

# THE KAPLANSKY TEST PROBLEMS FOR $\aleph_1$ -SEPARABLE GROUPS

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ABSTRACT. We answer a long-standing open question by proving in ordinary set theory, ZFC, that the Kaplansky test problems have negative answers for  $\aleph_1$ -separable abelian groups of cardinality  $\aleph_1$ . In fact, there is an  $\aleph_1$ -separable abelian group  $M$  such that  $M$  is isomorphic to  $M \oplus M \oplus M$  but not to  $M \oplus M$ . We also derive some relevant information about the endomorphism ring of  $M$ .

## INTRODUCTION

Kaplansky [15, pp. 12f] posed two test problems in order to “know when we have a satisfactory [structure] theorem. ... We suggest that a tangible criterion be employed: the success of the alleged structure theorem in solving an explicit problem.” The two problems were:

- (I) If  $A$  is isomorphic to a direct summand of  $B$  and conversely, are  $A$  and  $B$  isomorphic?
- (II) If  $A \oplus A$  and  $B \oplus B$  are isomorphic, are  $A$  and  $B$  isomorphic?

In fact, he says ([15, p. 75]) that he invented the problems “to show that Ulm’s theorem [a structure theory for countable abelian  $p$ -groups] could really be used”. For some other classes of abelian groups, such as finitely-generated groups, free groups, divisible groups, or completely decomposable torsion-free groups, the existence of a structure theory leads to an affirmative answer to the test problems. On the other hand, negative answers are taken as evidence of the absence of a useful classification theorem for a given class; Kaplansky says “I believe their defeat is convincing evidence that no reasonable invariants exist” [15, p. 75]. Negative answers to both questions have been proven, for example, for the class of uncountable abelian  $p$ -groups and for the class of countable torsion-free abelian groups.

Of particular interest is the method developed by Corner (cf. [1], [2],[4]) which, by realizing certain rings as endomorphism rings of groups, provides negative answers to both test problems (for a given class) as special cases of an even more extreme pathology. More precisely, Corner’s method — where applicable — yields, for any positive integer  $r$ , an abelian group  $G_r$  (in the class) such that for any positive integers  $m$  and  $k$ , the direct sum of  $m$  copies of  $G_r$  is isomorphic to the direct sum of  $k$  copies of  $G_r$  if and only if  $m$  is congruent to  $k$  mod  $r$ . (See, for example, [2] or [11, Thm 91.6, p. 145].) Then we obtain negative answers to both test problems by letting  $A = G_2$  ( $\cong G_2 \oplus G_2 \oplus G_2$ ) and  $B = G_2 \oplus G_2$ .

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Our focus here is on the class of  $\aleph_1$ -separable abelian groups (of cardinality  $\aleph_1$ ). We will prove, in ordinary set theory (ZFC), that both test problems have negative answers by deriving the Corner pathology:

**Theorem 0.1.** *For any positive integer  $r$  there is an  $\aleph_1$ -separable group  $M = M_r$  of cardinality  $\aleph_1$  such that for any positive integers  $m$  and  $k$ ,  $M^m$  is isomorphic to  $M^k$  if and only if  $m$  is congruent to  $k$  mod  $r$ .*

(Here  $M^m$  denotes the direct sum of  $m$  copies of  $M$ .) We do not determine the endomorphism ring of  $M$ , even modulo an ideal. However, we can derive a property of the endomorphism ring of  $M$  which is sufficient to imply the Corner pathology: see section 3.

A group  $M$  is called  $\aleph_1$ -separable [10, p. 184] (respectively, strongly  $\aleph_1$ -free) if it is abelian and every countable subset is contained in a countable free direct summand of  $M$  (resp., contained in a countable free subgroup  $H$  which is a direct summand of every countable subgroup of  $M$  containing  $H$ ). Obviously, an  $\aleph_1$ -separable group is strongly  $\aleph_1$ -free, so a negative answer to one of the test problems for the class of  $\aleph_1$ -separable groups implies a negative answer to the problem for the class of strongly  $\aleph_1$ -free groups. (It is independent of ZFC whether these classes are different for groups of cardinality  $\aleph_1$ : the weak Continuum Hypothesis ( $2^{\aleph_0} < 2^{\aleph_1}$ ) implies that there are strongly  $\aleph_1$ -free groups of cardinality  $\aleph_1$  which are not  $\aleph_1$ -separable; on the other hand, Martin's Axiom (MA) plus the negation of the Continuum Hypothesis ( $\neg$ CH) implies that every strongly  $\aleph_1$ -free group of cardinality  $\aleph_1$  is  $\aleph_1$ -separable; cf.[16] )

Dugas and Göbel [5] proved that  $\text{ZFC} + 2^{\aleph_0} < 2^{\aleph_1}$  implies that the Corner pathology exists for the class of strongly  $\aleph_1$ -free groups of cardinality  $\aleph_1$ ; in fact, they showed that there is a strongly  $\aleph_1$ -free group  $G$  whose endomorphism ring is an appropriate ring (the ring  $A = A_r$  of the next section). (See also [12].) This group  $G$  cannot be  $\aleph_1$ -separable since the endomorphism ring of an  $\aleph_1$ -separable group has too many idempotents. However, Thomé ([20] and [21]) showed that ZFC plus  $V = L$  (Gödel's Axiom of Constructibility) implies the Corner pathology for  $\aleph_1$ -separable groups of cardinality  $\aleph_1$ ; he did this by constructing an  $\aleph_1$ -separable  $G$  such that  $\text{End}(G)$  is a split extension of  $A$  by  $I$  (in the sense of [3, p. 277]), where  $I$  is the ideal of endomorphisms with a countable image.

It follows from known structure theorems for the class of  $\aleph_1$ -separable groups of cardinality  $\aleph_1$  under the hypothesis  $\text{MA} + \neg\text{CH}$  that the Dugas-Göbel and Thomé realization results are *not* theorems of ZFC (cf. [7] or [17]). The fact that there *are* positive structure theorems for the class of  $\aleph_1$ -separable groups assuming  $\text{MA} + \neg\text{CH}$  or the stronger Proper Forcing Axiom (PFA) — see, for example, [8] or [18] — led to the question of whether the Kaplansky test problems could have affirmative answers for this class assuming, say, PFA. Thomé [21] gave a negative answer to the second test problem in ZFC, using a result of Jónsson [14] for countable torsion-free groups; however, till now, the first test problem as well as the Corner pathology were open (in ZFC).

Our construction of the Corner pathology involves a direct construction of the pathological group  $M$  using a tree-like ladder system and a “countable template” which comes from the Corner example for countable torsion-free groups. A key role

is played by a paper of Göbel and Goldsmith [13] which — while it does not itself prove any new results about the Kaplansky test problems for strongly  $\aleph_1$ -free or  $\aleph_1$ -separable groups — provides the tools for creating a suitable template from the Corner example.

1. THE COUNTABLE TEMPLATE

Fix a positive integer  $r$ . For this  $r$ , let  $A = A_r$  be the countable ring constructed by Corner in [2]. (See also [11, p. 146].) Specifically,  $A$  is the ring freely generated by symbols  $\rho_i$  and  $\sigma_i$  ( $i = 0, 1, \dots, r$ ) subject to the relations

$$\rho_j \sigma_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{i=0}^r \sigma_i \rho_i = 1.$$

Then  $A$  is free as an abelian group, and  $\sigma_0 \rho_0, \dots, \sigma_r \rho_r$  are pairwise orthogonal idempotents. Moreover, if  $M$  is a right  $A$ -module, then  $M = M\sigma_0 \rho_0 \oplus M\sigma_1 \rho_1 \oplus \dots \oplus M\sigma_r \rho_r$  and  $M\sigma_i \rho_i \cong M$  because  $\sigma_i \rho_i \sigma_i : M \rightarrow M\sigma_i \rho_i$  and  $\rho_i \sigma_i \rho_i : M\sigma_i \rho_i \rightarrow M$  are inverses; therefore  $M \cong M^{r+1}$ .

Our construction will work for any countable torsion-free ring  $A$  whose additive subgroup is free; but hereafter  $A$  will denote the ring  $A_r$  just defined.

Corner shows that there is a torsion-free countable abelian group  $G$  whose endomorphism ring is  $A$ ; thus  $G$  is an  $A$ -module and hence  $G \cong G^{r+1}$ . Furthermore, he shows that  $G^\ell$  is not isomorphic to  $G^m$  if  $1 \leq \ell < m \leq r$ , and hence  $G^m$  is not isomorphic to  $G^k$  if  $m$  is not congruent to  $k \pmod r$ . We shall require these and further properties of  $G$ , which we summarize in the following:

**Proposition 1.1.** *There are countable free  $A$ -modules  $B \subseteq H$  such that  $G \cong H/B$  and  $B$  is the union of a chain of free  $A$ -modules,  $B = \bigcup_{n \in \omega} B_n$ , such that  $B_0 = 0$  and for all  $n \in \omega$ ,  $H/B_n$  and  $B_{n+1}/B_n$  are free  $A$ -modules of rank  $\omega$ . Moreover for any positive integers  $m$  and  $k$ , if  $m$  is not congruent to  $k \pmod r$ , then  $G^m \oplus \mathbf{Z}^{(\omega)}$  is not isomorphic to  $G^k \oplus \mathbf{Z}^{(\omega)}$ .*

The main work in proving Proposition 1.1 will be done in two lemmas from [13]. For the first one, we give a revised proof (cf. [13, p. 343]). We maintain the notation above.

**Lemma 1.2.** *The group  $G$  is the union,  $G = \bigcup_{n \geq 1} G_n$ , of an increasing chain of free  $A$ -modules.*

*Proof.* By [1, p. 699]  $G$  is the pure closure  $\langle G_1 \rangle_*$  in  $\hat{A}$  of a free  $A$ -module  $G_1 = \bigoplus_{i \in I} e_i A \oplus A$  containing  $A$ . Here  $\hat{A}$  is the natural, or  $\mathbf{Z}$ -adic, completion of  $A$  (cf. [1, p. 692]). We will define inductively  $G_n = \bigoplus_{i \in I} e_{i,n} A \oplus A$  such that  $G_n \supseteq G_{n-1}$  and for all  $i \in I$ ,  $ne_{i,n} + A = e_{i,n-1} + A$ . Let  $e_{i,1} = e_i$  for all  $i \in I$ . If  $G_{n-1} \subseteq G$  has been defined for some  $n > 1$ , then since  $A$  is dense in  $\hat{A}$ , there exists  $e_{i,n} \in \hat{A}$  such that  $ne_{i,n} + A = e_{i,n-1} + A$ ; say  $ne_{i,n} = e_{i,n-1} + a_i$ . By the definition of  $G$ ,  $e_{i,n} \in G$ . We need to show that  $\{e_{i,n} : i \in I\} \cup \{1\}$  is  $A$ -linearly independent. Suppose that  $\sum_{i \in I} e_{i,n} c_i + 1 \cdot c_0 = 0$  for some  $c_0, c_i \in A$ . Then  $\sum_{i \in I} ne_{i,n} c_i + nc_0 = 0$ , so  $\sum_{i \in I} e_{i,n-1} c_i + 1 \cdot (\sum_{i \in I} a_i c_i + nc_0) = 0$ . By the  $A$ -linear

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independence of  $\{e_{i,n-1} : i \in I\} \cup \{1\}$ , we can conclude that each  $c_i$  equals 0 and hence also  $c_0$  equals 0. This completes the definition of  $G_n$ .

It remains to prove that  $G \subseteq \bigcup_{n \geq 1} G_n$ . Let  $g \in G \setminus G_1$ . For some  $n > 1$ ,  $ng \in G_1$ . We claim that  $g \in G_n$ . Since  $ng \in G_{n-1}$ ,  $ng = \sum_{i \in I} e_{i,n-1} c_i + c_0$  for some  $c_i, c_0 \in A$ . Then

$$ng = \sum_{i \in I} (ne_{i,n} - a_i) c_i + c_0 = n \sum_{i \in I} e_{i,n} c_i + a'$$

for some  $a' \in A$ . Since  $A$  is pure in  $\hat{A}$ ,  $a' = na''$  for some  $a'' \in A$ . Thus  $g = \sum_{i \in I} e_{i,n} c_i + a'' \in G_n$ .  $\square$

The second lemma is proved in [13, Lemma 2.5] generalizing a result in [9, Lemma XII.1.4]. We state it here for the sake of completeness.

**Lemma 1.3.** *Let  $G$  be a countable  $A$ -module which is the union,  $G = \bigcup_{n \geq 1} G_n$ , of an increasing chain of free  $A$ -modules, then there exist countable free  $A$ -modules  $B \subseteq H$  such that  $G \cong H/B$  and  $B$  is the union of a chain of free  $A$ -modules,  $B = \bigcup_{n \geq 1} B_n$ , such that for all  $n \geq 1$ ,  $H/B_n$  and  $B_{n+1}/B_n$  are free  $A$ -modules.  $\square$*

PROOF OF PROPOSITION 1.1. The existence of  $H$ ,  $B$ , and the  $B_n$  is now an immediate consequence of Lemmas 1.2 and 1.3. All that is left to show is that if  $m$  is not congruent to  $k \pmod r$ , then  $G^m \oplus \mathbf{Z}^{(\omega)}$  is not isomorphic to  $G^k \oplus \mathbf{Z}^{(\omega)}$ . Since  $G^m$  is not isomorphic to  $G^k$ , it is enough to show that  $R_{\mathbf{Z}}(G^l \oplus \mathbf{Z}^{(\omega)}) = G^l$  for any  $l \in \omega$ . Here  $R_{\mathbf{Z}}(N)$  is the  $\mathbf{Z}$ -radical of  $N$ , that is,  $R_{\mathbf{Z}}(N) = \bigcap \{\ker(\varphi) : \varphi : N \rightarrow \mathbf{Z}\}$ . (See, for example, [9, pp. 289f].) To show that  $R_{\mathbf{Z}}(G^l \oplus \mathbf{Z}^{(\omega)}) = G^l$  it is enough to show that  $\text{Hom}(G^l, \mathbf{Z}) = 0$ , or, equivalently,  $\text{Hom}(G, \mathbf{Z}) = 0$ . This follows from Observation 2.7 of [13], but we give here a self-contained argument based on the notation of Lemma 1.2. Suppose  $\psi \in \text{Hom}(G, \mathbf{Z})$ ; we can regard  $\psi$  as an endomorphism of  $G$  by identifying  $\mathbf{Z}$  with the subgroup  $\langle 1 \rangle$  of  $A \subseteq G$  which is generated by the unit 1 of  $A$ . Since the endomorphism ring of  $G$  is  $A$ , there is  $a \in A$  such that  $\psi(g) = ga$  for all  $g \in G$ . By considering  $\psi(1) = 1a = a$ , we see that  $a \in \langle 1 \rangle$ . Now consider  $\psi(e_i)$  for any  $e_i$ ; since  $\psi(e_i) = e_i a$  and since  $e_i A \cap \langle 1 \rangle = \{0\}$  we see that  $a = 0$ .  $\square$

## 2. THE MAIN CONSTRUCTION

Fix a positive integer  $r$  and let  $A, H, B, B_n$  and  $G$  be as in Proposition 1.1. For each  $n \in \omega$ , fix a basis  $\{b_{n,i} + B_n : i \in \omega\}$  of  $B_{n+1}/B_n$  (as  $A$ -module). Also, fix a set of representatives  $\{h_i : i \in \omega\}$  for  $H/B$  where  $h_0 = 0$ ; thus each coset  $h + B$  equals  $h_i + B$  for a unique  $i \in \omega$ .

Fix a stationary subset  $E$  of  $\omega_1$  consisting of limit ordinals and a ladder system  $\{\eta_\delta : \delta \in E\}$ . That is, for every  $\delta$  in  $E$ ,  $\eta_\delta : \omega \rightarrow \delta$  is a strictly increasing function whose range is cofinal in  $\delta$ ; we shall also choose  $\eta_\delta$  so that its range is disjoint from  $E$ . Furthermore, we choose a ladder system which is *tree-like*, that is, for all  $\delta, \gamma \in E$  and  $n, m \in \omega$ ,  $\eta_\delta(n) = \eta_\gamma(m)$  implies that  $m = n$  and  $\eta_\delta(l) = \eta_\gamma(l)$  for all  $l < n$  (cf. [9, pp. 368, 386]).

Inductively define free  $A$ -modules  $M_\beta$  ( $\beta < \omega_1$ ) as follows: if  $\beta$  is a limit ordinal,  $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ ; if  $\beta = \alpha + 1$  where  $\alpha \notin E$ , let

$$M_\beta = M_\alpha \oplus \bigoplus_{i \in \omega} x_{\alpha,i} A.$$

If  $\beta = \delta + 1$  where  $\delta \in E$ , define an embedding  $\iota_\delta : B \rightarrow M_\delta$  by sending the basis element  $b_{n,i}$  to  $x_{\eta_\delta(n),i}$ . Essentially  $M_{\delta+1}$  will be defined to be the pushout of

$$\begin{array}{ccc} M_\delta & & \\ \uparrow \iota_\delta & & \\ B & \hookrightarrow & H \end{array}$$

but we will be more explicit in order to avoid the necessity of identifying isomorphic copies. Let  $y_{\delta,0} = 0$  and let  $\{y_{\delta,i} : i \in \omega \setminus \{0\}\}$  be a new set of distinct elements (not in  $M_\delta$ ). Then define  $M_{\delta+1}$  to be  $\{y_{\delta,i} + u : u \in M_\delta, i \in \omega\}$  where the operations on  $M_{\delta+1}$  extend those on  $M_\delta$  and are otherwise determined by the rules

$$\begin{aligned} y_{\delta,i} + y_{\delta,j} &= y_{\delta,k} + \iota_\delta(b) & \text{if } h_i + h_j &= h_k + b \\ y_{\delta,i}a &= y_{\delta,\ell} + \iota_\delta(b) & \text{if } h_i a &= h_\ell + b \end{aligned}$$

where  $b \in B$  and  $a \in A$ . Then there is an embedding  $\theta_\delta : H \rightarrow M_{\delta+1}$  extending  $\iota_\delta$  which takes  $h_i$  to  $y_{\delta,i}$  and induces an isomorphism of  $H/B$  with  $M_{\delta+1}/M_\delta$ .

This completes the inductive definition of the  $M_\beta$ . Let  $M = \bigcup_{\beta < \omega_1} M_\beta$ . Note that it follows from the construction that every element of  $M$  has a unique representation in the form

$$\sum_{j=1}^s y_{\delta_j, n_j} + \sum_{\ell=1}^t x_{\alpha_\ell, i_\ell} a_\ell$$

where  $\delta_1 < \delta_2 < \dots < \delta_s$  are elements of  $E$ ,  $n_j \in \omega \setminus \{0\}$ ,  $\alpha_\ell \in \omega_1 \setminus E$ ,  $i_\ell \in \omega$ ,  $a_\ell \in A$ , and the pairs  $(\alpha_\ell, i_\ell)$  ( $\ell = 1, \dots, t$ ) are distinct.

Since  $M$  is constructed to be an  $A$ -module,  $M$  is isomorphic to  $M^{r+1}$ . We claim that

(†)  $M$  is  $\aleph_1$ -separable; in fact for all  $\alpha < \omega_1$ ,  $M_{\alpha+1}$  is a free direct summand of  $M$ .

Assuming this for the moment, we can show that

(††)  $M^m$  is not isomorphic to  $M^k$  if  $m$  is not congruent to  $k \pmod r$ .

In brief this is because  $M^m$  and  $M^k$  are not quotient-equivalent (cf. [9, pp. 251f]) since for all  $\delta \in E$ ,  $(M_{\delta+1}/M_\delta)^m \oplus \mathbf{Z}^{(\omega)}$  is not isomorphic to  $(M_{\delta+1}/M_\delta)^k \oplus \mathbf{Z}^{(\omega)}$  by Proposition 1.1. In more detail, if there is an isomorphism  $\varphi : M^m \rightarrow M^k$ , then there is a closed unbounded subset  $C$  of  $\omega_1$  such that for  $\beta \in C$ ,  $\varphi[M_\beta^m] = M_\beta^k$ . Since  $E$  is stationary in  $\omega_1$ , there exist  $\delta \in C \cap E$ ; choose  $\beta > \delta$  such that  $\beta \in C$ . Then  $\varphi$  induces an isomorphism of  $M_\beta^m/M_\delta^m$  with  $M_\beta^k/M_\delta^k$ . Since  $M_\beta/M_{\delta+1}$  is free (of infinite rank) by (†), we can conclude that

$$\begin{aligned} (M_{\delta+1}/M_\delta)^m \oplus \mathbf{Z}^{(\omega)} &\cong (M_{\delta+1}^m/M_\delta^m) \oplus (M_\beta^m/M_{\delta+1}^m) \cong M_\beta^m/M_\delta^m \cong M_\beta^k/M_\delta^k \\ &\cong (M_{\delta+1}^k/M_\delta^k) \oplus (M_\beta^k/M_{\delta+1}^k) \cong (M_{\delta+1}/M_\delta)^k \oplus \mathbf{Z}^{(\omega)} \end{aligned}$$

which contradicts Proposition 1.1.

We are left with the task of proving (†). First we shall show that each  $M_{\alpha+1}$  is a direct summand of  $M$  by defining a projection  $\pi_\alpha$  of  $M$  onto  $M_{\alpha+1}$  (that is,  $\pi_\alpha|_{M_{\alpha+1}}$  is the identity). For every integer  $k$  there is a projection  $\rho_k : H \rightarrow B_{k+1}$  since  $H/B_{k+1}$  is free. Given  $\alpha$ , for each  $\delta \in E$  with  $\delta > \alpha$ , let  $k_\delta$  be the maximal integer  $k$  such that  $\eta_\delta(k) \leq \alpha$ . For each  $\delta \in E$ , we let  $\pi_\alpha$  act like  $\rho_{k_\delta}$  on the isomorphic copy,  $\theta_\delta[H]$ , of  $H$ . More precisely, for each element  $z$  of  $\theta_\delta[H]$ , define  $\pi_\alpha(z)$  to be  $\theta_\delta(\rho_{k_\delta}(\theta_\delta^{-1}(z)))$ ; if  $\nu \notin \bigcup\{\text{ran}(\eta_\delta) : \delta \in E\}$  and  $\nu > \alpha$ , define  $\pi_\alpha(x_{\nu,i}) = 0$ . Extend to an arbitrary element of  $M$  by additivity; this will define

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a homomorphism on  $M$  provided that  $\pi_\alpha$  is well-defined. It is easy to see, using the unique representation of elements, that the question of well-definition reduces to showing that the definition of  $\pi_\alpha(x_{\beta,i})$  for  $x_{\beta,i} \in \theta_\delta[H]$  is independent of  $\delta$ . If  $\beta \leq \alpha$ , then  $\pi_\alpha(x_{\beta,i}) = x_{\beta,i}$ . Say  $\beta > \alpha$  and  $\beta = \eta_\delta(n) = \eta_\gamma(n)$ ; by the tree-like property,  $\eta_\delta(m) = \eta_\gamma(m)$  for all  $m \leq n$ , and hence  $k_\delta = k_\gamma$ . Hence  $\pi_\alpha(x_{\beta,i})$  is well-defined because  $\rho_{k_\delta} = \rho_{k_\gamma}$  and thus  $\theta_\delta(\rho_{k_\delta}(\theta_\delta^{-1}(x_{\beta,i}))) = \theta_\gamma(\rho_{k_\gamma}(\theta_\gamma^{-1}(x_{\beta,i})))$ .

It remains to prove that each  $M_\beta$  is  $\aleph_1$ -free (as abelian group). Since  $A$  is free as abelian group, it suffices to show that  $M_{\delta+1}$  is a free  $A$ -module for every  $\delta \in E$ . We will inductively define  $S_n$  so that

$$B = \bigcup_{n \in \omega} S_n \cup \{x_{\nu,i} : \nu \in \delta \setminus (E \cup \bigcup \{\text{ran}(\eta_\mu) : \mu \in E \cap (\delta + 1)\}), i \in \omega\}$$

is an  $A$ -basis of  $M_{\delta+1}$ . Let  $S_0$  be the image under  $\theta_\delta$  of a basis of  $H$ . Fix a bijection  $\psi : \omega \rightarrow E \cap \delta$ ; also, for convenience, let  $\psi(-1) = \delta$ . Suppose that  $S_m$  has been defined for  $m \leq n$  so that  $\bigcup_{m \leq n} S_m$  is  $A$ -linearly independent and generates  $\bigcup \{\theta_{\psi(m)}[H] : -1 \leq m < n\}$ . Let  $\gamma = \psi(n)$  and let  $k = k_n$  be maximal such that  $\eta_\gamma(k) = \eta_{\psi(m)}(k)$  for some  $-1 \leq m < n$ . Notice that  $\{x_{\eta_\gamma(\ell),i} : \ell \leq k, i \in \omega\}$  is contained in the  $A$ -submodule generated by  $\bigcup_{m \leq n} S_m$ . Since  $H/B_{k+1}$  is  $A$ -free, we can write  $H = B_{k+1} \oplus C_k$  for some  $A$ -free module  $C_k$  ( $= \ker(\rho_k)$ ); let  $S_{n+1}$  be the image under  $\theta_\gamma$  of a basis of  $C_k$ . This completes the inductive construction. One can then easily verify that  $B$  is an  $A$ -basis of  $M_{\delta+1}$ ; indeed, the fact that  $\bigcup_{m \leq n} S_m$  is  $A$ -linearly independent can be proved by induction on  $n$ , using the unique representation of elements of  $M$  to show that if  $\sum_{i=1}^r z_i a_i \in \langle \bigcup_{m \leq n} S_m \rangle$ , where  $z_1, \dots, z_r$  are distinct elements of  $S_{n+1}$ , then  $a_i = 0$  for all  $i = 1, \dots, r$ .

### 3. THE ENDOMORPHISM RING OF $M$

While we cannot show that  $\text{End}(M)$  is a split extension of  $A$  by an ideal, we can obtain enough information about  $\text{End}(M)$  to imply the negative results on the Kaplansky test problems. (A similar idea is used in [19, p. 118].)

The ring  $A$  is naturally a subring of  $\text{End}(M)$ . We say that  $A$  is *algebraically closed* in  $\text{End}(M)$  when every finite set of ring equations with parameters from  $A$  (i.e., polynomials in several variables over  $A$ ) which is satisfied in  $\text{End}(M)$  is also satisfied in  $A$ .

**Proposition 3.1.** *If  $A = A_r$  is as in section 1, and  $A$  is algebraically closed in  $\text{End}(M)$ , then for any positive integers  $m$  and  $k$ ,  $M^m$  is isomorphic to  $M^k$  if and only if  $m$  is congruent to  $k \pmod r$ .*

*Proof.* Since  $M$  is an  $A$ -module,  $M \cong M^{r+1}$ . If  $M^\ell$  is isomorphic to  $M^n$  where  $1 \leq \ell < n \leq r$ , then  $\sum_{i=1}^\ell M\sigma_i\rho_i \cong \sum_{i=1}^n M\sigma_i\rho_i$ , so by Lemma 2 of [2], there are elements  $x$  and  $y$  of  $\text{End}(M)$  such that  $xy = \sum_{i=1}^\ell \sigma_i\rho_i$  and  $yx = \sum_{i=1}^n \sigma_i\rho_i$ . So by hypothesis, such elements  $x$  and  $y$  exist in  $A$ . We then obtain a contradiction as in [2, p. 45].  $\square$

**Proposition 3.2.** *If  $M$  is defined as in section 2, then  $A$  is algebraically closed in  $\text{End}(M)$ .*

*Proof.* For any  $\sigma \in \text{End}(M)$ , there is a closed unbounded subset  $C_\sigma$  of  $\omega_1$  such that for all  $\alpha \in C_\sigma$ ,  $\sigma[M_\alpha] \subseteq M_\alpha$ . For any  $\sigma_1, \dots, \sigma_n$  in  $\text{End}(M)$ , choose  $\alpha < \beta$  in  $C_{\sigma_1} \cap \dots \cap C_{\sigma_n}$  so that also  $\alpha \in E$ . Then each  $\sigma_i$  induces an endomorphism, also denoted  $\sigma_i$ , of  $M_\beta/M_\alpha$ . The endomorphism ring of  $M_\beta/M_\alpha$  is  $\text{End}(G \oplus \mathbf{Z}^{(\omega)})$  and restriction to  $G$  defines a natural homomorphism,  $\pi$ , of  $\text{End}(G \oplus \mathbf{Z}^{(\omega)})$  onto  $\text{End}(G) \cong A$  because  $\text{Hom}(G, \mathbf{Z}^{(\omega)}) = 0$ . If  $\sigma_i = a \in A$  (regarded as an element of  $\text{End}(M)$ ), then  $\pi(a) = a$ . Hence if  $\sigma_1, \dots, \sigma_m$  satisfy some ring equations over  $A$ , then so do  $\pi(\sigma_1), \dots, \pi(\sigma_m)$ .  $\square$

Propositions 3.1 and 3.2 provide an alternative proof of ( $\dagger\dagger$ ).

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