On full Souslin trees

Saharon Shelah*
Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, USA
and
Mathematics Department
University of Wisconsin – Madison
Madison, WI 53706, USA

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Abstract

In the present note we answer a question of Kunen (15.13 in [Mi91]) showing (in 1.7) that

it is consistent that there are full Souslin trees.

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0 Introduction

In the present paper we answer a combinatorial question of Kunen listed in Arnie Miller's Problem List. We force, e.g. for the first strongly inaccessible Mahlo cardinal λ, a full (see 1.1(2)) λ–Souslin tree and we remark that the existence of such trees follows from $\mathbf{V} = \mathbf{L}$ (if λ is Mahlo strongly inaccessible). This answers [Mi91, Problem 15.13].

Our notation is rather standard and compatible with those of classical textbooks on Set Theory. However, in forcing considerations, we keep the older tradition that

a stronger condition is the larger one.

We will keep the following conventions concerning use of symbols.

Notation 0.1 1. λ, μ will denote cardinal numbers and $\alpha, \beta, \gamma, \delta, \xi, \zeta$ will be used to denote ordinals.

2. Sequences (not necessarily finite) of ordinals are denoted by ν, η, ρ (with possible indexes).

3. The length of a sequence η is $\ell g(\eta)$.

4. For a sequence η and an ordinal $\alpha \leq \ell g(\eta)$, $\eta \restriction \alpha$ is the restriction of the sequence η to α (so $\ell g(\eta \restriction \alpha) = \alpha$). If a sequence ν is a proper initial segment of a sequence η then we write $\nu \vartriangleleft \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).

5. A tilde indicates that we are dealing with a name for an object in forcing extension (like $\underset{\sim}{x}$).

1 Full λ-Souslin trees

A subset T of ${}^{\alpha >}2$ is an α–tree whenever (α is a limit ordinal and) the following three conditions are satisfied:

- $\langle\rangle \in T$, if $\nu \vartriangleleft \eta \in T$ then $\nu \in T$,
- $\eta \in T$ implies $\eta^\frown\langle 0\rangle,\ \eta^\frown\langle 1\rangle \in T$, and
- for every $\eta \in T$ and $\beta < \alpha$ such that $\ell g(\eta) \leq \beta$ there is $\nu \in T$ such that $\eta \trianglelefteq \nu$ and $\ell g(\eta) = \beta$.

A λ–Souslin tree is a λ–tree $T \subseteq {}^{\lambda >}2$ in which every antichain is of size less than λ.

Definition 1.1 *1. For a tree $T \subseteq {}^{\alpha>}2$ and an ordinal $\beta \leq \alpha$ we let*

$$T_{[\beta]} \stackrel{\rm def}{=} T \cap {}^{\beta}2 \quad \textit{and} \quad T_{[<\beta]} \stackrel{\rm def}{=} T \cap {}^{\beta>}2.$$

If $\delta \leq \alpha$ is limit then we define

$$\lim_\delta T_{[<\delta]} \stackrel{\rm def}{=} \{\eta \in {}^{\delta}2 : (\forall \beta < \delta)(\eta{\restriction}\beta \in T)\}.$$

2. *An α–tree T is* full *if for every limit ordinal $\delta < \alpha$ the set $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ has at most one element.*

3. *An α–tree $T \subseteq {}^{\alpha>}2$ has* true height *α if for every $\eta \in T$ there is $\nu \in {}^{\alpha}2$ such that*

$$\eta \vartriangleleft \nu \quad \textit{and} \quad (\forall \beta < \alpha)(\nu{\restriction}\beta \in T).$$

We will show that the existence of full λ–Souslin trees is consistent assuming the cardinal λ satisfies the following hypothesis.

Hypothesis 1.2 **(a)** λ is strongly inaccessible (Mahlo) cardinal,

(b) $S \subseteq \{\mu < \lambda : \mu$ is a strongly inaccessible cardinal $\}$ is a stationary set,

(c) $S_0 \subseteq \lambda$ is a set of limit ordinals,

(d) for every cardinal $\mu \in S$, $\diamondsuit_{S_0 \cap \mu}$ holds true.

Further in this section we will assume that λ, S_0 and S are as above and we may forget to repeat these assumptions.

Let as recall that the diamond principle $\diamondsuit_{S_0\cap\mu}$ postulates the existence of a sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$ (called *a $\diamondsuit_{S_0\cap\mu}$–sequence*) such that $\nu_\delta \in {}^{\delta}2$ (for $\delta \in S_0 \cap \mu$) and

$$(\forall \nu \in {}^{\mu}2)[\text{ the set } \{\delta \in S_0 \cap \mu : \nu \restriction \delta = \nu_\delta\} \text{ is stationary in } \mu].$$

Now we introduce a forcing notion $\mathbb{Q}$ and its relative $\mathbb{Q}^*$ which will be used in our proof.

Definition 1.3 *1.* **A condition** *in $\mathbb{Q}$ is a tree $T \subseteq {}^{\alpha>}2$ of a true hight $\alpha = \alpha(T) < \lambda$ (see 1.1(3); so α is a limit ordinal) such that $\| \lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]} \| \leq 1$ for every limit ordinal $\delta < \alpha$,*

the order *on $\mathbb{Q}$ is defined by $T_1 \leq T_2$ if and only if*

$T_1 = T_2 \cap {}^{\alpha(T_1)>}2$ (so it is the end–extension order).

2. *For a condition $T \in \mathbb{Q}$ and a limit ordinal $\delta < \alpha(T)$, let $\eta_\delta(T)$ be the unique member of $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ if there is one, otherwise $\eta_\delta(T)$ is not defined.*

3. *Let $T \in \mathbb{Q}$. A function $f : T \longrightarrow \lim_{\alpha(T)}(T)$ is called* a witness for T *if $(\forall \eta \in T)(\eta \lhd f(\eta))$.*

4. **A condition** *in $\mathbb{Q}^*$ is a pair (T, f) such that $T \in \mathbb{Q}$ and $f : T \longrightarrow \lim_{\alpha(T)}(T)$ is a witness for T,*

   the order *on $\mathbb{Q}^*$ is defined by* $(T_1, f_1) \leq (T_2, f_2)$ *if and only if*

   $T_1 \leq_{\mathbb{Q}} T_2$ *and* $(\forall \eta \in T_1)(f_1(\eta) \trianglelefteq f_2(\eta))$.

Proposition 1.4 *1. If $(T_1, f_1) \in \mathbb{Q}^*$, $T_1 \leq_{\mathbb{Q}} T_2$ and*

($*$) *either $\eta_{\alpha(T_1)}(T_2)$ is not defined or it does not belong to* $\mathrm{rang}(f_1)$

then there is $f_2 : T_2 \longrightarrow \lim_{\alpha(T_2)}(T_2)$ such that $(T_1, f_1) \leq (T_2, f_2) \in \mathbb{Q}^*$.

2. *For every $T \in \mathbb{Q}$ there is a witness f for T.*

Proof Should be clear. ■

Proposition 1.5 *1. The forcing notion $\mathbb{Q}^*$ is $(< \lambda)$–complete, in fact any increasing chain of length $< \lambda$ has the least upper bound in $\mathbb{Q}^*$.*

2. *The forcing notion $\mathbb{Q}$ is strategically γ-complete for each $\gamma < \lambda$.*

3. *Forcing with $\mathbb{Q}$ adds no new sequences of length $< \lambda$. Since $\|\mathbb{Q}\| = \lambda$, forcing with $\mathbb{Q}$ preserves cardinal numbers, cofinalities and cardinal arithmetic.*

Proof 1) It is straightforward: suppose that $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is an increasing sequence of elements of $\mathbb{Q}^*$. Clearly we may assume that $\xi < \lambda$ is a limit ordinal and $\zeta_1 < \zeta_2 < \xi \Rightarrow \alpha(T_{\zeta_1}) < \alpha(T_{\zeta_2})$. Let $T_\xi = \bigcup\limits_{\zeta<\xi} T_\zeta$ and $\alpha = \sup\limits_{\zeta<\xi} \alpha(T_\zeta)$. Easily, the union is increasing and the T_ξ is a full α–tree. For $\eta \in T_\xi$ let $\zeta_0(\eta)$ be the first $\zeta < \xi$ such that $\eta \in T_\zeta$ and let $f_\xi(\eta) = \bigcup\{f_\zeta(\eta) : \zeta_0(\eta) \leq \zeta < \xi\}$. By the definition of the order on $\mathbb{Q}^*$ we get that the sequence $\langle f_\zeta(\eta) : \zeta_0(\eta) \leq \zeta < \xi \rangle$ is $\lhd$–increasing and hence $f_\xi(\eta) \in \lim_\alpha(T_\xi)$. Plainly, the function f_ξ witnesses that T_ξ has a true height α, and thus $(T_\xi, f_\xi) \in \mathbb{Q}^*$. It should be clear that (T_ξ, f_ξ) is the least upper bound of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$.

2) For our purpose it is enough to show that for each ordinal $\gamma < \lambda$ and a condition $T \in \mathbb{Q}$ the second player has a winning strategy in the following game $\mathcal{G}_\gamma(T, \mathbb{Q})$. (Also we can let Player I choose T_ξ for ξ odd.)

> The game lasts γ moves and during a play the players, called I and II, choose successively open dense subsets $\mathcal{D}_\xi$ of $\mathbb{Q}$ and conditions $T_\xi \in \mathbb{Q}$. At stage $\xi < \gamma$ of the game:
> Player I chooses an open dense subset $\mathcal{D}_\xi$ of $\mathbb{Q}$ and
> Player II answers playing a condition $T_\xi \in \mathbb{Q}$ such that
>
> $$T \leq_{\mathbb{Q}} T_\xi, \quad (\forall \zeta < \xi)(T_\zeta \leq_{\mathbb{Q}} T_\xi), \quad \text{and} \quad T_\xi \in \mathcal{D}_\xi.$$
>
> The second player wins if he has always legal moves during the play.

Let us describe the winning strategy for Player II. At each stage $\xi < \gamma$ of the game he plays a condition T_ξ and writes down on a side a function f_ξ such that $(T_\xi, f_\xi) \in \mathbb{Q}^*$. Moreover, he keeps an extra obligation that $(T_\zeta, f_\zeta) \leq_{\mathbb{Q}^*} (T_\xi, f_\xi)$ for each $\zeta < \xi < \gamma$.
So arriving to a non-limit stage of the game he takes the condition (T_ζ, f_ζ) he constructed before (or just (T, f), where f is a witness for T, if this is the first move; by 1.4(2) we can always find a witness). Then he chooses $T^*_\zeta \geq_{\mathbb{Q}} T_\zeta$ such that $\alpha(T^*_\zeta) = \alpha(T_\zeta) + \omega$ and $(T^*_\zeta)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$. Thus $\eta_{\alpha(T_\zeta)}(T^*_\zeta)$ is not defined. Now Player II takes $T_{\zeta+1} \geq_{\mathbb{Q}} T^*_\zeta$ from the open dense set $\mathcal{D}_{\zeta+1}$ played by his opponent at this stage. Clearly $\eta_{\alpha(T_\zeta)}(T_{\zeta+1})$ is not defined, so Player II may use 1.4(1) to choose $f_{\zeta+1}$ such that $(T_\zeta, f_\zeta) \leq_{\mathbb{Q}^*} (T_{\zeta+1}, f_{\zeta+1}) \in \mathbb{Q}^*$.
At a limit stage ξ of the game, the second player may take the least upper bound $(T'_\xi, f'_\xi) \in \mathbb{Q}^*$ of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ (exists by 1)) and then apply the procedure described above.
3) Follows from 2) above. ■

Definition 1.6 *Let $\underset{\sim}{\mathbf{T}}$ be the canonical $\mathbb{Q}$–name for a generic tree added by forcing with $\mathbb{Q}$:*

$$\Vdash_{\mathbb{Q}} \underset{\sim}{\mathbf{T}} = \bigcup \{T : T \in \underset{\sim}{G}_{\mathbb{Q}}\}.$$

It should be clear that $\underset{\sim}{\mathbf{T}}$ is (forced to be) a full λ–tree. The main point is to show that it is λ–Souslin and this is done in the following theorem.

Theorem 1.7 $\Vdash_{\mathbb{Q}}$ *"$\underset{\sim}{\mathbf{T}}$ is a λ–Souslin tree".*

PROOF Suppose that $\underset{\sim}{A}$ is a $\mathbb{Q}$–name such that

$$\Vdash_{\mathbb{Q}} \text{" } \underset{\sim}{A} \subseteq \underset{\sim}{\mathbf{T}} \text{ is an antichain ",}$$

and let T_0 be a condition in $\mathbb{Q}$. We will show that there are $\mu < \lambda$ and a condition $T^* \in \mathbb{Q}$ stronger than T_0 such that $T^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \underset{\sim}{\mathbf{T}}_{[<\mu]}$ " (and thus it forces that the size of $\underset{\sim}{A}$ is less than λ).

Let $\underset{\sim}{\mathbf{A}}$ be a $\mathbb{Q}$–name such that

$$\Vdash_{\mathbb{Q}} \text{" } \underset{\sim}{\mathbf{A}} = \{\eta \in \underset{\sim}{\mathbf{T}} : (\exists \nu \in \underset{\sim}{A})(\nu \trianglelefteq \eta) \text{ or } \neg(\exists \nu \in \underset{\sim}{A})(\eta \trianglelefteq \nu)\} \text{ ".}$$

Clearly, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{A}} \subseteq \underset{\sim}{\mathbf{T}}$ is dense open".
Let χ be a sufficiently large regular cardinal ($\beth_7(\lambda^+)^+$ is enough).

Claim 1.7.1 *There are $\mu \in S$ and $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that:*

(a) $\underset{\sim}{A}, \underset{\sim}{\mathbf{A}}, S, S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}$,

(b) $\|\mathfrak{B}\| = \mu$ *and* ${}^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$,

(c) $\mathfrak{B} \cap \lambda = \mu$.

Proof of the claim: First construct inductively an increasing continuous sequence $\langle \mathfrak{B}_\xi : \xi < \lambda \rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\underset{\sim}{A}, \underset{\sim}{\mathbf{A}}, S, S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}_0$ and for every $\xi < \lambda$

$$\|\mathfrak{B}_\xi\| = \mu_\xi < \lambda, \quad \mathfrak{B}_\xi \cap \lambda \in \lambda, \quad \text{and} \quad {}^{\mu_\xi \geq}\mathfrak{B}_\xi \subseteq \mathfrak{B}_{\xi+1}.$$

Note that for a club E of λ, for every $\mu \in S \cap E$ we have

$$\|\mathfrak{B}_\mu\| = \mu, \quad {}^{\mu>}\mathfrak{B}_\mu \subseteq \mathfrak{B}_\mu, \quad \text{and} \quad \mathfrak{B} \cap \lambda = \mu.$$

Let $\mu \in S$ and $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be given by 1.7.1. We know that $\diamondsuit_{S_0 \cap \mu}$ holds, so fix a $\diamondsuit_{S_0 \cap \mu}$–sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$.

Let

$$\begin{aligned} \underset{\sim}{\mathcal{I}} \stackrel{\text{def}}{=} \{T \in \mathbb{Q} : \ & T \text{ is incompatible (in } \mathbb{Q}) \text{ with } T_0 \quad \text{or:} \\ & T \geq T_0 \text{ and } T \text{ decides the value of } \underset{\sim}{\mathbf{A}} \cap {}^{\alpha(T)>}2 \text{ and} \\ & (\forall \eta \in T)(\exists \rho \in T)(\eta \trianglelefteq \rho \ \& \ T \Vdash_{\mathbb{Q}} \rho \in \underset{\sim}{\mathbf{A}})\}. \end{aligned}$$

Claim 1.7.2 *$\underset{\sim}{\mathcal{I}}$ is a dense subset of $\mathbb{Q}$.*

Proof of the claim: Should be clear (remember 1.5(2)).

Now we choose by induction on $\xi < \mu$ a continuous increasing sequence $\langle (T_\xi, f_\xi) : \xi < \mu \rangle \subseteq \mathbb{Q}^* \cap \mathfrak{B}$.

STEP: $i = 0$

T_0 is already chosen and it belongs to $\mathbb{Q} \cap \mathfrak{B}$. We take any f_0 such that $(T_0, f_0) \in \mathbb{Q}^* \cap \mathfrak{B}$ (exists by 1.4(2)).

STEP: limit ξ

Since ${}^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$, the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is in $\mathfrak{B}$. By 1.5(1) it has the least upper bound (T_ξ, f_ξ) (which belongs to $\mathfrak{B}$).

STEP: $\xi = \zeta + 1$

First we take (the unique) tree T^*_ξ of true height $\alpha(T^*_\xi) = \alpha(T_\zeta) + \omega$ such that $T^*_\xi \cap {}^{\alpha(T_\zeta)>}2 = T_\zeta$ and:

if $\alpha(T_\zeta) \in S_0$ and $\nu_{\alpha(T_\zeta)} \notin \mathrm{rang}(f_\zeta)$ then $(T^*_\xi)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta) \setminus \{\nu_{\alpha(T_\zeta)}\}$, otherwise $(T^*_\xi)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$.

Let $T_\xi \in \mathbb{Q} \cap \underset{\sim}{\mathcal{I}}$ be strictly above T^*_ξ (exists by 1.7.2). Clearly we may choose such T_ξ in $\mathfrak{B}$. Now we have to define f_ξ. We do it by 1.4, but additionally we require that

$$\text{if} \quad \eta \in T_\xi \quad \text{then} \quad (\exists \rho \in T_\xi)(\rho \lhd f_\xi(\eta) \ \& \ T \Vdash_{\mathbb{Q}} \text{“ } \rho \in \underset{\sim}{\mathbf{A}} \text{ ”}).$$

Plainly the additional requirement causes no problems (remember the definition of $\underset{\sim}{\mathcal{I}}$ and the choice of T_ξ) and the choice can be done in $\mathfrak{B}$.

There are no difficulties in carrying out the induction. Finally we let

$$T_\mu \stackrel{\rm def}{=} \bigcup_{\xi<\mu} T_\xi \quad \text{and} \quad f_\mu = \bigcup_{\xi<\mu} f_\xi.$$

By the choice of $\mathfrak{B}$ and μ we are sure that T_μ is a μ–tree. It follows from 1.5(1) that $(T_\mu, f_\mu) \in \mathbb{Q}^*$, so in particular the tree T_μ has enough μ branches (and belongs to $\mathbb{Q}$).

Claim 1.7.3 *For every $\rho \in \lim_\mu(T_\mu)$ there is $\xi < \mu$ such that*

$$(\exists \beta < \alpha(T_{\xi+1}))(T_{\xi+1} \Vdash_{\mathbb{Q}} \text{“ } \rho{\restriction}\beta \in \underset{\sim}{\mathbf{A}} \text{ ”}).$$

Proof of the claim: Fix $\rho \in \lim_\mu(T_\mu)$ and let

$$S^*_\nu \stackrel{\rm def}{=} \{\delta \in S_0 \cap \mu : \alpha(T_\delta) = \delta \ \text{ and } \ \nu_\delta = \rho{\restriction}\delta\}.$$

Plainly, the set S^*_ν is stationary in μ (remember the choice of $\bar{\nu}$). By the definition of the T_ξ's (and by $\rho \in \lim_\mu(T_\mu)$) we conclude that for every $\delta \in S^*_\nu$

if $\eta_\delta(T_{\delta+1})$ is defined then $\rho{\restriction}\delta \neq \eta_\delta(T_\mu) = \eta_\delta(T_{\delta+1})$.

But $\rho \restriction \delta = \nu_\delta$ (as $\delta \in S^*_\nu$). So look at the inductive definition: necessarily for some $\rho^*_\delta \in T_\delta$ we have $\nu_\delta = f_\delta(\rho^*_\delta)$, i.e. $\rho{\restriction}\delta = f_\delta(\rho^*_\delta)$. Now, $\rho^*_\delta \in T_\delta = \bigcup\limits_{\xi<\delta} T_\xi$ and hence for some $\xi(\delta) < \delta$, we have $\rho^*_\delta \in T_{\xi(\delta)}$. By Fodor lemma we find $\xi^* < \mu$ such that the set

$$S'_\nu \stackrel{\rm def}{=} \{\delta \in S^*_\nu : \xi(\delta) = \xi^*\}$$

is stationary in μ. Consequently we find ρ^* such that the set

$$S^+_\nu \stackrel{\rm def}{=} \{\delta \in S'_\nu : \rho^* = \rho^*_\delta\}$$

is stationary (in μ). But the sequence $\langle f_\xi(\rho^*) : \xi^* \leq \xi < \mu\rangle$ is $\trianglelefteq$–increasing, and hence the sequence ρ is its limit. Now we easily conclude the claim using the inductive definition of the (T_ξ, f_ξ)'s.

It follows from the definition of $\underset{\sim}{\mathbf{A}}$ and 1.7.3 that

$$T_\mu \Vdash_{\mathbb{Q}} \text{“ } \underset{\sim}{A} \subseteq T_\mu \text{ ”}$$

(remember that $\underset{\sim}{A}$ is a name for an antichain of $\underset{\sim}{\mathbf{T}}$), and hence

$$T_\mu \Vdash_{\mathbb{Q}} \text{“ } \|\underset{\sim}{A}\| < \lambda \text{ ”},$$

finishing the proof of the theorem. ■

Definition 1.8 *A λ–tree T is S_0–*full*, where $S_0 \subseteq \lambda$, if for every limit $\delta < \lambda$*

if $\delta \in \lambda \setminus S_0$ then $T_{[\delta]} = \lim_\delta(T)$,
if $\delta \in S_0$ then $\|T_{[\delta]} \setminus \lim_\delta(T)\| \leq 1$.

Corollary 1.9 *Assuming Hypothesis 1.2:*

1. *The forcing notion $\mathbb{Q}$ preserves cardinal numbers, cofinalities and cardinal arithmetic.*
2. *$\Vdash_{\mathbb{Q}}$ “ $\underset{\sim}{\mathbf{T}} \subseteq {}^{\lambda>}2$ is a λ–Souslin tree which is full and even S_0–full ”.*

   [So, in $\mathbf{V}^{\mathbb{Q}}$, in particular we have:

   for every $\alpha < \beta < \mu$, for all $\eta \in T \cap {}^{\alpha}2$ there is $\nu \in T \cap {}^{\beta}2$ such that $\eta \lhd \nu$, and for a limit ordinal $\delta < \lambda$, $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ is either empty or has a unique element (and then $\delta \in S_0$).]

PROOF By 1.5 and 1.7. ■

Of course, we do not need to force.

Definition 1.10 *Let $S_0, S \subseteq \lambda$. A sequence $\langle (C_\alpha, \nu_\alpha) : \alpha < \lambda$ limit$\rangle$ is called* a squared diamond sequence for (S, S_0) *if for each limit ordinal $\alpha < \lambda$*

(i) *C_α a club of α disjoint to S,*

(ii) *$\nu_\alpha \in {}^{\alpha}2$,*

(iii) *if $\beta \in \mathrm{acc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ and $\nu_\beta \lhd \nu_\alpha$,*

(iv) *if $\mu \in S$ then $\langle \nu_\alpha : \alpha \in C_\mu \cap S_0 \rangle$ is a diamond sequence.*

Proposition 1.11 *Assume (in addition to 1.2)*

(e) *there exist a squared diamond sequence for (S, S_0).*

Then there is a λ–Souslin tree $T \subseteq {}^{\lambda>}2$ which is S_0–full.

PROOF Look carefully at the proof of 1.7. ■

Corollary 1.12 *Assume that $\mathbf{V} = \mathbf{L}$ and λ is Mahlo strongly inaccessible. Then there is a full λ–Souslin tree.*

PROOF Let $S \subseteq \{\mu < \lambda : \mu$ is strongly inaccessible$\}$ be a stationary non-reflecting set. By Beller and Litman [BeLi80], there is a square $\langle C_\delta : \delta < \lambda$ limit$\rangle$ such that $C_\delta \cap S = \emptyset$ for each limit $\delta < \lambda$. As in Abraham Shelah Solovay [AShS 221, §1] we can have also the squared diamond sequence. ■

References

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