On full Souslin trees

Saharon Shelah∗
Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, USA
and
Mathematics Department
University of Wisconsin – Madison
Madison, WI 53706, USA

August 17, 2011

Abstract

In the present note we answer a question of Kunen (15.13 in [Mi91])
showing (in 1.7) that

it is consistent that there are full Souslin trees.

∗ We thank the NSF for partially supporting this research under grant #144-EF67.
Publication No 624.
0 Introduction

In the present paper we answer a combinatorial question of Kunen listed in Arnie Miller’s Problem List. We force, e.g. for the first strongly inaccessible Mahlo cardinal \( \lambda \), a full (see 1.1(2)) \( \lambda \)-Souslin tree and we remark that the existence of such trees follows from \( V = L \) (if \( \lambda \) is Mahlo strongly inaccessible). This answers [Mi91, Problem 15.13].

Our notation is rather standard and compatible with those of classical textbooks on Set Theory. However, in forcing considerations, we keep the older tradition that

\[ \text{a stronger condition is the larger one.} \]

We will keep the following conventions concerning use of symbols.

**Notation 0.1**  
1. \( \lambda, \mu \) will denote cardinal numbers and \( \alpha, \beta, \gamma, \delta, \xi, \zeta \) will be used to denote ordinals.

2. Sequences (not necessarily finite) of ordinals are denoted by \( \nu, \eta, \rho \) (with possible indexes).

3. The length of a sequence \( \eta \) is \( \ell g(\eta) \).

4. For a sequence \( \eta \) and an ordinal \( \alpha \leq \ell g(\eta) \), \( \eta|\alpha \) is the restriction of the sequence \( \eta \) to \( \alpha \) (so \( \ell g(\eta|\alpha) = \alpha \)). If a sequence \( \nu \) is a proper initial segment of a sequence \( \eta \) then we write \( \nu < \eta \) (and \( \nu \leq \eta \) has the obvious meaning).

5. A tilde indicates that we are dealing with a name for an object in forcing extension (like \( \tilde{x} \)).

1 Full \( \lambda \)-Souslin trees

A subset \( T \) of \( \alpha > 2 \) is an \( \alpha \)-tree whenever (\( \alpha \) is a limit ordinal and) the following three conditions are satisfied:

- \( \langle \rangle \in T \), if \( \nu < \eta \in T \) then \( \nu \in T \),
- \( \eta \in T \) implies \( \eta^{-\langle 0 \rangle}, \eta^{-\langle 1 \rangle} \in T \), and
- for every \( \eta \in T \) and \( \beta < \alpha \) such that \( \ell g(\eta) \leq \beta \) there is \( \nu \in T \) such that \( \eta \leq \nu \) and \( \ell g(\eta) = \beta \).

A \( \lambda \)-Souslin tree is a \( \lambda \)-tree \( T \subseteq \lambda > 2 \) in which every antichain is of size less than \( \lambda \).
Definition 1.1
1. For a tree $T \subseteq \alpha > 2$ and an ordinal $\beta \leq \alpha$ we let

$$T[\beta] \overset{\text{def}}{=} T \cap \beta^2$$

and

$$T[\varnothing > \beta] \overset{\text{def}}{=} T \setminus \beta^2.$$

If $\delta \leq \alpha$ is limit then we define

$$\lim_\delta T[\varnothing < \delta] \overset{\text{def}}{=} \{ \eta \in \delta^2 : (\forall \beta < \delta)(\eta|\beta \in T) \}.$$

2. An $\alpha$–tree $T$ is full if for every limit ordinal $\delta < \alpha$ the set $\lim_\delta T[\varnothing \varnothing < \delta] \setminus T[\delta]$ has at most one element.

3. An $\alpha$–tree $T \subseteq \alpha > 2$ has true height $\alpha$ if for every $\eta \in T$ there is $\nu \in \alpha^2$ such that

$$\eta \vartriangleleft \nu \quad \text{and} \quad (\forall \beta < \alpha)(\nu|\beta \in T).$$

We will show that the existence of full $\lambda$–Souslin trees is consistent assuming the cardinal $\lambda$ satisfies the following hypothesis.

Hypothesis 1.2 (a) $\lambda$ is strongly inaccessible (Mahlo) cardinal,
(b) $S \subseteq \{ \mu < \lambda : \mu$ is a strongly inaccessible cardinal $\}$ is a stationary set,
(c) $S_0 \subseteq \lambda$ is a set of limit ordinals,
(d) for every cardinal $\mu \in S$, $\diamondsuit_{S_0 \cap \mu}$ holds true.

Further in this section we will assume that $\lambda$, $S_0$ and $S$ are as above and we may forget to repeat these assumptions.

Let us recall that the diamond principle $\diamondsuit_{S_0 \cap \mu}$ postulates the existence of a sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$ (called a $\diamondsuit_{S_0 \cap \mu}$--sequence) such that $\nu_\delta \in \delta^2$ (for $\delta \in S_0 \cap \mu$) and

$$(\forall \nu \in \mu^2)[\{ \delta \in S_0 \cap \mu : \nu \upharpoonright \delta = \nu_\delta \} \text{ is stationary in } \mu].$$

Now we introduce a forcing notion $Q$ and its relative $Q^*$ which will be used in our proof.

Definition 1.3
1. A condition in $Q$ is a tree $T \subseteq \alpha > 2$ of a true height $\alpha = \alpha(T) < \lambda$ (see 1.1(3); so $\alpha$ is a limit ordinal) such that $\| \lim_\delta T[\varnothing < \delta] \setminus T[\delta] \| \leq 1$ for every limit ordinal $\delta < \alpha$,

the order on $Q$ is defined by $T_1 \leq T_2$ if and only if $T_1 = T_2 \cap \alpha(T_1)>2$ (so it is the end–extension order).
2. For a condition $T \in Q$ and a limit ordinal $\delta < \alpha(T)$, let $\eta_\delta(T)$ be the unique member of $\lim_\delta(T_{<\delta}) \setminus T_\delta$ if there is one, otherwise $\eta_\delta(T)$ is not defined.

3. Let $T \in Q$. A function $f : T \to \lim_\alpha(T)(T)$ is called a witness for $T$ if $(\forall \eta \in T)(\eta < f(\eta))$.

4. A condition in $Q^*$ is a pair $(T, f)$ such that $T \in Q$ and $f : T \to \lim_\alpha(T)(T)$ is a witness for $T$.

   The order on $Q^*$ is defined by $(T_1, f_1) \leq (T_2, f_2)$ if and only if $T_1 \leq Q T_2$ and $(\forall \eta \in T_1)(f_1(\eta) \leq f_2(\eta))$.

**Proposition 1.4** 1. If $(T_1, f_1) \in Q^*$, $T_1 \leq Q T_2$ and

   (a) either $\eta_\alpha(T_1)(T_2)$ is not defined or it does not belong to $\text{rang}(f_1)$

   then there is $f_2 : T_2 \to \lim_\alpha(T_2)(T_2)$ such that $(T_1, f_1) \leq (T_2, f_2) \in Q^*$.

2. For every $T \in Q$ there is a witness $f$ for $T$.

**Proof** Should be clear. \[
\]

**Proposition 1.5** 1. The forcing notion $Q^*$ is $(< \lambda)$–complete, in fact any increasing chain of length $< \lambda$ has the least upper bound in $Q^*$.

2. The forcing notion $Q$ is strategically $\gamma$-complete for each $\gamma < \lambda$.

3. Forcing with $Q$ adds no new sequences of length $< \lambda$. Since $\|Q\| = \lambda$, forcing with $Q$ preserves cardinal numbers, cofinalities and cardinal arithmetic.

**Proof** 1) It is straightforward: suppose that $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is an increasing sequence of elements of $Q^*$. Clearly we may assume that $\xi < \lambda$ is a limit ordinal and $\zeta_1 < \zeta_2 < \xi \Rightarrow \alpha(T_{\zeta_1}) < \alpha(T_{\zeta_2})$. Let $T_\xi = \bigcup_{\zeta < \xi} T_\zeta$ and $\alpha = \sup \alpha(T_\zeta)$. Easily, the union is increasing and the $T_\zeta$ is a full $\zeta$–tree. For $\eta \in T_\xi$ let $\zeta_0(\eta)$ be the first $\zeta < \xi$ such that $\eta \in T_\zeta$ and let $f_\xi(\eta) = \bigcup \{f_\xi(\eta) : \zeta_0(\eta) \leq \zeta < \xi\}$. By the definition of the order on $Q^*$ we get that the sequence $\langle f_\xi(\eta) : \zeta_0(\eta) \leq \zeta < \xi \rangle$ is $<\zeta$–increasing and hence $f_\xi(\eta) \in \lim_\alpha(T_\zeta)$. Plainly, the function $f_\xi$ witnesses that $T_\xi$ has a true height $\alpha$, and thus $(T_\xi, f_\xi) \in Q^*$. It should be clear that $(T_\xi, f_\xi)$ is the least upper bound of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$. 

"\[\]"
2) For our purpose it is enough to show that for each ordinal \( \gamma < \lambda \) and a condition \( T \in Q \) the second player has a winning strategy in the following game \( \mathcal{G}_\gamma(T, Q) \). (Also we can let Player I choose \( T_\xi \) for \( \xi \) odd.)

The game lasts \( \gamma \) moves and during a play the players, called I and II, choose successively open dense subsets \( D_\xi \) of \( Q \) and conditions \( T_\xi \in Q \). At stage \( \xi < \gamma \) of the game:
- Player I chooses an open dense subset \( D_\xi \) of \( Q \) and
- Player II answers playing a condition \( T_\xi \in Q \) such that

\[
T \leq Q T_\xi, \quad (\forall \zeta < \xi)(T_\zeta \leq Q T_\xi), \quad \text{and} \quad T_\xi \in D_\xi.
\]

The second player wins if he has always legal moves during the play.

Let us describe the winning strategy for Player II. At each stage \( \xi < \gamma \) of the game he plays a condition \( T_\xi \) and writes down on a side a function \( f_\xi \) such that \( (T_\xi, f_\xi) \in Q^* \). Moreover, he keeps an extra obligation that \( (T_\zeta, f_\zeta) \leq Q^* (T_\xi, f_\xi) \) for each \( \zeta < \xi < \gamma \).

So arriving to a non-limit stage of the game he takes the condition \( (T_\zeta, f_\zeta) \) he constructed before (or just \( (T, f) \), where \( f \) is a witness for \( T \), if this is the first move; by 1.4(2) we can always find a witness). Then he chooses \( T^*_\zeta \geq Q T_\zeta \) such that \( \alpha(T^*_\zeta) = \alpha(T_\zeta) + \omega \) and \( (T^*_\zeta)_{|\alpha(T^*_\zeta)} = \lim_{\alpha(T_\zeta)}(T_\zeta) \).

Thus \( \eta(\alpha(T_\zeta))(T^*_\zeta) \) is not defined. Now Player II takes \( T^*_\xi \geq Q T^*_\zeta \) from the open dense set \( D_{\xi+1} \) played by his opponent at this stage. Clearly \( \eta(\alpha(T_\xi))(T^*_\xi+1) \) is not defined, so Player II may use 1.4(1) to choose \( f_\xi+1 \) such that \( (T_\xi, f_\xi) \leq Q^* (T^*_\xi, f^*_\xi+1) \in Q^* \).

At a limit stage \( \xi \) of the game, the second player may take the least upper bound \( (T_\xi, f^*_\xi) \in Q^* \) of the sequence \( \langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle \) (exists by 1)) and then apply the procedure described above.

3) Follows from 2) above.

**Definition 1.6** Let \( T \) be the canonical \( Q \)-name for a generic tree added by forcing with \( Q \):

\[
\mathcal{M}_Q = \bigcup\{T : T \in \mathcal{G}_Q\}.
\]

It should be clear that \( T \) is (forced to be) a full \( \lambda \)-tree. The main point is to show that it is \( \lambda \)-Souslin and this is done in the following theorem.

**Theorem 1.7** \( \mathcal{M}_Q \) "\( T \) is a \( \lambda \)-Souslin tree".
PROOF  Suppose that $A$ is a $Q$–name such that

\[ \Downarrow Q " A \subseteq T \text{ is an antichain } " \]

and let $T_0$ be a condition in $Q$. We will show that there are $\mu < \lambda$ and a condition $T^* \in Q$ stronger than $T_0$ such that $T^* \Downarrow Q " A \subseteq T_{[< \mu]} "$ (and thus it forces that the size of $A$ is less than $\lambda$).

Let $A$ be a $Q$–name such that

\[ \Downarrow Q " A = \{ \eta \in T : (\exists \nu \in A)(\nu \not\leq \eta) \text{ or } \neg(\exists \nu \in A)(\eta \leq \nu) \} " \]

Clearly, $\Downarrow Q " A \subseteq T$ is dense open”.

Let $\chi$ be a sufficiently large regular cardinal ($\beth_7(\lambda^+)$ is enough).

Claim 1.7.1 There are $\mu \in S$ and $\mathcal{B} \prec (H(\chi), \in, <^*)$ such that:

(a) $A, A, S, S_0, Q, Q^*, T_0 \in \mathcal{B}$,
(b) $|\mathcal{B}| = \mu$ and $\mu > \mathcal{B} \subseteq \mathcal{B}$,
(c) $\mathcal{B} \cap \lambda = \mu$.

Proof of the claim: First construct inductively an increasing continuous sequence $\langle \mathcal{B}_\xi : \xi < \lambda \rangle$ of elementary submodels of $(H(\chi), \in, <^*)$ such that $A, A, S, S_0, Q, Q^*, T_0 \in \mathcal{B}_0$ and for every $\xi < \lambda$

\[ |\mathcal{B}_\xi| = \mu_\xi < \lambda, \quad \mathcal{B}_\xi \cap \lambda \in \lambda, \quad \text{and } \mu_\xi > \mathcal{B}_\xi \subseteq \mathcal{B}_{\xi+1}. \]

Note that for a club $E$ of $\lambda$, for every $\mu \in S \cap E$ we have

\[ |\mathcal{B}_\mu| = \mu, \quad \mu > \mathcal{B}_\mu \subseteq \mathcal{B}_\mu, \quad \text{and } \mathcal{B} \cap \lambda = \mu. \]

Let $\mu \in S$ and $\mathcal{B} \prec (H(\chi), \in, <^*)$ be given by 1.7.1. We know that $\diamond S_0 \cap \mu$ holds, so fix a $\diamond S_0 \cap \mu$–sequence $\bar{\eta} = (\eta_\delta : \delta \in S_0 \cap \mu)$.

Let

\[ T \\Downarrow \{ T \in Q : T \text{ is incompatible (in } Q \text{) with } T_0 \text{ or:} \]

\[ T \geq T_0 \text{ and } T \text{ decides the value of } A \cap \alpha(T) > 2 \text{ and } \]

\[ (\forall \eta \in T)(\exists \rho \in T)(\eta \leq \rho \& T \Downarrow Q \rho \in A) \} \]

Claim 1.7.2 $\mathcal{I}$ is a dense subset of $Q$. 

Proof of the claim: Should be clear (remember 1.5(2)).

Now we choose by induction on $\xi < \mu$ a continuous increasing sequence
\[(T_\xi, f_\xi) : \xi < \mu \subseteq Q^* \cap B.\]

**Step:** $i = 0$

$T_0$ is already chosen and it belongs to $Q \cap B$. We take any $f_0$ such that $(T_0, f_0) \in Q^* \cap B$ (exists by 1.4(2)).

**Step:** limit $\xi$

Since $\mu > B \subseteq B$, the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is in $B$. By 1.5(1) it has the least upper bound $(T_\xi, f_\xi)$ (which belongs to $B$).

**Step:** $\xi = \zeta + 1$

First we take (the unique) tree $T_\xi^*$ of true height $\alpha(T_\xi^*) = \alpha(T_\xi) + \omega$ such that $T_\xi^* \cap \alpha(T_\xi)^{>2} = T_\xi$ and:

- if $\alpha(T_\xi) \in S_0$ and $\nu_{\alpha(T_\xi)} \notin \text{rang}(f_\xi)$ then $(T_\xi^*)|_{\alpha(T_\xi)} \subseteq \{\nu_{\alpha(T_\xi)}\}$,
- otherwise $(T_\xi^*)|_{\alpha(T_\xi)} = \text{lim}_{\alpha(T_\xi)}(T_\xi)$.

Let $T_\xi \in Q \cap \check{I}$ be strictly above $T_\xi^*$ (exists by 1.7.2). Clearly we may choose such $T_\xi$ in $B$. Now we have to define $f_\xi$. We do it by 1.4, but additionally we require that

\[ \text{if } \eta \in T_\xi \text{ then } (\exists \rho \in T_\xi)(\rho \triangleleft f_\xi(\eta) \& T \models \mu \text{ “ } \rho \in A \text{ ”}). \]

Plainly the additional requirement causes no problems (remember the definition of $\check{I}$ and the choice of $T_\xi$) and the choice can be done in $B$.

There are no difficulties in carrying out the induction. Finally we let

\[ T_\mu \overset{\text{def}}{=} \bigcup_{\xi < \mu} T_\xi \quad \text{and} \quad f_\mu = \bigcup_{\xi < \mu} f_\xi. \]

By the choice of $B$ and $\mu$ we are sure that $T_\mu$ is a $\mu$–tree. It follows from 1.5(1) that $(T_\mu, f_\mu) \in Q^*$, so in particular the tree $T_\mu$ has enough $\mu$ branches (and belongs to $Q$).

**Claim 1.7.3** For every $\rho \in \text{lim}_\mu(T_\mu)$ there is $\xi < \mu$ such that

\[ (\exists \beta < \alpha(T_{\xi+1}))(T_{\xi+1} \models \mu \text{ “ } \rho \in A \text{ ”}). \]

**Proof of the claim:** Fix $\rho \in \text{lim}_\mu(T_\mu)$ and let

\[ S^*_\rho \overset{\text{def}}{=} \{\delta \in S_0 \cap \mu : \alpha(T_\delta) = \delta \quad \text{and} \quad \nu_\delta = \rho|\delta\}. \]

Plainly, the set $S^*_\rho$ is stationary in $\mu$ (remember the choice of $\check{\nu}$). By the definition of the $T_\xi$’s (and by $\rho \in \text{lim}_\mu(T_\mu)$) we conclude that for every $\delta \in S^*_\rho$.
if \( \eta(T_{\delta+1}) \) is defined then \( \rho|\delta \neq \eta(T_{\mu}) = \eta(T_{\delta+1}) \).

But \( \rho|\delta = \nu_{\delta} \) (as \( \delta \in S_{0}^* \)). So look at the inductive definition: necessarily for some \( \rho_{\delta}^{*} \in T_{\delta} \) we have \( \nu_{\delta} = f_{\delta}(\rho_{\delta}^{*}) \); i.e. \( \rho|\delta = f_{\delta}(\rho_{\delta}^{*}) \). Now, \( \rho_{\delta}^{*} \in T_{\delta} = \bigcup \xi<\delta T_{\xi} \) and hence for some \( \xi(\delta) < \delta \), we have \( \rho_{\delta}^{*} \in T_{\xi(\delta)} \). By Fodor lemma we find \( \xi^{*} < \mu \) such that the set

\[
S_{\nu}^{+} \overset{\text{def}}{=} \{ \delta \in S_{\nu}^{*} : \xi(\delta) = \xi^{*} \}
\]

is stationary in \( \mu \). Consequently we find \( \rho^{*} \) such that the set

\[
S_{\nu}^{*} \overset{\text{def}}{=} \{ \delta \in S_{\nu}^{*} : \rho^{*} = \rho_{\delta}^{*} \}
\]

is stationary (in \( \mu \)). But the sequence \( \langle f_{\xi}(\rho^{*}) : \xi^{*} \leq \xi < \mu \rangle \) is \( \subseteq \)-increasing, and hence the sequence \( \rho \) is its limit. Now we easily conclude the claim using the inductive definition of the \( (T_{\xi}, f_{\xi}) \)'s.

It follows from the definition of \( A \) and 1.7.3 that

\[ T_{\mu} \models_{Q} " A \subseteq T_{\mu} " \]

(remember that \( A \) is a name for an antichain of \( T \)), and hence

\[ T_{\mu} \models_{Q} " ||A|| < \lambda " , \]

finishing the proof of the theorem.

\[ \blacksquare \]

**Definition 1.8** A \( \lambda \)-tree \( T \) is \( S_{0} \)-full, where \( S_{0} \subseteq \lambda \), if for every limit \( \delta < \lambda \)

- if \( \delta \in \lambda \setminus S_{0} \) then \( T_{[\delta]} = \lim_{\delta}(T) \),
- if \( \delta \in S_{0} \) then \( ||T_{[\delta]} \setminus \lim_{\delta}(T)|| \leq 1 \).

**Corollary 1.9** Assuming Hypothesis 1.2:

1. The forcing notion \( Q \) preserves cardinal numbers, cofinalities and cardinal arithmetic.
2. \( \models_{Q} " T \subseteq \lambda>2 \) is a \( \lambda \)-Souslin tree which is full and even \( S_{0} \)-full ".

[So, in \( V^{Q} \), in particular we have:

for every \( \alpha < \beta < \mu \), for all \( \eta \in T \cap \alpha \) there is \( \nu \in T \cap \beta \) such that \( \eta < \nu \), and for a limit ordinal \( \delta < \lambda \), \( \lim_{\delta}(T_{[\delta]} \setminus T_{[\delta]}) \) is either empty or has a unique element (and then \( \delta \in S_{0} \)).]
Proof  By 1.5 and 1.7.  

Of course, we do not need to force.

**Definition 1.10**  Let $S_0, S \subseteq \lambda$. A sequence $\left( (C_\alpha, \nu_\alpha) : \alpha < \lambda \text{ limit} \right)$ is called a squared diamond sequence for $(S, S_0)$ if for each limit ordinal $\alpha < \lambda$

(i)  $C_\alpha$ a club of $\alpha$ disjoint to $S$,

(ii)  $\nu_\alpha \in \alpha^2$,

(iii)  if $\beta \in \text{acc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ and $\nu_\beta \triangleleft \nu_\alpha$,

(iv)  if $\mu \in S$ then $\langle \nu_\alpha : \alpha \in C_\mu \cap S_0 \rangle$ is a diamond sequence.

**Proposition 1.11**  Assume (in addition to 1.2)

(e)  there exist a squared diamond sequence for $(S, S_0)$.

Then there is a $\lambda$–Souslin tree $T \subseteq \lambda^{>2}$ which is $S_0$–full.

Proof  Look carefully at the proof of 1.7.

**Corollary 1.12**  Assume that $V = L$ and $\lambda$ is Mahlo strongly inaccessible. Then there is a full $\lambda$–Souslin tree.

Proof  Let $S \subseteq \{ \mu < \lambda : \mu$ is strongly inaccessible $\}$ be a stationary non-reflecting set. By Beller and Litman [BeLi80], there is a square $\langle C_\delta : \delta < \lambda \text{ limit} \rangle$ such that $C_\delta \cap S = \emptyset$ for each limit $\delta < \lambda$. As in Abraham Shelah Solovay [AShS 221, §1] we can have also the squared diamond sequence.

**References**

