

# Functorial Equations for Lexicographic Products \*

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## Abstract

We generalize the main result of [K–K–S] concerning the convex embeddings of a chain  $\Gamma$  in a lexicographic power  $\Delta^\Gamma$ . For a fixed non-empty chain  $\Delta$ , we derive necessary and sufficient conditions for the existence of non-empty solutions  $\Gamma$  to each of the lexicographic functorial equations

$$(\Delta^\Gamma)^{\leq 0} \simeq \Gamma, \quad (\Delta^\Gamma) \simeq \Gamma \quad \text{and} \quad (\Delta^\Gamma)^{< 0} \simeq \Gamma.$$

## 1 Introduction

Let us recall the definition of lexicographic products of ordered sets. Let  $\Gamma$  and  $\Delta_\gamma$ ,  $\gamma \in \Gamma$  be non-empty totally ordered sets. For every  $\gamma \in \Gamma$ , we fix a distinguished element  $0_\gamma \in \Delta_\gamma$ . The **support** of a family  $a = (\delta_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} \Delta_\gamma$  is the set of all  $\gamma \in \Gamma$  for which  $\delta_\gamma \neq 0_\gamma$ . We denote it by  $\text{supp}(a)$ . As a set, we define  $\mathbf{H}_{\gamma \in \Gamma}(\Delta_\gamma, 0_\gamma)$  to be the set of all families  $(\delta_\gamma)_{\gamma \in \Gamma}$  with well-ordered support (with respect to fixed distinguished elements  $0_\gamma$ ). To relax the notation, we shall write  $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$  instead of  $\mathbf{H}_{\gamma \in \Gamma}(\Delta_\gamma, 0_\gamma)$  once the distinguished elements  $0_\gamma$  have been fixed. Then the **lexicographic order** on  $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$  is defined as follows. Given  $a = (\delta_\gamma)_{\gamma \in \Gamma}$  and  $b = (\delta'_\gamma)_{\gamma \in \Gamma} \in \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ , observe that  $\text{supp}(a) \cup \text{supp}(b)$  is well-ordered. Let  $\gamma_0$  be the least of all elements  $\gamma \in \text{supp}(a) \cup \text{supp}(b)$  for which  $\delta_\gamma \neq \delta'_\gamma$ . We set  $a < b \Leftrightarrow \delta_{\gamma_0} < \delta'_{\gamma_0}$ . Then  $(\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma, <)$  is a totally ordered set, the **lexicographic product** (or **Hahn product**) of the ordered sets  $\Delta_\gamma$ . We shall always denote by  $0$  the sequence with empty support in  $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ .

Note that if all  $\Delta_\gamma$  are totally ordered abelian groups, then we can take the distinguished elements  $0_\gamma$  to be the neutral elements of the groups  $\Delta_\gamma$ . Defining addition on  $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$  componentwise, we obtain a totally ordered abelian group  $(\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma, +, 0 <)$ .

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**Lexicographic exponentiation of chains:** If  $\Delta = \Delta_\gamma$  for every  $\gamma \in \Gamma$ , we fix a distinguished element in  $\Delta$  (the same distinguished element for every  $\gamma \in \Gamma$ ), and denote it by  $0_\Delta$ . In this case we denote  $\prod_{\gamma \in \Gamma} \Delta_\gamma$  by  $\Delta^\Gamma$ , and call it the **lexicographic power**  $\Delta^\Gamma$  (with respect to  $0_\Delta$ ). In other words,  $\Delta^\Gamma$  is the set

$$\{s; s : \Gamma \rightarrow \Delta \text{ such that } \text{supp}(s) \text{ is well-ordered in } \Gamma\},$$

ordered lexicographically.

This exponentiation of chains has its own arithmetic. In this paper we study some of its aspects (cf. also [K] and [H–K–M]). Note that if  $\Gamma$  and  $\Delta$  are infinite ordinals, then lexicographic exponentiation does *not* coincide with ordinal exponentiation (cf. [H]).

Lexicographic powers appear naturally in many contexts. For example,  $\mathbb{N}^{\mathbb{N}}$  is the order type of the nonnegative reals, and  $\mathbb{Z}^{\mathbb{N}}$  that of the irrationals (cf. [R]). Also,  $2^\Gamma$  is (isomorphic to) the chain of all well-ordered subsets of  $\Gamma$ , ordered by inclusion. The chain  $2^{\mathbb{N}}$  has been studied in [H].

However, the main motivating example for us was that of generalized power series fields. If  $k$  is a real closed field and  $G$  a totally ordered divisible abelian group, then the field  $k((G))$  of power series with exponents in  $G$  and coefficients in  $k$  is again real closed. The unique order of  $k((G))$  is precisely the chain  $k^G$ . It was while studying such fields that our interest in the present problems arose. In [K–K–S], we considered the problem of defining an exponential function on  $K = k((G))$ , that is, an isomorphism  $f$  of ordered groups  $f : (K, +, 0, <) \rightarrow (K^{>0}, \cdot, 1, <)$ . We showed that the existence of  $f$  would imply that of a **convex embedding** (that is, an embedding with convex image) of the chain  $G^{<0}$  into the chain  $k^{G^{<0}}$ . On the other hand, we proved:

**Theorem 1** *Let  $\Gamma$  and  $\Delta$  be non-empty totally ordered sets without greatest element, and fix an element  $0_\Delta \in \Delta$ . Suppose that  $\Gamma'$  is a cofinal subset of  $\Gamma$  and that  $\iota : \Gamma' \rightarrow \Delta^\Gamma$  is an order preserving embedding. Then the image  $\iota\Gamma'$  is not convex in  $\Delta^\Gamma$ .*

Now for any ordered field  $k$ , the chain  $k$  has no last element. Similarly,  $G^{<0}$  has no last element if  $G$  is nontrivial and divisible. So, using Theorem 1 one establishes that no exponentiation is possible on generalized power series fields.

If we omit the conditions on  $\Gamma$  and  $\Delta$  in Theorem 1, the situation changes drastically. In this paper, we study conditions on the chains  $\Gamma$  and  $\Delta$  under which a convex embedding of  $\Gamma$  in  $\Delta^\Gamma$  exists. In particular, we seek for non-empty solutions  $\Gamma$  to the functorial equations:

$$(\Delta^\Gamma)^{\leq 0} \simeq \Gamma, \quad (\Delta^\Gamma) \simeq \Gamma, \quad \text{and} \quad (\Delta^\Gamma)^{< 0} \simeq \Gamma.$$

(if  $T$  is any totally ordered set and  $0 \in T$  is any element, we denote by  $T^{\leq 0}$  the initial segment (including 0), and by  $T^{< 0}$  the strict initial segment (excluding 0) determined by 0 in  $T$ ). None of the three equations hold if both  $\Delta$  and  $\Gamma$  have no

last element (for the first, this is trivial, and for the second and third it follows from Theorem 1). In Section 2 we start by proving a strong generalization of Theorem 1 (cf. Theorem 2). In Section 3, for each of the three functorial equations, we give simple characterizations of those chains  $\Delta$  for which non-empty solutions  $\Gamma$  exist. In Section 4 we study simultaneous solutions to all three equations.

## 2 Nonexistence of convex embeddings

In this section, we shall prove that Theorem 1 remains true in the case where  $\Delta$  is arbitrary, but  $0_\Delta$  is not the last element of  $\Delta$ . This will follow from the following more general result:

**Theorem 2** *Let  $\Gamma$  and  $\Delta_\gamma$ ,  $\gamma \in \Gamma$ , be non-empty totally ordered sets. For every  $\gamma \in \Gamma$ , fix an element  $0_\gamma$  which is not the last element in  $\Delta_\gamma$ . Suppose that  $\Gamma$  has no last element and that  $\Gamma'$  is a cofinal subset of  $\Gamma$ . Then there is no convex embedding*

$$\iota : \Gamma' \rightarrow \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma .$$

*Proof:* For every  $\gamma \in \Gamma'$ , we choose an element  $1_\gamma \in \Delta_\gamma$  such that  $1_\gamma > 0_\gamma$ . Take  $d = (d_\gamma)_{\gamma \in \Gamma}$ . If  $S$  is a well-ordered subset of  $\Gamma'$  such that  $d_\gamma = 0_\gamma$  for all  $\gamma \in S$ , then we set

$$d \oplus S := (d'_\gamma)_{\gamma \in \Gamma} \quad \text{with} \quad d'_\gamma = \begin{cases} d_\gamma & \text{for } \gamma \notin S \\ 1_\gamma & \text{for } \gamma \in S . \end{cases}$$

Observe that the support of  $d \oplus S$  is contained in  $\text{supp}(d) \cup S$  and thus, it is again well-ordered. Note also that

$$S' \subsetneq S \Rightarrow d \oplus S' < d \oplus S . \tag{1}$$

Indeed, let  $\gamma_0$  be the least element in  $S \setminus S'$ . Then  $(d \oplus S')_{\gamma_0} = 0_{\gamma_0} < 1_{\gamma_0} = (d \oplus S)_{\gamma_0}$ . On the other hand, if  $\gamma \in \Gamma$  and  $\gamma < \gamma_0$  then  $(d \oplus S')_\gamma = (d \oplus S)_\gamma$ : if  $\gamma \in S$  then  $\gamma \in S'$  (by minimality of  $\gamma_0$ ) and  $(d \oplus S')_\gamma = 1_\gamma = (d \oplus S)_\gamma$ ; if  $\gamma \notin S$  then  $\gamma \notin S'$  and  $(d \oplus S')_\gamma = d_\gamma = (d \oplus S)_\gamma$ .

Now suppose that  $\iota : \Gamma' \rightarrow \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$  is an order preserving embedding such that the image  $\iota\Gamma'$  is convex in  $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ . We wish to deduce a contradiction. The idea of the proof is the following. Let ON denote the class of ordinal numbers. We shall define an infinite  $\text{ON} \times \mathbb{N}$  matrix with coefficients in  $\Gamma'$ , such that each column  $(\gamma_\nu^{(n)})_{\nu \in \text{ON}}$  is a strictly increasing sequence in  $\Gamma'$ . Since  $\Gamma'$  is a *set*, every column of this matrix will provide a contradiction at the end of the construction (cf. figure).

$$\begin{pmatrix} \gamma_0^{(1)} & \cdots & \boxed{\gamma_0^{(n)}} & \boxed{\gamma_0^{(n+1)}} & \cdots \\ \vdots & & \vdots & \vdots & \\ \gamma_\nu^{(1)} & \cdots & \gamma_\nu^{(n)} & \gamma_\nu^{(n+1)} & \cdots \\ \vdots & & \vdots & \vdots & \\ \gamma_\mu^{(1)} & \cdots & \gamma_\mu^{(n)} = ? & \cdots & \cdots \\ \vdots & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

To get started, we have to define the first row of the matrix. We construct sequences  $\beta^{(n)}$ ,  $n \in \mathbb{N} \cup \{0\}$ , and  $\gamma_0^{(n)}$ ,  $n \in \mathbb{N}$ , in  $\Gamma'$ . We take an arbitrary  $\beta^{(0)} \in \Gamma'$ . Having constructed  $\beta^{(n)}$ , we choose  $\gamma_0^{(n+1)}$  and  $\beta^{(n+1)}$  as follows. Since  $\Gamma'$  has no last element, we can choose  $\mu^{(n)}, \nu^{(n)} \in \Gamma'$  such that  $\beta^{(n)} < \mu^{(n)} < \nu^{(n)}$ . Hence,

$$\iota\beta^{(n)} < \iota\mu^{(n)} < \iota\nu^{(n)} .$$

Let  $\sigma^{(n)} \in \Gamma$  be the least element in  $\text{supp } \iota\beta^{(n)} \cup \text{supp } \iota\mu^{(n)}$  for which

$$(\iota\beta^{(n)})_{\sigma^{(n)}} < (\iota\mu^{(n)})_{\sigma^{(n)}} , \quad (2)$$

and  $\tau^{(n)} \in \Gamma$  the least element in  $\text{supp } \iota\mu^{(n)} \cup \text{supp } \iota\nu^{(n)}$  for which

$$(\iota\mu^{(n)})_{\tau^{(n)}} < (\iota\nu^{(n)})_{\tau^{(n)}} . \quad (3)$$

Since  $\Gamma'$  is cofinal in  $\Gamma$ , we can choose  $\beta^{(n+1)} \in \Gamma'$  such that

$$\beta^{(n+1)} \geq \max\{\sigma^{(n)}, \tau^{(n)}\} .$$

Further, we set

$$d^{(n+1)} := (d_\gamma^{(n+1)})_{\gamma \in \Gamma} \quad \text{with} \quad d_\gamma^{(n+1)} = \begin{cases} (\iota\mu^{(n)})_\gamma & \text{for } \gamma \leq \beta^{(n+1)} \\ 0_\gamma & \text{for } \gamma > \beta^{(n+1)} . \end{cases}$$

Then by (2) and (3),

$$\iota\beta^{(n)} < d^{(n+1)} < \iota\nu^{(n)} .$$

Thus,  $d^{(n+1)} \in \iota\Gamma'$  by convexity, and we can set

$$\gamma_0^{(n+1)} := \iota^{-1}d^{(n+1)} .$$

Now for every  $n \in \mathbb{N}$  we have that  $\beta^{(n)} < \gamma_0^{(n+1)}$ , hence every well-ordered set  $S \subset \Gamma'$  with smallest element  $\gamma_0^{(n+1)}$  has the property that  $(\iota\gamma_0^{(n)})_\gamma = d_\gamma^{(n)} = 0_\gamma$  for all  $\gamma \in S$ ; and moreover,

$$\iota\gamma_0^{(n)} < \iota\gamma_0^{(n)} \oplus S < \iota\nu^{(n-1)} .$$

Thus,  $\iota\gamma_0^{(n)} \oplus S \in \iota\Gamma'$  by convexity. Suppose now that for some ordinal number  $\mu \geq 1$  we have chosen elements  $\gamma_\nu^{(n)} \in \Gamma'$ ,  $\nu < \mu$ ,  $n \in \mathbb{N}$ , such that for every fixed  $n$ , the sequence  $(\gamma_\nu^{(n)})_{\nu < \mu}$  is strictly increasing. Then we set

$$\gamma_\mu^{(n)} := \iota^{-1}(\iota\gamma_0^{(n)} \oplus \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}) \in \Gamma'$$

for every  $n \in \mathbb{N}$ . If  $\lambda < \mu$ , then  $\{\gamma_\nu^{(n+1)} \mid \nu < \lambda\} \subsetneq \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}$  and thus,  $\gamma_\lambda^{(n)} < \gamma_\mu^{(n)}$  by (1). So for every ordinal number  $\mu$ , the sequences  $(\gamma_\nu^{(n)})_{\nu < \mu}$  can be extended. We obtain strictly increasing sequences of arbitrary length, contradicting the fact that their length is bounded by the cardinality of  $\Gamma$ .  $\square$

**Corollary 3** *Assume that  $0_\Delta$  is not the last element of  $\Delta$ . If there is an embedding of  $\Gamma$  in  $\Delta^\Gamma$  with convex image, then  $\Gamma$  has a last element.*

### 3 Solutions to the Functorial equations

We start with a few easy remarks and lemmas. Throughout, fix a chain  $\Delta$  with distinguished element  $0_\Delta$ .

**Remark 4** 1) If  $0_\Delta$  is last in  $\Delta$  (respectively, least), then 0 is last in  $\Delta^\Gamma$  (respectively, least), for any non-empty chain  $\Gamma$ .

2) Let  $I$  be any chain, and  $C$  a non-empty convex subset of  $I$ . Let  $c \in C$ . Then the initial segment determined by  $c$  in  $C$  is a final segment of the initial segment determined by  $c$  in  $I$ .

**Remark 5** If  $\Delta^{<0_\Delta}$  has no last element, then also  $(\Delta^\Gamma)^{<0}$  has no last element, for any chain  $\Gamma$ : If not, let  $s$  be last in  $(\Delta^\Gamma)^{<0}$  and set  $\gamma = \min \text{supp}(s)$ . Then  $s(\gamma) = \delta < 0_\Delta$ . Take  $\delta < \delta' < 0_\Delta$ . Consider  $s'$  defined by  $s'(\gamma) = \delta'$  and  $s'(\gamma') = 0_\Delta$  if  $\gamma' \neq \gamma$ . Then  $s' \in (\Delta^\Gamma)^{<0}$ , but  $s' > s$ , contradiction.

**Lemma 6** *Let  $\Gamma$  and  $\Gamma'$  be chains, and suppose that  $\phi : \Gamma \rightarrow \Gamma'$  is a chain embedding. Then  $\phi$  lifts to a chain embedding*

$$\hat{\phi} : \Delta^\Gamma \rightarrow \Delta^{\Gamma'}.$$

Proof: For  $s \in \Delta^\Gamma$  and  $x \in \Gamma'$ , set

$$\hat{\phi}(s)(x) = \begin{cases} 0_\Delta & \text{if } x \notin \text{Im } \phi \\ s(\phi^{-1}(x)) & \text{if } x \in \text{Im } \phi. \end{cases}$$

(here,  $\text{Im } \phi$  denotes the image of  $\phi$ ). Now, it is straightforward to check the assertion of the lemma.  $\square$

In view of this lemma, if  $F$  is a subchain of a chain  $\Gamma$ , then there is a natural identification of  $\Delta^F$  as a subchain of  $\Delta^\Gamma$ .

**Lemma 7** *Let  $\Gamma$  be a chain and  $F$  a non-empty final segment of  $\Gamma$ . Then  $\Delta^F$  is convex in  $\Delta^\Gamma$  (and  $0 \in \Delta^F$ ).*

Proof: Let  $s_i \in \Delta^F$ , and set  $\gamma_i = \min \text{supp}(s_i) \in F$ , for  $i = 1, 2$ . Let  $s \in \Delta^\Gamma$  be such that  $s_1 < s < s_2$ . If  $s = 0$ , then  $s \in \Delta^F$ . So assume  $s \neq 0$  and set  $\gamma = \min \text{supp}(s)$ . Suppose that  $\gamma \notin F$ . If  $s > 0$ , then  $s(\gamma) > 0_\Delta$ . On the other hand,  $\gamma < \gamma_2$  (otherwise,  $\gamma \in F$ ). Thus,  $s > s_2$ , a contradiction. Similarly, we argue that if  $s < 0$ , then  $s < s_1$ , a contradiction. Hence,  $\min \text{supp}(s) \in F$ . Since  $F$  is a final segment of  $\Gamma$ , this implies that  $s \in \Delta^F$ , which proves our assertion.  $\square$

**Corollary 8** *Assume that  $\Gamma$  has a last element. Then  $\Delta$  embeds convexly in  $\Delta^\Gamma$ , such that  $0_\Delta$  is mapped to  $0 \in \Delta^\Gamma$ . If moreover  $0_\Delta$  is last in  $\Delta$ , then  $\Delta^F$  embeds as a final segment in  $\Delta^\Gamma$ , for any non-empty final segment  $F$  of  $\Gamma$ . Consequently, if  $\Gamma$  has a last element, and  $0_\Delta$  is last in  $\Delta$ , then  $\Delta$  embeds as a final segment in  $\Delta^\Gamma$ .*

Proof: The first assertion follows from Lemma 7, applied to the final segment consisting of the single last element of  $\Gamma$ . For the second assertion use Remark 4, parts 1) and 2).  $\square$

We now give a complete solution to the **first functorial equation**, and a sufficient condition for the existence of solutions  $\Gamma$  to the third functorial equation:

**Theorem 9** *There is always a non-empty solution  $\Gamma$  for the functorial equation  $(\Delta^\Gamma)^{\leq 0} \simeq \Gamma$ . If  $\Delta^{< 0_\Delta}$  has a last element, then there is also a non-empty solution  $\Gamma$  for  $(\Delta^\Gamma)^{< 0} \simeq \Gamma$ .*

Proof: Set  $\Gamma_0 := \Delta^{\leq 0_\Delta}$ . Since  $\Gamma_0$  has a last element,  $\Delta$  embeds convexly in  $\Delta^{\Gamma_0}$ . Consequently,  $\Gamma_0$  embeds as a final segment in  $\Gamma_1 := (\Delta^{\Gamma_0})^{\leq 0}$ . By induction on  $n \in \mathbb{N}$  we define  $\Gamma_n := (\Delta^{\Gamma_{n-1}})^{\leq 0}$ , and obtain an embedding of  $\Gamma_{n-1}$  as a final segment in  $\Gamma_n$ . We set  $\Gamma := \cup_{n \in \mathbb{N}} \Gamma_n$ .

Since every  $\Gamma_n$  is a final segment of  $\Gamma$ , every well-ordered subset  $S$  of  $\Gamma$  is already contained in some  $\Gamma_n$  (just take  $n$  such that the first element of  $S$  lies in  $\Gamma_n$ ). Hence, an element of  $(\Delta^\Gamma)^{\leq 0}$  with support  $S$  is actually an element of  $\Gamma_{n+1} = (\Delta^{\Gamma_n})^{\leq 0}$ , for some  $n$ . This fact gives rise to an order isomorphism of  $(\Delta^\Gamma)^{\leq 0}$  onto  $\Gamma$ .

To prove the second assertion, we set  $\Gamma_0 := \Delta^{< 0_\Delta}$ . Since  $\Gamma_0$  has a last element by assumption,  $\Delta$  embeds convexly in  $\Delta^{\Gamma_0}$ , and the same arguments as above work if we define  $\Gamma_n := (\Delta^{\Gamma_{n-1}})^{< 0}$ .  $\square$

**Remark 10** Note that  $\Gamma_0$  has a last element and embeds as a final segment in the constructed solution  $\Gamma$  (in both cases considered in the proof). Thus,  $\Gamma$  has a last element, and there is no contradiction to Theorem 2.

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Note that if  $0_\Delta$  is least in  $\Delta$ , then the first equation has the trivial solution  $\Gamma = \{0_\Delta\}$ .

We next turn to the **second functorial equation**.

**Remark 11** Suppose that  $0_\Delta$  is last in  $\Delta$ . Then the solution to the first equation given in Theorem 9 also solves the second equation. Indeed, in this case, 0 is last in  $\Delta^\Gamma$ , so  $(\Delta^\Gamma)^{\leq 0} = \Delta^\Gamma$ .

We also have the converse:

**Corollary 12** *Assume  $\Delta$  is a chain such that the functorial equation  $\Delta^\Gamma \simeq \Gamma$  has a non-empty solution  $\Gamma$ . Then  $0_\Delta$  is last in  $\Delta$ . Thus, the functorial equation  $\Delta^\Gamma \simeq \Gamma$  has a non-empty solution if and only if  $0_\Delta$  is last in  $\Delta$ .*

Proof: Assume  $0_\Delta$  is not last, and choose some element  $1_\Delta > 0_\Delta$ . This provides us with characteristic functions. If  $S \subset \Gamma$  is well-ordered, then let  $\chi_S \in \Delta^\Gamma$  denote the characteristic function on  $S$  defined by:

$$\chi_S(\gamma) = \begin{cases} 1_\Delta & \text{if } \gamma \in S \\ 0_\Delta & \text{if } \gamma \notin S. \end{cases}$$

Note that these characteristic functions reflect inclusion: if  $S$  is a proper well-ordered subset of  $S'$ , then  $\chi_S < \chi_{S'}$ . Now assume for a contradiction that  $i : \Gamma \simeq \Delta^\Gamma$ , and let  $\kappa = \text{card}(\Gamma)$ . We shall construct a strictly increasing sequence  $\{\gamma_\mu; \mu < \kappa^+\}$  in  $\Gamma$ .

Set  $\gamma_0 = i^{-1}(0)$ , and assume by induction that  $\{\gamma_\nu; \nu < \mu\}$  is defined, and strictly increasing in  $\Gamma$ . Then define

$$\gamma_\mu = i^{-1}(\chi_{\{\gamma_\nu; \nu < \mu\}}).$$

It follows that  $\chi_{\{\gamma_\lambda; \lambda < \nu\}} < \chi_{\{\gamma_\lambda; \lambda < \mu\}}$ , whenever  $\nu < \mu$ . Since  $i^{-1}$  is order preserving, it follows that  $\gamma_\nu < \gamma_\mu$  as required.  $\square$

We now turn to the **third functorial equation**. We deduce a simple criterion for the existence of solutions:

**Corollary 13** *Assume that  $0_\Delta$  is not the last element of  $\Delta$ . Then the functorial equation  $(\Delta^\Gamma)^{<0} \simeq \Gamma$  has a non-empty solution  $\Gamma$  if and only if  $\Delta^{<0_\Delta}$  has a last element.*

Proof: The “if” direction is just the second assertion of Theorem 9. So assume now that  $\Gamma$  is a non-empty solution. Assume for a contradiction that  $\Delta^{<0_\Delta}$  has no last element. Then by Remark 5  $(\Delta^\Gamma)^{<0}$  has no last element as well. Thus, the same holds for the solution  $\Gamma$ . This contradicts Theorem 2.  $\square$

## 4 Simultaneous Solutions

Recall that by Remark 11, the chain  $\Gamma$  given in Theorem 9 solves the first *and* the second functorial equations, if  $0_\Delta$  is last in  $\Delta$ . By  $\omega^*$  we denote the ordinal  $\omega$  with the reverse ordering.

**Theorem 14** *Assume that  $0_\Delta$  is last in  $\Delta$  and that  $\omega^*$  embeds as a final segment in  $\Delta$ . Then the solution  $\Gamma$  given in Theorem 9 to the first and second functorial equations solves  $(\Delta^\Gamma)^{<0} \simeq \Gamma$  as well.*

Proof: Recall that  $\Delta$  embeds as a final segment in the given solution  $\Gamma$ . Thus,  $\omega^*$  embeds as a final segment in  $\Gamma$  as well. In particular,  $\Gamma$  has a last element 0. Since  $\Delta^\Gamma = (\Delta^\Gamma)^{<0} \cup \{0\}$  and  $\Delta^\Gamma \simeq \Gamma$ , we find that  $(\Delta^\Gamma)^{<0} \simeq \Gamma \setminus \{0\}$ . But  $\Gamma \simeq \Gamma \setminus \{0\}$ , since  $\omega^*$  is a final segment of  $\Gamma$ .  $\square$

We now turn to the question of whether the sufficient conditions given in this last theorem is also necessary. We need to introduce a definition: Say that a solution  $\Gamma$  (to any of the three equations) is **special** if  $\Delta$  embeds as a final segment in  $\Gamma$ . Note that special solutions are necessarily non-empty.

**Proposition 15** *Every non-empty solution to  $\Gamma \simeq \Delta^\Gamma$  is special.*

Proof: Necessarily,  $0_\Delta$  is last in  $\Delta$  (by Corollary 12). Thus,  $\Gamma$  has a last element, so by Corollary 8,  $\Delta$  embeds as a final segment in  $\Delta^\Gamma$ , and thus in  $\Gamma$ .  $\square$

**Corollary 16** *Assume that  $\Delta$  is infinite and  $\Gamma$  is any non-empty chain which solves simultaneously*

$$(\Delta^\Gamma)^{<0} \simeq \Gamma \simeq \Delta^\Gamma.$$

*Then  $0_\Delta$  is last in  $\Delta$  and  $\omega^*$  embeds as a final segment in  $\Delta$ .*

Proof: Since  $\Gamma \simeq \Delta^\Gamma$ ,  $0_\Delta$  is last in  $\Delta$  (Corollary 12). Therefore, 0 is last in  $\Delta^\Gamma$  by Remark 4, and so also  $\Gamma$  has a last element 0. The assumptions imply that  $\Gamma \setminus \{0\} \simeq \Gamma$ . This is equivalent to the assertion that  $\omega^*$  embeds as a final segment in  $\Gamma$ . Now note that  $\Gamma$  is a special solution by Proposition 15, i.e.,  $\Delta$  embeds as a final segment of  $\Gamma$ . Since  $\Delta$  is infinite this implies that  $\omega^*$  embeds as a final segment in  $\Delta$ , as required.  $\square$

**Corollary 17** *Assume that  $\Delta$  is infinite. Then the following are equivalent:*

- (a)  $0_\Delta$  is last in  $\Delta$  and  $\omega^*$  embeds as a final segment in  $\Delta$ .
- (b) There exists a (special) simultaneous solution to all three equations.
- (c) There exists a (special) simultaneous solution to the second and third equations.



Proof: (a) implies (b) by Theorem 14. (b) implies (c) trivially. Finally, (c) implies (a) by Corollary 16.  $\square$

We conclude with the following question: *Are special solutions unique up to isomorphism?* We can give a partial answer to this last question:

**Proposition 18** *Assume that  $0_\Delta$  is last in  $\Delta$ . Let  $\Gamma = \cup \Gamma_n$  be the solution to the second equation given in Theorem 9. Then  $\Gamma$  embeds as a final segment in any other solution.*

Proof: Let  $\Gamma'$  be another solution. Then it is a special solution, by Proposition 15. So  $\Delta = \Gamma_0$  embeds as a final segment in  $\Gamma'$ . Since  $0_\Delta$  is last in  $\Delta$ ,  $\Gamma_1 = \Delta^{\Gamma_0}$  embeds as a final segment in  $\Delta^{\Gamma'}$ . By induction,  $\Gamma_n$  is a final segment of  $\Gamma'$  for every  $n \in \mathbb{N}$ . Thus,  $\Gamma$  embeds as a final segment in  $\Gamma'$  as well.  $\square$

### References

- [H] Hausdorff, F.: Grundzüge der Mengenlehre, Verlag von Veit, Leipzig (1914)
- [H–K–M] Holland, W. C. – Kuhlmann, S. – McCleary, S.: The Arithmetic of Lexicographic Exponentiation, preprint
- [K] Kuhlmann, S.: Isomorphisms of Lexicographic Powers of the Reals, Proc. Amer. Math. Soc. **123** (1995), 2657-2662
- [K–K–S] Kuhlmann, F.-V. – Kuhlmann, S. – Shelah, S.: Exponentiation in power series fields, Proc. Amer. Math. Soc. **125** (1997), 3177-3183
- [R] Rosenstein, J. G.: Linear orderings, Academic Press, New York - London (1982)

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