeta < \delta \leq \zeta \leq \zeta_1 < \zeta_2 < \ldots < \zeta_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda^+$ then

$$\bigcap_{\ell=1}^n A_{\xi_\ell, \varepsilon_\ell} \neq \emptyset \mod D^\zeta_\eta$$

$(\zeta) \ D^{\zeta+1}_\eta, I_{\delta,\zeta}, J_{\delta,\zeta}$ satisfies the requirement $(F)_2$

$(\eta) \ \nu_{1,\zeta} \in J_{\delta,\zeta} \Leftrightarrow \nu_{2,\zeta} \notin J_{\delta,\zeta}$ or $\nu_{1,\zeta}, \nu_{2,\zeta} \in w^0_{\delta,\zeta}$

$(\theta) \ D^\zeta_\eta$ is $D'_{\eta} + \{A_{\zeta_1, \varepsilon_0(\zeta_0)} : \zeta_1 < \zeta \}$ for some function $\varepsilon_\eta : [\zeta < \delta, \zeta] \Rightarrow \lambda$.

Note: For $\zeta = 0$, condition $(\varepsilon)$ holds by the induction hypothesis (i.e. clause $(D)$) and choice of $D'_{\eta}$ (and choice of the $A_{\xi, \varepsilon, \delta}$’s if for no $\delta_1, \vec{\eta} \upharpoonright \delta_1$ is a $\delta_1$-candidate).

$(\iota)$ if $\zeta < \zeta < \delta$ then:

$$w_{\delta,\zeta} = w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta} \text{ are defined as in } (F)_2$$

$$I_{\delta,\zeta}^\zeta = w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$$

$$J_{\delta,\zeta}^\zeta \ = \{\eta \in T_\delta : \delta \in w^2_{\delta,\zeta} \text{ and for some } \delta\text{-candidate } \vec{\eta} \text{ we have } \eta_\lambda = \eta \}$$

hence $(\forall i < \lambda)(\exists \delta' \in S \cap \delta)[\eta_i \upharpoonright \delta' \in w_{\delta',\zeta}]$ and $\{i < \lambda : (\exists \delta' \in S \cap \delta)[\eta_i \upharpoonright \delta' \in J_{\delta',\zeta}]\}$ belongs to $D'_{\eta}$. 


So toward contradiction suppose \(\eta\) and \(\eta\) Cae 1.

Observe

Let \(A\) by the assumption \(i\) and \(\eta\) is a \(\delta\)-candidate with \(\eta = \rho\).

Note that \(\eta^\delta < \delta \rho\).

Let \(w = w_{\delta, \zeta} = I_{\delta, \zeta} = \bigcup_{n<\omega} w_n^\zeta\), so \(|w_{\delta, \zeta}| \leq \lambda\) (note that this is the first "time" we deal with \(\zeta\)).

We need: to choose \(J_{\alpha, \zeta} \cap w_{\delta, \zeta}\) so that the cases of condition (\(\zeta\)) (i.e. (F)2) for \(\tilde{\eta}^\delta, \rho \in w\) hold and condition (\(\eta\)) (i.e. (C) for \(\nu_{1, \zeta}, \nu_{2, \zeta}\)) holds.

Let \(w_{\delta, \zeta} = \{\rho \in w_{\delta, \zeta} : \tilde{\eta}^\rho\) is well defined\}, (so \(w'_{\delta, \zeta} \subseteq w_{\delta, \zeta}\)). Let \(w'_{\delta, \zeta} = \{\rho[\zeta, \varepsilon] : \varepsilon < \varepsilon^* \leq \lambda\}\). Now we define \(D^\zeta_0\) as \(D^\zeta_{\tilde{\eta}^\rho[\zeta, \varepsilon]} + A_{\zeta, \varepsilon}\), clearly "legal".

Let \(A'_{\zeta, \varepsilon} = \{i < \lambda : i \in A_{\zeta, \varepsilon}\) and \(i > \varepsilon^*\) and \(\eta_{i, \varepsilon}^{\rho[\zeta, \varepsilon]} \not\in \{\eta_{i, \varepsilon}^{\rho[\zeta, \varepsilon]} : \varepsilon < \varepsilon^* \leq \lambda\}\) and \(\eta_{i, \varepsilon}^{\rho[\zeta, \varepsilon]} \not\in \nu_{1, \zeta}, \nu_{2, \zeta}\}\).

Observe

\[(*)_1 A'_{\zeta, \varepsilon} \setminus A_{\varepsilon}^1 \text{ is not stationary by Fodor's lemma as } \{\eta_{i, \varepsilon}^{\rho[\zeta, \varepsilon]} : i < \lambda\}\) is with no repetition.

Now we shall prove that

\[(*)_2 \text{ the sets } \{\eta_{i, \varepsilon}^{\rho[\zeta, \varepsilon]} : i \in A_{\varepsilon}^1\} \text{ for } \varepsilon > \varepsilon^* \text{ are pairwise disjoint.}\]

So toward contradiction suppose \(i_1 \in A_{\varepsilon_1}^1, i_2 \in A_{\varepsilon_2}^1, \varepsilon_1 < \varepsilon_2 < \varepsilon^*\) and \(\eta_{i_1}^{\rho[\zeta, \varepsilon_1]} = \eta_{i_2}^{\rho[\zeta, \varepsilon_2]}\) and try to get a contradiction.

**Case 1:** \(i_2 > i_1\).

As \(i_1 \in A_{i_1}^1\) we have \(i_1 > \varepsilon_1\) similarly \(i_2 > \varepsilon_2\) but \(\varepsilon_1 < \varepsilon_2\) so \(i_2 > \varepsilon_2 > \varepsilon_1\), and by the assumption \(i_2 > i_1\). So \(\eta_{i_1}^{\rho[\zeta, \varepsilon_1]}\) belongs to the set \(\{\eta_{i_1}^{\rho[\zeta, \varepsilon]} : \varepsilon < i_2 \text{ & } i < i_2\}\) so \(\eta_{i_1}^{\rho[\zeta, \varepsilon_2]} \not\in \rho_{i_1}^{\rho[\zeta, \varepsilon_1]}\) as \(\eta_{i_2}^{\rho[\zeta, \varepsilon_2]}\) does not belong to this set as \(i_2 \in A_{\varepsilon_2}^1\).
Case 2: \( i_2 < i_1 \).

As \( i_2 \in A'_{\zeta, \varepsilon_2} \) necessarily \( \varepsilon_2 < i_2 \). So \( \varepsilon_2 < i_2 < i_1 \) so \( \eta_{i_2}^{[\zeta, \varepsilon_2]} \in \{ \eta_{i_1}^{[\zeta]} : \varepsilon < i_1 \} \) but \( \eta_{i_2}^{[\zeta, \varepsilon_1]} \) does not belong to this set as \( i_1 \in A'_{\varepsilon_1} \) hence \( \eta_{i_1}^{[\zeta, \varepsilon_1]} \), \( \eta_{i_2}^{[\zeta, \varepsilon_2]} \) cannot be equal.

Case 3: \( i_1 = i_2 \).

As \( i_1 \in A'_{\varepsilon_1} \) we have \( i_1 \in A_{\zeta, \varepsilon_1} \) similarly \( i_2 \in A_{\zeta, \varepsilon_2} \) but those sets are disjoint; a contradiction.

So (\( \ast \))\(_2 \) holds.

Now define \( w_{\eta}^{\zeta, \ell} \) for \( \ell = 1, 2, n < \omega \) by induction on

\[
n : w_{\eta}^{\zeta, \ell} = \{ \nu_{\ell, \zeta} \}
\]

\[
w_{n+1}^{\zeta, \ell} = \{ \eta_{i}^{[\zeta, \varepsilon]} : \rho[\zeta, \varepsilon] \in w_{n}^{\zeta, \ell} \text{ and } i \in A'_{\varepsilon} \text{ and } \varepsilon < \varepsilon^* \}.
\]

Let \( w^{\zeta, \ell} = \bigcup_{n<\omega} w_{n}^{\zeta, \ell} \), now by (\( \ast \))\(_2 \), \( w^{\zeta, 1} \cap w^{\zeta, 2} = \emptyset \) (note the clause \( \eta_{i}^{[\zeta, \varepsilon]} \neq \nu_{1, \zeta} \) in the definition of \( A'_{\varepsilon} \)).

So we define

\[
J_{\delta, \zeta} = w^{\zeta, 2}.
\]

Now it is easy to check clause (F), i.e. (\( \zeta \)) and we have finished the induction on \( \zeta < \zeta_\delta \). Now choose \( D_{\eta} \) to satisfy clause (G) and to extend \( \bigcup_{\zeta < \zeta_\delta} D_{\eta}^{\zeta} \), so we are done.

\[\square_{1.1}\]

* * *

(606)
2.1 Theorem. Assume $\lambda > \text{cf}(\lambda)$. Let $\mu = 2^\lambda, \kappa = \text{Min}\{\kappa : 2^\kappa > \mu}\}$. There is a Hausdorff space $X$ with a clopen basis with $|X| = 2^\kappa$ such that for $Y \subseteq \lambda$ closed $|Y| < |X| \Rightarrow |Y| < \lambda$.

Proof. For $\lambda$ singular we should replace the filter $D_0$ on $\lambda$. So let $\lambda = \sum_{j < \text{cf}(\lambda)} \lambda_j, \lambda_j$ strictly increasing $\bar{\lambda} = \langle \lambda_j : j < \text{cf}(\lambda) \rangle$. Let $D^*_\bar{\lambda} = \{A \subseteq \lambda : \text{for every } j < \text{cf}(\lambda) \text{ large enough, the set } A \cap \lambda_j^+ \text{ contains a club of } \lambda_j^+ \}$. We can find a partition $\langle A^j_\alpha : \alpha < \lambda_j^+ \rangle$ of $\lambda_j^+ \setminus \lambda_j$ to stationary sets; let us stipulate $A^j_\alpha = \emptyset$ when $\lambda \leq \alpha < \lambda$ and let $\bar{A}^* = \bigcup_{j < \text{cf}(\lambda)} A^j_\alpha : \alpha < \lambda$ (so $A_\alpha \neq \emptyset$ mod $D^*_\lambda$ and $\alpha < \beta < \lambda \Rightarrow A_\alpha \cap A_\beta = \emptyset$). Let $\{f_\xi : \xi < \mu \times \kappa\}$ be a family of functions from $\lambda$ to $\lambda$ such that if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\{\alpha < \lambda : f_\xi(\alpha) = \varepsilon_\ell \text{ for } \ell = 1, \ldots, n\}$ is not empty (exists by [EK]). Now for $\xi < \mu \times \kappa$ and $\varepsilon < \lambda$ we let $A^j_{\xi,\varepsilon} = \cup\{A_\alpha : f_\xi(\alpha) = \varepsilon\}$. Clearly $\xi < \mu \times \kappa$ & $\varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow A_{\xi,\varepsilon_1} \cap A_{\xi,\varepsilon_2} = \emptyset$, and also: if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\bigcap_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset$ mod $D^*_\lambda$. Let $D_0$ be a maximal filter on $\lambda$ extending $D^*_\lambda$ and still satisfying $\bigcap_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset$ mod $D_0$ for $n, \xi_\ell, \varepsilon_\ell (\ell < n)$ as above.

Now the proof proceeds as before. All is the same except in stage H where we use $\lambda$ regular, $D_0$ contains all clubs of $\lambda$.

The point is that we define $A^j_\varepsilon$ as before, the main question is: why $A^j_\varepsilon = A_\varepsilon$ mod $D^*_\lambda$.

Choose $j^* < \text{cf}(\lambda)$ such that:

$$\varepsilon < \lambda_{j^*}.$$

So it is enough to show

$$(*) \text{ if } j^* \leq j < \text{cf}(\lambda) \text{ then } A^j_\varepsilon \cap [\lambda_j, \lambda_j^+] = A_\varepsilon \cap [\lambda_j, \lambda_j^+] \text{ mod } D^*_\lambda$$
(where $D_{\lambda^+}$-the club filter on $\lambda^+$).

Looking at the definition of $A'_{\zeta, \varepsilon}$,

$$A'_{\zeta, \varepsilon} \cap [\lambda_j, \lambda^+] = \{ i \in [\lambda_j, \lambda^+] : i \in A_{\zeta, \varepsilon} \cap [\lambda_j, \lambda^+] \}
\text{ and } \eta_{i_1}^{\rho'_{\zeta, \varepsilon}} \notin \{ \eta_{i_1}^{\rho'_{\zeta, \varepsilon}} : \varepsilon_1 < i \text{ and } i_1 < i \} \text{ and } \eta_i^{\rho_{\zeta, \varepsilon}} \neq \nu_{i, \zeta} \}
$$

as $\langle \eta_i^{\rho_{\zeta, \varepsilon}} : \lambda_j \leq i < \lambda_j^+ \rangle$ is with no repetition and Fodor’s theorem holds (can formulate the demand on $D$). Just check that the use of $A'_{\zeta, \varepsilon}$ in §1 still works.

2.2 Conclusion: If $\lambda \geq \aleph_0$, $\kappa = \text{Min}\{\kappa : 2^\kappa > 2^\lambda\}$, then there is a $T_3$-space $\lambda, |X| = 2^\kappa$ with no closed subspace of cardinality $\in [\lambda, 2^\kappa]$.

**2.3 Theorem.** For $\lambda \geq \aleph_0$ there is a $T_3$ space $X$ with clopen basis such that: no closed subspace has cardinality in $[\lambda, 2^{2^\lambda}]$.

**Proof.** For $\lambda = \aleph_0$ it is known so let $\lambda > \aleph_0$. Like the proof of 1.1 with $\kappa = 2^\mu$.

The only problem is that $T_\delta = \delta \mu$ may have cardinality $> 2^\mu$ so we have to redefine a $\delta$-candidate (as there are too many $\eta_i | \gamma$ to code) and in the crucial Stages G and H we have the list $\{(\rho_{1, \varepsilon}^\delta, \rho_{2, \varepsilon}^\delta) : \varepsilon < |T_\delta|\}$ but possibly $|T_\delta| > 2^\mu$. Still $|T_\delta| \leq \mu[\delta] \leq 2^\mu$; so instead dedicating one $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$ to deal with any such pair we just do it for each “kind” of pairs such that the number of kinds is $\leq \mu$, (but we can deal with all of them at once).

**Stage $B'$.**

Let $Cd : \mu \to H_{<\lambda^+}(\mu)$ be such that for every $x \in H_{<\lambda^+}(\mu)$ for $\mu$ ordinals $\alpha < \mu$ we have $Cd(\alpha) = x$.

**Stage $C'$.**

For limit $\delta \leq \kappa$ we call $\bar{\eta}$ a $\delta$-candidate if:

(a) $\bar{\eta} = \langle \eta_i : i \leq \lambda \rangle$
(b) \( \eta_i \in T_\delta \)

(c) for some \( \gamma, \langle \eta_i \mid \gamma : i < \lambda \rangle \) is with no repetition

(d) for odd \( \beta < \delta \) we have
\[
Cd(\eta_\lambda(\beta)) = \langle \langle \eta_i(\beta - 1), \eta_i(\beta) \rangle : i < \lambda \rangle
\]

(e) \( Cd(\eta_\lambda(0)) = \{ \langle i, j, \gamma, \eta_i(\gamma), \eta_j(\gamma) \rangle : i < j < \lambda \text{ and for some } i_1 < j_1 < \lambda, \gamma \text{ minimal such that } \eta_{i_1}(\gamma) \neq \eta_{j_1}(\gamma) \} \)

(f) \( \eta_\lambda(0) > \sup \{ \eta_i(0) : i < \lambda \} \).

So

\( \ast \_1 \) if \( \langle \eta_i : i < \lambda \rangle \) is a \( \delta_1 \)-candidate, \( \delta_0 < \delta_1 \) limit and \( (\exists \gamma < \delta_0)(\langle \eta_i \mid \gamma : i < \lambda \rangle \) with no repetitions then \( \langle \eta_i \mid \delta_0 : i < \lambda \rangle \) is a \( \delta_0 \)-candidate

\( \ast \_2 \) if \( \eta_i \in T_\kappa \) for \( i < \kappa \) are pairwise distinct then for \( 2^\mu \) sequences \( \eta_\lambda \in T_\kappa \) we have \( \langle \eta_i : i \leq \lambda \rangle \) is a \( \kappa \)-candidate.

**Stage \( H^\prime \):**

For each \( \varepsilon < |T_\delta| \) we can choose \( v_{\delta, \varepsilon} = \cup \{ v_{\delta, \varepsilon, n} : n < \omega \} \) where we define \( v_{\delta, \varepsilon, n} \) by induction on \( n \) as follows:

\[
v_{\delta, \varepsilon, 0} = \{ v_{\delta, \varepsilon, 1}^0, v_{\delta, \varepsilon, 1}^1 \}, v_{\delta, \varepsilon, n+1} = v_{\delta, \varepsilon, n} \cup \{ \eta^\rho : \rho \in v_{\delta, \varepsilon, n} \text{ and } \bar{\rho} \text{ is a } \delta \text{-candidate such that } \eta_\lambda^\rho = \rho \}. \]

We choose \( u_\varepsilon = u_{\delta, \varepsilon} \in [\delta]^{\leq \lambda} \) such that: if \( \bar{\eta} \) is a \( \delta \)-candidate satisfying \( \eta_\lambda \in v_{\delta, \varepsilon} \) (so \( \eta_i \in v_{\delta, \varepsilon} \) for \( i < \lambda \)) then \( 0 \in u_\varepsilon \) \& \( i < j < \lambda \Rightarrow \min \{ \gamma : \eta_i(\gamma) \neq \eta_j(\gamma) \} \in u_\varepsilon \).

As \( |T_\delta| \leq 2^\mu \) and \( \mu^\lambda = \mu \) by Engelking Karlowic [EK] there are functions \( H^\delta_\kappa : T_\delta \rightarrow \mathcal{H}_{<\lambda^+}(\mu) \) for \( \kappa \in [\zeta_{<\delta}, \zeta_\delta] \) such that for every \( w \in [T_\delta]^{\lambda} \) and \( h : w \rightarrow \mathcal{H}_{<\lambda^+}(\mu) \) there is \( \kappa \in [\zeta_{<\delta}, \zeta_\delta] \) such that \( h \subseteq H^\delta_\kappa \).

As \( \mu = \mu^\lambda = |\mathcal{H}_{<\lambda^+}(\mu)| \), without loss of generality \( |\text{Rang}(H^\delta_\kappa)| \leq \lambda \) (divide \( H^\delta_\kappa \) to \( \leq 2^\lambda = \mu \) functions).

For each \( \varepsilon < |T_\delta| \) let \( h^\varepsilon_\delta : v_{\delta, \varepsilon} \rightarrow \mathcal{H}_{<\lambda^+}(\mu) \) be \( h^\varepsilon_\delta(\eta) = (h^\varepsilon_{\delta, 0}(\eta), h^\varepsilon_{\delta, 1}(\eta), h^\varepsilon_{\delta, 2}(\eta)) \) where

\[
h^\varepsilon_{\delta, 0}(\eta) = \text{otp}(\{ \nu \in w^\varepsilon_\delta : \nu <^\varepsilon_\delta \eta \}, \leq^\varepsilon_\delta)
\]

\[
h^\varepsilon_{\delta, 1}(\eta) = \{ (\gamma, \eta(\gamma)) : \gamma \in u_{\delta, \varepsilon} \}
\]

\[
h^\varepsilon_{\delta, 2}(\eta) = \text{ truth value of } \eta \in v_{\delta, \varepsilon, 0}
\]

(the function \( h^\varepsilon_\delta \) belongs to \( \mathcal{H}_{<\lambda^+}(\mu) \) as \( |v_{\delta, \varepsilon}| \leq \lambda \)); let
\[ \Upsilon_\varepsilon = \min \{ \Upsilon \in [\zeta_{<\delta}, \zeta_\delta) : h_\delta^\varepsilon \subseteq H_\Upsilon^\delta \} \]

(well defined). Let \( \gamma_\Upsilon^\delta = \sup \{ \gamma < \lambda^+ : \gamma \) is the first cardinal in some sequence \( \lambda \) from \( \text{Rang}(H_\Upsilon^\delta) \}, \) let \( g_\Upsilon^\delta \) be a one-to-one function from \( \gamma_\Upsilon^\delta \) into \( \lambda \).

Next we can define the \( D^\Upsilon_\eta \) for \( \eta \) a \( \delta \)-candidate; for \( \Upsilon < \mu \):

\[ D^\Upsilon_{\eta+1} = D^\Upsilon_\eta + A_{\Upsilon, \gamma_\Upsilon^\delta}. \]

In Stage \( \Upsilon \in [\zeta_{<\delta}, \zeta_\delta) \) we deal with all \( \varepsilon < |T_\delta| \) such that \( \Upsilon_\varepsilon = \Upsilon \). Now we treat the choice of \( I_{\delta, \zeta}, J_{\delta, \zeta}, w_{\delta, \zeta} \). We can finish as before (but dealing with many cases at once). \( \square_{2.3} \)
REFERENCES.


