A NON-REFLEXIVE WHITEHEAD GROUP

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Abstract. We prove that it is consistent that there is a non-reflexive Whitehead group, in fact one whose dual group is free. We also prove that it is consistent that there is a group A such that $\mathrm{Ext}(A, \mathbb{Z})$ is torsion and $\mathrm{Hom}(A, \mathbb{Z}) = 0$. As an application we show the consistency of the existence of new co-Moore spaces.

0. Introduction

This paper is motivated by a theorem and a question due to Martin Huber. He proved [8] in ZFC that if A is $\aleph_1$-coseparable (that is, $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$), then A is reflexive (that is, the natural map of A to its double dual $A^{**} = \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Z}), \mathbb{Z})$ is an isomorphism). He asked whether it is provable in ZFC that every Whitehead group A (i.e., $\mathrm{Ext}(A, \mathbb{Z}) = 0$) is reflexive. This is true in any model where every Whitehead group is free. It is also true for Whitehead groups of cardinality $\aleph_1$ in a model of MA + $\neg$CH (because they are $\aleph_1$-coseparable: cf. [4, Cor. XII.1.12]). Moreover, it is true in the original models of GCH where there are non-free Whitehead groups (cf. [12], [13], [4, Thm. XII.1.9]). It was left as an open question in [4, p. 455] whether every Whitehead group is reflexive. Here we give a strong negative answer:

Theorem 0.1. *It is consistent with ZFC that there is a strongly non-reflexive strongly $\aleph_1$-free Whitehead group A of cardinality $\aleph_1$.*

A group A is strongly non-reflexive if A is not isomorphic to A^{**}. In fact, the example A has the property that A^* is free of rank $\aleph_2$ (i.e., isomorphic to $\mathbb{Z}^{(\aleph_2)}$) so A^{**} is isomorphic to the product $\mathbb{Z}^{\aleph_2}$; it is therefore not isomorphic to A since its cardinality is $2^{\aleph_2} > \aleph_1$. (See Theorem 1.5 and Corollary 1.6 of section 1.)

If $\mathrm{Ext}(A, \mathbb{Z}) = 0$, then A is separable ([5, Thm 99.1]) and hence A^* is non-zero. However, using the same methods we can also prove:

Theorem 0.2. *It is consistent with ZFC that there is a non-free strongly $\aleph_1$-free group A of cardinality $\aleph_1$ such that* $\mathrm{Ext}(A, \mathbb{Z})$ *is torsion and* $\mathrm{Hom}(A, \mathbb{Z}) = 0$.

It is not a theorem of ZFC that there is a non-free torsion-free group A such that $\mathrm{Ext}(A, \mathbb{Z})$ is torsion. Indeed, in any model where every Whitehead group is free—a hypothesis which is consistent with CH or $\neg$CH (cf. [11])—if A is not free, then $\mathrm{Ext}(A, \mathbb{Z})$ is not torsion ([7], [3], [4, Thm. XII.2.4]).

Theorems 1.5 and 0.2 provide new examples of possible co-Moore spaces (see section 6). In particular, we answer a question in [6, p. 46] by showing that it is consistent that

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for any $n \geq 2$ there is a co-Moore space of type (F, n) where F is a free group of rank $\aleph_2$.

The models for both theorems result from a finite support iteration of c.c.c. posets and are models of ZFC + ¬CH. (Other methods will be needed to obtain consistency with CH.) We begin the iteration with a poset which yields "generic data" from which the group A is defined; we then iterate the natural posets which insure that $\mathrm{Ext}(A, \mathbb{Z}) = 0$ (resp. $\mathrm{Ext}(A, \mathbb{Z})$ is torsion). The hard work is in proving that $\mathrm{Hom}(A, \mathbb{Z})$ is as claimed. We define the forcing and the group more precisely in the next section and then prove their properties in the succeeding sections.

1. The basic construction

The group-theoretic construction is a generalization of that in [4, XII.3.4]. Let E be a stationary and co-stationary subset of ω_1 consisting of limit ordinals, and for each $\delta \in E$, let η_δ be a ladder on δ, that is, a strictly increasing function $\eta_\delta : \omega \to \delta$ whose range approaches δ. Let F be the free abelian group with basis $\{x_\nu : \nu \in \omega_1\} \cup \{z_{\delta,n} : \delta \in E, n \in \omega\}$. Let g be a function from $E \times \omega$ to the integers ≥ 1. Let u be a function from $E \times \omega$ to the subgroup $\langle x_\nu : \nu \in \omega_1 \rangle$ generated by $\{x_\nu : \nu \in \omega_1\}$ such that $u(\delta, n)$ belongs to $\langle x_\nu : \nu < \eta_\delta(n) \rangle$. Let K be the subgroup of F generated by $\{w_{\delta,n} : \delta \in E, n \in \omega\}$ where

$$w_{\delta,n} = 2^{g(\delta,n)} z_{\delta,n+1} - z_{\delta,n} - x_{\eta_\delta(n)} - u(\delta, n). \tag{1.1}$$

Let $A = F/K$. Then clearly A is an abelian group of cardinality $\aleph_1$. Notice that because the right-hand side of (1.1) is 0 in A, we have for each $\delta \in E$ and $n \in \omega$ the following relations in A:

$$2^{g(\delta,n)} z_{\delta,n+1} = z_{\delta,n} + x_{\eta_\delta(n)} + u(\delta, n) \tag{1.2}$$

and

$$2^{\sum_{j=0}^{n} g(\delta,j)} z_{\delta,n+1} = z_{\delta,0} + \sum_{k=0}^{n} 2^{\sum_{j=0}^{k-1} g(\delta,j)} (x_{\eta_\delta(k)} + u(\delta, k)) \tag{1.3}$$

Here, and occasionally in what follows, we abuse notation and write, for example, $z_{\delta,n+1}$ instead of $z_{\delta,n+1} + K$ for an element of A. For each $\alpha < \omega_1$, let A_α be the subgroup of A generated by

$$\{x_\nu : \nu < \alpha\} \cup \{z_{\delta,n} : \delta \in E \cap \alpha, n \in \omega\}. \tag{1.4}$$

Then, by (1.3), for each $\delta \in E$, $z_{\delta,0} + A_\delta$ is non-zero and divisible in $A_{\delta+1}/A_\delta$ by 2^m for all $m \in \omega$. Thus $A_{\delta+1}/A_\delta$ is not free and hence A is not free. (In fact $\Gamma(A) \supseteq \tilde{E}$.) Moreover, A is strongly $\aleph_1$-free; in fact, for every $\alpha < \omega_1$, using Pontryagin's Criterion [4, IV.2.3] we can show that $A/A_{\alpha+1}$ is $\aleph_1$-free for all $\alpha \in \omega_1 \cup \{-1\}$.

We begin with a model V of ZFC where GCH holds, choose $E \in V$, and define the group A in a generic extension V^{Q_0} using generic ladders η_δ, and generic u and g. Specifically:

Definition 1.1. *Let Q_0 be the set of all finite functions q such that* $\mathrm{dom}(q)$ *is a finite subset of E and for all $\gamma \in \mathrm{dom}(q)$, $q(\gamma)$ is a triple $(\eta^q_\gamma, u^q_\gamma, g^q_\gamma)$ where for some $r^q_\gamma \in \omega$:*

- *η^q_γ is a strictly increasing function: $r^q_\gamma \to \gamma$;*
- *$u^q_\gamma : \{\gamma\} \times r^q_\gamma \to \langle x_\nu : \nu \in \omega_1 \rangle$ such that for all $n < r^q_\gamma$, $u^q_\gamma(\gamma, n) \in \langle x_\nu : \nu < \eta_\gamma(n) \rangle$; and*

- $g^q_\gamma : \{\gamma\} \times r^q_\gamma \to \{n \in \omega : n \geq 1\}$.

The partial ordering is defined by: $q_1 \leq q_2$ if and only if $q_1 \subseteq q_2$; note that we follow the convention that stronger conditions are larger. Clearly Q_0 is c.c.c. and hence E remains stationary and co-stationary in a generic extension. We now do an iterated forcing to make A a Whitehead group. We begin by defining the basic forcing that we will iterate.

Definition 1.2. *Given a homomorphism $\psi : K \to \mathbb{Z}$, let Q_ψ be the poset of all finite functions q into $\mathbb{Z}$ satisfying:*

There are $\delta_0 < \delta_1 < ... < \delta_m$ in E and $\{r_\ell : \ell \leq m\} \subseteq \omega$ such that $\mathrm{dom}(q) =$

$$\{z_{\delta_\ell,n} : \ell \leq m, n \leq r_\ell\} \cup \{x_\nu : \nu \in I_q\}$$

where $I_q \subset \omega_1$ is finite and is such that for all $\ell \leq m$

$$n < r_\ell \Rightarrow u(\delta_\ell, n) \in \langle x_\nu : \nu \in I_q\rangle \text{ and } \eta_{\delta_\ell}(n) \in I_q \Leftrightarrow n < r_\ell \tag{1.5}$$

and for all $\ell \leq m$ and $n < r_\ell$, $u(\delta_\ell, n) \in \langle x_\nu : \nu \in I_q\rangle$ and

$$\psi(w_{\delta_\ell,n}) = 2^{g(\delta,n)} q(z_{\delta_\ell,n+1}) - q(z_{\delta_\ell,n}) - q(x_{\eta_{\delta_\ell}(n)}) - q(u(\delta_\ell, n)). \tag{1.6}$$

(Compare with (1.1). The definition of $q(u(\delta_\ell, n))$ is the obvious one, given that q should extend to a homomorphism.) Moreover, we require of q that for all $\ell \neq j$ in $\{0, ..., m\}$,

$$\eta_{\delta_j}(k) \neq \eta_{\delta_\ell}(i) \text{ for all } k \geq r_j \text{ and } i \in \omega. \tag{1.7}$$

We will denote $\{\delta_0, ..., \delta_m\}$ by $\mathrm{cont}(q)$ and r_ℓ by $\mathrm{num}(q, \delta_\ell)$. The partial ordering on Q_ψ is inclusion.

Proposition 1.3. *(i) For every $\delta \in E$ and $k \in \omega$, $D_{\delta,k} = \{q \in Q_\psi : \delta \in \mathrm{cont}(q)$ and $k \leq \mathrm{num}(q, \delta)\}$ is dense in Q_ψ*

(ii) Q_ψ is c.c.c.

Before proving Proposition 1.3, we prove a lemma:

Lemma 1.4. *Given $\{\delta_0, ..., \delta_m\} \in E$, integers r'_ℓ for $\ell \leq m$ and a finite subset I' of ω_1, there are integers $r''_\ell \geq r'_\ell$ for all $\ell \leq m$ and a finite subset I'' of ω_1 containing I' such that for all $\ell \leq m$:*

(a) $\eta_{\delta_\ell}(n) \in I'' \iff n < r''_\ell$; and

(b) for all $n < r''_\ell$, $u(\delta_\ell, n) \in \langle x_\nu : \nu \in I''\rangle$.

Proof. The proof is by induction on $m \geq 0$. If $m = 0$ we can take

$$r''_0 = \max\{r'_0, \max\{k + 1 : \eta_{\delta_0}(k) \in I'\}\}$$

and take I'' to be a minimal extension of $I' \cup \{\eta_{\delta_0}(n) : n < r''_0\}$ satisfying (b); then (a) holds because $u(\delta, n) \in \langle x_\nu : \nu < \eta_\delta(n)\rangle$. If $m > 0$, without loss of generality we can assume that $\delta_0 < \delta_1 < ... < \delta_m$. Let

$$r''_m = \max\{r'_m, \max\{k + 1 : \eta_{\delta_m}(k) \in I'\}, \min\{k : \eta_{\delta_m}(k) > \delta_{m-1}\}\}.$$

As in the case $m = 0$, there exists $\tilde{I}$ containing I' such that (a) and (b) hold for $\tilde{I}$ for $\ell = m$. Then apply the inductive hypothesis to $\{\delta_0, ..., \delta_{m-1}\}$, $\tilde{I}$, and the r'_ℓ ($\ell < m$) to obtain r''_ℓ for $\ell < m$ and a minimal I''. ∎

For $q \in Q_\psi$ and $\alpha \in \omega_1$, let $q \restriction \alpha$ denote the restriction of q to

$$\operatorname{dom}(q) \cap (\{z_{\delta,n} : \delta < \alpha, n \in \omega\} \cup \{x_\nu : \nu < \alpha\}).$$

Say that τ occurs in q if $\tau \in \operatorname{cont}(q)$ or $x_\tau \in \operatorname{dom}(q)$.

PROOF OF PROPOSITION 1.3. (i) Given $\delta \in E$, $k \in \omega$ and $p \in Q_\psi$, we need $q \geq p$ such that $q \in D_{\delta,k}$. Let $\operatorname{cont}(p) = \{\delta_0, ..., \delta_m\}$. We consider two cases. The first is that $\delta \in \operatorname{cont}(p)$, that is, $\delta = \delta_j$ for some $j \leq m$. We can assume that $k > \operatorname{num}(p, \delta_j)$. Apply Lemma 1.4 with $I' = I_p$, $r'_j = k$, and $r'_\ell = \operatorname{num}(p, \delta_\ell)$ for $\ell \neq j$ to obtain I'' and r''_ℓ. Then we can define q to be the extension of p with $\operatorname{cont}(q) = \operatorname{cont}(p)$ and $\operatorname{num}(q, \delta_\ell) = r''_\ell$ and $I_q = I''$. Since (1.5) and (1.7) hold, we can inductively define $q(x_{\eta_{\delta_\ell}(i)})$ and $q(z_{\delta_\ell,i+1})$ for $r'_\ell \leq i < r''_\ell$ (setting $q(x_\nu) = 0$ for $\nu \in I_q \setminus \cup\{\operatorname{rge}(\eta_{\delta_\ell} : \ell \leq m\}$ if not already defined) so that (1.6) holds. Note that (1.7) continues to hold.

The second case is when $\delta \notin \operatorname{cont}(p)$. Let $\delta_{m+1} = \delta$. Choose r'_ℓ for $\ell \leq m+1$ so that $r'_\ell \geq \operatorname{num}(p, \delta_\ell)$ for $\ell \leq m$ and such that (1.7) holds, that is, $\eta_{\delta_j}(n) \neq \eta_{\delta_\ell}(i)$ for all $n \geq r'_j$ and $i \in \omega$ for all $j \neq \ell \in \{0, ..., m+1\}$. Apply Lemma 1.4 to $\{\delta_0, ..., \delta_{m+1}\}$, I_p, and the r'_ℓ to obtain r''_ℓ for $\ell \leq m+1$ and I''. Let $I_q = I''$ and $\operatorname{num}(q, \delta_\ell) = r''_\ell$. For $\ell \leq m$ define $q(x_{\eta_{\delta_\ell}(i)})$, $q(z_{\delta_\ell,i+1})$ and $u(\delta_\ell, i)$ for $\operatorname{num}(p, \delta_\ell) \leq i < r''_\ell$ by induction on i as in the first case. Define $q(z_{\delta, r''_{m+1}+1}) = 0$ and define $q(z_{\delta,n})$ for $n \leq r''_{m+1}$ by "downward induction", i.e.

$$q(z_{\delta,n}) = 2^{g(\delta,n)} q(z_{\delta,n+1}) - q(x_{\eta_\delta(n)}) - q(u(\delta, n)) - \psi(w_{\delta,n}).$$

(Setting $q(x_\tau) = 0$ where not already defined, we can assume $q(x_{\eta_\delta(n)})$ and $q(u(\delta, n))$ are defined.)

(ii) Consider an uncountable subset $\{q_\nu : \nu \in \omega_1\}$ of Q_ψ. By the Δ-system lemma we can assume that $\{\operatorname{cont}(q_\nu) : \nu \in \omega_1\}$ forms a Δ-system, i.e., there is a finite subset Δ of E such that for all $\nu \neq \mu$, $\operatorname{cont}(q_\nu) \cap \operatorname{cont}(q_\mu) = \Delta$. By renumbering an uncountable subset, we can assume that for all ν, if $\delta \in \operatorname{cont}(q_\nu) \setminus \Delta$, then $\delta > \nu$. Furthermore, by passing to a subset and using (i) we can assume that if $\delta \in \operatorname{cont}(q_\nu) \setminus \Delta$ and $\eta_\delta(n) < \nu$, then $n < \operatorname{num}(q_\nu, \delta)$. By Fodor's Lemma we can assume that there exists $\gamma \geq \max \Delta$ such that for all ν and n, if $\delta \in \operatorname{cont}(q_\nu)$ and $\eta_\delta(n) < \nu$, then $\eta_\delta(n) < \gamma$ and moreover such that if $\tau \in I_{q_\nu}$ and $\tau < \nu$, then $\tau < \gamma$. We can also assume that for all μ, ν, $q_\mu \restriction \mu = q_\nu \restriction \nu$. If we pick $\mu < \nu$ such that $\gamma < \mu$ and whenever τ occurs in q_μ, then $\tau < \nu$, then we will have that $q_\mu \cup q_\nu \in Q_\psi$. Notice that (1.7) will be satisfied: if $\delta \in \operatorname{cont}(q_\mu) \setminus \Delta$ and $\rho \in \operatorname{cont}(q_\nu) \setminus \Delta$ and $k \geq \operatorname{num}(q_\mu, \delta)$ and $m \geq \operatorname{num}(q_\nu, \rho)$, then $\mu \leq \eta_\delta(k) < \nu \leq \eta_\rho(m)$; moreover, if $i \in \omega$ and $\eta_\rho(i) < \nu$, then $\eta_\rho(i) < \gamma < \mu \leq \eta_\delta(k)$. Similarly it follows that (1.5) holds. ∎

Now $P = \left\langle P_i, \dot{Q}_i : 0 \leq i < \omega_2 \right\rangle$ is defined to be a finite support iteration of length ω_2 so that for every $i \geq 1 \Vdash_{P_i} \dot{Q}_i = Q_{\dot{\psi}_i}$ where $\Vdash_{P_i} \dot{\psi}_i$ is a homomorphism: $K \to \mathbb{Z}$ and the enumeration of names $\{\dot{\psi}_i : 1 \leq i < \omega_2\}$ is chosen so that if G is P-generic and $\psi \in V[G]$ is a homomorphism: $K \to \mathbb{Z}$, then for some $i \geq 1$, $\dot{\psi}_i$ is a name for ψ in V^{P_i}. Then P is c.c.c. and in $V[G]$ every homomorphism from K to $\mathbb{Z}$ extends to one from F to $\mathbb{Z}$. This means that $\operatorname{Ext}(A, \mathbb{Z}) = 0$, that is, A is a Whitehead group (see, for example, [4, p. 8]). We claim moreover that:

Theorem 1.5. *In $V[G]$ A^* ($= \operatorname{Hom}(A, \mathbb{Z})$) is free of cardinality $\aleph_2$.*

As a consequence we can conclude:

Corollary 1.6. *In $V[G]$ A is strongly non-reflexive.*

Proof. Since A^* is isomorphic to $\mathbb{Z}^{(\aleph_2)}$, A^{**} is isomorphic to $\mathbb{Z}^{\aleph_2}$ and hence not isomorphic to A because its cardinality is different. We remark also that A^{**} is not slender, but A is slender since it is a Whitehead group — see [4, Prop. XII.1.3, p. 345]). ∎

The next three sections are devoted to a proof of Theorem 1.5. The fact that A^* has cardinality $2^{\aleph_1}$ is a consequence of a result of Chase [1, Thm. 5.6]; by standard arguments it can be seen that in $V[G]$ $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Let $G_\nu = \{p \restriction \nu : p \in G\}$, so that G_ν is P_ν-generic. To prove that $\operatorname{Hom}(A, \mathbb{Z})^{V[G]}$ is free, it suffices to prove that:

(I) $\operatorname{Hom}(A, \mathbb{Z})^{V[G_1]} = 0$;

(II) for every limit $\beta \leq \omega_2$, $\operatorname{Hom}(A, \mathbb{Z})^{V[G_\beta]} = \bigcup_{i<\beta} \operatorname{Hom}(A, \mathbb{Z})^{V[G_i]}$;

and

(III) for all $i < \omega_2$, $\operatorname{Hom}(A, \mathbb{Z})^{V[G_{i+1}]} / \operatorname{Hom}(A, \mathbb{Z})^{V[G_i]}$ is free, and in fact is either 0 or $\mathbb{Z}$.

We shall prove (I) immediately, and then prove the other two parts in the next three sections.

PROOF OF (I): Notice first that, by (1.3), if $h \in \operatorname{Hom}(A, \mathbb{Z})$ and $h(x_\mu) = 0$ for all $\mu \in \omega_1$, then h is identically zero. So suppose, to obtain a contradiction, that there exists a Q_0-name $\dot{h}$ and $p \in G_1$ such that

$$p \Vdash \dot{h} \in \operatorname{Hom}(A, \mathbb{Z}) \wedge \dot{h}(x_\mu) = m$$

for some $\mu \in \omega_1$ and some non-zero integer m. Choose d such that 2^d does not divide m. For each $\delta \in E$ there exists $p_\delta \geq p$ and $c_\delta \in \mathbb{Z}$ such that

$$p_\delta \Vdash \dot{h}(z_{\delta,0}) = c_\delta.$$

By Fodor's Lemma and a Δ-system argument, there exist $\delta_1 \neq \delta_2 > \mu$ such that $c_{\delta_1} = c_{\delta_2}$, and if (for convenience of notation) we let $p^i = p_{\delta_i}$, $r^{p^1}_{\delta_1} = r^{p^2}_{\delta_2} = r$, $\eta_{\delta_1}(n) = \eta_{\delta_2}(n)$, $u(\delta_1, n) = u(\delta_2, n)$ for all $n < r$ and p^1 and p^2 are compatible. Then there is a condition $q \in Q_0$ such that $p^i \leq q$ for $i = 1, 2$ and

$$q \Vdash \eta_{\delta_1}(r) = \eta_{\delta_2}(r) \wedge g(\delta_1, r) = d = g(\delta_2, r) \wedge u(\delta_1, r) = x_\mu \wedge u(\delta_2, r) = 0.$$

Now consider a generic extension $V[G'_1]$ where $q \in G'_1$. By subtracting (1.3) for $n = r$ and $\delta = \delta_2$ from (1.3) for $n = r$ and $\delta = \delta_1$ and applying h we obtain that (in $V[G'_1]$) 2^d divides $h(u(\delta_1, r)) - h(u(\delta_2, r)) = h(x_\mu) - h(0) = m$. But this is a contradiction of the choice of d. ∎

2. Preliminaries

Before beginning the proof proper of (II) and (III), we prove a crucial Proposition that we will need. For a fixed $m \in \omega$ and $S \subseteq E$, let $Z_m[S]$ denote the pure closure in A of the subgroup generated by $\{z_{\delta,m} + K : \delta \in S\}$. For $t \in \omega$, let $Z_{m,t}[S]$ denote $Z_m[S] + 2^t A$.

Proposition 2.1. *In $V[G]$, for all $m, t \in \omega$ and all stationary $S \subseteq E$, $A/Z_{m,t}[S]$ is a finite group.*

Proof. The proof is by contradiction. Suppose that $q^* \in G$ such that for some $m, t \in \omega$ and some $\dot{S}$

$$q^* \Vdash_P \dot{S} \text{ is a stationary subset of } E \text{ and } A/Z_{m,t}[\dot{S}] \text{ is infinite.}$$

Let S' be the set of all $\delta \in E$ such that q^* does not force "$\delta \notin \dot{S}$"; then $S' \in V$ is a stationary subset of E. For each $\delta \in S'$ choose $p_\delta \geq q^*$ such that $p_\delta \Vdash \delta \in \dot{S}$.

We can assume that each p_δ satisfies:

(†) $0 \in \mathrm{dom}(p_\delta)$; $\delta \in \mathrm{dom}(p_\delta(0))$; for each $j \in \mathrm{dom}(p_\delta)$, $p_\delta(j)$ is a function in V and not just a name; $r_\gamma^{p_\delta(0)}$ $(= r_\delta)$ is independent of $\gamma \in \mathrm{dom}(p_\delta(0))$; if $j \in \mathrm{dom}(p_\delta) \setminus \{0\}$, $\gamma \in \mathrm{cont}(p_\delta(j))$ implies $\gamma \in \mathrm{dom}(p_\delta(0))$ and $\mathrm{num}(p_\delta(j), \gamma)$ $(= r'_{\delta,j})$ is $\leq r_\delta$ and independent of γ. Moreover, if $\gamma \in \mathrm{dom}(p_\delta(0))$ and $\gamma > \delta$, then $\eta_\gamma(r'_{\delta,j}) > \delta$.

When we say that "ν occurs in p" we mean that $p(\nu)$ is non-empty, or ν occurs in $p(j)$ for some $j > 0$ or $\nu \in \mathrm{dom}(p(0)) \cup \{\eta_\gamma(n) : \gamma \in \mathrm{dom}(p(0)),\ n < r_\gamma^{p(0)}\}$, or $u(\gamma, n) \notin \langle x_\mu : \mu \neq \nu\rangle$ for some $n < r_\gamma^{p(0)}$. Without loss of generality we can assume (passing to a subset of S') that

(††) there exists τ such that for all $\delta \in S'$, $\delta > \tau$ and every ordinal $< \delta$ which occurs in p_δ is less than τ; $\{\mathrm{dom}(p_\delta) : \delta \in S'\}$ forms a Δ-system, whose root we denote C (i.e., $\mathrm{dom}(p_{\delta_1}) \cap \mathrm{dom}(p_{\delta_2}) = C = \{0, \mu_1, ..., \mu_d\}$ for all $\delta_1 \neq \delta_2$ in S'); r_δ $(= r^*)$ and $r'_{\delta,j}$ $(= r'_j)$ are independent of δ. Moreover, for every $j \in C$, $\{\mathrm{dom}(p_\delta(j)) : \delta \in S'\}$ forms a Δ-system and for all $\delta_1 \neq \delta_2$ in S', $p_{\delta_1}(j)$ and $p_{\delta_2}(j)$ agree on $\mathrm{dom}(p_{\delta_1}(j)) \cap \mathrm{dom}(p_{\delta_2}(j))$, so $p_{\delta_1}(j) \restriction \delta_1 = p_{\delta_2}(j) \restriction \delta_2$. Also, $\mathrm{dom}(p_\delta(0)) \cap \delta$ and $p_\delta(0) \restriction (\mathrm{dom}(p_\delta(0)) \cap \delta)$ are independent of δ. Finally, $\eta_\delta^{p_\delta(0)}(n)$ $(=\zeta_n)$, $g^{p_\delta(0)}(\delta, n)$ $(= g(n))$ and $u^{p_\delta(0)}(\delta, n)$ are independent of δ for each $n < r^*$.

Let p^* denote the "heart" of $\{p_\delta : \delta \in S'\}$; that is, $\mathrm{dom}(p^*) = C$ and for all $\mu \in C$, $\mathrm{dom}(p^*(\mu)) = \mathrm{dom}(p_{\delta_1}(\mu)) \cap \mathrm{dom}(p_{\delta_2}(\mu))$ $(= C_\mu$, say) for $\delta_1 \neq \delta_2 \in S'$; and $p^*(\mu) = p_{\delta_1}(\mu) \restriction C_\mu = p_{\delta_1}(\mu) \restriction \delta_1$.

The conditions in (††) insure that if $\delta_1 < \delta_2$ are members of S' such that every ordinal which occurs in p_{δ_1} is $< \delta_2$, then p_{δ_1} and p_{δ_2} are almost compatible; however, there may be problems in determining a value for $p_{\delta_\ell}(j)(x_{\zeta_n})$ for $r'_j \leq n < r^*$ (independent of $\ell = 1, 2$); it is because of these that the following argument is necessary.

We can assume that $r^* \geq m$ and that for all $\delta \in S'$ $\delta \notin \mathrm{dom}(p^*(0))$. Choose $M \geq t$ such that $g(n) \leq M$ for all $n < r^*$. Let

$$N = 2^{(r^*+1)M}.$$

To obtain a contradiction, it suffices to prove that p^* forces:

(∇) $A/Z_{m,t}[S]$ is a group of cardinality $\leq N^d$

This is a contradiction since $p^* \geq q^*$. If p^* does not force (∇), then there is a finite subset Θ of ω_1 and a condition $p^{**} \geq p^*$ such that p^{**} forces

($\nabla\nabla$)$(\langle x_\nu : \nu \in \Theta\rangle + Z_{m,t}[S])/Z_{m,t}[S]$ has cardinality $> N^d$.

(Note that it follows from (1.3) that A/p^tA is generated by $\langle x_\nu : \nu \in \Theta\rangle$.) We can assume that if ν occurs in p^{**}, then $\nu \in \Theta$. Let T be the subset of $\langle x_\nu : \nu \in \Theta\rangle$ composed of all elements of the form $\sum_{\nu \in \Theta} c_\nu x_\nu + x_\beta$ where $0 \leq c_\nu < 2^t$. Let $\theta = 2^{|\Theta|t}$; so T has $\theta > N^d$ elements; list them as $\{\tau_\ell : \ell < \theta\}$. Now choose elements $\{\delta_\ell : \ell < \theta\}$ of S' listed in increasing order and such that the smallest, δ_0, is larger than $\max\Theta$ and such that every ordinal which occurs in p_{δ_ℓ} is $< \delta_{\ell+1}$. Moreover, we can choose them so that for any $\ell < \theta$, the "common part" of p_{δ_ℓ} and p^{**} is p^*; that is, $\mathrm{dom}(p_{\delta_\ell}) \cap \mathrm{dom}(p^{**}) = C = \mathrm{dom}(p^*)$ and for all $\mu \in C$, $\mathrm{dom}(p_{\delta_\ell}(\mu)) \cap \mathrm{dom}(p^{**}(\mu)) = \mathrm{dom}(p^*(\mu))$. (So, in particular, $\delta_\ell \notin \mathrm{dom}(p^{**}(0))$.)

Choose new ordinals α_ℓ for $-1 \leq \ell < \theta$ such that

$$\alpha_{-1} < \alpha_0 < \delta_0 < \alpha_1 < \delta_1 < ... < \delta_\ell < \alpha_{\ell+1} < \delta_{\ell+1} < ...$$

Moreover, we make the choice so that for all ℓ, α_ℓ is larger than any ordinal $< \delta_\ell$ which occurs in any p_{δ_k}. There is a condition $q_0 \in Q_0$ which extends $p^{**}(0)$ and each $p_{\delta_\ell}(0)$ ($\ell < \theta$) such that q_0 forces for all $\ell < \theta$:

$$\eta_{\delta_\ell}(r^*) = \alpha_{-1}; \eta_{\delta_\ell}(r^*+1) = \alpha_\ell; u(\delta_\ell, r^*) = \tau_\ell; \text{ and } g(\delta_\ell, r^*) = t.$$

This q_0 will force versions of (1.5) and (1.7) for the δ_ℓ. Also choose q_0 to force values for $\eta_\gamma(r^*)$ for any $\gamma \in \bigcup_{\ell<\theta}(\mathrm{dom}(p_{\delta_\ell}(0)) - \{\delta_\ell\})$ so that (1.5) and (1.7) hold for $p_{\delta_\ell}(\mu) \cup p_{\delta_{\ell'}}(\mu)$ for any $\ell, \ell' < \theta$ and any $\mu \in C$. We claim that

> (IV.1) There is a subset W of $\{0, ..., \theta - 1\}$ of size at least $\theta \cdot N^{-d}$ and a condition $q \in P_{\omega_2}$ which extends p^{**} and p_{δ_ℓ} for every $\ell \in W$ and satisfies $q(0) = q_0$

Assuming (IV.1), let us deduce a contradiction, which will prove that p^* forces ($\bigtriangledown$). Work in a generic extension $V[G']$ such that $q \in G'$. For $\ell_1 \neq \ell_2$ in W we have $\tau_{\ell_1} - \tau_{\ell_2} \in \langle x_\nu : \nu \in \Theta\rangle \cap Z_{m,t}[S]$ because

$$z_{\delta_{\ell_1}, r^*} - z_{\delta_{\ell_2}, r^*} = \tau_{\ell_1} - \tau_{\ell_2} + 2^t a$$

for some $a \in A$ and, letting $e = \sum_{n=m}^{r^*-1} g(n)$,

$$2^e(z_{\delta_{\ell_1}, r^*} - z_{\delta_{\ell_2}, r^*}) = z_{\delta_{\ell_1}, m} - z_{\delta_{\ell_2}, m} \in Z_m[S]$$

so since $Z_m[S]$ is pure-closed, $\tau_{\ell_1} - \tau_{\ell_2} + 2^t a \in Z_m[S]$, and hence $\tau_{\ell_1} - \tau_{\ell_2} \in Z_{m,t}[S]$. Therefore $(\langle x_\nu : \nu \in \Theta\rangle + Z_{m,t}[S])/Z_{m,t}[S]$ has cardinality at most

$$2^{|\Theta|t}/|W| = \theta/|W| \leq N^d$$

which is a contradiction of the choice of p^{**}.

In order to prove (IV.1) we define inductively, for $1 \leq n \leq d+1$, a condition $q_n \in P_{\mu_n}$ (where μ_n is as in the enumeration of C for $n \leq d$ and $\mu_{d+1} = \omega_2$) such that $q_n \geq p^{**} \restriction \mu_n$ and for $n' < n$, $q_n \restriction \mu_{n'} \geq q_{n'}$. We also define a subset W_n of W of size at least $\theta \cdot N^{-(n-1)}$ such that for each $\ell \in W_n$, $q_n \geq p_{\delta_\ell} \restriction \mu_n$. (So in the end we let $q = q_{d+1}$ and $W = W_{d+1}$.)

To begin, let $W_1 = \{0, ..., \theta - 1\}$ and let q_1 be any common extension of q_0 and the $p_{\delta_\ell} \restriction \mu_1$. (There is no problem finding such an extension.) Suppose now that q_n and W_n have been defined for some $n \geq 1$. Choose $\tilde{q}_n \geq q_n$ in P_{μ_n} such that $\tilde{q}_n$ decides for all $\ell \in W_n$ the value of $\psi_{\mu_n}(w_{\gamma,k})$ for all $\gamma \in \mathrm{dom}(p_{\delta_\ell(0)})$ and all $k \leq r^*$. For each $\ell \in W_n$ fix $s_\ell \in V$ such that $s_\ell \in Q_{\mu_n}$ extends $p_{\delta_\ell}(\mu_n)$ and satisfies $\mathrm{num}(s_\ell, \delta_\ell) = r^* + 1$, $s_\ell(x_{\alpha_{-1}}) < 2^M$, and $s_\ell(x_{\zeta_k}) < 2^M$ for all $k < r^*$. (This is possible by the proof of Proposition 1.3 since we only need to find solutions to the equations (1.2) modulo 2^Msince $g^{p_{\delta_\ell}(0)}(\delta_\ell, k) \leq M$ for $k \leq r^*$.)

Define an equivalence relation $\equiv_n$ on W_n by: $\ell_1 \equiv_n \ell_2$ iff $s_{\ell_1} \cup s_{\ell_2}$ is a function. By choice of M and N, there is an equivalence class, W_{n+1}, of size at least $|W_n|/N$. For $\ell \in W_{n+1}$ we can define a common extension $q_{n+1}(\mu_n)$ of $p^{**}(\mu_n)$ and the $p_{\delta_\ell}(\mu_n)$ and let $q_{n+1} \restriction \mu_n = \tilde{q}_n$. This completes the inductive construction. ∎

For any abelian group H, let $\nu(H)$ be the Chase radical of H: the intersection of the kernels of all homomorphisms of H into an $\aleph_1$-free group (cf. [2]). Then $H/\nu(H)$ is $\aleph_1$-free ([2, Prop. 1.2], [4, p. 290]). Let $\mathrm{cl}(Z_m[S])$ be defined by: $\mathrm{cl}(Z_m[S])/Z_m[S] = \nu(A/Z_m[S])$, so in particular $A/\,\mathrm{cl}(Z_m[S])$ is $\aleph_1$-free. Notice also that every homomorphism from A to $\mathbb{Z}$ is determined on $\mathrm{cl}(Z_m[S])$ by its values on $\{z_{\delta,m} + K : \delta \in S\}$.

Corollary 2.2. *In $V[G]$, for all $m \in \omega$ and stationary $S \subseteq E$, $A/\,\mathrm{cl}(Z_m[S])$ is a finite rank free group.*

Proof. If not, then since $A/\,\mathrm{cl}(Z_m[S])$ is $\aleph_1$-free, it contains a free pure subgroup of countably infinite rank. Let $\{a_n + \mathrm{cl}(Z_m[S]) : n \in \omega\}$ be a basis of such a subgroup. For any $n \neq m$, $a_n + Z_{m,1}[S] \neq a_m + Z_{m,1}[S]$ since 2 does not divide $a_n - a_m$ mod $Z_m[S]$ (or even mod $\mathrm{cl}(Z_m[S])$). Therefore $\{a_n + Z_{m,1}[S] : n \in \omega\}$ is an infinite subset of $A/Z_{m,1}[S]$, which contradicts Proposition 2.1. ∎

For the next Corollary we will need the following:

Lemma 2.3. *The Chase radical, $\nu(H)$, of a torsion-free group H is absolute for generic extensions.*

Proof. We give an absolute construction of $\nu(H)$ using the fact that a torsion-free group is $\aleph_1$-free if and only if every finite rank subgroup is finitely-generated (cf. [5, Thm. 19.1]), that is, if and only if the pure closure of every finitely-generated subgroup is finitely-generated. For any group H', let $\mu(H')$ be the sum of all finite rank subgroups G of H' which are not free but are such that every subgroup of G of smaller rank is free; it is easy to see that for such G, $\nu(G) = G$ and hence $\mu(H') \subseteq \nu(H')$. Moreover, the definition of $\mu(H')$ is absolute. Now define $\nu_\beta(H)$ by induction: $\nu_0(H) = 0$, $\nu_{\alpha+1}(H)/\nu_\alpha(H) = \mu(H/\nu_\alpha(H))$,and for limit ordinals β, $\nu_\beta(H) = \cup_{\alpha<\beta}\nu_\alpha(H)$. It follows by induction that $\nu_\alpha(H) \subseteq \nu(H)$ for all $\alpha \leq \omega_1$. We claim that $\nu(H) = \nu_{\omega_1}(H)$; it suffices to prove that $H/\nu_{\omega_1}(H)$ is $\aleph_1$-free. If not, then there is a finite rank subgroup of $H/\nu_{\omega_1}(H)$ which is not finitely-generated. We can choose one, G, of minimal rank, so all of its subgroups of smaller rank are free; say G is the pure closure of $\{a_1+\nu_{\omega_1}(H), ..., a_n+\nu_{\omega_1}(H)\}$; but then for some $\alpha < \omega_1$, the pure closure of $\{a_1+\nu_\alpha(H), ..., a_n+\nu_\alpha(H)\}$ is not free, but still has the property that every subgroup of smaller rank is free; hence $\{a_1, ..., a_n\} \subseteq \nu_{\alpha+1}(H)$, which is a contradiction. ∎

Corollary 2.4. *If $h \in \mathrm{Hom}(A, \mathbb{Z})^{V[G]}$ and for some $i \in \omega_2$, $m \in \omega$ and some stationary set $S \in V[G_i]$, the sequence $(h(z_{\delta,m} + K) : \delta \in S)$ belongs to $V[G_i]$, then h belongs to $V[G_i]$.*

Proof. Suppose h, S and m are as in the hypotheses. First we claim that $h \restriction Z_m[S]$ belongs to $V[G_i]$. Indeed we can define $h \restriction Z_m[S]$ in $V[G_i]$ as follows: $h(a) = k$ if for some $n \neq 0$, na belongs to the subgroup generated by $\{z_{\delta,m} + K : \delta \in S\}$ and $h(na) = nk$; and otherwise $h(a) = \xi$ for some fixed $\xi \notin \mathbb{Z}$. In fact, the second case never occurs because $h(a)$ is defined in $V[G]$. Next we claim that $h \restriction \mathrm{cl}(Z_m[S])$ belongs to $V[G_i]$. The proof is similar in principle, using the inductive construction of $\mathrm{cl}(Z_m[S])$ given by the proof of Lemma 2.3. But then by Corollary 2.2, h is determined by only finitely many more values, so also h belongs to $V[G_i]$. ∎

For the next corollary we introduce some notation that will be used in section 4. Let $\varphi_i \in V[G_{i+1}]$ denote the generic function for Q_i; thus φ_i is a homomorphism: $F \to \mathbb{Z}$

extending $\psi_i : K \to \mathbb{Z}$, where ψ_i is the interpretation in $V[G_i]$ of the name $\dot{\psi}_i$. The canonical map $\mathrm{Hom}(K, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z})$ sends ψ_i to a short exact sequence

$$\mathcal{E}_i : \ 0 \to \mathbb{Z} \xrightarrow{\iota} B_i \xrightarrow{\pi} A \to 0$$

and there is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & K & \xrightarrow{\iota'} & F & \longrightarrow & A & \to & 0 \\ & & \downarrow^{\psi_i} & & \downarrow^{\sigma_i} & & \downarrow^{1_A} & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{\iota} & B_i & \xrightarrow{\pi} & A & \to & 0 \end{array}$$

where ι and ι' are inclusion maps. Moreover, for all $z \in F$, $(\pi \circ \sigma_i)(z) = z + K \in A$. Then φ_i gives rise to a splitting $\rho_i \in \mathrm{Hom}(B_i, \mathbb{Z})^{V[G_{i+1}]}$ defined by $\rho_i(\sigma_i(z)) = \varphi_i(z)$. Thus $\rho_i \circ \iota = 1_{\mathbb{Z}}$ (the identity on $\mathbb{Z}$) and also $\rho_i \circ \sigma_i = \varphi_i$.

Corollary 2.5. *If $f \in \mathrm{Hom}(B_i, \mathbb{Z})^{V[G]}$ and for some $m \in \omega$ and some stationary set $S \in V[G_i]$, the sequence $(f(\sigma_i(z_{\delta,m})) : \delta \in S)$ belongs to $V[G_i]$, then f belongs to $V[G_i]$.*

Proof. If Z' is defined to be the pure subgroup of B_i generated by $\{\sigma_i(z_{\delta,m}) : \delta \in S\} \cup \{\iota(1)\}$ and C' is such that $\nu(B_i/Z') = C'/Z'$, then π induces an isomorphism of $C'/\mathrm{rge}(\iota)$ with $\mathrm{cl}(Z_m[S])$. Hence $B_i/C' \cong A/\ \mathrm{cl}(Z_m[S])$ is finite rank free; therefore, arguing as in Corollary 2.4, f belongs to $V[G_i]$. ∎

3. Proof of (II)

We divide the proof of (II) into three cases according to the cofinality of β. The case of cofinality ω_2 (i.e., $\beta = \omega_2$) is trivial since any function from A (which has cardinality $\aleph_1$) to $\mathbb{Z}$ must belong to $V[G_i]$ for some $i < \beta$.

Let $\dot{h}$ be a P_β-name and $p \in G_\beta$ such that

$$p \Vdash_{P_\beta} \dot{h} \in \mathrm{Hom}(A, \mathbb{Z}).$$

Then for each $\delta \in E$ there is $p_\delta \in G_\beta$ and $k_\delta \in \mathbb{Z}$ such that $p_\delta \geq p$ and $p_\delta \Vdash_{P_\beta} \dot{h}(z_{\delta,0} + K) = k_\delta$.

Suppose that the cofinality of β is ω, and fix an increasing sequence $(\beta_n : n \in \omega)$ whose sup is β. Then there is $n \in \omega$ and a stationary subset S_1 of E, belonging to $V[G_\beta]$, such that for $\delta \in S_1$, $p_\delta \in G_{\beta_n}$. Without loss of generality there is $p^* \in G_{\beta n}$ such that p^* forces

$$S = \{\delta \in E : \exists p_\delta \in \dot{G}_{\beta_n} \text{ and } k_\delta \in \mathbb{Z} \text{ s.t. } p_\delta \Vdash_{P_\beta} \dot{h}(z_{\delta,0} + K) = k_\delta\} \text{ is stationary.}$$

Then $(h(z_{\delta,0} + K) : \delta \in S)$ belongs to $V[G_{\beta_n}]$, so $h \in \mathrm{Hom}(A, \mathbb{Z})^{V[G_{\beta n}]}$, by Corollary 2.4.

The final, and hardest, case is when the cofinality of β is ω_1. Fix an increasing continuous sequence $(\beta_\nu : \nu < \omega_1)$ whose sup is β. Then there is $\nu \in \omega_1$ and a stationary subset S of ω_1 such that for $\delta \in S$, $p_\delta \upharpoonright \beta_\delta \in G_{\beta_\nu}$.

For any $t \geq 1$, $(Z_{0,t}[S \setminus \alpha] : \alpha < \omega_1)$ is a non-increasing sequence of groups. Since the groups $A/Z_{0,t}[S \backslash \alpha]$ are finite, it follows that there is a countable ordinal α_t such that for $\gamma, \alpha \geq \alpha_t$, $Z_{0,t}[S \setminus \alpha] = Z_{0,t}[S \setminus \gamma]$. Therefore there is a countable ordinal α_* and a countable subset Y of A such that for all $t \geq 1$, $\alpha_t \leq \alpha_*$, and Y contains representatives of all the elements of $A/Z_{0,t}[S \setminus \alpha_*]$. Increasing ν if necessary, we can assume that we can compute $h(y)$ in $V[G_{\beta_\nu}]$ for all $y \in Y$.

We claim that h belongs to $V[G_{\beta_\nu}]$. In pursuit of a contradiction, suppose that there are $a \in A$, conditions $q_1, q_2 \in P_\beta / G_{\beta_\nu}$ and integers $c_1 \neq c_2$ such that $q_\ell \Vdash_{P_\beta} \dot{h}(a) = c_\ell$ for $\ell = 1, 2$. Choose t sufficiently large such that 2^t does not divide $c_2 - c_1$ and choose $\mu < \omega_1$ such that $q_1, q_2 \in P_{\beta_\mu}$. For some $y \in Y$, $a - y \in Z_{0,t}[S \setminus \alpha_*] = Z_{0,t}[S \setminus \beta_\mu]$. Thus

$$a - y = z + 2^t a'$$

for some $a' \in A$ and z in the pure closure of the subgroup generated by $\{z_{\delta_j,0} : j = 1, ..., n\}$ for some $\delta_1, ..., \delta_n \in S \setminus \beta_\mu$. For $\ell = 1, 2$ there is an upper-bound $r_\ell \in P_\beta$ of $\{q_\ell, p_{\delta_1}, ..., p_{\delta_n}\}$. Then r_1 and r_2 force the same value, b, to $h(z)$ (because they are both $\geq p_{\delta_j}$ for $j = 1, ..., n$) and the same value, k, to $h(y)$ (because it is determined in $V[G_{\beta_\nu}]$). Therefore

$$r_\ell \Vdash 2^t \text{ divides } c_\ell - k - b.$$

So for $\ell = 1, 2$, the integer $c_\ell - k - b$ is divisible by 2^t. But this contradicts the choice of t.

4. Proof of (III)

We continue with the notation from the end of section 2; so $\mathcal{E}_i \in V[G_i]$. Suppose that $\mathcal{E}_i$ represents a torsion element of $\mathrm{Ext}(A, \mathbb{Z})$, of order $e \geq 1$, that is, there is a homomorphism $g_i : B_i \to \mathbb{Z}$ such that $g_i \restriction \mathbb{Z} = e1_\mathbb{Z}$, or more precisely, $g_i \circ \iota = e1_\mathbb{Z}$. (We consider the zero element to be torsion of order 1.) Then $e\rho_i - g_i$ is a homomorphism from B_i to $\mathbb{Z}$ which is identically 0 on $\mathbb{Z}$, so it induces a homomorphism $\theta_i \in \mathrm{Hom}(A, \mathbb{Z})^{V[G_{i+1}]}$ (that is, $\theta_i \circ \pi = e\rho_i - g_i$) which is a new element of A^* — that is, it is not in $V[G_i]$. To prove (III) it will suffice to prove that if there is an element h of A^* which is in $V[G_{i+1}]$ but not in $V[G_i]$, then $\mathcal{E}_i$ is torsion, and in that case h is an integral multiple of θ_i modulo $(A^*)^{V[G_i]}$.

Given such an h, let $h' = h \circ \pi : B_i \to \mathbb{Z}$. Clearly $h' \in V[G_{i+1}] - V[G_i]$. We claim that:

(III.1) For some integer c, $h' - c\rho_i$ belongs to $V[G_i]$.

Let us see first why this Claim implies the desired conclusion. Note that $c \neq 0$ since h' does not belong to $V[G_i]$. Since $(h' - c\rho_i) \restriction \mathbb{Z} = -c1_\mathbb{Z}$, we conclude that in $V[G_i]$, $\mathcal{E}_i$ is torsion, of order e dividing $-c$; let $g_i \in \mathrm{Hom}(B_i, \mathbb{Z})^{V[G_i]}$ such that $g_i \restriction \mathbb{Z} = e1_\mathbb{Z}$. Let θ_i be induced by $e\rho_i - g_i$, as above. Say $c = ne$; then $h' - c\rho_i + ng_i$ belongs to $V[G_i]$ and is identically 0 on $\mathbb{Z}$ so it induces a homomorphism $f \in \mathrm{Hom}(A, \mathbb{Z})^{V[G_i]}$. By composing both sides with π one sees that $h = n\theta_i + f$.

We shall now work on the proof of (III.1). Let F' be the subgroup of F generated by $\{x_\nu : \nu < \omega_1\}$. We work in $V[G_i]$. For any countable ordinal $\alpha \in \omega_1 - E$, define

$$Q_{i,\alpha} = \{q \in Q_i : z_{\delta,n} \in \mathrm{dom}(q) \Rightarrow \delta < \alpha \text{ and } x_\nu \in \mathrm{dom}(q) \Rightarrow \nu < \alpha\}.$$

Then $Q_{i,\alpha}$ is a complete subforcing of Q_i. In particular,

$$V[G_{i+1}] = V[G_i][G_{i+1,\alpha}][H_{i+1,\alpha}]$$

where $G_{i+1,\alpha}$ is $Q_{i,\alpha}$-generic over $V[G_i]$ and $H_{i+1,\alpha}$ is $Q_i/G_{i+1,\alpha}$-generic over $V[G_i][G_{i+1,\alpha}]$. We claim:

(III.2) There is a countable ordinal $\alpha \in \omega_1 - E$ such that in $V[G_i][G_{i+1,\alpha}]$ there is an assignment to every $y \in F'$ of a function $\xi_y : \mathbb{Z} \to \mathbb{Z}$ such that for all $y \in F'$ $\Vdash_{Q_i/G_{i+1,\alpha}} \dot{h}(y + K) = \xi_y(\dot{\varphi}_i(y))$.

Let us see first why this implies (III.1). First we assert that the following consequence of (III.2) holds in $V[G_i][G_{i+1,\alpha}]$:

(III.2.1) There is an integer c such that for every $k \in \mathbb{Z}$, and every $\beta \geq \alpha$, $\xi_{x_\beta}(k) - \xi_{x_\beta}(0) = kc$.

To see this, let $\beta, \gamma \geq \alpha$ with $\beta \neq \gamma$, and let $k \in \mathbb{Z}$. By the proof of Proposition 1.3, there are conditions $q_1, q_2 \in Q_i/G_{i+1,\alpha}$ such that

$$q_1 \Vdash \dot{\varphi}_i(x_\beta) = k \wedge \dot{\varphi}_i(x_\gamma) = 0$$

and

$$q_2 \Vdash \dot{\varphi}_i(x_\beta) = 0 \wedge \dot{\varphi}_i(x_\gamma) = k.$$

Let $y = x_\beta + x_\gamma$. By (III.2) and the fact that $\dot{h}$ and φ_i are homomorphisms,

$$q_1 \Vdash \xi_{x_\beta}(k) + \xi_{x_\gamma}(0) = \xi_y(k)$$

which implies that $\xi_{x_\beta}(k) + \xi_{x_\gamma}(0) = \xi_y(k)$ holds in $V[G_i][G_{i+1,\alpha}]$. Similarly, reasoning with q_2, we can conclude that $\xi_{x_\gamma}(k) + \xi_{x_\beta}(0) = \xi_y(k)$ holds in $V[G_i][G_{i+1,\alpha}]$. Thus $\xi_{x_\beta}(k) - \xi_{x_\beta}(0) = \xi_{x_\gamma}(k) - \xi_{x_\gamma}(0)$ in $V[G_i][G_{i+1,\alpha}]$; we denote this value by $\Xi(k)$. If we can prove that for all k, $\Xi(k) = k\Xi(1)$, then we can let $c = \Xi(1)$. Again, let $\beta, \gamma \geq \alpha$ with $\beta \neq \gamma$ and this time let $y = kx_\beta + x_\gamma$. Using conditions $q_3 \Vdash \dot{\varphi}_i(x_\beta) = 1 \wedge \dot{\varphi}_i(x_\gamma) = 0$ and $q_2 \Vdash \dot{\varphi}_i(x_\beta) = 0 \wedge \dot{\varphi}_i(x_\gamma) = k$ we conclude that

$$k\xi_{x_\beta}(1) + \xi_{x_\gamma}(0) = k\xi_{x_\beta}(0) + \xi_{x_\gamma}(k)$$

from which it follows that $k\Xi(1) = \Xi(k)$. This proves (III.2.1)

Now work in $V[G_{i+1}]$; we have

$$h(x_\beta + K) = \xi_{x_\beta}(\varphi_i(x_\beta)) = c\varphi_i(x_\beta) + \xi_{x_\beta}(0)$$

for $\beta \geq \alpha$. Since $((h' - c\rho_i) \circ \sigma_i)(x) = h(x + K) - c\varphi_i(x)$ for $x \in F'$, it follows that $h' - c\rho_i \restriction \{\sigma_i(x_\beta) : \beta \geq \alpha\}$ belongs to $V[G_i][G_{i+1,\alpha}]$. Moreover, for $\beta < \alpha$, $\varphi_i(x_\beta)$ is determined in $V[G_i][G_{i+1,\alpha}]$, and hence so are $h(x_\beta + K) = \xi_{x_\beta}(\varphi_i(x_\beta))$ and $(h' - c\rho_i)(\sigma_i(x_\beta))$. Therefore $h' - c\rho_i$ belongs to $V[G_i][G_{i+1,\alpha}]$ (since it is determined by its values on $\{\sigma_i(x_\beta) : \beta \in \omega_1\} \cup \{\iota(1)\}$). Let $f = h' - c\rho_i$. For each $\delta \in E$, there exist $p_\delta \in G_{i+1,\alpha}$ and $k_\delta \in \mathbb{Z}$ such that

$$p_\delta \Vdash_{Q_i} \dot{f}(\sigma_i(z_{\delta,m})) = k_\delta.$$

Since $Q_{i,\alpha}$ and $\mathbb{Z}$ are countable, there exist $p \in G_{i+1,\alpha}$, $k \in \mathbb{Z}$, and a stationary $S \in V[G_i]$ such that for $\delta \in S$, $p \Vdash_{Q_i} \dot{f}(\sigma_i(z_{\delta,m})) = k$. Then the (constant) sequence $(f(\sigma_i(z_{\delta,m})) : \delta \in S)$ belongs to $V[G_i]$, so by Corollary 2.5, f belongs to $V[G_i]$.

So it remains to prove (III.2). Work in $V[G_i]$. Let

$$D_{i,\alpha} = \{q \in Q_i : \forall\delta \in (\mathrm{cont}(q) - \alpha)\forall n \in \omega[(\eta_\delta(n) < \alpha) \Rightarrow x_{\eta_\delta(n)} \in \mathrm{dom}(q)]\}.$$

Then $D_{i,\alpha}$ is a dense subset of Q_i. We claim that it is true in $V[G_i]$ that:

(III.3) there is a countable ordinal $\alpha \in \omega_1 - E$ such that for every $y \in F'$, $t, c_1, c_2 \in \mathbb{Z}$, and $q_1, q_2 \in D_{i,\alpha}$ with $q_1 \restriction \alpha = q_2 \restriction \alpha$, if

$$q_\ell \Vdash_{Q_i} \dot{\varphi}_i(y) = t \wedge \dot{h}(y + K) = c_\ell$$

for $\ell = 1, 2$, then $c_1 = c_2$.

Clearly this implies (III.2). Indeed, we define $\xi_y(t)$ to be c if there is a $q \in D_{i,\alpha}$ such that $q \restriction \alpha \in G_{i+1,\alpha}$ and $q \Vdash_{Q_i/G_{i+1,\alpha}} \dot{\varphi}_i(y) = t \wedge \dot{h}(y+K) = c$ and otherwise $\xi_y(t) = 0$. By (III.3), $\xi_y(t)$ is well-defined.

PROOF OF (III.3). The proof is by contradiction and uses some of the methods of the proof of Proposition 2.1. So suppose that for every $\alpha \in \omega_1 - E$ there are $y^\alpha \in F'$, $t^\alpha, c_1^\alpha, c_2^\alpha \in \mathbb{Z}$, and $q_1^\alpha, q_2^\alpha \in D_{i,\alpha}$ such that $q_1^\alpha \restriction \alpha = q_2^\alpha \restriction \alpha$ and $q_\ell^\alpha \Vdash_{Q_i} \dot{\varphi}_i(y^\alpha) = t^\alpha \wedge \dot{h}''(y) = c_\ell^\alpha$ where $c_1^\alpha \neq c_2^\alpha$ for $\ell = 1, 2$. Then, by Fodor's Lemma and counting, there is a $p_0 \in G_i$, $t, c_1, c_2 \in \mathbb{Z}$, $\tilde{q} \in V$ and names $\dot{S}$, $\dot{q}_\ell^\alpha$, $\dot{y}^\alpha$ such that

$$p_0 \Vdash_{P_i} \dot{S} \text{ is a stationary subset of } \omega_1 - E \text{ s.t. for all } \alpha \in \dot{S}, \\ \dot{t}^\alpha = t, \dot{c}_1^\alpha = c_1, \dot{c}_2^\alpha = c_2 \text{ and } \dot{q}_1^\alpha \restriction \alpha = \dot{q}_2^\alpha \restriction \alpha = \tilde{q}$$

and moreover such that p_0 forces the names to be a counterexample to (III.3), as above.

There is a stationary subset $S' \subseteq \omega_1 - E$ such that for every $\alpha \in S'$ there is a condition $p_\alpha \geq p_0$ in P_i which forces $\alpha \in \dot{S}$ and forces values (elements of V) to $\dot{q}_\ell^\alpha$ and to $\dot{y}^\alpha$. Moreover, we can suppose that the $p_\alpha \cup \{(i, q_\ell^\alpha)\} \in P_{i+1}$ ($\ell = 1, 2$) are as in (†) [cf. proof of Proposition 2.1] and that $\{p_\alpha : \alpha \in S'\}$ is as in (††) [with α in place of δ, but since $\alpha \notin E$, the last sentence does not apply]. Let p^* be the heart of $\{p_\alpha : \alpha \in S'\}$. We can also assume that $\{q_\ell^\alpha : \alpha \in S'\}$ forms a Δ-system with heart $\tilde{q}$ (for $\ell = 1, 2$).

For each $\delta \in E$, there is $\hat{p}_\delta \in P_{i+1}$ and $k_\delta \in \mathbb{Z}$ such that $\hat{p}_\delta \restriction i \geq p^*$, $\hat{p}_\delta(i) \geq \tilde{q}$ and $\hat{p}_\delta \Vdash \dot{h}(z_{\delta,0}) = k_\delta$. There is a stationary $\hat{S} \subseteq E$ such that $\{\hat{p}_\delta : \delta \in \hat{S}\}$ satisfies (†) and (††); in particular, $\hat{r} = r_\delta^{\hat{p}_\delta(0)}$ for $\delta \in \hat{S}$ and $\eta_\delta^{\hat{p}_\delta(0)}(n)$, $g^{\hat{p}_\delta(0)}(\delta, n) = g(n)$ and $u^{\hat{p}_\delta(0)}(\delta, n)$ are independent of δ for each $n < \hat{r}$. Moreover we can assume that there is $k \in \mathbb{Z}$ such that $k_\delta = k$ for all $\delta \in \hat{S}$. Let $\hat{p}^*$ be the heart of $\{\hat{p}_\delta : \delta \in \hat{S}\}$ (so $\hat{p}^* \geq p^* \cup \{(i, \tilde{q})\}$).

Choose m such that 2^m does not divide $c_1 - c_2$. Let $M \geq \max(\{g(n) : n < \hat{r}\} \cup \{m\})$ and let

$$N = 2^{1+(\hat{r}+1)M(d+1)}$$

(where d is the size of the domain of $\hat{p}^* - \{0\}$). Choose

$$\alpha_0 < \ldots < \alpha_{N-1} < \gamma < \delta_0 < \ldots < \delta_{N-1}$$

where $\alpha_j \in S'$, $\delta_j \in \hat{S}$, every ordinal which occurs in $\hat{p}^*$ is $< \alpha_0$, and for all $j \leq N-1$ every ordinal which occurs in p_{α_j} or in $q_\ell^{\alpha_j}$ ($\ell = 1, 2$) is less than α_{j+1} (where α_N is taken to be γ); and for all $j < N-1$, every ordinal which occurs in p_{δ_j} is less than δ_{j+1}. Then there is a condition $q_0 \in Q_0$ which extends $\hat{p}^*(0)$ and each $p_{\alpha_j}(0)$ and $p_{\delta_j}(0)$ such that q_0 forces for all $j < N$:

$$\eta_{\delta_j}(\hat{r}) = \gamma;\ \eta_{\delta_j}(\hat{r}+1) = \delta_{j-1} + 1;\ u(\delta_j, \hat{r}) = y^{\alpha_j};\ \text{and } g(\delta_j, \hat{r}) = 2^m$$

where $\delta_{-1} = \gamma + 1$.

As in the proof of Proposition 2.1, there is a condition $q' \in P_i$ and a subset W' of N of size $\geq 2^{1+(\hat{r}+1)M}$ such that $q'(0) = q_0$, $q' \geq p_{\alpha_j}$ for all $j \leq N-1$, and $q' \geq p_{\delta_j} \restriction i$ for $j \in W'$. Repeating the argument one more time and using the facts that $q_1^{\alpha_j}$ and $q_2^{\alpha_j}$ force the same value to $\varphi(y^{\alpha_j})$ and that $q_1^{\alpha_j} \restriction \alpha_j = q_2^{\alpha_j} \restriction \alpha_j = \tilde{q}$, there is a subset $W = \{j, j_o\}$ of W' such that for any function $f : W \to \{1, 2\}$ there is a condition $q_* \in P_{i+1}$ such that $q_* \restriction i = q'$ and $q_*(i)$ is an upper bound of $\{p_{\delta_j}(i), p_{\delta_{j_o}}(i)\} \cup \{q_1^{\alpha_j}, q_2^{\alpha_{j_o}}\}$. In a generic extension $V[G']$ where $q_* \in G'$ we have (since $h(z_{\delta_j,0} + K) = h(z_{\delta_{j_o},0} + K) = k$ and $g(\delta, n)$ and $u(\delta, n)$ are independent of $\delta \in \hat{S}$ for $n < \hat{r}$) that 2^m divides

$$h(u(\delta_j, \hat{r})) - h(u(\delta_{j_o}, \hat{r})) = h(y^{\alpha_j} + K) - h(y^{\alpha_{j_o}} + K) = c_1 - c_2$$

which is a contradiction of the choice of m. This proves (III.3) and thus finally completes the proof of Theorem 1.5.

5. Theorem 0.2

To prove Theorem 0.2 we use a variant of the iterated forcing that is described in section 1. Let Q_0 and Q_ψ be as defined there. We shall use a finite support iteration $P' = \left\langle P'_i, \dot{Q}_i : 0 \leq i < \omega_2 \right\rangle$; the $\dot{Q}_i$ are defined inductively. We consider an enumeration, as before, of names $\{\dot{\psi}_i : i < \omega_2\}$ for functions from K to $\mathbb{Z}$. In V^{P_i} we define

$$\dot{Q}_i = \begin{cases} \{0\} & \text{if the s.e.s } \mathcal{E}_i \text{ is torsion} \\ Q_{\dot{\psi}_i} & \text{otherwise} \end{cases}$$

We claim that if G is P'-generic, then in $V[G]$ (i) $\mathrm{Ext}(A, \mathbb{Z})$ is torsion and (ii) $\mathrm{Hom}(A, \mathbb{Z}) = 0$.

To see why (i) holds, consider $\psi \in \mathrm{Hom}(K, \mathbb{Z})$. For some $i \in \omega_2$, $\dot{\psi}_i$ is a name for ψ. In $V[G_i]$ either ψ represents a torsion element of $\mathrm{Ext}(A, \mathbb{Z})$ or else, by construction, in $V[G_{i+1}]$ $\psi = \varphi | K$ for some $\varphi \in \mathrm{Hom}(F, \mathbb{Z})$, in which case ψ represents the zero element of $\mathrm{Ext}(A, \mathbb{Z})$.

To prove (ii), it suffices to show for all $i \in \omega_2$ that if $h \in \mathrm{Hom}(A, \mathbb{Z})^{V[G_{i+1}]}$, then $h \in \mathrm{Hom}(A, \mathbb{Z})^{V[G_i]}$. If not, then $\dot{Q}_i \neq \{0\}$; but then by the arguments in section 4 it follows that $\mathcal{E}_i$ is torsion, so $\dot{Q}_i = \{0\}$, a contradiction.

6. co-Moore spaces

Following [7] we call a topological space X a *co-Moore space of type* (G, n), where $n \geq 1$, if its reduced integral cohomology groups satisfy

$$\tilde{H}^i(X, \mathbb{Z}) = \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geq 2$, application of the Universal Coefficient Theorem shows that

(♣) there exist B_1 and B_2 such that $G \cong \mathrm{Hom}(B_1, \mathbb{Z}) \oplus \mathrm{Ext}(B_2, \mathbb{Z})$ where $\mathrm{Ext}(B_1, \mathbb{Z}) = 0 = \mathrm{Hom}(B_2, \mathbb{Z})$.

Conversely, if G satisfies (♣), then there is a co-Moore space of type (G, n) for any $n \geq 2$ (cf. [7, Thm. 5], [6]). A sufficient condition for G to be of the form (♣) is that $G = D \oplus C$ where C is compact and D is isomorphic to a direct product of copies of $\mathbb{Z}$ ([7, Thm. 5]). In a model of ZFC where every W-group is free, this condition is necessary (cf. [7, Thm. 3(a)] and [3, Thm. 2.20]); in particular the (torsion-free) rank of C is of the form 2^μ for some infinite cardinal μ. However, as a consequence of our proofs we have:

Corollary 6.1. *It is consistent with ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ that there is a group A of cardinality $\aleph_1$ such that* $\mathrm{Hom}(A, \mathbb{Z}) = 0$ *but* $\mathrm{Ext}(A, \mathbb{Z})$ *does not admit a compact topology.*

Corollary 6.2. *It is consistent with ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ that for any $n \geq 2$ there is a co-Moore space of type (F, n) where F is the free abelian group of rank $\aleph_2$.*

Corollary 6.3. *It is consistent with ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ that for any $n \geq 2$ there is a co-Moore space of type (C, n) for some uncountable torsion divisible group C.*

Compare Corollary 6.3 with [6, 2.5 and 2.6]. The conclusions of the corollaries are not provable in ZFC. Moreover, by an easy modification we can replace $\aleph_2$ in the corollaries by any regular cardinal greater than $\aleph_1$. (Note that by [1, Thm. 5.6], $\mathrm{Hom}(B_1, \mathbb{Z})$ cannot be the free group of rank $\aleph_1$ if $\mathrm{Ext}(B_1, \mathbb{Z}) = 0$.)

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