[to appear in the *Journal of Symbolic Logic*]

Superdestructibility: A Dual to Laver's Indestructibility

Joel David Hamkins†

Saharon Shelah‡

Abstract. After small forcing, any $<\kappa$-closed forcing will destroy the supercompactness and even the strong compactness of κ.

In a delightful argument, Laver [L78] proved that any supercompact cardinal κ can be made indestructible by $<\kappa$-directed closed forcing. This indestructibility, however, is evidently not itself indestructible, for it is always ruined by small forcing: in [H96] the first author recently proved that small forcing makes any cardinal superdestructible; that is, any further $<\kappa$-closed forcing which adds a subset to κ will destroy the measurability, even the weak compactness, of κ. What is more, this property holds higher up: after small forcing, any further $<\kappa$-closed forcing which adds a subset to λ will destroy the λ-supercompactness of κ, provided λ is not too large (his proof needed that $\lambda < \aleph_{\kappa+\delta}$, where the small forcing is $<\delta$-distributive). In this paper, we happily remove this limitation on λ, and show that after small forcing, the supercompactness of κ is destroyed by any $<\kappa$-closed forcing. Indeed, we will show that even the strong compactness of κ is destroyed. By doing so we answer the questions asked at the conclusion of [H96], and obtain the following attractive complement to Laver indestructibility:

Main Theorem. *After small forcing, any $<\kappa$-closed forcing will destroy the supercompactness and even the strong compactness of κ.*

We will provide two arguments. The first, similar to but generalizing the Superdestruction Theorem of [H96], will show that supercompactness is destroyed; the second, by a different technique, will show fully that strong compactness is destroyed. Both arguments will rely fundamentally on the Key Lemma, below, which was proved in [H96]. Define that a set or sequence is *fresh* over V when it is not in V but every initial segment of it is in V.

† The first author's research has been supported in part by the College of Staten Island and a grant from The City University of New York PSC-CUNY Research Award Program.

‡ The second author's research has been supported by The Israel Science Foundation, administered by the Israel Academy of Sciences and Humanities. This is publication 618 in his independent numbering system.

modified:1998-10-05

618 revision:1998-10-05

Key Lemma. *Assume that $|\mathbb{P}| = \beta$, that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\beta$-closed, and that $\mathrm{cof}(\lambda) > \beta$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ adds no fresh subsets of λ, and no fresh λ-sequences.*

While in [H96] it is proved only that no fresh sets are added, the following simple argument shows that no fresh sequences can be added: given a sequence in δ^λ, code it in the natural way with a binary sequence of length $\delta\lambda$, by using λ many blocks of length δ, each with one 1. The binary sequence corresponds to a subset of the ordinal $\delta\lambda$, which, since $\mathrm{cof}(\delta\lambda) = \mathrm{cof}(\lambda)$, cannot be fresh. Thus, the original λ-sequence cannot be fresh.

Let us give now the first argument. We will use the notion of a θ-club to extend the inductive proof of the Superdestruction Theorem [H96] to all values of λ.

Theorem. *After small forcing, any $<\kappa$-closed forcing which adds a subset to λ will destroy the λ-supercompactness of κ.*

Proof: Suppose that $|\mathbb{P}| < \kappa$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\kappa$-closed. Suppose that $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic, and that $\mathbb{Q} = \dot{\mathbb{Q}}_g$ adds a new subset $A \subseteq \lambda$, with λ minimal, so that $A \in V[g][G]$ but $A \notin V[g]$. By the closure of $\mathbb{Q}$, we know that $\mathrm{cof}(\lambda) \geq \kappa$. Suppose, towards a contradiction, that κ is λ-supercompact in $V[g][G]$. Let $P_\kappa\lambda$ denote $(P_\kappa\lambda)^{V[g][G]}$, which is also $(P_\kappa\lambda)^{V[g]}$.

Lemma. *Every normal fine measure on $P_\kappa\lambda$ in $V[g][G]$ concentrates on $(P_\kappa\lambda)^V$.*

Proof: Let us begin with some definitions. Fix a regular cardinal θ such that $|\mathbb{P}| < \theta < \kappa$. A set $C \subseteq P_\kappa\lambda$ is *unbounded* iff for every $\sigma \in P_\kappa\lambda$ there is $\tau \in C$ such that $\sigma \subseteq \tau$. A set $D \subseteq P_\kappa\lambda$ is *θ-directed* iff whenever $B \subseteq D$ and $|B| < \theta$ then there is some $\tau \in D$ such that $\sigma \subseteq \tau$ for every $\sigma \in B$. The set C is *θ-closed* iff every θ-directed $D \subseteq C$ with $|D| < \kappa$ has $\cup D \in C$. Finally, C is a *θ-club* iff C is both θ-closed and unbounded.

Claim. *A normal fine measure on $P_\kappa\lambda$ contains every θ-club.*

Proof: Work in any model $\bar{V}$. Suppose that C is a θ-club in $P_\kappa\lambda$ and that μ is a normal fine measure on $P_\kappa\lambda$. Let $j : \bar{V} \to M$ be the ultrapower by μ. It is well known that $j \text{ " } \lambda$ is a seed for μ in the sense that $X \in \mu \leftrightarrow j \text{ " } \lambda \in j(X)$ for $X \subseteq P_\kappa\lambda$. By elementarity $j(C)$ is a θ-club in M and $j \text{ " } C \subseteq j(C)$. (We know $j \text{ " } C \in M$ because M is closed under $\lambda^{<\kappa}$ sequences in $\bar{V}$.) Also, it is easy to check that $j \text{ " } C$ is θ-directed. Thus, by the definition of θ-club, we know $\cup(j \text{ " } C) \in j(C)$. But

$$\cup(j \text{ " } C) = \bigcup_{\sigma\in C} j(\sigma) = \bigcup_{\sigma\in C} (j \text{ " } \sigma) = j \text{ " } \lambda.$$

Thus, $j \text{ " } \lambda \in j(C)$ and so $C \in \mu$. $\square_{\text{Claim}}$

modified:1998-10-05

618 revision:1998-10-05

Now let $C = (P_\kappa\lambda)^V$. We will show that C is a θ-club in $V[g][G]$. First, let us show that C is unbounded. If $\sigma \in P_\kappa\lambda$ in $V[g][G]$, then actually $\sigma \in V[g]$, and so $\sigma = \dot\sigma_g$ for some $\mathbb{P}$-name $\dot\sigma \in V$. We may assume that $[\![\, |\dot\sigma| < \check\kappa \,]\!] = 1$ and consequently $\sigma \subseteq \{\, \alpha \mid [\![\, \alpha \in \dot\sigma \,]\!] \neq 0 \,\} \in C$; so σ is covered as desired. To show that C is θ-closed, suppose in $V[g][G]$ that $D \subseteq C$ has size less than κ and is θ-directed. We have to show that $\cup D \in C$. It suffices to show that $\cup D \in V$ since $C = P_\kappa\lambda \cap V$. Since $\mathbb{Q}$ is $<\kappa$-closed, we know that $D \in V[g]$, and thus $D = \dot D_g$ for some name $\dot D \in V$. In V let $D_p = \{\, \sigma \in C \mid p \Vdash \check\sigma \in \dot D \,\}$. It follows that $D = \cup_{p\in g} D_p$. There must be some $p \in g$ such that D_p is $\subseteq$-cofinal in D; for if not, then for each $p \in g$ we may choose $\sigma_p \in D$ such that D_p contains no supersets of σ_p. Since D is θ-directed and $|g| < \theta$ there is some $\sigma \in D$ such that $\sigma_p \subseteq \sigma$ for all $p \in g$. But σ must be forced into D by some condition $p \in g$, so $\sigma \in D_p$ for some $p \in g$, contradicting the choice of σ_p. So we may fix some $p \in g$ such that D_p is $\subseteq$-cofinal in D. But in this case $\cup D_p = \cup D$ and since $D_p \in V$ we conclude $\cup D \in V$. Thus C is a θ-club in $V[g][G]$, and the lemma is proved. $\square_{\text{Lemma}}$

Let us now continue with the theorem. Since κ is λ-supercompact in $V[g][G]$ there must be an embedding $j : V[g][G] \to M[g][j(G)]$ which is the ultrapower by a normal fine measure μ on $P_\kappa\lambda$.

Lemma. $P(\lambda)^M = P(\lambda)^V$.

Proof: ($\supseteq$). By the previous lemma we know that $(P_\kappa\lambda)^V \in \mu$ and so $j\text{ ``}\lambda \in j((P_\kappa\lambda)^V) = (P_\kappa\lambda)^M$. Since $M$ is transitive, it follows that $j\text{ ``}\lambda \in M$. And obtaining this fact was the only reason for proving the previous lemma. Now if $B \subseteq \lambda$ and $B \in V$ then $j(B) \in M$, and since B is constructible from $j(B)$ and $j\text{ ``}\lambda$ it follows that $B \in M$ as well.

($\subseteq$). Now we prove the converse. By induction we will show that $P(\delta)^M \subseteq V$ for all $\delta \leq \lambda$. Suppose that $B \subseteq \delta$ and $B \in M$ and every initial segment of B is in V. By the Key Lemma it follows that $B \in V$ unless $\mathrm{cof}(\delta) < \kappa$. So suppose $\mathrm{cof}(\delta) < \kappa$. By the closure of $\mathbb{Q}$ we know in this case that $B \in V[g]$ and so $B = \dot B_g$ for some name $\dot B \in V$. We may view $\dot B$ as a function from δ to the set of antichains of $\mathbb{P}$. Since $\dot B$ may be coded with a subset of δ, we know $\dot B \in M$ by the previous direction of this lemma. Thus, both B and $\dot B$ are in M and g is M-generic. Since $B = \dot B_g$ in $M[g]$ there is in M a condition $p \in g$ such that $p \Vdash \dot B = \check B$. That is, p decides every antichain of $\dot B$ in a way that makes it agree with B. Use p to decide $\dot B$ in V and conclude that $B \in V$. This completes the induction. $\square_{\text{Lemma}}$

618 revision:1998-10-05 modified:1998-10-05

Now we are nearly done. Consider again the new set $A \subseteq \lambda$ such that $A \in V[g][G]$ but $A \notin V[g]$. Since j is a λ-supercompact embedding, we know $A \in M[g][j(G)]$. Since the $j(G)$ forcing is $<j(\kappa)$-closed, we know $A \in M[g]$. Therefore $A = \dot{A}_g$ for some name $\dot{A} \in M$. Viewing $\dot{A}$ as a function from λ to the set of antichains in $\mathbb{P}$, we can code $\dot{A}$ with a subset of λ, and so by the last lemma we know $\dot{A} \in V$. Thus, $A = \dot{A}_g \in V[g]$, contradicting the choice of A. $\square_{\text{Theorem}}$

Corollary. *By first adding in the usual way a generic subset to β and then to λ, where $\mathrm{cof}(\lambda) > \beta$, one destroys all supercompact cardinals between β and λ.*

In fact, one does not even need to add them in the usual way. This is because the proof of the theorem does not really use the full $<\kappa$-closure of $\mathbb{Q}$. Rather, if $\mathbb{P}$ has size β, then we only need that $\mathbb{Q}$ is $\leq\beta$-closed and adds no new elements of $P_\kappa\lambda$. Thus, we have actually proved the following theorem.

Theorem. *After any forcing of size $\beta < \kappa$, any further $\leq\beta$-closed forcing which adds a subset to λ but no elements to $P_\kappa\lambda$ will destroy the λ-supercompactness of κ.*

This improvement is striking when β is small, having the consequence that after adding a Cohen real, any countably-closed forcing which adds a subset to some minimal λ destroys all supercompact cardinals up to λ.

Let us now give the second argument, which will improve the previous results with a different technique and establish fully that strong compactness is destroyed.

Theorem. *After small forcing, any $<\kappa$-closed forcing which adds a λ-sequence will destroy the λ-strong compactness of κ.*

Proof: Define that a cardinal κ is λ-*measurable* iff there is a κ-complete (non κ^+-complete) uniform measure on λ. Necessarily $\kappa \leq \mathrm{cof}(\lambda)$. This notion is studied in [K72].

Lemma. *Assume that $|\mathbb{P}| < \kappa \leq \lambda$, that $\dot{\mathbb{Q}}$ adds a new λ-sequence over $V^{\mathbb{P}}$, λ minimal, and that κ is λ-measurable in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ must add a fresh λ-sequence over V.*

Proof: This lemma is the heart of the proof. Assume the hypotheses of the lemma. So $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{s}$ is a λ-sequence of ordinals not in $V^{\mathbb{P}}$, and $\dot{\mu}$ is a κ-complete uniform measure on λ. Without loss of generality, we may assume that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is a complete boolean algebra on an ordinal. Suppose now that $g * G$ is V-generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\mathbb{Q} = \dot{\mathbb{Q}}_g$, and $s = \dot{s}_{g*G}$.

In $V[g]$, let $T = \{\, u \in \mathrm{ORD}^{<\lambda} \mid [\![\, \check{u} \subseteq \dot{s} \,]\!]^{\mathbb{Q}} \neq 0 \,\}$. Thus, under inclusion, T is a tree with λ many levels, and $\mathbb{Q}$ adds the λ-branch s. For $u \in T$, let $b_u = [\![\, u \subseteq \dot{s} \,]\!]^{\mathbb{Q}}$.

618 revision:1998-10-05 modified:1998-10-05

Thus, b_u is an ordinal. Let $I = \{ \langle \ell(u), b_u \rangle \mid u \in T \}$, where $\ell(u)$ denotes the length of u, and define $\langle \alpha, b_u \rangle \lhd \langle \alpha', b_{u'} \rangle$ when $\alpha' < \alpha$ and $b_u \leq_{\mathbb{Q}} b_{u'}$. Since $u \supset v \leftrightarrow \langle \ell(u), b_u \rangle \lhd \langle \ell(v), b_v \rangle$ it follows that $\langle T, \supset \rangle \cong \langle I, \lhd \rangle$, and consequently I is also a tree, under the relation $\lhd$, with λ many levels. Furthermore, the αth level of I consists of pairs of the form $\langle \alpha, \beta \rangle$. For $p \in \mathbb{P}$ let us define that $a \lhd_p b$ when $p \Vdash a \lhd b$. Thus, $\lhd = \cup_{p \in g} \lhd_p$.

In $V[g][G]$ let $b_\gamma = \langle \gamma, b_{s \restriction \gamma} \rangle$. Thus, $b_\gamma \in I$, and if $\gamma < \zeta$ then $b_\zeta \lhd b_\gamma$ and so there is some $r \in g$ such that $b_\zeta \lhd_r b_\gamma$. Since there are fewer than κ many such r, for each γ there must be an r which works for μ-almost every ζ. But then again, since there are relatively few r, it must be that there is some $r^* \in g$ which has this property for μ-almost every γ. So, fix $r^* \in g$ such that for μ-almost every γ, for μ-almost every ζ, we have $b_\zeta \lhd_{r^*} b_\gamma$. Fix also a condition $\langle p_0, q_0 \rangle \in g * G$ forcing r^* to have this property. Let $t = \big\langle b_\gamma \mid \gamma < \lambda \ \&\ \text{for } \mu\text{-a.e. } \zeta,\ b_\zeta \lhd_{r^*} b_\gamma \big\rangle$. Thus, t is a partial function from λ to pairs of ordinals, and $\mathrm{dom}(t) \in \mu$. In particular, $\mathrm{dom}(t)$ is unbounded in λ.

We will argue that t is fresh over V. First, notice that $t \notin V[g]$ since in $V[g]$ knowing t we could read off the branch s. Thus, $t \notin V$.

Nevertheless, we will argue that every initial segment of t is in V. Suppose $\delta < \lambda$, and let $t_\delta = t \restriction \delta$. By the minimality of λ it follows that $t_\delta \in V[g]$, and so there is a $\mathbb{P}$-name $\dot{t}_\delta$ and a condition $\langle p_1, q_1 \rangle \in g * G$, stronger than $\langle p_0, q_0 \rangle$, forcing this name to work. Assume towards a contradiction that $t_\delta \notin V$, and that this is forced by p_1. Then, for each $r \in \mathbb{P}$ below p_1 we may choose $\gamma_r < \delta$ such that r does not decide $t(\gamma_r)$ (or whether γ_r is in the domain of t). But, nevertheless, for each r either for μ-almost every ζ, $b_\zeta \lhd_{r^*} b_{\gamma_r}$ or else for μ-almost every ζ, $b_\zeta \not\lhd_{r^*} b_{\gamma_r}$ (but not both). In the first case it follows that $t(\gamma_r) = b_{\gamma_r}$, and in the second it follows that $\gamma_r \notin \mathrm{dom}(t)$. Since there are relatively few r, by intersecting these sets of ζ we can find a single ζ which acts, with respect to the γ_r, exactly the way μ-almost every ζ acts. Fix such a ζ. Thus, for each r we have either $b_\zeta \lhd_{r^*} b_{\gamma_r}$, and consequently $t(\gamma_r) = b_{\gamma_r}$, or else $\gamma_r \notin \mathrm{dom}(t)$ (but not both). Notice that ζ and b_ζ are just some particular ordinals. Fix some condition $\langle p^*, q^* \rangle$ below $\langle p_1, q_1 \rangle$ forcing ζ and b_ζ to have the property we mention in the sentence before last. Now we will argue that this is a contradiction. Let $\gamma = \gamma_{p^*}$. There are two cases. First, it might happen that $b_\zeta \lhd_{r^*} \langle \gamma, \beta \rangle$ for some ordinal β. Such a situation can be observed in V. In this case, $\langle p^*, q^* \rangle$ forces $\beta = b_{s \restriction \gamma}$ and therefore, by the assumption on ζ, it also forces $t(\gamma) = \langle \gamma, \beta \rangle$. Since $\dot{t}_\delta$ is a $\mathbb{P}$-name, it follows that $p^* \Vdash \dot{t}_\delta(\check{\gamma}) = \langle \check{\gamma}, \check{\beta} \rangle$,

modified:1998-10-05

618 revision:1998-10-05

contrary to the choice of $\gamma = \gamma_{p^*}$. Alternatively, in the second case, it may happen that $b_\zeta \not\triangleleft_{r^*} \langle \gamma, \beta \rangle$ for every β. In this case, by the assumption on ζ, it must be that $\langle p^*, q^* \rangle$ forces that $\gamma \notin \mathrm{dom}(t)$. Again, since $\dot{t}_\delta$ is a $\mathbb{P}$-name, it follows that $p^* \Vdash \gamma \notin \mathrm{dom}(\dot{t}_\delta)$, contrary again to the choice of $\gamma = \gamma_{p^*}$. Thus, in either case we reach a contradiction, and so we have proven that $\mathbb{P} * \dot{\mathbb{Q}}$ must add a fresh λ-sequence. $\square_{\text{Lemma}}$

Lemma. *If $\kappa \leq \mathrm{cof}(\lambda)$ and κ is λ-strongly compact, then κ is λ-measurable.*

Proof: Let $j : V \to M$ be the ultrapower map witnessing that κ is λ-strongly compact. By our assumption on $\mathrm{cof}(\lambda)$, it follows that $\sup j \text{ " } \lambda < j(\lambda)$. Let $\alpha = (\sup j \text{ " } \lambda) + \kappa$, and let μ be the measure germinated by the seed α. That is, $X \in \mu$ iff $\alpha \in j(X)$. Since $\alpha < j(\lambda)$ it follows that μ is a measure on λ. Since $j(\beta) < \alpha$ for all $\beta < \lambda$ it follows that μ is uniform. Since $\mathrm{cp}(j) = \kappa$ it follows that μ is κ-complete. For $\gamma < \kappa$, let $B_\gamma = \{ \beta \mid \gamma < \mathrm{cof}(\beta) < \kappa \}$. Since $\mathrm{cof}(\alpha) = \kappa$ in M, it follows that $\alpha \in j(B_\gamma)$ and consequently $B_\gamma \in \mu$ for every $\gamma < \kappa$. Since $\cap_\gamma B_\gamma = \varnothing$, it follows that μ is not κ^+-complete, as desired. $\square_{\text{Lemma}}$

Remark. Ketonen [K72] has proved that if κ is λ-measurabile for every regular λ above κ, then κ is strongly compact. This cannot, however, be true level-by-level, since if $\kappa < \lambda$ are both measurable, with measures μ and ν, then $\mu \times \nu$ is a κ-complete, non-κ^+-complete, uniform measure on $\kappa \times \lambda$. Thus, in this situation, κ will be λ-measurable, even when it may not be even κ^+-strongly compact. But the previous lemma establishes that the direction we need does indeed hold level-by-level.

Let us now finish the proof of the theorem. Suppose that $V[g][G]$ is a forcing extension by $\mathbb{P} * \dot{\mathbb{Q}}$, where $|\mathbb{P}| < \kappa$ and $\mathbb{Q}$ is $<\kappa$-closed. Let λ be least such that $\mathbb{Q}$ adds a new λ-sequence not in $V[g]$. Necessarily, $\kappa \leq \lambda$ and λ is regular. By the Key Lemma $V[g][G]$ has no λ-sequences which are fresh over V. Thus, by the first lemma κ is not λ-measurable in $V[g][G]$. Therefore, by the second lemma, κ is not λ-strongly compact in $V[g][G]$. $\square_{\text{Theorem}}$

So the proof actually establishes that after small forcing of size $\beta < \kappa$, any $\leq\beta$-closed forcing which adds a new λ-sequence for some minimal λ, with $\lambda \geq \kappa$, will destroy the λ-measurability of κ. This subtlety about adding a λ-sequence as opposed to a *subset* of λ has the following intriguing consequence, which is connected with the possibilities of changing the cofinalities of very large cardinals.

modified:1998-10-05

618 revision:1998-10-05

Corollary. *Suppose that κ is λ-measurable. Then after forcing with $\mathbb{P}$ of size $\beta < \kappa$, any $\leq\beta$-closed $\mathbb{Q}$ which adds a λ-sequence, but no shorter sequences, must necessarily add subsets to λ.*

Proof: Such forcing will destroy the λ-measurability of κ. Hence, it must add subsets to λ. $\square_{\text{Corollary}}$

Bibliography

[H96] Joel David Hamkins, *Small Forcing Makes Any Cardinal Superdestructible*, Journal of Symbolic Logic (in press)

[K72] Jussi Ketonen, *Strong Compactness and Other Cardinal Sins*, Annals of Mathematical Logic **5** p. 47-76 (1972)

[L78] Richard Laver, *Making the Supercompactness of κ Indestructible Under κ-Directed Closed Forcing*, Isreal Journal Math **29** p. 385-388 (1978)

modified:1998-10-05

618 revision:1998-10-05