NORMS ON POSSIBILITIES II: MORE CCC IDEALS ON 2^ω

A. ROSŁANOWSKI AND S. SHELAH

Abstract. We use the method of *norms on possibilities* to answer a question of Kunen and construct a ccc σ–ideal on 2^ω with various closure properties and distinct from the ideal of null sets, the ideal of meager sets and their intersection.

0. Introduction. In the present paper we use the method of *norms on possibilities* to answer a question of Kunen (see 0.1 below) and construct a ccc σ–ideal on 2^ω with various closure properties and distinct from the ideal of null sets, the ideal of meager sets and their intersection. The method we use is, in a sense, a generalization of the one studied systematically in [13] (the case of creating pairs). However, as the main desired property of the forcing notion we construct is satisfying the ccc, we do not use the technology of that paper (where the forcing notions were naturally proper not–ccc) and our presentation does not require familiarity with the previous part.

The following problem was posed by Kunen and for some time stayed open.

1991 *Mathematics Subject Classification.*

The first author thanks the Hebrew University and the Lady Davis Foundation for the Golda Meir Postdoctoral Fellowship. The research of the second author was partially supported by "Basic Research Foundation" of the Israel Academy of Sciences and Humanities. Publication 628 of the second author.

Kunen's Problem 0.1 (see [8, Question 1.2] or [9, Problem 5.5])**.** Does there exist a ccc σ–ideal $\mathcal{I}$ of subsets of 2^ω which:

> has a Borel basis, is index–invariant, translation–invariant and is neither the meager ideal, nor the null ideal nor their intersection?

We give a positive answer to this problem and, in fact, we construct a large family of ideals with the desired properties. Naturally, these ideals are obtained from ccc forcing notions. However, the construction presented in Section 4 is purely combinatorial and no forcing techniques are needed for it.

Let us recall the definitions of the properties required from $\mathcal{I}$ in 0.1.

Definition 0.2.

(1) If $(\mathcal{X},+)$ is a commutative group then the product space $\mathcal{X}^\omega$ is equipped with the product group structure. The Cantor space 2^ω carries the structure of Polish group generated by the addition modulo 2 on $2=\{0,1\}$. We equip the Baire space ω^ω with a group operation interpreting ω as the group $\mathbb{Z}$ of integers.

(2) An ideal $\mathcal{I}$ of subsets of a group $(\mathcal{X},+)$ is *translation–invariant* if

$$(\forall x\in\mathcal{X})(\forall A\in\mathcal{I})(A+x\stackrel{\rm def}{=}\{a+x: a\in A\}\in\mathcal{I}).$$

(3) An ideal $\mathcal{I}$ of subsets of the product space $\mathcal{X}^\omega$ is *index–invariant* if for every embedding $\pi:\omega\stackrel{1-1}{\longrightarrow}\omega$ and every set $A\in\mathcal{I}$ we have

$$\pi_*(A)\stackrel{\rm def}{=}\{x\in\mathcal{X}^\omega: x\circ\pi\in A\}\in\mathcal{I}.$$

(4) The ideal $\mathcal{I}$ as above is *permutation–invariant* if it satisfies the demand above when we restrict ourselves to permutations π of ω only.

(5) An ideal $\mathcal{I}$ on a Polish space $\mathcal{X}$ has a Borel basis if every set $A\in\mathcal{I}$ is contained in a Borel set $B\in\mathcal{I}$.
[In this situation we may say that $\mathcal{I}$ is a Borel ideal.]

(6) A Borel σ-ideal $\mathcal{I}$ on a Polish space $\mathcal{X}$ is ccc if the quotient Boolean algebra $\mathrm{BOREL}(\mathcal{X})/\mathcal{I}$ of Borel subsets of $\mathcal{X}$ modulo $\mathcal{I}$ satisfies the ccc. (Equivalently, there is no uncountable family of disjoint Borel subsets of $\mathcal{X}$ which are not in $\mathcal{I}$.)

There has been some partial answers to 0.1 already. Kechris and Solecki [7] showed that ccc σ–ideals generated by closed sets (i.e. with Σ^0_2–basis) are essentially like the meager ideal. It was shown in [12] how Souslin ccc forcing notions may produce nice σ–ideals on the Baire space ω^ω (those notes were presented in [1, pp 193–203]). That method provided an answer to 0.1 if one replaced the demand that the ideal in question is on 2^ω by allowing it to be on ω^ω. It seems that the approach presented there is not applicable if we want to stay in the Cantor space. Our method here, though similar to the one there, is more direct.

Notation: Our notation is rather standard and compatible with that of classical textbooks on Set Theory (like Jech [5] or Bartoszyński Judah [1]). However in forcing we keep the convention that *a stronger condition is the larger one.*

Notation 0.3.

(1) $\mathbb{R}^{\geq 0}$ stands for the set of non-negative reals. The integer part of a real $r \in \mathbb{R}^{\geq 0}$ is denoted by $\lfloor r \rfloor$.
(2) For a set X, $[X]^{\leq\omega}$, $[X]^{<\omega}$ and $\mathcal{P}(X)$ will stand for families of countable, finite and all, respectively, subsets of the set X. The family of k-element subsets of X will be denoted by $[X]^k$. The set of all finite sequences with values in X is called $X^{<\omega}$ (so domains of elements of $X^{<\omega}$ are integers).
(3) The Cantor space 2^ω and the Baire space ω^ω are the spaces of all functions from ω to 2, ω, respectively, equipped with natural (Polish) topology.
(4) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a dot above (e.g. $\dot{\tau}$, $\dot{X}$).

Basic Notation: In this paper $\mathbf{H}$ will stand for a function with domain ω such that $(\forall m \in \omega)(|\mathbf{H}(m)| \geq 2)$. We usually assume that $0 \in \mathbf{H}(m)$ (for all $m \in \omega$). Moreover, we assume that at least $\mathbf{H} \in \mathcal{H}(\aleph_1)$ (the family of hereditarily countable sets) or, what is more natural, even $\mathbf{H}(i) \in \omega \cup \{\omega\}$ (for $i \in \omega$).

1. Semi–creating triples.

In [13] we explored a general method of building forcing notions using *norms on possibilities.* We studied weak creating pairs and their two specific cases: creating pairs and tree–creating pairs. For our applications here, we have to modify the general schema introducing an additional operation. In our presentation we will not refer the reader to [13], but familiarity with that paper may be of some help in getting a better picture of the method.

The definition 1.1 of semi–creatures and semi–creating triples is not covered by the general case as presented in [13, 1.1]. But one should notice a close relation of it to the case discussed in [13, 1.2] under an additional demand that the considered creating pair is forgetful (see [13, 1.2.5]).

Definition 1.1. Let $\mathbf{H}:\omega\longrightarrow\mathcal{H}(\aleph_1)$.

(1) A quadruple $t=(\mathbf{dom},\mathbf{sval},\mathbf{nor},\mathbf{dis})$ is *a semi–creature for* $\mathbf{H}$ if
- $\mathbf{dom}\in[\omega]^{<\omega}\setminus\{\emptyset\}$,
- $\mathbf{sval}\subseteq\prod\limits_{i\in\mathbf{dom}}\mathbf{H}(i)$ is non-empty,
- $\mathbf{nor}\in\mathbb{R}^{\geq 0}\cup\{\infty\}$,
- $\mathbf{dis}\in\mathcal{H}(\aleph_1)$, and
- $\mathbf{sval}=\prod\limits_{i\in\mathbf{dom}}\mathbf{H}(i)$ if and only if $\mathbf{nor}=\infty$.

The family of all semi–creatures for $\mathbf{H}$ will be denoted by SCR[$\mathbf{H}$].

(2) In the above definition we write $\mathbf{dom}=\mathbf{dom}[t]$, $\mathbf{sval}=\mathbf{sval}[t]$, $\mathbf{nor}=\mathbf{nor}[t]$ and $\mathbf{dis}=\mathbf{dis}[t]$.

(3) Suppose that $K\subseteq\mathrm{SCR}[\mathbf{H}]$. A function $\Sigma:[K]^{<\omega}\longrightarrow\mathcal{P}(K)$ is *a semi–composition operation on* K if for each $\mathcal{S}\in[K]^{<\omega}$ and $t\in K$:
  (a) if for each $s\in\mathcal{S}$ we have $s\in\Sigma(\mathcal{S}_s)$, $\mathcal{S}_s\in[K]^{<\omega}$, then $\Sigma(\mathcal{S})\subseteq\Sigma(\bigcup\limits_{s\in\mathcal{S}}\mathcal{S}_s)$,
  (b) $t\in\Sigma(t)$, $\Sigma(\emptyset)=\emptyset$, and
  (c) if $t\in\Sigma(\mathcal{S})$ (so $\Sigma(\mathcal{S})\neq\emptyset$) then
    (α) $\mathbf{dom}[t]=\bigcup\limits_{s\in\mathcal{S}}\mathbf{dom}[s]$,
    (β) $v\in\mathbf{sval}[t]\ \&\ s\in\mathcal{S}\quad\Rightarrow\quad v{\restriction}\mathbf{dom}[s]\in\mathbf{sval}[s]$, and
    (γ) $s_1,s_2\in\mathcal{S}\ \&\ s_1\neq s_2\quad\Rightarrow\quad\mathbf{dom}[s_1]\cap\mathbf{dom}[s_2]=\emptyset$.

(4) A mapping $\Sigma^\perp:K\longrightarrow[K]^{<\omega}\setminus\{\emptyset\}$ is called *a semi–decomposition operation on* K if for each $t\in K$:
  (a)$^\perp$ if $\mathcal{S}=\{s_0,\ldots,s_k\}\in\Sigma^\perp(t)$ and $\mathcal{S}_i\in\Sigma^\perp(s_i)$ (for $i\leq k$) then $\mathcal{S}_0\cup\ldots\cup\mathcal{S}_k\in\Sigma^\perp(t)$,
  (b)$^\perp$ $\{t\}\in\Sigma^\perp(t)$,
  (c)$^\perp$ if $\mathcal{S}\in\Sigma^\perp(t)$ then
    $(\alpha)^\perp$ $\mathbf{dom}[t]=\bigcup\limits_{s\in\mathcal{S}}\mathbf{dom}[s]$,
    $(\beta)^\perp$ $\{v\in\prod\limits_{i\in\mathbf{dom}[t]}\mathbf{H}(i):(\forall s\in\mathcal{S})(v{\restriction}\mathbf{dom}[s]\in\mathbf{sval}[s])\}\subseteq\mathbf{sval}[t]$.

(5) If Σ is a semi–composition operation on $K\subseteq\mathrm{SCR}[\mathbf{H}]$ and $\Sigma^\perp$ is a semi–decomposition on K then $(K,\Sigma,\Sigma^\perp)$ is called *a semi–creating triple for* $\mathbf{H}$.

(6) If we omit $\mathbf{H}$ this means that either $\mathbf{H}$ should be clear from the context or we mean *for some* $\mathbf{H}$.

Remark 1.2.

(1) In the definition of semi–creatures above, **dom** stands for domain, **sval** for semi–values, **nor** is for norm and **dis** for distinguish. The last plays a role of an additional parameter and it may be forgotten sometimes (compare [13, 1.1.2]). One should notice that the difference with [13, 1.1.1, 1.2.1] is in **dom** (which in case of creatures of [13, 1.2] is always an interval). This additional freedom has some price: we put a slightly more restrictive demands on **val**, so we have **sval** here. We could have more direct correspondence between **val** of [13, 1.1] and **sval** here, but that would complicate notation only.
(2) Note that in 1.1(3) we allow $\Sigma(\mathcal{S}) = \emptyset$ even if $\mathcal{S} \neq \emptyset$. In applications we will say what are the values of $\Sigma(\mathcal{S})$ only if it is a non-empty set; so not defining $\Sigma(\mathcal{S})$ means that the value is $\emptyset$. The demand 1.1(3c) appears to simplify 1.3 below only; we could have used there pos, defined like in [13, 1.1.6].
(3) The main innovation here is the additional operation $\Sigma^\perp$, which will play a crucial role in getting the ccc. If it is trivial (i.e. $\Sigma^\perp(t) = \{\{t\}\}$) then we are almost in the case of creating pairs of [13, 1.2].

Definition 1.3. Assume $(K, \Sigma, \Sigma^\perp)$ is a semi–creating triple for $\mathbf{H}$. We define forcing notions $\mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ and $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ as follows.

1) **Conditions** in $\mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ are sequences $(w, t_0, t_1, t_2, \dots)$ such that

(a) $w \in \prod\limits_{i \in \mathrm{dom}(w)} \mathbf{H}(i)$ is a finite function,
(b) each t_i belongs to K and $\omega = \mathrm{dom}(w) \cup \bigcup\limits_{i\in\omega} \mathbf{dom}[t_i]$ is a partition of ω (so $i < j < \omega$ implies $\mathrm{dom}(w) \cap \mathbf{dom}[t_i] = \mathbf{dom}[t_i] \cap \mathbf{dom}[t_j] = \emptyset$).

The relation $\leq_\emptyset$ on $\mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ is given by: $(w_1, t^1_0, t^1_1, t^1_2, \dots) \leq_\emptyset (w_2, t^2_0, t^2_1, t^2_2, \dots)$ if and only if $(w_2, t^2_0, t^2_1, t^2_2, \dots)$ can be obtained from $(w_1, t^1_0, t^1_1, t^1_2, \dots)$ by applying finitely many times the following operations (in the description of the operations we say what are their legal results for a condition $(w, t_0, t_1, t_2, \dots) \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$).

Deciding the value for $(w, t_0, t_1, t_2, \dots)$:
a legal result is a condition $(w^*, t^*_0, t^*_1, t^*_2, \dots) \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ such that for some finite $A \subseteq \omega$ (possibly empty) we have
$w \subseteq w^*$, $\mathrm{dom}(w^*) = \mathrm{dom}(w) \cup \bigcup\limits_{i\in A} \mathbf{dom}[t_i]$, $w^* \restriction \mathbf{dom}[t_i] \in \mathbf{sval}[t_i]$ (for $i \in A$) and $\{t^*_0, t^*_1, \dots\} = \{t_i : i \in \omega \setminus A\}$.

Applying Σ to $(w, t_0, t_1, t_2, \dots)$:

a legal result is a condition $(w, t^*_0, t^*_1, t^*_2, \dots) \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ such that for some disjoint sets $A_0, A_1, A_2, \dots \in [\omega]^{<\omega}$ we have

$$t^*_i \in \Sigma(t_j : j \in A_i) \quad \text{for each } i \in \omega.$$

Applying $\Sigma^\perp$ to $(w, t_0, t_1, t_2, \dots)$:
a legal result is a condition $(w, t^*_0, t^*_1, t^*_2, \dots) \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ such that for some disjoint non-empty finite sets $A_0, A_1, A_2, \dots \subseteq \omega$ we have $\{t^*_j : j \in A_i\} \in \Sigma^\perp(t_i)$.

2) If $p = (w, t_0, t_1, t_2, \dots) \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ then we let $w^p = w$, $t^p_i = t_i$.

3) Finally we define $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ as the collection of all $p \in \mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$ such that

(c) $(\forall i \in \omega)(\mathbf{nor}[t^p_i] \neq \infty)$ and $\lim\limits_{i\to\infty} \mathbf{nor}[t^p_i] = \infty$,

and the relation $\leq$ on $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ is the restriction of $\leq_\emptyset$.

Proposition 1.4. *If (K, Σ) is a semi–creating pair then $\mathbb{Q}_\emptyset(K, \Sigma, \Sigma^\perp)$, $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ are forcing notions (i.e. the relation $\leq_\emptyset$ is transitive).*

Remark 1.5.

(1) Like in [13], one may consider various variants of the demand 1.3(3)(c).
(2) Note that in a condition in $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ we allow only those $t_i \in K$ for which $\mathbf{nor}[t_i] \neq \infty$. We could restrict K to semi–creatures with finite norm, but presence of t which give no restrictions will make some definitions simpler.
(3) One should notice a close relation of $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ here to forcing notions of type $\mathbb{Q}^*_\infty(K, \Sigma)$ in [13, 1.2]. What occurs in applications here, is that (in interesting cases) the forcing notion $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ is equivalent to some $\mathbb{Q}^*_\infty(K', \Sigma')$.

As in the present paper we are interested mainly in σ–ideals on the real line, let us show how our forcing notions introduce ideals on $\prod\limits_{i\in\omega} \mathbf{H}(i)$. Later we will look more closely at $\mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ as a forcing notion.

Definition 1.6. Assume $(K, \Sigma, \Sigma^\perp)$ is a semi–creating triple for $\mathbf{H}$.

(1) For a condition $p \in \mathbb{Q}^+_\infty(K, \Sigma, \Sigma^\perp)$ we define its *total possibilities* as

$$\mathrm{POS}(p) = \{x \in \prod_{i\in\omega} \mathbf{H}(i) : w^p \subseteq x \ \&\ (\forall j \in \omega)(x{\restriction}\mathbf{dom}[t^p_j] \in \mathbf{sval}[t^p_j])\}.$$

(2) Let $\dot{W}$ be a $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$–name such that

$$\Vdash_{\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)} \dot{W} = \bigcup\{w^p : p \in \Gamma_{\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)}\}$$

(compare to [13, 1.1.13]).

(3) Let $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ be the ideal of subsets of $\prod\limits_{i\in\omega}\mathbf{H}(i)$ generated by those Borel sets $B\subseteq\prod\limits_{i\in\omega}\mathbf{H}(i)$ for which $\Vdash_{\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)}$ "$\dot{W}\notin B$".
[Remember that $\mathbf{H}(i)$ is countable (for each $i\in\omega$); so considering the product topology on $\prod\limits_{i\in\omega}\mathbf{H}(i)$ (with each $\mathbf{H}(i)$ discrete) we get a Polish space.]

Proposition 1.7. *Assume (K,Σ,Σ^+) is a semi–creating triple for $\mathbf{H}$.*

(1) *If $p,q\in\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$, $p\leq q$ then $\mathrm{POS}(q)\subseteq\mathrm{POS}(p)$, $\mathrm{POS}(p)$ is a non-empty closed subset of $\prod\limits_{i\in\omega}\mathbf{H}(i)$ and $p\Vdash$ "$\dot{W}\in\mathrm{POS}(p)$".*

(2) *$\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ is a σ–ideal, $\prod\limits_{i\in\omega}\mathbf{H}(i)\notin\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ (in fact, $\mathrm{POS}(p)\notin\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ for $p\in\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$).*

(3) *If the forcing notion $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$ satisfies the ccc then the ideal $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ satisfies the ccc.*

To make sure that the ideal $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ is invariant we have to assume natural invariance properties of the semi–creating triple.

Definition 1.8. Assume that $\mathbf{H}$ is a constant function, say $\mathbf{H}(i)=\mathcal{X}$, and we have a commutative group operation $+$ on $\mathcal{X}$. Let $(K,\Sigma,\Sigma^\perp)$ be a semi–creating triple for $\mathbf{H}$. We say that

(1) $(K,\Sigma,\Sigma^\perp)$ is *directly +–invariant* if for each $t\in K$ and $v\in\mathcal{X}^{\mathbf{dom}[t]}$ there is a unique semi–creature $s\in K$ (called $t+v$) such that

$$\mathbf{dom}[s]=\mathbf{dom}[t],\quad \mathbf{nor}[s]=\mathbf{nor}[t],$$

$$\mathbf{sval}[s]=\mathbf{sval}[t]+v=\{w+v : w\in\mathbf{sval}[t]\},$$

and $\Sigma^\perp(t+v)=\{\{s+(v\restriction\mathbf{dom}[s]) : s\in\mathcal{S}\} : \mathcal{S}\in\Sigma^\perp(t)\}$ (for $t\in K$, $v\in\mathcal{X}^{\mathbf{dom}[t]}$),
and $\Sigma(t+(v\restriction\mathbf{dom}[t]) : t\in\mathcal{S})=\{s+v : s\in\Sigma(\mathcal{S})\}$ (for $\mathcal{S}\in[K]^{<\omega}\setminus\{\emptyset\}$ and $v:\bigcup\limits_{s\in\mathcal{S}}\mathbf{dom}[s]\longrightarrow\mathcal{X}$),

(2) (K,Σ) is *directly permutation–invariant* if for each $t\in K$ and an embedding $\pi: X \stackrel{1-1}{\longrightarrow} \omega$ such that $\mathbf{dom}[t]\subseteq X\subseteq\omega$ there is a unique semi–creature $s\in K$ (called $\pi(t)$) such that

$$\mathbf{dom}[s]=\pi[\mathbf{dom}[t]],\quad \mathbf{nor}[s]=\mathbf{nor}[t],$$

$$\mathbf{sval}[s]=\{w\circ\pi^{-1}: w\in\mathbf{sval}[t]\},$$

and $\Sigma^\perp(\pi(t))=\{\{\pi(s): s\in\mathcal{S}\}:\mathcal{S}\in\Sigma^\perp(t)\}$ (for $t\in K$, $\pi:\omega\stackrel{1-1}{\longrightarrow}\omega$),
and $\Sigma(\pi(t): t\in\mathcal{S})=\{\pi(s): s\in\Sigma(\mathcal{S})\}$ (for $\mathcal{S}\in[K]^{<\omega}$, $\pi:\omega\stackrel{1-1}{\longrightarrow}\omega$).

Proposition 1.9. *Assume that* $\mathbf{H}$ *is a constant function (and* $\mathbf{H}(i)=\mathcal{X}$ *for* $i\in\omega$*), and we have a commutative group operation* $+$ *on* $\mathcal{X}$ *(so then* $\prod\limits_{i\in\omega}\mathbf{H}(i)=\mathcal{X}^\omega$ *becomes a commutative group too). Let* $(K,\Sigma,\Sigma^\perp)$ *be a semi–creating triple for* $\mathbf{H}$*.*

(1) *If* $(K,\Sigma,\Sigma^\perp)$ *is directly* $+$*–invariant then the ideal* $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ *is translation invariant (in the product group* $\mathcal{X}^\omega$*).*
(2) *If* $(K,\Sigma,\Sigma^\perp)$ *is directly permutation–invariant then* $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ *is permutation invariant (see 0.2(4)).*

Proof. 1) Note that if $v\in\mathcal{X}^\omega$ then the mapping

$$(w,t_0,t_1,\ldots)\mapsto(w+(v\restriction \mathrm{dom}(w)),t_0+(v\restriction\mathbf{dom}[t_0]),t_1\restriction(\mathbf{dom}[t_1]),\ldots)$$

is an automorphism of the forcing notion $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$.
2) Similarly. □

In general it is not clear if the ideal $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$ contains any non-empty set. One can easily formulate a demand ensuring this (like [13, 3.2.7]), but there is no need for us to deal with it, as anyway we want to finish with a ccc ideal.

2. Getting ccc. In this section we show how we can make sure that the forcing notion $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$ satisfies the ccc. The method for this (and the required properties) are quite simple and they will allow us to conclude more properties of the ideal $\mathbb{I}_\infty(K,\Sigma,\Sigma^\perp)$. The main difficulty will be to construct an example meeting all the appropriate demands (and this will be done in the next section).

Definition 2.1. Assume that $(K,\Sigma,\Sigma^\perp)$ is a semi–creating triple for $\mathbf{H}$.

(1) We say that $(K,\Sigma,\Sigma^\perp)$ *is linked* if for each $t_0,t_1\in K$ such that

$$\mathbf{nor}[t_0],\mathbf{nor}[t_1]>1 \quad\text{and}\quad \mathbf{dom}[t_0]=\mathbf{dom}[t_1]$$

there is $s\in\Sigma(t_0)\cap\Sigma(t_1)$ with

$$\mathbf{nor}[s]\geq\min\{\mathbf{nor}[t_0],\mathbf{nor}[t_1]\}-1.$$

(2) The triple $(K,\Sigma,\Sigma^\perp)$ is called *semi–gluing* if for each $t_0,\ldots,t_n\in K$ such that $k<\ell\leq n \quad\Rightarrow\quad \mathbf{dom}[t_k]\cap\mathbf{dom}[t_\ell]=\emptyset$ there is $s\in\Sigma(t_0,\ldots,t_n)$ with

$$\mathbf{nor}[s]\geq\min\{\mathbf{nor}[t_k]:k\leq n\}-1.$$

(3) We say that $(K,\Sigma,\Sigma^\perp)$ *has the cutting property* if for every $t\in K$ with $\mathbf{nor}[t]>1$ and for each non-empty $z\subsetneq\mathbf{dom}[t]$ there are $s_0,s_1\in K$ such that
   (α) $\mathbf{dom}[s_0]=z$, $\mathbf{dom}[s_1]=\mathbf{dom}[t]\setminus z$,
   (β) $\mathbf{nor}[s_\ell]\geq\mathbf{nor}[t]-1$ (for $\ell=0,1$) and
   (γ) $\{s_0,s_1\}\in\Sigma^\perp(t)$.

Remark 2.2.

(1) Semi–gluing triples $(K,\Sigma,\Sigma^\perp)$ correspond to gluing creating pairs (as defined in [13, 2.1.7(2)]).
(2) Note that in 2.1(3) we do not require that $\mathbf{nor}[s_\ell]\neq\infty$. However, if $t\in K$ satisfies $\mathbf{nor}[t]\neq\infty$ and s_0,s_1 are as in 2.1(3) then necessarily at least one of the s_ℓ's has to have the same property.

Definition 2.3. A forcing notion $\mathbb{Q}$ is *σ-$*$–linked* if for every $n\in\omega$ there is a partition $\langle A_i:i\in\omega\rangle$ of $\mathbb{Q}$ such that

$$\text{if}\quad q_0,\ldots,q_n\in A_i \quad\text{then}\quad (\exists q\in\mathbb{Q})(q_0\leq q\ \&\ \ldots\ \&\ q_n\leq q).$$

Theorem 2.4. *Assume that $(K,\Sigma,\Sigma^\perp)$ is a linked semi–gluing semi–creating triple with the cutting property. Then the forcing notion $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$ is σ-$*$–linked.*

Proof. Fix $n\in\omega$. For a finite function $w\in\prod\limits_{i\in\mathrm{dom}(w)}\mathbf{H}(i)$ let

$$A_w\stackrel{\rm def}{=}\{p\in\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp): w^p=w\ \&\ (\forall i\in\omega)(\mathbf{nor}[t^p_i]>n+5)\}.$$

Clearly $\bigcup\{A_w: w\in\prod\limits_{i\in\mathrm{dom}(w)}\mathbf{H}(i), \mathrm{dom}(w)\in[\omega]^{<\omega}\}$ is a dense subset of $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$, so it is enough to show the following claim.

Claim 2.4.1. *Let* $w\in\prod\limits_{i\in\mathrm{dom}(w)}\mathbf{H}(i)$, $\mathrm{dom}(w)\in[\omega]^{<\omega}$ *and* $p_0,\ldots,p_n\in A_w$. *Then the conditions* $p_0,\ldots,p_n$ *have a common upper bound (in the forcing notion* $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$*).*

Proof of the claim: Suppose $p_0,\ldots,p_n\in A_w$ (so in particular $w=w^{p_0}=\ldots=w^{p_n}$). Choose an increasing sequence $\langle m_i: i<\omega\rangle\subseteq\omega$ such that $\mathrm{dom}(w)\subseteq m_0$ and for every $\ell\leq n$ and each $i\in\omega$:

$$(\exists j\in\omega)(\mathbf{dom}[t^{p_\ell}_j]\subseteq m_0),\quad (\exists j\in\omega)(\mathbf{dom}[t^{p_\ell}_j]\subseteq[m_i,m_{i+1}))\quad\text{and}$$
$$(\forall j\in\omega)(\mathbf{dom}[t^{p_\ell}_j]\cap m_i\neq\emptyset\ \Rightarrow\ \mathbf{dom}[t^{p_\ell}_j]\subseteq m_{i+1})\quad\text{and}$$
$$(\forall j\in\omega)(\mathbf{dom}[t^{p_\ell}_j]\setminus m_i\neq\emptyset\ \Rightarrow\ \mathbf{nor}[t^{p_\ell}_j]>n+5+i).$$

Fix $\ell\leq n$ for a moment.
For each $j\in\omega$ and $i\in\omega$ such that $\mathbf{dom}[t^{p_\ell}_j]\cap m_i\neq\emptyset$ and $\mathbf{dom}[t^{p_\ell}_j]\cap[m_i,m_{i+1})\neq\emptyset$ use 2.1(3) to choose $s^{\ell,0}_j,s^{\ell,1}_j\in K$ such that

- $\mathbf{dom}[s^{\ell,0}_j]=\mathbf{dom}[t^{p_\ell}_j]\cap m_i$, $\mathbf{dom}[s^{\ell,1}_j]=\mathbf{dom}[t^{p_\ell}_j]\cap[m_i,m_{i+1})$,
- $\mathbf{nor}[s^{\ell,0}_j]\geq\mathbf{nor}[t^{p_\ell}_j]-1$, $\mathbf{nor}[s^{\ell,1}_j]\geq\mathbf{nor}[t^{p_\ell}_j]-1$,
- $\{s^{\ell,0}_j,s^{\ell,1}_j\}\in\Sigma^\perp(t^{p_\ell}_j)$.

Let $\mathcal{S}^\ell_0$ consist of all $t^{p_\ell}_j$ (for $j<\omega$) such that $\mathbf{dom}[t^{p_\ell}_j]\subseteq m_0$ and all $s^{\ell,0}_j$ such that

$$\mathbf{dom}[t^{p_\ell}_j]\cap m_0\neq\emptyset\neq\mathbf{dom}[t^{p_\ell}_j]\cap[m_0,m_1).$$

It should be clear that elements of $\mathcal{S}^\ell_0$ have disjoint domains and $\bigcup\limits_{s\in\mathcal{S}^\ell_0}\mathbf{dom}[s]=m_0\setminus\mathrm{dom}(w)$. Use 2.1(2) to find $r^\ell_0\in\Sigma(\mathcal{S}^\ell_0)$ such that $\mathbf{nor}[r^\ell_0]>n+3$ (remember the definition of A_w). Note that necessarily $\mathbf{nor}[r^\ell_0]\neq\infty$ as there is j such that $\mathbf{dom}[t^{p_\ell}_j]\subseteq m_0$ (remember 1.1(3) and 1.3(3)). Similarly, for each $i>0$ we take $\mathcal{S}^\ell_i$ to be the collection of all $t^{p_\ell}_j$ such that $\mathbf{dom}[t^{p_\ell}_j]\subseteq[m_{i-1},m_i)$ and all $s^{\ell,1}_j$ such that

$$\mathbf{dom}[t^{p_\ell}_j]\cap m_{i-1}\neq\emptyset\neq\mathbf{dom}[t^{p_\ell}_j]\cap[m_{i-1},m_i),$$

and all $s^{\ell,0}_j$ such that

$$\mathbf{dom}[t^{p_\ell}_j]\cap m_i\neq\emptyset\neq\mathbf{dom}[t^{p_\ell}_j]\cap[m_i,m_{i+1}).$$

Now apply 2.1(2) to get $r^\ell_i\in\Sigma(\mathcal{S}^\ell_i)$ such that $\mathbf{nor}[r^\ell_i]>n+2+i$ (remember the choice of the sequence $\langle m_i: i<\omega\rangle$; note that, like before, $\mathbf{nor}[r^\ell_i]\neq\infty$).

It should be clear that $(w,r^\ell_0,r^\ell_1,r^\ell_2,\ldots)$ is a condition in $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$ stronger than p_ℓ. Moreover, for each $\ell\leq n$,

$$\mathbf{dom}[r^\ell_0]=m_0\setminus\mathrm{dom}(w)\quad\text{and}\quad\mathbf{dom}[r^\ell_{i+1}]=[m_i,m_{i+1}).$$

By 2.1, for each $i\in\omega$ we find $s^*_i\in K$ such that

$$s^*_i\in\Sigma(r^0_i)\cap\ldots\cap\Sigma(r^n_i)\quad\text{ and }\quad \mathbf{nor}[s^*_i]>i+2.$$

Look at $(w,s^*_0,s^*_1,s^*_2,\ldots)$ — it is a condition in $\mathbb{Q}^+_\infty(K,\Sigma,\Sigma^\perp)$ stronger than all $p_0,\ldots,p_n$. The claim and the theorem are proved. □

3. The example. In this section we construct a semi–creating triple with all nice properties defined and used in the previous section.

Example 3.1. Let $(\mathcal{X},+)$ be a commutative group, $2\le|\mathcal{X}|\le\omega_0$, and let $\mathbf{H}(i)=\mathcal{X}$. Then there is a semi–creating triple $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ for $\mathbf{H}$ such that

1. the forcing notion $\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is not trivial,
2. $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is directly $+$–invariant, directly permutation–invariant,
3. it is linked and semi–gluing,
4. it has the cutting property.

Construction: Let $\mathcal{K}$ consist of all pairs (z,Δ) such that

(a) z is a non-empty finite subset of ω, and
(b) Δ is a non-empty set of non-empty partial functions such that $\mathrm{dom}(\eta)\subseteq z$ and $\mathrm{rng}(\eta)\subseteq\mathcal{X}$ for $\eta\in\Delta$.

For $(z,\Delta)\in\mathcal{K}$ we define

$$\begin{aligned} v(z,\Delta) &\stackrel{\rm def}{=} \{x\in\mathcal{X}^z:\neg(\exists\eta\in\Delta)(\eta\subseteq x)\}\\ n(z,\Delta) &\stackrel{\rm def}{=} \max\{k\in\omega: \text{for every } \Delta'\subseteq\Delta \text{ there is } \Delta''\subseteq\Delta' \text{ such that}\\ &\qquad\text{elements of } \Delta'' \text{ have pairwise disjoint domains}\\ &\qquad\text{and } |\bigcup_{\eta\in\Delta''}\mathrm{dom}(\eta)|\ge k\cdot|\Delta'|\}.\end{aligned}$$

Note that the set in the definition of $n(z,\Delta)$ contains 0. Elements of $\mathcal{K}$ and the two functions n,v are the main ingredients of our construction. Before we define $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ let us show some properties of $(\mathcal{K},n,v)$.

Claim 3.1.1.

1. *If $(z,\Delta)\in\mathcal{K}$, $n(z,\Delta)>0$ then $v(z,\Delta)\neq\emptyset$.*
2. *For each non-empty $z\in[\omega]^{<\omega}$ there is Δ such that $(z,\Delta)\in\mathcal{K}$ and $n(z,\Delta)=|z|$.*

Proof of the claim: 1) Suppose $n(z,\Delta)>0$. Then we know that for each $\Delta'\subseteq\Delta$ there is $\Delta''\subseteq\Delta'$ such that

$$\eta_0,\eta_1\in\Delta''\ \ \&\ \ \eta_0\neq\eta_1\quad\Rightarrow\quad \mathrm{dom}(\eta_0)\cap\mathrm{dom}(\eta_1)=\emptyset$$

and $|\bigcup\limits_{\eta\in\Delta''}\operatorname{dom}(\eta)|\geq|\Delta'|$. Thus we may use the marriage theorem of Hall (see [3]) and choose a system of distinct representatives of $\{\operatorname{dom}(\eta):\eta\in\Delta\}$. So for $\eta\in\Delta$ we have $x_\eta\in\operatorname{dom}(\eta)$ such that

$$\eta_0,\eta_1\in\Delta\ \ \&\ \ \eta_0\neq\eta_1\quad\Rightarrow\quad x_{\eta_0}\neq x_{\eta_1}.$$

Let $w\in\mathcal{X}^z$ be such that $w(x_\eta)\in\mathcal{X}\setminus\{\eta(x_\eta)\}$ for $\eta\in\Delta$. Clearly $w\in v(z,\Delta)$.

2) Take $\Delta=\{\mathbf{0}_z\}$, where $\mathbf{0}_z$ is a function on z with constant value 0.

Claim 3.1.2.

(1) *Suppose that* $(z,\Delta)\in\mathcal{K}$, $\emptyset\neq z^*\subseteq z$ *and*

$$\Delta^*\stackrel{\rm def}{=}\{\eta\restriction z^*:\eta\in\Delta\ \&\ |\operatorname{dom}(\eta)\cap z^*|\geq\frac{1}{2}|\operatorname{dom}(\eta)|\}\neq\emptyset.$$

Then $(z^*,\Delta^*)\in\mathcal{K}$ *and* $n(z^*,\Delta^*)\geq\lfloor\frac{1}{2}n(z,\Delta)\rfloor$.

(2) *Suppose that* $(z_0,\Delta_0),\ldots,(z_n,\Delta_n)\in\mathcal{K}$ *are such that the sets* z_k *(for* $k\leq n$*) are pairwise disjoint. Let* $z=\bigcup\limits_{k\leq n}z_k$, $\Delta=\bigcup\limits_{k\leq n}\Delta_k$. *Then* $(z,\Delta)\in\mathcal{K}$ *and* $n(z,\Delta)\geq\min\{n(z_k,\Delta_k):k\leq n\}$.

(3) *Assume* $(z,\Delta_0),(z,\Delta_1)\in\mathcal{K}$, $\Delta=\Delta_0\cup\Delta_1$. *Then* $(z,\Delta)\in\mathcal{K}$ *and* $n(z,\Delta)\geq\min\{\lfloor\frac{1}{2}n(z,\Delta_0)\rfloor,\lfloor\frac{1}{2}n(z,\Delta_1)\rfloor\}$.

Proof of the claim: 1) It should be clear that $(z^*,\Delta^*)\in\mathcal{K}$. Suppose that $\Delta'\subseteq\Delta*$. For each $\nu\in\Delta'$ fix $\eta_\nu\in\Delta$ such that

$$\nu=\eta\restriction z^*\quad\text{ and }\quad|\operatorname{dom}(\eta_\nu)\cap z^*|\geq\frac{1}{2}|\operatorname{dom}(\eta_\nu)|.$$

Loot at $\Delta^+\stackrel{\rm def}{=}\{\eta_\nu:\nu\in\Delta'\}$. By the definition of $n(z,\Delta)$ we find $\Delta^{++}\subseteq\Delta^+$ such that elements of Δ^{++} have pairwise disjoint domains and

$$|\bigcup_{\eta_\nu\in\Delta^{++}}\operatorname{dom}(\eta_\nu)|\geq(z,\Delta)\cdot|\Delta^+|.$$

So now look at $\Delta''\stackrel{\rm def}{=}\{\nu\in\Delta':\eta_\nu\in\Delta^{++}\}$. Clearly, elements of Δ'' have pairwise disjoint domains and

$$|\bigcup_{\nu\in\Delta''}\operatorname{dom}(\nu)|\geq\frac{1}{2}|\bigcup_{\eta_\nu\in\Delta^{++}}\operatorname{dom}(\eta_\nu)|\geq\frac{n(z,\Delta)}{2}\cdot|\Delta^+|\geq\lfloor\frac{1}{2}n(z,\Delta)\rfloor\cdot|\Delta^+|.$$

2) Suppose that $\Delta'\subseteq\Delta$. For $k\leq n$ let $\Delta'_k=\Delta'\cap\Delta_k$ and choose $\Delta''_k\subseteq\Delta'_k$ such that the sets $\operatorname{dom}(\eta)$ (for $\eta\in\Delta''_k$) are pairwise disjoint and

$$|\bigcup_{\eta\in\Delta''_k}\operatorname{dom}(\eta)|\geq n(z_k,\Delta_k)\cdot|\Delta'_k|.$$

Let $\Delta''=\bigcup\limits_{k\le n}\Delta''_k$. Clearly the elements of Δ'' have pairwise disjoint domains. Moreover,

$$\begin{aligned}|\bigcup_{\eta\in\Delta''}\mathrm{dom}(\eta)| &= \sum_{k\le n}\sum_{\eta\in\Delta''_k}|\mathrm{dom}(\eta)|\ge\sum_{k\le n}n(z_k,\Delta_k)\cdot|\Delta'_k|\\ &\ge|\Delta'|\cdot\min\{n(z_k,\Delta_k):k\le m\}.\end{aligned}$$

3) Let $\Delta'\subseteq\Delta$. Let $\ell<2$ be such that $|\Delta'\cap\Delta|\ge\frac{1}{2}|\Delta'|$. Now we may choose $\Delta''\subseteq\Delta'\cap\Delta_\ell$ such that all members of Δ'' have pairwise disjoint domains and

$$|\bigcup_{\eta\in\Delta''}\mathrm{dom}(\eta)|\ge n(z,\Delta_\ell)\cdot|\Delta'\cap\Delta_\ell|\ge\lfloor\frac{1}{2}n(z,\Delta_\ell)\rfloor\cdot|\Delta'|.$$

Now we are ready to define the triple $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$. A semi–creature $t\in\mathrm{SCR}[\mathbf{H}]$ is taken to $K_{3.1}$ if

- $\mathbf{dis}[t]=(z_t,\Delta_t)$, where either $\emptyset\ne z_t\in[\omega]^{<\omega}$, $\Delta_t=\emptyset$ or $(z_t,\Delta_t)\in\mathcal{K}$ and $n(z_t,\Delta_t)\ge 1$,
- $\mathrm{dom}[t]=z_t$,
- if $(z_t,\Delta_t)\in\mathcal{K}$ then

$$\mathbf{nor}[t]=\log_8(n(z_t,\Delta_t))\quad\text{ and }\quad\mathbf{sval}[t]=v(z_t,\Delta_t),$$

- if $\Delta_t=\emptyset$ then $\mathbf{nor}[t]=\infty$ and $\mathbf{sval}[t]=\mathcal{X}^{z_t}$.

For semi–creatures $t_0,\dots,t_n\in K_{3.1}$ with disjoint domains, $\Sigma_{3.1}(t_0,\dots,t_n)$ consists of all $t\in K_{3.1}$ such that

$$z_t=\bigcup_{\ell\le n}z_{t_\ell}\quad\text{ and }\quad\Delta_t\supseteq\bigcup_{\ell\le n}\Delta_{t_\ell}$$

(so in particular $\mathbf{dom}[t]=\bigcup\limits_{\ell\le n}\mathbf{dom}[t_\ell]$ and $\mathbf{sval}[t]\subseteq\{v\in\mathcal{X}^{\mathbf{dom}[t]}:(\forall\ell\le n)(v\restriction\mathbf{dom}[t_\ell]\in\mathbf{sval}[t_\ell])\}$). It should be clear that $\Sigma_{3.1}$ is a semi–composition operation on $K_{3.1}$ (i.e. the demands (a)–(c) of 1.1(3) are satisfied). Next, for a semi–creature $t\in K_{3.1}$ we define $\Sigma^\perp_{3.1}(t)$ as follows. It consists of all sets $\{s_0,\dots,s_n\}\subseteq K_{3.1}$ such that

$$z_t=\bigcup_{\ell\le n}z_{s_\ell}\quad\text{ and }\quad(\forall\eta\in\Delta_t)(\exists\ell\le n)(\eta\restriction z_{s_\ell}\in\Delta_{s_\ell})$$

(note that this implies that at least one Δ_{s_ℓ} is non-empty, provided $\Delta_t\ne\emptyset$). Again, $\Sigma^\perp_{3.1}$ is a semi–decomposition operation on $K_{3.1}$ (i.e. clauses $(a)^\perp$–$(c)^\perp$ of 1.1(4) hold). Consequently, $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is a semi–creating triple for $\mathbf{H}$, and plainly it is directly +–invariant and directly permutation–invariant. It follows from 3.1.1 that the forcing notion $\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$

is not trivial. To show that the triple $(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ has the cutting property suppose that $t \in K_{3.1}$, $\mathbf{nor}[t] > 1$ and $\emptyset \neq z \subsetneq \mathbf{dom}[t]$. Let

$$\begin{aligned}\Delta_0 &= \{\eta\restriction z : \eta \in \Delta_t \ \&\ |\operatorname{dom}(\eta)\cap z| \geq \tfrac{1}{2}|\operatorname{dom}(\eta)|\},\\ \Delta_1 &= \{\eta\restriction(\mathbf{dom}[t]\setminus z) : \eta \in \Delta_t \ \&\ |\operatorname{dom}(\eta)\setminus z| \geq \tfrac{1}{2}|\operatorname{dom}(\eta)|\}\end{aligned}$$

and let $s_0, s_1 \in K_{3.1}$ be such that $\mathbf{dis}[s_0] = (z, \Delta_0)$, $\mathbf{dis}[s_1] = (\mathbf{dom}[t] \setminus z, \Delta_1)$. Easily $\{s_0, s_1\} \in \Sigma^{\perp}_{3.1}$ is as required in 2.1(3) (remember 3.1.2(1)). Similarly, it follows from 3.1.2(3) that $(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ is linked, and we immediately conclude from 3.1.2(2) that it is semi–gluing.

Conclusion 3.2. *Let $(\mathcal{X}, +)$ be a finite Abelian group. Let $(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ be the semi–creating pair constructed in 3.1 for $\mathbf{H}(i) = \mathcal{X}$ $(i \in \omega)$. Then $\mathbb{I}_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ is:*

> *a Borel ccc index–invariant translation–invariant σ–ideal of subsets of $\mathcal{X}^\omega$ which is neither the meager ideal nor the null ideal nor their intersection.*

Proof. By 1.9 and 1.7 + 2.4 we know that $\mathbb{I}_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ is a Borel ccc permutation–invariant translation–invariant σ–ideal of subsets of $\mathcal{X}^\omega$. Still we have to show the following claim.

Claim 3.2.1. *$\mathbb{I}_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ is index–invariant.*

Proof of the claim: Let $\pi : \omega \xrightarrow{1-1} \omega$ be an embedding. Let

$$\mathbb{Q}_\pi \stackrel{\rm def}{=} \{p \in \mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1}) : (\forall i \in \omega)(\mathbf{dom}[t^p_i] \cap \operatorname{rng}(\pi) = \emptyset \ \text{ or } \ \mathbf{dom}[t^p_i] \subseteq \operatorname{rng}(\pi))\}.$$

Like in 2.4.1 one shows that $\mathbb{Q}_\pi$ is a dense subset of $\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ (remember that if $t_0, \ldots, t_n \in K_{3.1}$ have disjoint domains and norms ∞ then there is $t \in \Sigma_{3.1}(t_0, \ldots, t_n)$ with $\mathbf{nor}[t] = \log_8(n + 1)$). Define a mapping $f_\pi : \mathbb{Q}_\pi \longrightarrow \mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ by

$$f_\pi(p) = (w^p \circ \pi, \langle \pi^{-1}(t^p_i) : i \in \omega,\ \mathbf{dom}(t^p_i) \subseteq \operatorname{rng}(\pi)\rangle) \qquad \text{for } p \in \mathbb{Q}_\pi.$$

(It should be clear that $f_\pi(p)$ is a condition in $\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$). We claim that f_π is a projection, and for this we have to show that

(α) if $p, q \in \mathbb{Q}_\pi$, $p \leq q$ then $f_\pi(p) \leq f_\pi(q)$, and
(β) if $p \in \mathbb{Q}_\pi$, $r \in \mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^{\perp}_{3.1})$ are such that $f_\pi(p) \leq r$ then there is $q \in \mathbb{Q}_\pi$ such that $p \leq q$ and $r \leq f_\pi(q)$.

So suppose that $p, q \in \mathbb{Q}_\pi$, $p \leq q$. Thus the condition q can be constructed from p by repeating finitely many times the operations described in 1.3: deciding the value, applying $\Sigma_{3.1}$ and applying $\Sigma^{\perp}_{3.1}$. Now we would like to apply the same procedures but restricting all creatures involved to $\mathrm{rng}(\pi)$ (remember $p, q \in \mathbb{Q}_\pi$). However it requires some extra care: in the original procedure we may build at some moment a semi–creature t such that $\mathbf{dom}[t] \cap \mathrm{rng}(\pi) \neq \emptyset \neq \mathbf{dom}[t] \setminus \mathrm{rng}(\pi)$. What should be the result s of restricting t to $\mathrm{rng}(\pi)$? We should take $z_s = z_t \cap \mathrm{rng}(\pi)$, but we cannot take just $\Delta_s = \{\eta{\restriction}z_s : \eta \in \Delta_t \ \&\ \mathrm{dom}(\eta) \cap z_s \neq \emptyset\}$, as later (in some application of $\Sigma^\perp_{3.1}$) some functions $\eta \in \Delta_t$ may be restricted to sets disjoint from $\mathrm{rng}(\pi)$ (look at the definition of $\Sigma^\perp_{3.1}$). But we know where we are going to finish: the active elements are $\nu \in \Delta_{t^q_i}$ for those i that $\mathbf{dom}[t^q_i] \subseteq \mathrm{rng}(\pi)$. So, whenever in our procedure of building q from p we create a semi–creature $t \in K_{3.1}$ such that $\mathbf{dom}[t] \cap \mathrm{rng}(\pi) \neq \emptyset$, we replace it by s such that $z_s = z_t \cap \mathrm{rng}(\pi)$ and $\Delta_s \subseteq \{\eta{\restriction}z_s : \eta \in \Delta_t \ \&\ (\exists \nu \in \bigcup\limits_{i\in\omega} \Delta_{t^q_i})(\nu \subseteq \eta{\restriction}z_s)\}$ is defined as follows. Let $\eta \in \Delta_t$. We ask if there is a sequence of $\langle \eta_i, t_i : i \leq k\rangle$ such that $\eta_0 = \eta$, $\eta_i \in \Delta_{t_i}$, $\eta_i \supseteq \eta_{i+1}$, $\mathrm{dom}(\eta_k) \subseteq \mathrm{rng}(\pi)$, $t_0 = t$, $t_k = t^q_j$ (for some j), $t_0, \ldots, t_k \in K_{3.1}$ appear at the successive levels of the construction (of q from p) after the one we consider. If the answer is positive we take $\eta{\restriction}z_s$ to Δ_s, otherwise not.

We ignore any t with $\mathrm{dom}[t] \cap \mathrm{rng}(\pi) = \emptyset$ and whenever we apply deciding the values we replace the respective sequence w (extending w^p) by $w{\restriction}\mathrm{rng}(\pi)$. This procedure results in transforming the sequence

$$(w^p{\restriction}\mathrm{rng}(\pi), t^p_i : \mathbf{dom}[t^p_i] \subseteq \mathrm{rng}(\pi))$$

by successive applications of legal operations (and getting results with domains included in $\mathrm{rng}(\pi)$). The final sequence is $(w^q, t^q_i : \mathbf{dom}[t^q_i] \subseteq \mathrm{rng}(\pi))$. Now we use π^{-1} to carry this procedure to $\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1})$ and we get a witness for $f_\pi(p) \leq f_\pi(q)$.

To show clause (β) suppose that $p \in \mathbb{Q}_\pi$, $r \in \mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1})$ are such that $f_\pi(p) \leq r$. Let $w^q = (w^r \circ \pi^{-1}) \cup (w^p \setminus \mathrm{rng}(\pi))$, and $t^q_i = t^p_i$ whenever $\mathbf{dom}[t^p_i] \cap \mathrm{rng}(\pi) = \emptyset$, and

$$\{t^q_i : i \in \omega,\ \mathbf{dom}[t^p_i] \subseteq \mathrm{rng}(\pi)\} = \{\pi(t^r_j) : j \in \omega\}.$$

It should be clear that $q = (w^q, t^q_0, t^q_1, \ldots) \in \mathbb{Q}_\pi$ is stronger than p and $f_\pi(q) = r$.

Thus the mapping f_π induces a complete embedding g^*_π of the complete Boolean algebra $\mathrm{BA}(\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1}))$ determined by $\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1})$ into $\mathrm{BA}(\mathbb{Q}_\pi)$ (the last is, of course, isomorphic to $\mathrm{BA}(\mathbb{Q}^+_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1}))$). Assume that $A \in \mathbb{I}_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1})$ is a Borel set. We want to show that $\pi_*(A) \in \mathbb{I}_\infty(K_{3.1}, \Sigma_{3.1}, \Sigma^\perp_{3.1})$. So suppose otherwise. Then we find a condition $p \in \mathbb{Q}_\pi$ such that $p \Vdash \dot{W} \in \pi_*(A)$ or, in other words, $p \Vdash \dot{W} \circ \pi \in A$.

But $\dot{W}\circ\pi$ is the image of $\dot{W}$ under the embedding g^*_π, so $f_\pi(p)\Vdash \dot{W}\in A$, a contradiction. Thus the claim is proved.

To finish the proof of the conclusion we have to check that the ideal $\mathbb{I}_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is really new. For this it is enough to show the following.

Claim 3.2.2.

(1) *For each* $p\in\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$, $\mathrm{POS}(p)$ *is a nowhere dense set.*

(2) *For every* $p\in\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ *there is* $q\geq p$ *such that* $\mathrm{POS}(q)$ *is a null subset of* $\mathcal{X}^\omega$.

Proof of the claim: 1) Should be clear.

2) As $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is semi–gluing we may assume that

$$(\forall i\in\omega)(|\,\mathbf{dom}[t^p_i]|>(3+i)\cdot|\mathcal{X}|^{2\cdot(3+i)}\ \text{ and }\ \mathbf{nor}[t^p_i]>5).$$

For each $i\in\omega$ choose a family Δ_i of partial functions with disjoint domains contained in $\mathbf{dom}[t^p_i]$ such that

$$|\Delta_i|=|\mathcal{X}|^{2\cdot(3+i)}\quad\text{ and }\quad \eta\in\Delta_i\ \Rightarrow|\eta|=3+i.$$

Let $s_i\in K_{3.1}$ be such that $\mathbf{dis}[s_i]=(z_{t^p_i},\Delta_i)$. Clearly $\mathbf{nor}[s_i]=\log_8(3+i)$, so $q_0=(w^p,s_0,s_1,s_2,\dots)\in\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$. Note that

$$\mathrm{Leb}(\{v\in\mathcal{X}^\omega:(\forall\nu\in\Delta_i)(\nu\not\subseteq v)\})=(1-\frac{1}{|\mathcal{X}|^{3+i}})^{|\mathcal{X}|^{2\cdot(3+i)}}\leq e^{-|\mathcal{X}|^{3+i}}$$

(where Leb stands for the (product) Lebesgue measure on $\mathcal{X}^\omega$). Hence $\mathrm{Leb}(\mathrm{POS}(q_0))=0$. Since $(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ is linked (and $\mathbf{dom}[s_i]=\mathbf{dom}[t^p_i]$) we find a condition q stronger than both p and q_0. As $\mathrm{POS}(q)\subseteq\mathrm{POS}(q_0)$, we are done. □

4. Making the example (perhaps) nicer. The ideal $\mathbb{I}_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ constructed in the previous section may be considered as a not so nice solution to 0.1. First note that there is no explicit relation between the ideal and the sets $\mathrm{POS}(p)$. One would like to have here that the family of all sets $\mathrm{POS}(p)$ is a basis of some ccc topology on $\mathcal{X}^\omega$ and the ideal is the ideal of meager (with respect to this topology) subsets of $\mathcal{X}^\omega$, or at least be able to apply Category Base approach of Morgan (see [10], [11]). But it is not clear if we may get this in our example. Moreover, the complexity of the incompatibility relation in $\mathbb{Q}^+_\infty(K_{3.1},\Sigma_{3.1},\Sigma^\perp_{3.1})$ seems to be above Σ^1_1, so the forcing notion is not really nice. However, we may modify our example a little bit to get more nice properties of the forcing notion. But the price to pay is that we go slightly out of the schema of *norms on possibilities*.

Definition 4.1.

(1) Let $\mathcal{N}$ consists of all sequences $\bar{n} = \langle n^0_m, n^1_m : m < \omega\rangle$ such that
- $4 < n^0_m \leq n^1_m < n^0_{m+1} < \omega$ for each $m \in \omega$, and
- $2^{2(m^*+2)} \cdot \sum\limits_{m<m^*} n^0_m \cdot n^1_m < n^0_{m^*}$ for each $m^* \in \omega$, and
- $\lim\limits_{m\to\infty} (n^1_m)^{\frac{1}{2\cdot n^0_m}} = \infty$.

(2) Let $\bar{n} = \langle n^0_m, n^1_m : m < \omega\rangle \in \mathcal{N}$. We define a forcing notion $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ (where $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\aleph_1)$) as follows.
Conditions are sequences $p = (w, \sigma_0, \sigma_1, \sigma_2, \dots)$ such that
(a) w, σ_j (for $j \in \omega$) are finite functions with pairwise disjoint domains, $w \in \prod\limits_{i\in\mathrm{dom}(w)} \mathbf{H}(i)$, $\sigma_j \in \prod\limits_{i\in\mathrm{dom}(\sigma_j)} \mathbf{H}(i)$,
(b) for some $m^* = m^*(p) < \omega$ there is a partition $\langle V^p_m : m^* \leq m < \omega\rangle$ of ω such that for each $m \geq m^*$

$$|V^p_m| \leq n^1_m \cdot 2^{m^*} \quad \text{and} \quad (\forall j \in V^p_m)(|\mathrm{dom}(\sigma_j)| \geq \frac{n^0_m}{2^{m^*}}).$$

For a condition $p = (w, \sigma_0, \sigma_1, \sigma_2, \dots)$ we let

$$\mathrm{POS}(p) = \{\eta \in \prod_{i\in\omega} \mathbf{H}(i) : w \subseteq \eta \ \&\ (\forall j \in \omega)(\sigma_j \not\subseteq \eta)\}.$$

The order is given by: $p \leq q$ if and only if $\mathrm{POS}(q) \subseteq \mathrm{POS}(p)$.

(3) We will keep the convention that a condition $p \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ is $(w^p, \sigma^p_0, \sigma^p_1, \dots)$.

Proposition 4.2. *Let $\bar{n} \in \mathcal{N}$.*

(1) *$\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ is a forcing notion.*
(2) *For $p, q \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$, $p \leq q$ if and only if $w^p \subseteq w^q$ and for each $i \in \omega$*

$$(\exists j \in \omega)(\sigma^q_j \subseteq \sigma^p_i) \text{ or } (\exists m \in \mathrm{dom}(w^q) \cap \mathrm{dom}(\sigma^p_i))(w^q(m) \neq \sigma^p_i(m)).$$

Proof. 2) Assume that $\mathrm{POS}(q) \subseteq \mathrm{POS}(p)$.
First note that necessarily $w^q \cup w^p$ is a function. Suppose that $k \in \mathrm{dom}(w^p) \setminus \mathrm{dom}(w^q)$. If $k \notin \bigcup\limits_{i\in\omega} \mathrm{dom}(\sigma^q_i)$ then we may easily construct a function in $\mathrm{POS}(q) \setminus \mathrm{POS}(p)$. So for some i we have $k \in \mathrm{dom}(\sigma^q_i)$. Take $\ell \in \mathrm{dom}(\sigma^q_i) \setminus \{k\}$ and build a function $\eta \in \mathrm{POS}(q)$ such that $\eta(\ell) \neq \sigma^q_i(\ell)$, $\eta(k) \neq w^q(k)$. Then $\eta \notin \mathrm{POS}(p)$, a contradiction showing that $w^p \subseteq w^q$.
Suppose now that $i \in \omega$ is such that for no $j \in \omega$, $\sigma^q_j \subseteq \sigma^p_i$. Then for each $j \in \omega$ such that $\mathrm{dom}(\sigma^q_j) \cap \mathrm{dom}(\sigma^p_i) \neq \emptyset$, either there is $k \in \mathrm{dom}(\sigma^q_j) \cap \mathrm{dom}(\sigma^p_i)$

such that $\sigma^q_j(k) \neq \sigma^p_i(k)$ or $\mathrm{dom}(\sigma^q_j) \setminus \mathrm{dom}(\sigma^p_i) \neq \emptyset$. Thus we may build a partial function $\eta \in \prod\limits_{i\in\mathrm{dom}(\eta)} \mathbf{H}(i)$ with $\mathrm{dom}(\eta) = \bigcup\limits_{j\in\omega} \mathrm{dom}(\sigma^q_j)$ and such that

$$\eta\restriction \mathrm{dom}(\sigma^p_i) \subseteq \sigma^p_i \quad \text{ and } \quad (\forall j\in\omega)(\sigma^q_j \not\subseteq \eta).$$

As any member of $\prod\limits_{i\in\omega} \mathbf{H}(i)$ extending $\eta \cup w^q$ is in $\mathrm{POS}(q)$ (and thus in $\mathrm{POS}(p)$) we conclude that the functions w^q and σ^p_i are incompatible, i.e.

$$(\exists m \in \mathrm{dom}(w^q)\cap \mathrm{dom}(\sigma^p_i))(w^q(m) \neq \sigma^p_i(m)).$$

The converse implication is even easier. □

Let us recall that a partial order $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is Souslin (Borel, respectively) if $\mathbb{Q}$, $\leq_{\mathbb{Q}}$ and the incompatibility relation $\perp_{\mathbb{Q}}$ are Σ^1_1 (Borel, respectively) subsets of $\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}$. On Souslin forcing notions and their applications see Judah Shelah [4], Goldstern Judah [2] and Judah Rosłanowski Shelah [6] (the results of these three and many other papers on the topic are presented in Bartoszyński Judah [1] too). A new systematic treatment of definable forcing notions is presented in a forthcoming paper [15] (several results there are applicable to forcing notions defined in this paper).

Theorem 4.3. *Suppose that $\bar{n} \in \mathcal{N}$ and $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\aleph_1)$.*

1. *The forcing notion $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ is σ-$*$–linked.*
2. *Conditions $p_0, p_1 \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ are compatible (in $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$) if and only if $\mathrm{POS}(p_0)\cap\mathrm{POS}(p_1) \neq \emptyset$.*
3. *The forcing notion $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ is Souslin ccc.*

Proof. 1) For a condition $p \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ let $m^*(p)$ and $\langle V^p_m : m^*(p) \leq m < \omega\rangle$ be given by 4.1(2b).
Fix $n \in \omega$.
For $m^* \in \omega$, $w \in \prod\limits_{i\in\mathrm{dom}(w)} \mathbf{H}(i)$, $\mathrm{dom}(w) \in [\omega]^{<\omega}$ and sequences

$$\bar{V} = \langle V_m : m^* \leq m < m^*{+}n{+}2\rangle \subseteq [\omega]^{<\omega} \quad \text{ and } \quad \bar{\sigma} = \langle \sigma_j : j \in \bigcup_{m=m^*}^{m^*+n+1} V_m\rangle$$

we let

$$A^{\bar{V},\bar{\sigma}}_{m^*,w} = \{p \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}} : \ m^*(p) = m^* \ \&\ (\forall m \in [m^*, m^*{+}n{+}2))(V^p_m = V_m) \ \&$$
$$(\forall j \in \bigcup_{m=m^*}^{m^*+n+1} V_m)(\sigma^p_j = \sigma_j) \ \&\ w^p = w\}.$$

We want to show that any $n+1$ members of $A^{\bar{V},\bar{\sigma}}_{m^*,w}$ have a common upper bound in $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$. For this we will need the following two technical observations.

Claim 4.3.1. *Suppose that* $p_0,\dots,p_n\in\mathbb{Q}^{\mathbf{H}}_{\bar n}$, $\max\{m^*(p_0),\dots,m^*(p_n)\}\le m^*$ *and* $U\subseteq\bigcup\{V^{p_\ell}_m\times\{\ell\}:\ell\le n,\ m^*+n+2\le m\}$ *is finite. Then there are pairwise disjoint sets* $u_{j,\ell}$ *for* $(j,\ell)\in U$ *such that*

$$u_{j,\ell}\subseteq\mathrm{dom}(\sigma^{p_\ell}_j)\quad\textit{and}\quad |u_{j,\ell}|\ge\frac{n^0_m}{2^{m^*+n+1}},$$

where m *is such that* $j\in V^{p_\ell}_m$.

Proof of the claim: Let $y(j,\ell)=\lfloor\frac{n^0_m}{2^{m^*+n+1}}\rfloor+1$ (for $(j,\ell)\in U$ and m such that $j\in V^{p_\ell}_m$). For $(j,\ell)\in U$ and $y<y(j,\ell)$ let $a^y_{j,\ell}=\mathrm{dom}(\sigma^{p_\ell}_j)$. We want to apply Hall's theorem to the system $\langle a^y_{j,\ell}:(j,\ell)\in U\ \&\ y<y(j,\ell)\rangle$. So suppose that $\mathcal{A}\subseteq\{(j,\ell,y):(j,\ell)\in U\ \&\ y<y(j,\ell)\}$. For some $\ell^*\le n$ we have $|\{(j,\ell,y)\in\mathcal{A}:\ell=\ell^*\}|\ge\frac{|\mathcal{A}|}{n+1}$. Now, remembering that

$$|a^y_{j,\ell}|\ge\frac{n^0_m}{2^{m^*(p_\ell)}}\ge\frac{n^0_m}{2^{m^*}}>y(j,\ell)\cdot 2^n,$$

we easily conclude that

$$|\bigcup\{a^y_{j,\ell^*}:(j,\ell^*,y)\in\mathcal{A}\}|\ge|\{(j,\ell,y)\in\mathcal{A}:\ell=\ell^*\}|\cdot 2^n\ge\frac{|\mathcal{A}|}{n+1}\cdot 2^n\ge|\mathcal{A}|.$$

Consequently we may apply the Hall theorem (see [3]) and choose a system $\langle k^y_{j,\ell}:(j,\ell)\in U\ \&\ y<y(j,\ell)\rangle$ of distinct representatives for $\langle a^y_{j,\ell}:(j,\ell)\in U,\ y<y(j,\ell)\rangle$ (so $k^y_{j,\ell}\in\mathrm{dom}(\sigma^{p_\ell}_j)$). Now let $u_{j,\ell}=\{k^y_{j,\ell}:y<y(j,\ell)\}$ (for $(j,\ell)\in U$). It should be clear that these sets are as required.

Claim 4.3.2. *Suppose* $p_0,\dots,p_n\in\mathbb{Q}^{\mathbf{H}}_{\bar n}$, $\max\{m^*(p_0),\dots,m^*(p_n)\}\le m^*$. *There is a sequence* $\langle u_{j,\ell}:\ell\le n,\ j\in\bigcup\{V^{p_\ell}_m:m\ge m^*+n+2\}\rangle$ *of pairwise disjoint sets such that*

$$u_{j,\ell}\subseteq\mathrm{dom}(\sigma^{p_\ell}_j)\quad\textit{and}\quad |u_{j,\ell}|\ge\frac{n^0_m}{2^{m^*+n+1}},$$

where m *is such that* $j\in V^{p_\ell}_m$.

Proof of the claim: By 4.3.1 we know that for each finite $U\subseteq\bigcup\{V^{p_\ell}_m\times\{\ell\}:\ell\le n,\ m^*+n+2\le m\}$ we can find a sequence $\langle u_{j,\ell}:(j,\ell)\in U\rangle$ with the respective properties. Of course, for each $U^*\subseteq U$ the restricted sequence $\langle u_{j,\ell}:(j,\ell)\in U^*\rangle$ will have those properties too. Moreover, for each finite U the number of all possible sequences is finite. Consequently we may use König lemma and conclude that there exists $\langle u_{j,\ell}:\ell\le n,\ j\in\bigcup\{V^{p_\ell}_m:m\ge m^*+n+2\}\rangle$ as desired.

Claim 4.3.3. *Suppose* $p_0,\dots,p_n\in A^{\bar V,\bar\sigma}_{m^*,w}$. *Then the conditions* $p_0,\dots,p_n$ *have a common upper bound in* $\mathbb{Q}^{\mathbf{H}}_{\bar n}$.

Proof of the claim: Using 4.3.2 choose a sequence

$$\langle u_{j,\ell}:\ell\leq n\ \&\ j\in\bigcup\{V^{p_\ell}_m: m\geq m^*+n+2\}\rangle$$

of pairwise disjoint sets such that $u_{j,\ell}\subseteq\mathrm{dom}(\sigma^{p_\ell}_j)$ and $|u_{j,\ell}|\geq\frac{n^0_m}{2^{m^*+n+1}}$, where m is such that $j\in V^{p_\ell}_m$. Let $\sigma^q_{j,\ell}=\sigma^{p_\ell}_j\restriction u_{j,\ell}$ and let w^q be a finite function such that $w^q\in\prod\limits_{i\in\mathrm{dom}(w^q)}\mathbf{H}(i)$,

$$\mathrm{dom}(w^q)=\mathrm{dom}(w)\cup\bigcup\{\mathrm{dom}(\sigma_j): j\in\bigcup_{m=m^*}^{m^*+n+1}V_m\}$$

and $w\subseteq w^q$ and if $k\in\mathrm{dom}(\sigma_j)$, $j\in V_m$, $m^*\leq m<m^*+n+2$ then $\sigma_j(k)\neq w^q(k)$. Look at the sequence

$$q=(w^q,\sigma^q_{j,\ell}:\ell\leq n,\ j\in\bigcup\{V^{p_\ell}_m: m^*+n+2\leq m\}).$$

It is a condition in $\mathbb{Q}^{\mathbf{H}}_{\bar n}$ with $m^*(q)=m^*+n+2$ and $V^q_m=\{(j,\ell):\ell\leq n\ \&\ j\in V^{p_\ell}_m\}$ witnessing the clause 4.1(2b). (To be strict, one should re-enumerate all $\sigma^q_{j,\ell}$ to have single indexes, but that is not a problem.) Immediately by the choice of the $\sigma^q_{j,\ell}$'s and w^q one concludes

$$\mathrm{POS}(q)\subseteq\mathrm{POS}(p_0)\cap\ldots\cap\mathrm{POS}(p_n),$$

finishing the proof of the claim.

Since there are countably many possibilities for indexes $(\bar V,\bar\sigma,m^*,w)$ (in $A^{\bar V,\bar\sigma}_{m^*,w}$), the first part of the theorem follows from 4.3.3.

2) Suppose $\eta\in\mathrm{POS}(p_0)\cap\mathrm{POS}(p_1)$. Let $m^*\geq m^*(p_0)+m^*(p_1)+3$ be such that $2^{m^*+1}>|\mathrm{dom}(w^{p_0})\cup\mathrm{dom}(w^{p_1})|$. For $\ell<2$, $m\in[m^*(p_\ell),m^*)$ and $j\in V^{p_\ell}_m$ choose an integer $k^\ell_j\in\mathrm{dom}(\sigma^{p_\ell}_j)$ such that $\eta(k^\ell_j)\neq\sigma^{p_\ell}_j(k^\ell_j)$. Let

$$a=\mathrm{dom}(w^{p_0})\cup\mathrm{dom}(w^{p_1})\cup\{k^\ell_j:\ell<2,\ j\in\bigcup_{m=m^*(p_\ell)}^{m^*-1}V^{p_\ell}_m\}.$$

Plainly,

$$|a|\leq|\mathrm{dom}(w^{p_0})\cup\mathrm{dom}(w^{p_1})|+2^{m^*+1}\cdot\sum_{m<m^*}n^1_m<\frac{n^0_{m^*}}{2^{m^*}}$$

(remember 4.1(1)). Now for $\ell<2$, $m\geq m^*$ and $j\in V^{p_\ell}_m$ let $\sigma^*_{\ell,j}=\sigma^{p_\ell}_j\restriction(\mathrm{dom}(\sigma^{p_\ell}_j)\setminus a)$. Note that (for relevant ℓ,j,m)

$$|\mathrm{dom}(\sigma^*_{\ell,j})|>\frac{n^0_m}{2^{m^*(p_\ell)}}-\frac{n^0_{m^*}}{2^{m^*}}>4\frac{n^0_m}{2^{m^*}}.$$

Like in 4.3.3, use 4.3.2 to get a sequence

$$(\sigma^q_{j,\ell} : \ell < 2,\ j \in \bigcup\{V^{p_\ell}_m : m^* \leq m\})$$

such that $\sigma^q_{j,\ell} \subseteq \sigma^*_{\ell,j}$, $|\operatorname{dom}(\sigma^q_{j,\ell})| \geq \frac{n^0_m}{2^{m^*}}$. Next we let $w^q = \eta\restriction a$ and as in 4.3.2 we see that

$$q = (w^q, \sigma^q_{j,\ell} : \ell < 2,\ j \in \bigcup\{V^{p_\ell}_m : m^* \leq m\}) \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$$

is a condition stronger than both p_0 and p_1. (But note that we cannot claim that $\eta \in \mathrm{POS}(q)$.).

3) Let $\mathcal{Y}$ be the space of all sequences $(w, \sigma_0, \sigma_1, \dots)$ of finite functions such that $w \in \prod\limits_{i\in\operatorname{dom}(w)} \mathbf{H}(i)$, $\sigma_j \in \prod\limits_{i\in\operatorname{dom}(\sigma_j)} \mathbf{H}(i)$. The space $\mathcal{Y}$, equipped with the product topology of discrete spaces, is a Polish space. Now, $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ is a Σ^1_1 subset of $\mathcal{Y}$, as to express that $(w, \sigma_0, \sigma_1, \dots) \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ we need to say that "*there is* a partition $\langle V_m : m^* \leq m < \omega\rangle$ of ω as in 4.1(2b)" and the rest of the demands is Borel. The relation $\leq_{\mathbb{Q}^{\mathbf{H}}_{\bar{n}}}$ is clearly Σ^1_1 if one uses 4.2(2) to express it. Finally, to deal with the incompatibility relation look at the proof of clause 2) above. To say that conditions p_0, p_1 are compatible we have to say that, for m^* as there, we can find points $k^\ell_j \in \operatorname{dom}(\sigma^{p_\ell})$ (for $\ell < 2$, j as there) and a function w^q such that $w^{p_0} \cup w^{p_1} \subseteq w^q$ and $w^q(k^\ell_j) \neq \sigma^{p_\ell}_j(k^\ell_j)$. □

Remark 4.4. Note that one can easily choose a dense suborder $\mathbb{Q}^* \subseteq \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ such that $\mathbb{Q}^*$ is Borel ccc: take to $\mathbb{Q}^*$ those conditions p for which $|\operatorname{dom}(\sigma_j)| = \lfloor\frac{n^0_m}{2^{m^*(p)}}\rfloor + 1$ for $j \in V^p_m$, $m \geq m^*(p)$.

Before we introduce an ideal related to $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ let us recall some basic notions of the Category Base technique.

Definition 4.5 (Morgan, see [10], [11])**.**

(1) A family $\mathcal{C}$ of subsets of a space $\mathcal{X}$ is called *a category base on* $\mathcal{X}$ if
 - (a) $\mathcal{X} = \bigcup\mathcal{C}$,
 - (b) for every $\mathcal{D} \subseteq \mathcal{C}$ consisting of disjoint sets and such that $0 < |\mathcal{D}| < |\mathcal{C}|$ and for any $A \in \mathcal{C}$:
   if $(\exists B \in \mathcal{C})(B \subseteq A \cap \bigcup\mathcal{C})$ then $(\exists D \in \mathcal{D})(\exists B \in \mathcal{C})(B \subseteq A \cap D)$, and
   if $\neg(\exists B \in \mathcal{C})(B \subseteq A \cap \bigcup\mathcal{C})$ then $(\exists B \in \mathcal{C})(B \subseteq A \setminus \bigcup\mathcal{D})$.

(2) Let $\mathcal{C}$ be a category base on $\mathcal{X}$. A set $X \subseteq \mathcal{X}$ is called
 - $\mathcal{C}$*–singular* if $(\forall A \in \mathcal{C})(\exists B \in \mathcal{C})(B \subseteq A \setminus X)$,

- $\mathcal{C}$–*meager* if X can be covered by a countable union of $\mathcal{C}$–singular sets.

We say that a set $X \subseteq \mathcal{X}$ has *the $\mathcal{C}$–Baire property* if for every $A \in \mathcal{C}$ there is $B \in \mathcal{C}$, $B \subseteq A$ such that either $B \cap X$ or $B \setminus X$ is $\mathcal{C}$–meager.

(3) For a category base $\mathcal{C}$ on $\mathcal{X}$, the family of all $\mathcal{C}$–meager sets will be denoted by $\mathcal{M}_{\mathcal{C}}$ and the family of subsets of $\mathcal{X}$ with the $\mathcal{C}$–Baire property will be called $\mathcal{B}_{\mathcal{C}}$.

Remark 4.6. A category base, $\mathcal{C}$–meager sets and sets with $\mathcal{C}$–Baire property generalize the notions of a topology, meager sets and sets with the Baire property (with respect to the topology). Several results true in the topological case remain true for the category base approach. In particular, $\mathcal{B}_{\mathcal{C}}$ is a σ-field of subsets of $\mathcal{X}$ closed under the Souslin operation $\mathcal{A}$, and $\mathcal{M}_{\mathcal{C}}$ is a σ-ideal of subsets of $\mathcal{X}$. For a systematic study of the category base method we refer the reader to Morgan [11].

Conclusion 4.7. *Let $\bar{n} \in \mathcal{N}$ and $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\aleph_1)$.*

(1) *The family $\mathcal{C}^{\mathbf{H}}_{\bar{n}} \stackrel{\rm def}{=} \{{\rm POS}(p) : p \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}\}$ is a category base on $\prod\limits_{i\in\omega} \mathbf{H}(i)$ such that no member of $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$ is $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–meager.*

(2) *Each Σ^1_1–subset of $\prod\limits_{i\in\omega} \mathbf{H}(i)$ has the $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–Baire property. For every set $A \subseteq \prod\limits_{i\in\omega} \mathbf{H}(i)$ with the $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–Baire property there is a Π^0_3–subset B of A such that $A \setminus B$ is $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–meager.*

(3) *The σ-ideal $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$ of $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–meager subsets of $\prod\limits_{i\in\omega} \mathbf{H}(i)$ is a Borel ccc σ-ideal (in fact, any member of $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$ can be covered by a Σ^0_3–set from $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$).*

(4) *Suppose that $\mathbf{H}(i) = \mathcal{X}$ (for $i \in \omega$) and $\mathcal{X}$ is a finite group. Then $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$ is a Borel ccc translation–invariant index–invariant σ-ideal of subsets of $\mathcal{X}^\omega$, which is neither the ideal of meager sets, nor the ideal of null sets nor their intersection. Moreover, the formula* "a real r a is a Borel code for a subset of $\mathcal{X}^\omega$ and the set $\#r$ coded by r is in $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$" *is Σ^1_2.*

Proof. 1)–3) Should be clear if you remember 4.3(2,3). Note that each POS(p) is closed.

4) Plainly, the ideal $\mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$ is translation–invariant. To estimate the complexity of the formula "$A \in \mathcal{M}_{\mathcal{C}^{\mathbf{H}}_{\bar{n}}}$" use remark 4.4. The remaining assertions follow from the following two observations.

Claim 4.7.1. *If $A \subseteq \mathcal{X}^\omega$ is $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–singular and $\pi:\omega \xrightarrow{1-1} \omega$ is an embedding then $\pi_*(A)$ is $\mathcal{C}^{\mathbf{H}}_{\bar{n}}$–singular.*

Proof of the claim: Let $p \in \mathbb{Q}^{\mathbf{H}}_{\bar{n}}$. Passing to a stronger condition if needed we may assume that

$$(\forall j\in\omega)(\operatorname{dom}(\sigma^p_j)\subseteq \operatorname{rng}(\pi)\quad\text{or}\quad \operatorname{dom}(\sigma^p_j)\cap\operatorname{rng}(\pi)).$$

Let $q=(w^p\circ\pi, \sigma^p_j\circ\pi:\ j\in\omega, \operatorname{dom}(\sigma^p_j)\subseteq\operatorname{rng}(\pi))$. Plainly $q\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$, so we find $r\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ such that $\mathrm{POS}(r)\subseteq\mathrm{POS}(q)\setminus A$. Let $m^*=m^*(p)+m^*(r)+1$. Choose w^{p^*} such that

(1) $\operatorname{dom}(w^{p^*})=\operatorname{dom}(w^p)\cup\pi[\operatorname{dom}(w^r)]\cup\bigcup\{\operatorname{dom}(\sigma^p_j):\operatorname{dom}(\sigma^p_j)\cap\operatorname{rng}(\pi)=\emptyset\ \&\ j\in\bigcup\limits_{m=m^*(p)}^{m^*-1}V^p_m\}\cup\bigcup\{\pi[\operatorname{dom}(\sigma^r_i)]: i\in\bigcup\limits_{m=m^*(r)}^{m^*-1}V^r_m\}$,

(2) $w^p{\restriction}(\omega\setminus\operatorname{rng}(\pi))\cup w^r\circ\pi^{-1}\subseteq w^{p^*}$, and

(3) $\sigma^p_j\not\subseteq w^{p^*}$, $\sigma^r_i\circ\pi^{-1}\not\subseteq w^{p^*}$ (for j,i as in 1) above).

Next choose $\sigma^{p^*}_j$ such that $\langle\sigma^{p^*}_j: j<\omega\rangle$ enumerates the set

$$\{\sigma^p_j:\operatorname{rng}(\sigma^p_j)\cap\operatorname{rng}(\pi)=\emptyset\ \&\ j\in\bigcup_{m\geq m^*}V^p_m\}\cup\{\sigma^r_j\circ\pi^{-1}: j\in\bigcup_{m\geq m^*}V^r_m\}.$$

Plainly, $p^*=(w^{p^*},\sigma^{p^*}_0,\sigma^{p^*}_1,\ldots)\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ and $\mathrm{POS}(p^*)\subseteq\mathrm{POS}(p)\setminus\pi_*(A)$.

Claim 4.7.2. *For each $p\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ there is $q\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ such that $\mathrm{POS}(q)\subseteq\mathrm{POS}(p)$ and $\mathrm{POS}(q)$ is nowhere dense and null.*

Proof of the claim: Take $m^*>m^*(p)+5$ such that

$$(\forall k\geq m^*)(n^1_k>|\mathcal{X}|^{2\cdot n^0_k})$$

and choose a sequence $\langle\sigma_{k,j}: k\geq m^*, j<n^1_k\rangle$ of finite functions with pairwise disjoint domains and such that $|\operatorname{dom}(\sigma_{k,j})|=n^0_k$ and

$$\operatorname{dom}(\sigma_{k,j})\cap(\operatorname{dom}(w^p)\cup\bigcup\{\operatorname{dom}(\sigma^p_i): i\in\bigcup_{m=m^*(p)}^{m^*-1}V^p_m\})=\emptyset.$$

Let $q=(w^p,\sigma_{k,j}: k\geq m^*, j<n^1_k)$. Easily $q\in\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ and the conditions p,q are compatible (compare the proof of 4.3(1)). Note that for each $k\geq m^*$

$$\mathrm{Leb}(\{\eta\in\mathcal{X}^\omega:(\forall j<n^1_k)(\sigma_{k,j}\not\subseteq\eta)\})=(1-\tfrac{1}{|\mathcal{X}|^{n^0_k}})^{n^1_k}<$$
$$(1-\tfrac{1}{|\mathcal{X}|^{n^0_k}})^{|\mathcal{X}|^{2\cdot n^0_k}}\leq e^{-|\mathcal{X}|^{n^0_k}}.$$

Hence $\mathrm{POS}(q)$ is a nowhere dense null set and now we easily finish. □

Remark 4.8. There are several possible variants of the forcing notions $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$, each of them doing the job. These forcing notions will be presented in a subsequent paper [14], where we will systematically study forcing properties of ccc partial orders build by the method of norms on possibilities. In particular, we will show there that we may get different forcing properties of $\mathbb{Q}^{\mathbf{H}}_{\bar{n}}$ (for different $\bar{n}$), thus showing that the corresponding ideals are not isomorphic. Let us note that the forcing notions which appeared in the present paper are not σ–centered and do add Cohen reals. The proof of these facts and more general statements will be presented in [14].

References

[1] Bartoszyński, T. and Judah, H., *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts (1995).

[2] Goldstern, M. and Judah, H., *Iteration of Souslin Forcing, Projective Measurability and the Borel Conjecture*, Israel Journal of Mathematics 78 (1992), 335–362.

[3] Hall, P., *On representatives of subsets*, Journal of the London Mathematical Society 10 (1935), 26–30.

[4] Ihoda, J. (Judah, H.) and Shelah, S., *Souslin forcing*, The Journal of Symbolic Logic 53 (1988), 1188–1207.

[5] Jech, T., *Set theory* Academic Press, New York (1978).

[6] Judah, H., Rosłanowski, A. and Shelah, S., *Examples for Souslin Forcing*, Fundamenta Mathematicae 144 (1994), 23–42.

[7] Kechris, A. S. and Solecki S., *Approximation of analytic by Borel sets and definable countable chain conditions.* Israel Journal of Mathematics, 89 (1995), 343–356.

[8] Kunen, K., *Random and Cohen Reals*, In K. Kunen and J. E. Vaughan, editors, *Handbook of Set–Theoretic Topology*, pages 887–911. Elsevier Science Publishers B.V. (1984).

[9] Miller, A., *Arnie Miller's problem list*, In H. Judah, editor, *Set Theory of the Reals*, volume 6 of Israel Mathematical Conference Proceedings, pages 645–654. Proceedings of the Winter Institute held at Bar–Ilan University, Ramat Gan (January 1991).

[10] Morgan, J.C. II, *Baire category from an abstract viewpoint*, Fundamenta Mathematicae 94 (1977), 13–23.

[11] Morgan, J.C. II, *Point set theory*, volume 131 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc, New York (1990).

[12] Rosłanowski, A. and Judah, H., *The ideals determined by Souslin forcing notions*, unpublished notes, 1992.

[13] Rosłanowski, A. and Shelah, S., *Norms on possibilities I: forcing with trees and creatures*, Memoirs of the AMS (submitted).

[14] Shelah, S., *Norms on possibilities III: ccc forcing notions.*

[15] Shelah, S., *Non-elementary proper forcing notions.*

ANDRZEJ ROSŁANOWSKI
INSTITUTE OF MATHEMATICS

SAHARON SHELAH
INSTITUTE OF MATHEMATICS

THE HEBREW UNIVERSITY
OF JERUSALEM
91904 JERUSALEM, ISRAEL
and
MATHEMATICAL INSTITUTE
OF WROCLAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
ROSLANOW@MATH.HUJI.AC.IL

THE HEBREW UNIVERSITY
OF JERUSALEM
91904 JERUSALEM, ISRAEL
and
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08854
USA
SHELAH@MATH.HUJI.AC.IL