CHOICELESS POLYNOMIAL TIME LOGIC: INABILITY TO EXPRESS

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Dedicated to my friend Yuri Gurevich

Abstract. We prove for the logic $\widetilde{C}PT$ime (the logic from the title) a sufficient condition for two models to be equivalent for any set of sentences which is "small" (certainly any finite set), parallel to the Ehrenfeucht Fraïssé games. This enables us to show that sentences cannot express some properties in the logic $\widetilde{C}PT$ime and prove 0-1 laws for it.

Key words and phrases. Finite model theory; Computer Science; Polynomial time logic; choiceless; games.

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Anotated Content

§0 Introduction

We deal here with choiceless polynomial time logic, introduced under the name $\tilde{C}PT$ime in Blass Gurevich Shelah [BGSh 533]; actually we deal with several versions. Knowledge of [BGSh 533], which is phrased with ASM (abstract state machine), is not required except when we explain in §4 how the definitions here and there fit. See there more on background, in particular, on ASM-s and logic capturing P Time. The aim of this logic is to capture statements on a (finite) model M computable in polynomial time and space <u>without</u> arbitrary choices. So we are not allowed to choose a linear order on M, but if P is a unary predicate from the vocabulary τ_M of M and P^M has $\leq \log_2(\|M\|)$ elements then we are allowed to "create" the family of all subsets of P^M, and if e.g. $(|P^M|)! \leq \|M\|$ we can create the family permutations of P^M. Note that a statement of the form $\tilde{C}PT$ime captures what can be computed in polynomial time without arbitrary choices" is a thesis not a theorem. For a given model M, we consider the elements of M as urelements, and build inductively $N_t = N_t[M]$, with $N_0 = M, N_{t+1} \subseteq N_t[M] \cup \mathscr{P}(N_t[M])$ but the definition is uniform and the size of $N_t[M]$ should not be too large, with the major case being: it has a polynomial bound.

So we should have a specific guide Υ telling us how to create N_{t+1} from N_t, hence we should actually write $N_t[M, \Upsilon]$. In the simplest version (we called it pure) essentially $\Upsilon = \{\psi_\ell(x, \bar{y}) : \ell < m_0\}$ and N_t is a transitive finite set with M the set of urelements, $N_0 = M, N_{t+1} = N_t \cup \{\{a : N_t \models \psi_\ell(a, \bar{b})\} : \bar{b} \in {}^{\ell g(\bar{y})}(N_t)$ and $\ell < m_0\}$; where each ψ_ℓ is a first order formula in the vocabulary of M plus the membership relation $\in$, (i.e. $\tau_M \cup \{\in\}$,) and N_t has the relations of M and the relation $\in \restriction N_t$. We stop when $N_t = N_{t+1}$, in the "nice" cases after polynomial time and space, and then can ask "$N \models \chi$"? getting yes or no.

We consider several versions of the definition of the logic; this should help us to see "what is the true logic capturing polynomially computable statements without making arbitrary choices".

Our aim here is to deal with finite models and processes, but we dedicate a separate place at the end to some remarks on infinitary ones and set theory. We also comment in this section on classical model theoretic roots; both, of course, can be ignored.

For a logic $\mathscr{L}$ it is important to develop methods to analyze what it can say. Usual applications of such methods are to prove that:

- (a) no sentence in $\mathscr{L}$ expresses some property (which another given logic can express so we can prove that they are really different)
- (b) certain pairs of models are equivalent
- (c) zero-one laws.

For first order logic we use mostly elimination of quantifiers, E.F. (Ehrenfeucht-Fraïssé) games (see [CK] or [Ho93] or Ebbinghaus and Flum [EbFl95]) and ? CK ?
others; we are most interested in relatives of E.F. games. In finite model theory people have worked on the logic $L_{\omega,k}$ (usually denoted by $L^k_{\omega,\omega}$) of first order

formulas in which every subformula has $\leq k$ free variables, a relative of the $L_{\lambda,\kappa}$ logics (see e.g. [Di]). We can use even such formulas in $L_{\infty,\omega}$ but for equivalence of finite models this does not matter. For $L_{\infty,\kappa}$ the relatives of E.F. games are called pebble games (see [EbFl95]). ? Di ?

The E.F. criterion (see Karp [Ka]) for the $L_{\infty,\omega}$-equivalence of two models M_1, M_2 of the same vocabulary τ is the existence of a non-empty family $\mathscr{F}$ of partial one-to-one functions from M_1 to M_2 such that for every $f \in \mathscr{F}, i \in \{1,2\}$ and $a \in M_i$ there is $g \in \mathscr{F}$ extending f such that $i = 1 \Rightarrow a \in \text{Dom}(g)$ and $i = 2 \Rightarrow a \in \text{Rang}(g)$ (a particular case is $M = N$; there, of course, M, N are equivalent but the question is when are $(M,a),(M,b)$ equivalent). Note that for finite models and even for countable models, this criterion implies isomorphism. But if we restrict ourselves to first order sentences of quantifier depth $< k$ we can replace $\mathscr{F}$ by $\langle \mathscr{F}_\ell : \ell < k \rangle$ and above we say that for every $f \in \mathscr{F}_{\ell+1}, j \in \{1,2\}, a \in M_j$ there is $g \in \mathscr{F}_\ell$ as there (this is the original E.F. game). For $L_{\infty,k}$ this does not work but without loss of generality, $f \in \mathscr{F}$ & $A \subseteq \text{Dom}(f) \Rightarrow f \restriction A \in \mathscr{F}$, and above we restrict ourselves to $f \in \mathscr{F}$ with $|\text{Dom}(f)| < k$. Now probably the simplest models are those with equality only: so every permutation of the (universe of the) model is an automorphism. So using this group it is proved in [BGSh 533] that $\tilde{C}PT$ime cannot say much on such models, thus showing that the $\tilde{C}PT$ime does not capture P Time logic, in fact odd/even is not captured.

But in our case suppose we are given $\mathscr{F}$, a family of partial isomorphisms from M_1 to M_2, we have to create such a family for $N_t[M_1,\Upsilon], N_t[M_2,\Upsilon]$.
We answer here some questions of [BGSh 533]: get a 0-1 law, show that $\tilde{C}PT$ime + counting does not capture P Time.

Note that if P^M is small enough then we can have e.g. $\text{Per}(P^M)$, the group of permutations of P^M, as a member of $N_t = N_t[M,\Upsilon]$ for t large enough: just in N_{3t} we may have the set of partial permutations of P^M of cardinality $< t$.

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<u>Notation</u>:
1) Natural numbers are denoted by $i, j, k, \ell, m, n, r, s, t$. We identify a natural number t with $\{s : s < t\}$ so $0 = \emptyset$ and we may allow $t = \infty =$ the set of natural numbers (ω for set theorists). Let $[m] = \{1, \dots, m\}$.
2) Let τ denote a vocabulary, i.e. a set of predicates P (each predicate with a given arity $\mathbf{n}(P)$); we may also have function symbols, but then it is natural to interpret them as partial functions, so better to treat them as relations and avoid function symbols. We may attach to each predicate $P \in \tau$ a group $\mathbf{g}_P$ permutations of $\{0, \dots, \mathbf{n}(P) - 1\}$ telling under what permutations of its argument places the predicate P is supposed to be preserved; if not specified $\mathbf{g}_P$ is a trivial group.
3) Let P, Q, R denote predicate symbols.
4) Formulas are denoted by $\varphi, \psi, \theta, \chi$; usually θ, χ are sentences.
5) Let $\mathscr{L}$ denote a logic and $\mathscr{L}(\tau)$ denote the resulting language, the set of $\mathscr{L}$-formulas in the vocabulary τ.

6) Let M, N denote models and let $\tau_M = \tau(M)$ denote the vocabulary of M and let P^M denote the interpretation of the predicate $P \in \tau_M$. So P^M is a relation on M with arity $\mathbf{n}(P)$ and if $\mathbf{g}^P = \mathbf{g}^\tau_P$ is not trivial and $\sigma \in \mathbf{g}_P, \langle a_\ell : \ell < \mathbf{n}(P)\rangle \in {}^{\mathbf{n}(P)}M$, then $\langle a_\ell : \ell < \mathbf{n}(P)\rangle \in P^M \Leftrightarrow \langle a_{\sigma(\ell)} : \ell < \mathbf{n}(P)\rangle \in P^M$. Models are finite if not said otherwise.
7) Let $|M|$ be the universe (= set of elements) of M and $\|M\|$ the cardinality of $|M|$, but abusing notation we may say "of M".
8) Let $\mathbf{t}, \mathbf{s}$ denote functions from the set of natural numbers to the set of natural numbers ≥ 1 and let $\mathbf{T}$ denote a family of such functions. We may write $\mathbf{t}(M)$ instead of $\mathbf{t}(\|M\|)$.
9) For a function f and a set A let $f"(A) = \{f(x) : x \in A \text{ and } x \in \text{ Dom}(f)\}$.
[Why do we not write $f(A)$? Because f may act both on A and on its elements; this occurs for functions like $G(f)$, defined and investigated is §2.]

0.1 Discussion: Would we really like to allow $\mathbf{t}(M)$ to depend not just on $\|M\|$? For the definition of the logic we have no problems, also for the criteria of equivalence of M_1, M_2 (except saying $\|M_1\| = \|M_1\|$ & $\mathbf{t}_1 = \mathbf{t}_2 \Rightarrow \mathbf{t}_1(M_1) = \mathbf{t}_2(M_2)$). But this is crucial in proving a weak version of the 0-1 law, say for random graphs, to overcome this we need to "compensate" with more assumptions.

§1 The Choiceless polynomial time logic presented

Here we present the logic. How does we "compute" when a model M satisfies a "sentence" θ? Note that the computation should not take too long and use too much "space" (in the "main" case: polynomial).

Informally, we start with a model M with each element an atom=ure-element, we successively define $N_t[M]$ and $N^+_t[M]$, an expansion of $N_t[M], t$ running on the stages of the "computation"; to get $N_{t+1}[M]$ from $N_t[M]$ we add few families of subsets of N_t, each family for some ψ consist of those defined by a formula $\psi(-, \bar{a})$ for some $\bar{a}$ from $N_t[M]$, and we update few relations or functions, by defining them from those of the previous stage. Those are $P_{t,\ell}$ for $\ell < m_1$. We may then check if a target condition holds, then we stop getting an answer: the model satisfies θ or fails by checking if some sentence χ holds; the natural target condition is when we stop to change. Note that each stage increases the size of $N_t[M]$ by at most a (fixed) power, however in $\|M\|$ steps we may have constructed a model of size $2^{\|M\|}$ but this is against our intentions. So we shall have a function $\mathbf{t}$ in $\|M\|$, normally polynomial, whose role is that when we have wasted too much resources (e.g. $\|N_t[M]\| + t$) we should stop the computation even if the target condition has not occurred, in this case we still have to decide what to do.

This involves some parameters. First a logic $\mathscr{L}$ telling us which formulas are allowable

(a) in the inductive step (the $\psi_\ell(-, \bar{y}) - s$)

(b) in stating "the target condition" (we use standard ones: $N^+_{t+1} = N^+_t$ or $P_0 = P_2$ or $c_0 = c_2$) and the χ telling us if the answer is yes or no.

Second, a family $\mathbf{T}$ of functions $\mathbf{t}$ (with domain the family of finite models) which tell us when we should stop having arrived to the limit of allowable resources (clearly the normal family $\mathbf{T}$ is $\{\mathbf{t}_k : k < \omega\}$ where $\mathbf{t}_k(M) = \|M\|^k$). So for Υ which describes the induction step (essentially how to get $N_{t+1}[M]$ from $N_t[M]$), χ telling us the answer and $\mathbf{t} \in \mathbf{T}$ we have a sentence $\theta_{\Upsilon,\chi,\mathbf{t}}$. We still have some variants, getting a logic $\mathscr{L}^{\mathbf{T}}_\iota[\mathscr{L}^*]$ for each $\iota \in \{1,2,3,4,5,6,7,11,22\}$.

In the case $\iota = 1$ we ignore $\mathbf{t}$, so $M \models \theta_{\Upsilon,\chi,\mathbf{t}}$ iff for some $t < \infty$ the target condition $N^+_t = N^+_{t+1}$ holds, or let $t = \infty$ and for this t, χ is satisfied by N^+_t. This is a very smooth definition, but we have lost our main goal. We may restrict ourselves to "good" sentences for which we always stop in "reasonable" time and space.

In the case $\iota = 2$, possibly we stop because of $\mathbf{t}$ before the target condition holds; in this case we say "$\theta_{\Upsilon,\chi,\mathbf{t}}$ is undefined for M". The case $\iota = 3$ is like $\iota = 2$ but we restrict ourselves to the so called "standard $\mathbf{T}$", where in $N_t[M]$ we have the natural numbers $< t$, so we can ignore the "time" as a resource as always $\|N_t[M]\| \geq t$. The case $\iota = 4$, is like $\iota = 3$ but instead stopping when $\|N_t[M]\|$ is too large, we stop when at least one of the families of sets added to $N_t[M]$ to form $N_{t+1}[M]$ is too large. For $\iota = 5$, is like the case $\iota = 2$, but an additional reason for stopping is $t > \mathbf{t}(\|M\|)$. The case $\iota = 6$ is as the case $\iota = 2$ separating the bounds on "space" (that is $\|N_t[M]\|$) and "time" (that is t), the case $\iota = 7$ is similar, not stopping for $N^+_t = N^+_{t+1}$. The cases $\iota = 11, \iota = 22$ are like $\iota = 1, \iota = 2$ respectively, but using $N_t[M, \Upsilon, \mathbf{t}]$ (see Definition 1.1(c)) instead $N_t[M, \Upsilon]$ but for $\iota = 22$ we separate $\mathbf{t}$ to two functions.

We treat as our main case $\iota = 3$, see more 1.7.

More formally

1.1 Definition. 1) We are given a model M, with vocabulary $\tau = \tau_{[0]}, \tau$ finite and $\in$ not in τ, let $\tau^+ = \tau_{[1]} = \tau \cup \{\in\}$. Considering the elements of M as atoms = urelements, we define $V_t[M]$ by induction of $t : V_0[M] = (M, \in\restriction M)$ and clearly with $\in\restriction M$ being empty (as we consider the members of M as atoms = "urelements"). Next $V_{t+1}[M]$ is the model with universe $V_t[M] \cup \{a : a \subseteq V_t[M]\}$ (by our assumption on "urelements" we have $a \subseteq V_t[M] \Rightarrow a \notin M$) with the predicates and individual constants and function symbols of τ interpreted as in M (so function symbols in τ are always interpreted as partial functions) and $\in^{V_{t+1}[M]}$ is $\in\restriction V_{t+1}[M]$.

2)

- (A) We say $\Upsilon = (\bar\psi, \bar\varphi, \bar P)$ is an inductive scheme for the logic $\mathscr{L}_{\text{f.o.}}$ or the language $\mathscr{L}_{\text{f.o.}}(\tau)$ (where $\mathscr{L}_{\text{f.o.}}$ is first order logic) if: letting $m_0 = \ell g(\bar\psi), m_1 = \ell g(\bar\varphi)$ and $\tau_{[2]} = \tau_{[2]}[\Upsilon] = \tau_{[1]} \cup \{P_k : k < m_1\}$ we have
  - (a) $\bar P = \langle P_k : k < m_1\rangle$ is a sequence (with no repetitions) of predicates and function symbols not in $\tau_{[1]}$ (notationally we treat an n-place function symbol as $(n+1)$-predicate); where P_k is an $\mathbf{m}^\Upsilon(P_k)$-place predicate. Let $\mathbf{m}^\Upsilon_1$ be the function giving this information and whether P_k is a predicate or a function symbol (so have domain $\{0, \ldots, m_1 - 1\}$)

(b) $\bar{\psi} = \langle \psi_\ell : \ell < m_0 \rangle, \psi_\ell = \psi_\ell(x; \bar{y}_\ell)$ is first order in the vocabulary $\tau_{[2]}$,

(c) $\bar{\varphi} = \langle \varphi_k : k < m_1 \rangle, \varphi_k = \varphi_k(\bar{x}_k)$ is first order in the vocabulary τ_2 with $\ell g(\bar{x}) = \mathbf{m}(P_k)$, moreover $\bar{x} = \langle x_i : i < \mathbf{m}(P_k) \rangle$.

(B) We say Υ is simple if

(d) each P_k is unary predicate and and each $\varphi_k(x)$ appears among the ψ_ℓ's (with empty $\bar{y}_\ell$) and for every hereditary model $N^* \subseteq \mathbf{V}_\infty[M]$, and a $\tau_{[2]}$–expansion N^+ of N^* we have $\{a : N^+ \models \varphi_\ell(a)\}$ is a member of N^* or is $\emptyset$

(C) We may write $m_0^\Upsilon, m_1^\Upsilon, \psi_\ell^\Upsilon, \mathbf{m}^\Upsilon, \mathbf{m}_1^\Upsilon, \varphi_\ell^\Upsilon, \psi_\ell^\Upsilon, P_\ell^\Upsilon$. We let

$$\Psi^\Upsilon = \{\psi_\ell(x, \bar{y}_\ell) : \ell < m_0^\Upsilon\}.$$

(D) We say Υ is predicative if each P_k is a predicate; we may restrict ourselves to this case for the general theorems. We say Υ is pure if $m_1 = 0$

(E) Υ is monotonic if $y \in x$ is ψ_ℓ for some $\ell < m_0^\Upsilon$ (this will cause N_t below to grow); no big loss if we restrict ourselves to such Υ. It is strongly monotonic if in addition each $\varphi_k(\bar{x})$ has the form $P_k(\bar{x}) \vee \varphi'_k(\bar{x})$ (this will cause also each P_k to grow)

(F) Υ is i.c. if each P_k is (informally an individual constants scheme) a zero place function symbol; in this case if P_k is well defined we may write it as c_k.

3) For $M, \tau = \tau_{[0]}, \tau_{[1]}, \tau_{[2]}$, and $\Upsilon = (\bar{\psi}, \bar{\varphi}, \bar{P})$ as above, we shall define by induction on t a submodel $N_t = N_t[M]$ of $V_t[M]$ and $\bar{P}_t = \langle P_{t,k} : k < m_1 \rangle$ and $N_t^+[M]$ and $\mathscr{P}_{t,\ell}$ (for $\ell < m_0$) as follows; more exactly we are defining $N_t[M, \Upsilon], P_{t,k}[M, \Upsilon]$ for $k < m_1$, $\mathscr{P}_{t,k}[M, \Upsilon]$ for $k < m_0$.

We let $N_t^+[M, \Upsilon] = (N_t[M, \Upsilon], P_{t,0}[M, \Upsilon], \dots, P_{t,m_1-1}[M, \Upsilon])$.

<u>Case 1: $t = 0$</u>: $N_t[M] = V_0[M]$ and $P_{t,k} = \emptyset$ (an $\mathbf{m}^\Upsilon(k)$-place relation).

<u>Case 2: $t + 1$</u>: $N_{t+1}[M]$ is the submodel of $V_{t+1}[M]$ with set of elements the transitive closure of $M \cup \bigcup_{\ell < m_0} \mathscr{P}_{t,\ell}[M]$ where we define $\mathscr{P}_{t,k}[M]$ and $P_{t+1,\ell}$ by:

$$\mathscr{P}_{t,\ell}[M] = \left\{ \{a \in N_t[M] : N_t^+[M] \models \psi_\ell(a, \bar{b})\} : \bar{b} \in {}^{(\ell g(\bar{y}_i))}(N_t[M]) \right\}$$

$$P_{t,k} = \{\bar{a} \in {}^{\mathbf{m}(k)}(N_t[M]) : N_t^+[M] \models \varphi_k[\bar{a}]\}$$

but if P_k is a function symbol,

$$P_{t,\ell} = \{\bar{a}\char`\^\langle b \rangle \in N_t[M] : N_t^+[M] \models \varphi_\ell(\bar{a}, b) \ \& \ (\exists! y)\varphi_\ell(\bar{a}, y)\}$$

(so if Υ is simple, then for $t > 1$, $\psi_k(x,\bar y) \in \mathscr{L}_{\text{f.o.}}(\tau_{[2]})$ is actually $\psi'_k(x,\bar y,\bar c_{t-1}) \in \mathscr{L}_{\text{f.o.}}(\tau_{[1]})$).

<u>Case 3</u>: $t = \infty$

$$N_t = \bigcup_{s<t} N_s, P_{t,k} = \bigcup_{s<t} P_{s,k}.$$

3A) If in addition $\mathbf{t} \in \mathbf{T}$ and Υ is monotonic (see part (2) clause (E)) we define $N_t[M,\Upsilon,\mathbf{t}]$, $P_{t,\ell}[M,\Upsilon,\mathbf{t}]$ and $\mathscr{P}_{t,\ell}[M,\Upsilon,\mathbf{t}]$ as in part (3) except that the universe of $N_{t+1}[M,\Upsilon,\mathbf{t}]$ is the transitive closure of $M \cup \bigcup\{\mathscr{P}_{t,\ell}[M,\Upsilon,\mathbf{t}] : \ell < m_0$ and $\mathscr{P}_{t,\ell}[M,\Upsilon,\mathbf{t}]$ has at most $\mathbf{t}(\|M\|)$ members$\}$.
4) We say Υ is standard if some ψ_i guarantees that $t \subseteq N_t[M]$ (so a natural number s belongs to $N_t[M]$ iff $s < t$; remember that we identify the natural number t with the set $\{0,1,\dots,t-1\}$).
5) Let q.d.(φ) be the quantifier depth of the formula φ.
6) We may replace above first order logic by another logic $\mathscr{L}$. We let $\mathscr{L}_{\text{f.o.}}$ denote first order, $\mathscr{L}_{\text{card}}$ is defined just like first order logic except that we demand that Υ is standard and defining inductively what is a formula, we allow the formation of formulas the form $|\{x : \theta(x,\bar y)\}| = s$. We let $\mathscr{L}_{\text{card},\mathbf{T}}$ (on $\mathbf{T}$ see below) be defined just like $\mathscr{L}_{\text{f.o.}}$ but for each $\mathbf{t} \in T$ we allow the quantifier $(Q_{\mathbf{t}}x)\varphi(x,\bar y)$ with

$$N \models (Q_{\mathbf{t}}x)\varphi(x,\bar a) \quad \text{iff} \quad \mathbf{t}(|\text{ure}(N)|) < |\{b : N \models \varphi(b,\bar a]\}|,$$

where ure(N) is the set of urelements of N.
6A) Lastly, let $\mathscr{L}_{\text{f.o.+na}}$ be like $\mathscr{L}_{\text{f.o.}}$ but we add one atomic formula $|\text{atoms}| = x$ being interpreted as: the number of atoms is x. So this can be expressed in $\mathscr{L}_{\text{card},\mathbf{T}}$ where $\mathbf{T} = \{\text{id}\}, \text{id}(n) = n$.

1.2 Remark. Alternatively to $\mathscr{L}_{\text{card}}$: have a quantifier

$$(Q^{\text{eq}}x_1,x_2)(\varphi_2(x_1,\bar y_1),\varphi_2(x_2,\bar y_2)),$$

which says that $|\{x : \varphi_2(x,\bar y_1)\}| = |\{x : \varphi_2(x,\bar y_2)\}|$.

In the definition below the reader can concentrate on $\iota = 3$. The "$t \ge 2$ & …" is not a serious matter.

1.3 Definition. Let $\mathbf{T}$ be a set of functions $\mathbf{t} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ and $\mathscr{L}^*$ be a logic ($\mathscr{L}_{\text{f.o.}}$ or $\mathscr{L}_{\text{f.o.}+na}$ or $\mathscr{L}_{\text{card}}$ usually) and let τ be a vocabulary. If $\mathbf{t}$ is constantly ∞ we may write ∞.

We define for $\iota = 1,2,3,4,5,6,7,11,22$ the logic $\mathscr{L}^{\mathbf{T}}_\iota[\mathscr{L}^*]$ below. For all of those logics the set of sentences for a vocabulary τ called $\mathscr{L}^{\mathbf{T}}_\iota[\mathscr{L}^*](\tau)$ is a subset of $\Theta = \Theta_\tau = \Theta_\tau[\mathscr{L}^*,\mathbf{T}] = \{\theta_{\Upsilon,\chi,\mathbf{t}} : \Upsilon$ an inductive scheme for $\mathscr{L}^*(\tau), \chi \in \mathscr{L}^*(\tau)$ and $\mathbf{t} \in \mathbf{T}\}$, (equal if not said otherwise). Also for most of those logics we define the stopping time $t_\iota[M,\Upsilon,\mathbf{t}]$ or $t_\iota[M,\Upsilon]$ (if $\mathbf{t}$ does not matter). The satisfaction relation for $\mathscr{L}^{\mathbf{T}}_\iota[\mathscr{L}^*](\tau)$ is denoted by $\models_\iota$. Also we write $\theta_{\Upsilon,\chi}$ instead $\theta_{\Upsilon,\chi,\mathbf{t}}$ if $\mathbf{t}$ does not matter. (We may let Dom$(\mathbf{t})$ be the set of relevant structures, see 0.1).

Case 1: $\iota = 1$.

We let

$$t_\iota[M,\Upsilon] = \text{ Min}\{t : t \geq 2 \text{ and } N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\}.$$

(If there is no such $t \in \mathbb{N}$ we let it be ∞ (i.e., ω for set theorists) and we could also have used "undefined"; note that $\mathbf{t}$ does not appear).

$M \models_\iota \theta_{\Upsilon,\chi}$ iff $N^+_t[M,\Upsilon] \models \chi$ for $t = t_\iota[M,\Upsilon]$.

Case 2: $\iota = 2$.

We let $t_\iota[M,\Upsilon,\mathbf{t}] = \text{ Min}\{t : \|N_{t+1}[M,\Upsilon]\| + (t+1) > \mathbf{t}(\|M\|)$ or $t \geq 2$ & $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\}$ and

(a) if for $t = t_\iota[M,\Upsilon,\mathbf{t}]$ we have $t \geq 2$ & $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]$ then $\theta_{\Upsilon,\chi,\mathbf{t}}$ is true or false in M iff $N^+_t[M,\Upsilon] \models \chi$ or $N^+_t[M,\Upsilon] \models \neg\chi$ respectively and we write $M \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}}$ or $M \models_\iota \neg\theta_{\Upsilon,\chi,\mathbf{t}}$ respectively, (so $\neg\theta_{\Upsilon,\chi,\mathbf{t}}$ is equivalent to $\theta_{\Upsilon,\neg\chi,\mathbf{t}}$).

(b) if $t_\iota[M,\Upsilon,t] = \mathbf{t}(\|M\|) + 1$ we say "$M \models_\iota \theta_{\Upsilon,\chi,t}$ is undefined" and we say "the truth value of $\theta_{\Upsilon,\chi,\mathbf{t}}$ in M is undefined".

Case 3: $\iota = 3$.

As in Case 2 but we restrict ourselves to standard Υ, see Definition 1.1(4), and let $t_\iota[M,\Upsilon,\mathbf{t}] = \text{ Min}\{t : \|N_{t+1}[M,\Upsilon]\| > \mathbf{t}(\|M\|)$ or $t \geq 2$ & $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\}$ and define $\models_\iota$ as in Case 2.

Case 4: $\iota = 4$.

As in Case 2 but we restrict ourselves to standard Υ and let:

$$\begin{aligned} t_\iota[M,\Upsilon,\mathbf{t}] = \text{ Min}\{t :& \text{for some } k < m_0 \text{ the set } \mathscr{P}_{t+1,k}[M,\Upsilon] \\ & \text{has } > \mathbf{t}(\|M\|) \text{ members } \underline{\text{or}} \\ & t \geq 2 \ \& \ N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\} \end{aligned}$$

(so it can be ∞; but by a choice of e.g. φ_0 we can guarantee $\mathscr{P}_{t,0} = \{0,\dots,t-1\}$ so that this never happens) and define $\models_\iota$ as in Case 2.

Case 5: $\iota = 5$.

As in Case 2 but we restrict ourselves to standard Υ and $t_\iota[M,\Upsilon,\mathbf{t}] = \text{ Min}\{t : t > \mathbf{t}(\|M\|)$ or for some $k < m_0$ the set $\mathscr{P}_{t+1,k}[M,\Upsilon]$ has $> \mathbf{t}(\|M\|)$ members or $t > 2$ & $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon,\mathbf{t}]\}$.

Case 6: $\iota = 6$.

As in the case $\iota = 2$ but

$$\begin{aligned} t_\iota[M,\Upsilon,\mathbf{t}] = \text{ Min}\{t : t > \mathbf{t}^{tm}(\|M\|) \text{ or } \|N_t[M,\Upsilon]\| > \mathbf{t}^{\rm sp}(\|M\|) \\ \text{or } N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\} \end{aligned}$$

where

$$\mathbf{t}^{\rm tm}(n) = \mathbf{t}(2n), \mathbf{t}^{\rm sp}(n) = \mathbf{t}(2n+1)$$

and, of course, *tm*, *sp* stand for time and space, respectively. So we may replace $\mathbf{t}$ by two functions $\mathbf{t}^{\rm tm}, \mathbf{t}^{\rm sp}$ and write sentences as $\theta_{\Upsilon,\chi,\mathbf{t}',\mathbf{t}''}$ (similarly for $\iota = 7, 22$).

Case 7: $\iota = 7$.

We define $t_\iota[M,\Upsilon,\mathbf{t}] = $ Min$\{t : t > \mathbf{t}^{\rm tm}(\|M\|)$ or $\|N_t[M,\Upsilon]\| > \mathbf{t}^{\rm sp}(\|M\|)$ or ≥ 2 & $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]\}$

and let:

$$M \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}} \text{ \underline{iff} } N_t[M,\Upsilon] \models \chi \text{ for } t = t_\iota[M,\Upsilon].$$

(so unlike cases 2-6, the truth value is always defined)

Case 11: $\iota = 11$.

As in the case $\iota = 1$, but we use $N_t[M,\Upsilon,\mathbf{t}], \bar{P}_t[M,\Upsilon,\mathbf{t}]$ (see Definition 1.1(3A)).

Case 22: $\iota = 22$.

As in the case $\iota = 2$, but we use $N_t[M,\Upsilon,\mathbf{t}^{\rm wd}], \bar{P}_t[M,\Upsilon,\mathbf{t}^{\rm wd}]$ where $\mathbf{t}^{\rm wd} \in \mathbf{T}$ is defined by $\mathbf{t}^{\rm wd}(n) = \mathbf{t}(2n)$ (wd for width) and define t_ι by

$$\begin{aligned} t_\iota[M,\Upsilon,\mathbf{t}] = \text{Min}\{t : &N_{t+1}[M,\Upsilon,\mathbf{t}^{\rm wd}] + (t+1) > \mathbf{t}^{\rm ht}(\|M\|) \text{ or} \\ &t \geq 2 \ \& \ N^+_t[M,\Upsilon,\mathbf{t}^{\rm wd}] = N^+_{t+1}[M,\Upsilon,\mathbf{t}^{\rm wd}]\} \end{aligned}$$

where (ht for height)

$$\mathbf{t}^{\rm ht} \in \mathbf{T} \text{ is defined by } \mathbf{t}^{\rm ht}(n) = \mathbf{t}(2n+1).$$

We may write $\mathbf{t}^{\rm wd}, \mathbf{t}^{\rm ht}$ instead of $\mathbf{t}$.

1.4 Remark. Alternatively:

Case $10+\iota$: For $\iota = 1,\dots,7$.

Like the case ι but we use $N_t[M,\Upsilon,\mathbf{t}], \bar{P}_t[M,\Upsilon,\mathbf{t}]$ and let $\mathbf{t}_{10+\iota} = \mathbf{t}_\iota$.

Case $20+\iota$: $\iota = 1,\dots,7$.

As in the cases $\iota = 1,\dots,7$, but we use $N_t[M,\Upsilon,\mathbf{t}^{\rm wd}], \bar{P}_t[M,\Upsilon,\mathbf{t}^{\rm wd}]$, where $\mathbf{t}^{\rm wd} \in \mathbf{T}$ is defined by $\mathbf{t}(n) = \mathbf{t}(2n)$ (wd for width) and we replace $\mathbf{t}$ by $\mathbf{t}^{\rm ht}, \mathbf{t}^{\rm ht}(n) = \mathbf{t}(2n+1)$, where for $x =$ tm,sp we derive from $\mathbf{t}^{\rm ht}$ the functions $\mathbf{t}^{{\rm ht},x}_{20+\iota} = (\mathbf{t}^{\rm ht})^x$.

1.5 Definition. 1) In Definition 1.3 we say "$\theta_{\Upsilon,\chi,\mathbf{t}}$ is ι-good" when: for every finite model M one of the following occurs:

(a) $\iota = 1$ and $t_\iota[M,\Upsilon] < \infty$

(b) $\iota \in \{2,3,4,5,6,7\}$ and in the definition of $t_\iota[M,\Upsilon,\mathbf{t}]$ always the last possibility occurs and not any of the previous ones

(c) $\iota = 11$ as in the case $\iota = 1$ using $N_t[M,\Upsilon,\mathbf{t}], \bar{P}_t[M,\Upsilon,\mathbf{t}]$

(d) $\iota = 22$, as for $\iota = 2$ using $N_t[M,\Upsilon,\mathbf{t}], \bar{P}_t[M,\Upsilon,\mathbf{t}]$.

2) Let $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}^*)^{\rm good}$ be the logic $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}^*)$ restricted to ι-good sentences.
3) If in $\theta_{\Upsilon,\chi,\mathbf{t}}$ we omit χ we mean $P_{t,0}$ = the set of atoms.
4) We say that $M \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}}$ is in a good way if this case of part (1) holds

1.6 Remark. We can replace in cases ι = 2,3,4 clause (b), the statement $N^+_t[M,\Upsilon] = N^+_{t+1}[M,\Upsilon]$ by a sentence χ_1.

1.7 Discussion: 0) Note that considering several versions should help to see how canonical is our logic.
1) The most smooth variant for our purpose is $\iota = 4$, and the most natural choice is $\mathscr{L}^* = \mathscr{L}_{\rm card}$ or $\mathscr{L}^* = L_{{\rm card},\mathbf{T}}$, but we are not less interested in the choice $\mathscr{L}^* = \mathscr{L}_{\rm f.o.}, \mathscr{L}^* = \mathscr{L}_{{\rm f.o.}+na}$. From considering the motivation the most natural $\mathbf{T}$ is $\{n^m : m < \omega\}$, and $\iota = 3$.
2) For e.g. $\iota = 1,2,3$ some properties of M can be "incidentally" expressed by the logic, as the stopping time gives us some information concerning cardinality. For example let Y be a complicated set of natural numbers, e.g. non-recursive, and let $\mathbf{t}^* \in \mathbf{T}$ be: $\mathbf{t}(\|M\|)$ is $\|M\| + 10$ if $\|M\| \in Y$ and $\mathbf{t}(\|M\|) = \|M\| + 6$ if $\|M\| \notin Y$. We can easily find $\theta = \theta_{\Upsilon,\chi,\mathbf{t}^*}$, with Υ a standard induction scheme such that it stops exactly for $t = 8$ and χ saying nothing (or if you like saying that there are 8 natural numbers). Clearly for $\iota = 2,3$ we have $M \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}^*}$ if $\|M\| \in Y$ and "not $M \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}^*}$" if $\|M\| \notin Y$. Of course, more generally, we could first compute some natural number from M and then compare it with $\mathbf{t}(\|M\|)$. This suggests preferring the option $\models_\iota$ undefined in clause (b) of case 2, Definition 1.3 rather than false.
3) If you like set theory, you can let t be any ordinal; but this is a side issue here; see end of §5.

Implicit in 1.3 (and an alternative to 1.3) is (note: an (M,Υ)-candidate $(N,\bar{P})$ is what looks like a possible $N_t[M,\Upsilon]$ and a $(\Upsilon,\mathbf{t})$-successor of it is what looks like $N_{t+1}[M,\Upsilon]$):

1.8 Definition. Let M,Υ as in Definition 1.3 be given.
1) We say $(N,\bar{P})$ is an M-candidate or (M,Υ)-candidate if:

(a) N is a finite transitive submodel of $\bigcup_t V_t[M]$ which includes M, expanded by the relations of M (so it is a $(\tau_M)_{[1]}$-model)

(b) $\bar{P} = \langle P_k : k < m_1 \rangle, P_k$ an $\mathbf{m}^\Upsilon(k)$-relation on N or a partial $(\mathbf{m}^\Upsilon(k) - 1)$-place function on N when P_k is a predicate or a function symbol respectively.

In fact here the only information from Υ used is $\mathbf{m}_1^\Upsilon$ of Υ, so we may write "$(M, \mathbf{m}_1^\Upsilon)$-candidate".
2) We say $(N', \bar{P}')$ is the $(\Upsilon, \mathbf{t})$-successor of $(N, \bar{P})$ if $(N', \bar{P}'), (N, \bar{P})$ satisfies what $N_{t+1}^+[M, \Upsilon, \mathbf{t}], N_t^+[M, \Upsilon, \mathbf{t}]$ satisfies in Definition 1.1(3A), so

$$|N'| = \text{ is the transitive closure of } M \cup \bigcup_{\ell < m_1} A_\ell[N, \Upsilon, \mathbf{t}],$$

where $A_\ell = A_\ell[N, \Upsilon, \mathbf{t}]$ is $\mathscr{P}_\ell[N, \Upsilon] = \big\{\{a : (N, \bar{c}) \models \psi_\ell(a, \bar{b})\} : \bar{b} \in {}^{\ell g(\bar{y}_\ell)}N\big\}$ if this family has $\le \mathbf{t}(M)$ members or is equal to N <u>and</u> A_ℓ is empty otherwise.
2A) We say $(N', \bar{P})$ is the Υ-successor of $(N, \bar{P})$ if $(\bar{N}', \bar{P}'), (N, \bar{P})$ satisfies what $N_{t+1}^+[M, \Upsilon], N_t^+[M, \Upsilon]$ satisfies in Definition 1.1(3); this means just that $(N', \bar{P}')$ is the (Υ, ∞)-successor of $(N, \bar{P})$.
2B) If Υ is pure (i.e. $m_1^\Upsilon = 0$), actually only $\bar{\psi}^\Upsilon$ count and we may replace Υ by $\bar{\psi}^\Upsilon$.
2C) We say that $(N, \bar{P})$ is a $(M, \Upsilon)^+$-candidate if it is an (M, Υ)-candidate and the sets $\emptyset, |M|$ (= set of atoms) belongs to N.
3) We define $N_t = N_t[M, \Upsilon, \mathbf{t}]$ and $\bar{P}_t = \bar{P}_t[M, \Upsilon, \mathbf{t}]$ by induction on t as follows:

for $t = 0$ it is M (i.e. with $P_{t,k} = \emptyset$),
for $t + 1, (N_{t+1}, \bar{P}_{t+1})$ is the $(\Upsilon, \mathbf{t})$–successor of $(N_t, \bar{P}_t)$, see below 1.9,
for $t = \infty$ we take the union.

1.9 Claim. *1) If $(N, \bar{P})$ is an (M, Υ)-candidate, it has exactly one $(\Upsilon, \mathbf{t})$-successor (and exactly one Υ-successor).*
2) The pair $(N_t[M, \Upsilon, \infty], \bar{P}_t[M, \Upsilon, \infty])$ defined in Definition 1.8(3), is equal to the pair $(N_t[M, \Upsilon], \bar{P}_t[M, \Upsilon])$ defined in Definition 1.1(3).
3) If Υ is monotonic, $(N', \bar{P}')$ the Υ-successor of $(N, \bar{P})$ where both are $(M, \Upsilon)^+$-candidates, <u>then</u> $N \subseteq N'$; if Υ is also standard then $N \subset N'$.
4) If Υ is strongly monotonic (see Definition 1.1(2)(E)) and $(N', \bar{P}')$ is the Υ-successor of $(N, \bar{P})$ both are $(M, \Upsilon)^+$-candidates, <u>then</u> $N \subseteq N'$ and $P_\ell \subseteq P'_\ell$ for $\ell < m_1^\Upsilon$.

There are many obvious inclusions between the variants of logics by natural translations. We mention the following claim which tells us that there is no real harm if we restrict ourselves to pure Υ's.

1.10 Claim. *1) Assume the Υ is an inductive scheme in $\mathscr{L}_{\text{f.o.}}(\tau^+), \chi$ a sentence in $\mathscr{L}_{\text{f.o.}}(\tau^+)$. <u>Then</u> we can find a <u>pure</u> inductive scheme Υ^* in $\mathscr{L}_{\text{f.o.}}(\tau^+)$ and r^*, r^{**} and $\mathbf{p}^*$ and sentences θ^*, χ_1^* and formulas $\varphi^*(x), \varphi_k^*(\bar{x}_k)$ for $k < m_1^\Upsilon, \ell g(\bar{x}_\ell) = \mathbf{m}^\Upsilon(k)$ in $\mathscr{L}_{\text{f.o.}}(\tau)$ such that:*

$\boxtimes$ *for every τ-model and t we have, if $t^* = r^{**} + r^* t$ then:*

(a) *the set $N_t[M, \Upsilon]$ is $\{a \in N_{t^*}[M, \Upsilon^*] : N_{t^*}[M, \Upsilon^*] \models \varphi^*[a]\}$,*

(b) *$P_{t,k}[M, \Upsilon] = \{\bar a \in {}^m(N_{t^*}[M, \Upsilon^*]) : N_{t^*}[M, \Upsilon^*] \models \varphi^*_k[\bar a]\}$, where $m = \mathbf{m}^\Upsilon(k)$,*

(c) *$N_t[M, \Upsilon] \models \chi$ iff $N_{t^*}[M, \Upsilon^*] \models \chi^*$,*

(d) *$N^+_t[M, \Upsilon] = N^+_{t+1}[M, \Upsilon]$ (i.e. it stops) iff $N_{t^*}[M, \Upsilon^*] \models \theta^*$ iff $N_{t^*}[M, \Upsilon^*] = N_{t^*+1}[M, \Upsilon^*]$,*

(e) *$N_{t^*}[M, \Upsilon^*]$ has exactly $\mathbf{p}^*(\|N_t[M, \Upsilon]\|)$ elements, and $N_{t^*+r}[M, \Upsilon^*]$ has $\le \mathbf{p}^*(\|N_t[M, \Upsilon]\|)$ when $r < r^*$ and $\mathbf{p}^*$ an integer polynomial,*

(f) *if Υ is standard then so is Υ^*.*

2) Similarly using a logic $\mathscr{L}^*$ if it is closed under first order operations and substitutions.

Remark. We can similarly deal with $N_t[M, \Upsilon, \mathbf{t}]$, but then we have to deal with some form of cardinality quantifiers, etc.

Proof. For simplicity we assume that Υ is standard. Now for every $(M, \mathbf{m}^\Upsilon_1)$–candidate $(N, \bar P)$ we shall define a M-candidate $N^* = N^*_{(N,\bar P)}$. We shall have $N^+_{r^{**}+r^*t}[M, \Upsilon^*]$ is $N^*_{N^+_t[M,\Upsilon]}$.

The set of natural numbers of N^* is $\{s : s < r^{**} + r^* t\}$. The universe of N^* is the union of the following sets:

(a) N

(b) $\{\{x, r^* t\} : x \in N\}$
(used to define N),
let $a_{x,k,m} =: \{x, r^*t + 1 + k, r^*t + 1 + m^\Upsilon_1 + m\}$ for $x \in N, k < m^\Upsilon_1, m < \mathbf{m}^\Upsilon(k)$
(used to help to code $P_{t,k}$)

(c) $\{\{a_{x_m,k,m} : m < \mathbf{m}^\Upsilon(k)\} \cup \{r^*t + 1 + k, r^*t + 1 + m^\Upsilon_1 + i\} :$
$k < m^\Upsilon_1, x_0, \ldots, x_{\mathbf{m}^\Upsilon(k)-1} \in N$ and $i \in \{0, 1\}$ and
$i = 1 \Leftrightarrow \langle x_m : m \in \mathbf{m}^\Upsilon(k)\rangle \in P_{t,k}\}$

(d) some more elements to take care of the

“ $N_{t^*}[M, \Upsilon^*]$ has exactly $\mathbf{p}^*(\|N_t[M, \Upsilon]\|)$ elements ”

(if we agree to $N_{t^*}[M, \Upsilon^*]$ has exactly $\mathbf{p}(\|N_t[M, \Upsilon^*], t\|)$ then this is not necessary).

The rest should be clear. $\square_{1.10}$

We try to sort out some of the relations between these logics by checking when two variants of a sentence say related things, for quite many ι_1, ι_2.

1.11 Claim. *Let $\iota_1, \iota_2 \in \{1, \dots, 7\}$ and $\theta_\ell = \theta_{\Upsilon,\chi,\mathbf{t}_\ell} \in \mathscr{L}^{\mathbf{T}}(\mathscr{L}^*)(\tau)$ for $\ell = 1, 2$ and consider*

- (α) *θ_2 is ι_2-good implies θ_1 is ι_1-good,*
- (β) *for every (finite) τ-model $M, (M \models_{\iota_2} \theta_2) \Rightarrow (M \models_{\iota_1} \theta_1)$,*
- $(\beta)^-$ *if θ_ℓ is ι_ℓ-good for $\ell = 1, 2$ then $M \models_{\iota_2} \theta_2 \Rightarrow M \models_{\iota_1} \theta_1$.*

In the following clauses we list cases which holds under various conditions.

$(A)(\alpha) + (\beta)$ *if $\iota_1 = \iota_2$ and $\mathbf{t}_1 < \mathbf{t}_2$,*

$(B)(\alpha) + (\beta)$ *if $\iota_1 = 2, \iota_2 = 3, \Upsilon$ is standard and $\mathbf{t}_1 \geq 2\mathbf{t}_2$,*

$(C)(\alpha) + (\beta)$ *if $\iota_1 = 3, \iota_2 = 2, \Upsilon$ is standard and $\mathbf{t}_1 \geq \mathbf{t}_2$,*

$(D)(\alpha) + (\beta)$ *if $\iota_1 = 4, \iota_2 = 3, \Upsilon$ is standard and $\mathbf{t}_1 \geq \mathbf{t}_2$,*

$(E)(\alpha) + (\beta)^-$ *if $\iota_1 = 3, \iota_2 = 4, \Upsilon$ is standard and $\mathbf{t}_1$ is large enough,*

$(F)(\alpha) + (\beta)$ *if $\iota_1 = 5, \iota_2 = 3, \Upsilon$ is standard and $\mathbf{t}_1 \geq \mathbf{t}_2$,*

$(G)(\alpha) + (\beta)$ *if $\iota_1 = 3, \iota_2 = 5, \Upsilon$ is standard and $(\forall n)[\mathbf{t}_1(n) \geq n + m_0^\Upsilon \mathbf{t}_2(n)\mathbf{t}_2(n)]$,*

$(H)(\alpha) + (\beta)^-$ *if $\iota_1 = 1, \iota_2 = 2$,*

$(I)(\alpha) + (\beta)^-$ *if $\iota_1 = 2, \iota_2 = 1$ and $\mathbf{t}_1$ is large enough,*
(i.e. $\mathbf{t}_1(n) > \mathrm{Max}\{t_{\iota_2}[M, \Upsilon, \mathbf{t}_2] : M$ a τ-model with universe $[n]$ and $t_{\iota_2}[M, \Upsilon, \mathbf{t}_2] < \infty\}$; note that Max is taken on a finite set)

$(J)(\alpha) + (\beta)$ *if $\iota_1 = 6, \iota_2 = 2$ and $\mathbf{t}_1(2n) \geq \mathbf{t}_2(n), \mathbf{t}_1(2n+1) \geq \mathbf{t}_2(n)$,*

$(K)(\alpha) + (\beta)$ *if $\iota_1 = 2, \iota_2 = 6$ and $\mathbf{t}_2 \geq \mathbf{t}_1(2n) + \mathbf{t}_1(2n+1)$,*

$(L)(\alpha) + (\beta)$ *if $\iota_1 = 7, \iota_2 = 6$ and $\mathbf{t}_1 \geq \mathbf{t}_2$*
(note: after "good" stopping, nothing changes) $\mathbf{t}_1$ is large enough

Proof. Straightforward.

1.12 Conclusion. 1) Assume that $\mathbf{T}$ satisfies

$$(*)\quad (\forall \mathbf{s} \in \mathbf{T})(\forall m)(\exists \mathbf{t} \in \mathbf{T})(\forall n)(\mathbf{t}(n) \geq n + m(\mathbf{s}(n))^2).$$

Then the logics $\mathscr{L}_\iota^{\mathbf{T}}(\mathscr{L}^*)^{\mathrm{good}}$ for $\iota = 2, 3, 4, 5, 6$ are weakly equivalent where $\mathscr{L}^1, \mathscr{L}^2$ are weakly equivalent if $\mathscr{L}^1 \leq_{\mathrm{wk}} \mathscr{L}^2$ and $\mathscr{L}^2 \leq_{\mathrm{wk}} \mathscr{L}^1$; where $\mathscr{L}^1 \leq_{\mathrm{wk}} \mathscr{L}^2$ means that for every sentence $\theta_1 \in \mathscr{L}^1$ there is $\theta_2 \in \mathscr{L}^2$ such that for every M we have[1] $M \models \theta_1$ implies $M \models \theta_2$
2) If in addition $\mathbf{T}$ consists of integer polynomials and $\mathscr{L}_* \in \{\mathscr{L}_{\mathrm{f.o.+na}}, \mathscr{L}_{\mathrm{card}}\}$ we can add $\iota = 7$.

1.13 Remark. In part 1.12(2) we can replace the assumption on $\mathbf{t}$ demanding only that:

[1]note that if teh truth value of θ_1 in M is undefined, then the implication is trivial.

(∗) for every $\mathbf{t} \in \mathbf{T}$ view $\mathbf{t}$ as $(\mathbf{t}^{\rm tm}, \mathbf{t}^{\rm sp})$ there is $\mathbf{s} \in \mathbf{T}$ such that we can "compute" $\mathbf{t}^{\rm tm}(n), \mathbf{t}^{\rm sp}(n)$ is $\mathbf{s}^{\rm tm}(n)$ time, $\mathbf{s}^{\rm sp}(n)$ space in the relevant sense (using the given $\mathscr{L}_*$, etc.).

1.14 Claim. *The family of good sentences in the logic $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}^*_{\rm f.o.})$, that is the logic $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}^*_{\rm f.o.})^{\rm good}$, is closed under the following: the Boolean operation, $(\exists x)$, and substitution (that is, up to equivalence) when at least one of the following holds*

$(*)_1$ $\iota = 1$

$(*)_2$ $\iota \in \{2,3,5\}$ *and* $\mathbf{T}$ *satisfies* (∗) *of 1.12*

$(*)_3$ $\iota = 11$

$(*)_4$ $\iota = 22$ *and for each* $\mathbf{t} \in \mathbf{T}, \mathbf{T}[\mathbf{t}^{\rm wd}] =: \{\mathbf{s}^{\rm ht} : \mathbf{s} \in \mathbf{T}, \mathbf{s}^{\rm wd} = \mathbf{t}^{\rm wd}\}$ *satisfies* (∗) *of 1.12.*

Proof. Straight.

§2 The general systems of partial isomorphisms

Though usually our aim is to compare two models M_1, M_2 we first concentrate on one model M; this, of course, gives shorter definitions.

Our aim is to have a family $\mathscr{F}$ of partial automorphisms as in Ehrenfeucht-Fraïssé games (actually Karp), of the model M we analyze, not total automorphism which is too restrictive. But this family has to be lifted to the N_t's. Hence their domains (and ranges) may and should sometimes contain an element of high rank. It is natural to extend each $f \in \mathscr{F}$ to $G_t(f)$, a partial automorphism of N_t. So we should not lose anything when we get up on t. The solution is $I \subseteq \{A : A \subseteq M\}$ closed downward and $\mathscr{F}$ (could have used $\langle \mathscr{F}_\ell : \ell < m_1\rangle$), a family of partial automorphisms of M. So every $x \in N_t$ will have a support $A \in I$ and for $f \in \mathscr{F}$, its action on A determines its action on $x, (G_t(f)(x)$ in this section notation). It is not unreasonable to demand that there is the smallest support, still this demand is somewhat restrictive (or we have to add imaginary elements as in [Sh:a] or [Sh:c], not a very appetizing choice here).

But how come we in stage $t+1$ succeed to add "all sets $X = X_{\ell,\bar b}$" definable by $\psi_\ell(x,\bar b)$ for some sequence $\bar b \in {}^{\ell g(\bar b)}N_t$? Let m be such that $\bar b = \langle b_1,\dots,b_m\rangle$.

The parameters $b_1,\dots,b_m$ each has a support say $A_1,\dots,A_m$ resp., all in I; so when we have enough mappings in the family $\mathscr{F}$, the new set has in some sense the support $A = \bigcup\limits_{\ell=1}^m A_\ell$, in the sense that suitable partial mappings act as expected. So if $y \in N_t$ has support B (BRy in this section notation), $f \in \mathscr{F}, A \cup B \subseteq$ Dom(f) and $f \restriction A =$ id$_A$, then the mapping $G_t(f)$ which f induces in N_t will satisfy $y \in X_{\ell,\bar b} \Leftrightarrow (G(f))(y) \in X_{i,\bar b}$.

But we are not allowed to increase the family of possible supports and A though a kind of support is probably too large: in general, I is not closed under unions.

But, if we add $X = X_{\ell,\bar b}$ we have to add all "similar" $X' = X_{\ell,\bar b'}$. Recall that necessarily our strategy is to look for a support $A' \in I$ for $X_{\ell,\bar b}$. So we like to find $A' \in I$ which is a support of X, that is such that that if $f \in \mathscr{F}, A \cup A' \subseteq \mathrm{Dom}(f)$, then $f \restriction A$ induces a mapping of $X_{i,\bar b}$ to some $X_{i,\bar b'}$, which when $f \restriction A' = \mathrm{id}_{A'}$, satisfies that $X_{i,\bar b'}$ will be equal to $X_{i,\bar b}$ thus justifying the statement "A' supports X." How? We use our bound on the size of the computation. So we need a dichotomy: either there is $A' \in I$ as above or the number of sets $X_{i,\bar b'}$ defined by $\psi_i(x, \bar b')$ varying $\bar b'$ is too large!!
On this dichotomy hangs the proof.

However, we do not like to state this as a condition on N_t but rather on M. We do not "know" how $\psi_\ell(x, \bar b')$ will act but for any possible A' this induces an equivalence relation on the set of images of A' (for this $\mathscr{F}$ has to be large enough).

Actually, we can ignore the ψ_ℓ's and develop set theory of elements demanding each has a support in I through $\mathscr{F}$. Now we break the proof to definition and claims.

We consider several variants of the logic: the usual variant to make preservation clear, and the case with the cardinality quantifier. We use one $\mathscr{F}$ but we could have used $\langle \mathscr{F}_\ell : \ell \le k' \rangle$; in this case actually, for much of the treatment only $\mathscr{F}_0$ would count. The relevant basic family of partial automorphisms is defined in 2.1. Note that the case with cardinality logic, with a stronger assumption is clearer, if you like to concentrate on it, ignore 2.1(4) and read in Definition 2.3 only part (1), ignore 2.9 but read 2.10, ignore 2.17, 2.20 but read 2.18, ignore 2.22, 2.24 but read 2.23.

2.1 The Main Definition. 1) We say $\mathscr{Y} = (M, I, \mathscr{F})$ is a k-system if

- (A) I is a non empty family of subsets of $|M|$ (the universe of the model M) closed under subsets and each singleton belongs to it
  [hint: intended as the possible supports of elements $N_t[M, \Upsilon]$ and as first approximation to the possible supports of the partial automorphisms of M, where M is the model of course; the intention is that M is a finite model]
  Let $I[m] =: \{\bigcup_{\ell=1}^{m} A_\ell : A_\ell \in I \text{ for } \ell = 1, \dots, m\}$
- (B) $\mathscr{F}$ is a non empty family of partial automorphisms of M such that $f \in \mathscr{F}$ & $A \subseteq \mathrm{Dom}(f)$ & $A \in I \Rightarrow f"(A) \in I$ (recall $f"(A) = \{f(x) : x \in A \cap \mathrm{Dom}(f)\}$; $\mathscr{F}$ is closed under inverse (i.e. $f \in \mathscr{F}_\ell \Rightarrow f^{-1} \in \mathscr{F}_\ell$) and composition and restriction (hence, together with (D) clearly $B \in I[k] \Rightarrow \mathrm{id}_B \in \mathscr{F}$)
- (C) if $f \in \mathscr{F}$ then $\mathrm{Dom}(f)$ is the union of $\le k$ members of I
- (D) if $f \in \mathscr{F}$ and $A_1, \dots, A_{k-1}, A_k \in I$ and $\ell \in \{1, \dots, k-1\} \Rightarrow A_\ell \subseteq \mathrm{Dom}(f)$, then for some $g \in \mathscr{F}$ we have

$$f \restriction \bigcup_{\ell=1}^{k-1} A_\ell \subseteq g$$

$$A_k \subseteq \text{Dom}(g)$$

2) Assume $\mathscr{Y}$ is a k-system and $B \in I[m], m \leq k-2$ and $A \in I$

(α) let $H_{\mathscr{Y}}(B,A) = \{g \in \mathscr{F} : \text{Dom}(g) \supseteq B \cup A \text{ and } \text{id}_B \subseteq g\}$

(β) $\mathscr{E}_{\mathscr{Y}}(B,A)$ is the family of equivalence relations E on $H_{\mathscr{Y}}(B,A)$ such that:

(*) if $g_1, g_2, g_3, g_4 \in H_{\mathscr{Y}}(B,A)$ and $f \in \mathscr{F}$ satisfies $\text{id}_B \subseteq f, g"_1(A) \cup g"_2(A) \subseteq \text{Dom}(f)$ and $g_3 \supseteq f \circ (g_1 \restriction (B \cup A)), g_4 \supseteq f \circ (g_2 \restriction (B \cup A)$ then $g_1 E g_2 \Leftrightarrow g_3 E g_4$.

(this tells you in particular that only $g \restriction A$ matter for determining g/E)

3)

(α) Let $H_{I,m} = H_{\mathscr{Y},m}$ be the family of functions h from $[m] = \{1, \ldots, m\}$ to $\text{Seq}_I = \{\bar{a} : \bar{a}$ list with no repetitions some $A \in I$; we can look at $\bar{a}$ as a one-to-one function from $[0, \ell g\bar{a})$ onto $A\}$; of course for $f \in \mathscr{F}$ and $\bar{a}, \bar{b} \in \text{Seq}_I$ the meaning of $f(\bar{a}) = \bar{b}$ is $\ell g(\bar{a}) = \ell g(\bar{b})$ and $f(a_i) = b_i$ for $i < \ell g(\bar{a})$. Let $\text{Seq}_{I,A} = \{\bar{a} \in \text{Seq}_I : \text{Rang}(\bar{a}) = A\}$. For $m = 1$ we may identify $h : [m] \to \text{Seq}_I$ with $h(1)$ so $H_{I,1}$ is identified with Seq_I.

(β) for $h \in H_{I,m}$ and $f \in \mathscr{F}$ such that $\bigcup_{i \in [m]} \text{Rang}(h(i)) \subseteq \text{Dom}(f)$ we define $h' = f * h$ as the following function: $\text{Dom}(h') = [m]$ and $h'(i) = f \circ (h(i))$

(γ) let $2r + s \leq k$; for $B \in I[s]$ and $1 \leq m \leq r$ let $E^0_{I,B,m} = E^0_{\mathscr{Y},B,m}$ be the following 2-place relation on $\{h : h : [m] \to \text{Seq}_I\}$:
$h_1 E^0_{I,B,m} h_2$ iff for some $f \in \mathscr{F}$, $\text{id}_B \subseteq f$ and $h_2 = f * h_1$.
If $B = \emptyset$ we may omit it; similarly $m = 1$

(δ) for $2m + s \leq k$ and $B \in I[s]$ let $\mathscr{E}_{I,m}(B) = \mathscr{E}_{\mathscr{Y},m}(\emptyset)$ be the family of equivalence relations such that for some $h^* \in H_{I,m}$, E is an equivalence relation on the set $\{h \in H_{I,m} : h E^0_{I,m} h^*\}$ which satisfies:

(*) if $h_1, h_2, h_3, h_4 \in H_{I,m}$, $f \in \mathscr{F}$, $\text{id}_B \subseteq f$, $h_2 = f * h_1, h_4 = f * h_3$ and $\bigcup\{\text{Rang}(h_1(i)) \cup \text{Rang}(h_1(i)) : i \in [m]\} \subseteq \text{Dom}(f)$,
then $h_1 E h_3 \equiv h_2 E h_4$.
If $B = \emptyset$ we may omit it. If $m = 1$ we may omit it.

4) We say that a k-system $\mathscr{Y} = (M, I, \mathscr{F})$ is $(\mathbf{t}, r)$-dichotomical[2] when

⊠ if $1 \leq m \leq r$ and $E \in \mathscr{E}_{I,m}(\emptyset)$, then $(\beta)_1 \vee (\beta)_2$ where

$(\beta)_1$ there is $A \in I$ which satisfies:
if $h_1, h_2 \in H_{I,m}$, $f \in \mathscr{F}$, $\text{id}_A \subseteq f$ and $h_2 = f * h_1$ then $h_1 E h_2$

$(\beta)_2$ the number of E-equivalence classes is $> \mathbf{t}(\|M\|)$.

---

[2]note that this is how from "there are not too many" we get "there is a support in I"

If we omit r (and write $\mathbf{t}$-dichotomical) we mean $r = [k/2]$, $k \geq 3$.

Note that in parts (3), (4) without loss of generality h is one-to-one.

* * *

2.2 Remark. 1) However, if we shall deal with $\mathscr{L}^* = \mathscr{L}_{\text{card}}$ or $\mathscr{L}^* = L_{\text{card},\mathbf{T}}$ we naturally have to require that $f \in \mathscr{F}$ preserve more. Whereas the $(\mathbf{t}, r)$-dichotomy is used to show that either we try to add too many sets to some $N_t[M, \Upsilon]$ or we have support, the "counting (k, r)-system" assure us that the lifting of f preserves the counting quantifier, and the medium $(\mathbf{t}, r)$-dichotomy will be used in the maximal successor, see 2.24.

2) It causes no harm real in 2.1(3)(γ), (δ) and similarly later, to restrict ourselves to e.g. $r + s \leq k/100, k > 400$.

2.3 Definition. 1) We say $\mathscr{Y} = (M, I, \mathscr{F})$ is a counting (or super) (k, r)-system if:

$\mathscr{Y}$ is a k-system and

$(*)_1$ Assume that $0 \leq m \leq r$ and for $\ell = 1, 2$ we have $B_\ell \in I[m]$ and $E_\ell \in \mathscr{E}_{\mathscr{Y},1}(B_\ell)$. If $f \in \mathscr{F}, f$ maps B_1 onto B_2 and f maps E_1 to E_2 (see 2.4(1)), then $|\text{Dom}(E_1)/E_1| = |\text{Dom}(E_2)/E_2|$.

(This should be good for analyzing the model $N_t[M, \Upsilon, \mathbf{t}]$). If we omit r (write counting k-system) we mean $r = k - 2, k \geq 3$.
2) We say that the k-system $\mathscr{Y} = (M, I, \mathscr{F})$ is medium $(\mathbf{t}, k, r)$-system if

$(*)_2$ Assume that $1 \leq m \leq r$ and for $\ell = 1, 2$ we have $B_\ell \in I[m]$ and $E_\ell \in \mathscr{E}_{\mathscr{Y},1}(B_\ell)$. If $f \in \mathscr{F}, f$ maps B_1 onto B_2 and f maps E_1 to E_2 (see 2.4), then $|\text{Dom}(E_1)/E_1| = |\text{Dom}(E_1)/E_2|$ or both are $> \mathbf{t}(\|M\|)$.

3) We omit r if $r = k - 2 \geq 1$ (see 2.8 below).

Note that 2.4 is closed to 2.8 and 2.7(2)-(4).

2.4 Observation. Let $\mathscr{Y} = (M, I, \mathscr{F})$ is a k-system.
1)

(α) if $2m + s \leq k$ and $B \in I[s]$, then $E^0_{\mathscr{Y},B,m}$ is an equivalence relation satisfying $(*)$ of Definition 2.1(3)(δ)

(β) the following two conditions on $B \in I[s], m \leq (k - s)/2, s \leq k$ and $\mathscr{G}$ are equivalent:

(i) $\mathscr{G}$ is an equivalence class of $E^0_{I,B,m}$

(ii) $\mathscr{G}$ is the domain of some $E \in \mathscr{E}_{I,m}(B)$

(γ) if $k^* = 2m + s \leq k$ and $\mathscr{F}^* = \{f \restriction A : f \in \mathscr{F}, A \in I[k^*]\}$, and $\mathscr{Y}^* = (M, I, \mathscr{F}^*)$ then $\mathscr{Y}^*$ is a k^*-system and for each $B \in I[s]$ we have $\mathscr{E}_{\mathscr{Y},m}(B) = \mathscr{E}_{\mathscr{Y}^*,m}(B)$ and $E^0_{\mathscr{Y},B,m} = E^0_{\mathscr{Y}^*,B,m}$.

(δ) if $B_1, B_2 \in I[k-2], f \in \mathscr{F}, f$ maps B_1 onto B_2 and $g'_1, g''_1 \in \text{Seq}_I$, $\text{Rang}(g'_1) \cup \text{Rang}(g'_2) \subseteq \text{Dom}(f)$, then $g'_2 =: f \circ g'_1$ belongs to Seq_I and $g''_2 = f \circ g''_1$ belongs to Seq_I and $g'_1 E^0_{I,B_1,1} g''_1 \Leftrightarrow g'_2 E^0_{I,B_2,1} g''_2$

(ε) If $B_1 \in I[k-2], A_2 \in I, f \in \mathscr{F}, B_1 \subseteq \text{Dom}(f), B_2 = f``(B_1)$, then we can define $F^{\text{eq}}_{f,B_1,B_2}(E) \in \mathscr{E}_{I,1}(B_2)$ for $E \in \mathscr{E}_{I,1}(B_1)$ (the image of E by f) by: if $f \restriction B_1 \subseteq g \in \mathscr{F}, \bar{a}_1, \bar{a}_2 \in \text{Seq}_I, g$ maps $\bar{a}_1$ to $\bar{a}^*_1$ and g maps $\bar{a}_2$ to $\bar{a}^*_2$ then $\bar{a}_1 E \bar{a}_2 \Leftrightarrow \bar{a}^*_1 F^{\text{eq}}_{f,B_1,B_2}(E) \bar{a}^*_2$.

2) Let $2r + s \leq k$,
(δ), (ε), parallely to part (1) with $m \leq r, h_\ell \in H_{I,m}, B_\ell \in I[s]$.

Proof. Straight, e.g.:
Part (1), Clause (α): We use: $\mathscr{F}$ contains id_C wherever $C \in I[k]$ (for reflexivity), $\mathscr{F}$ closed under inverting (for symmetry) and is closed under composition (for transitivity).

$\square_{2.4}$

2.5 Discussion 1) In the system $\mathscr{Y} = (M, I, \mathscr{F})$ we deal only with partial automorphisms of M, we need to lift them to the models $N_t[M, \Upsilon]$ or actually $N^+_t[M, \Upsilon]$ or $N^+_t[M, \Upsilon, \mathbf{t}]$ appearing in Definition 1.3; this motivates the following definition 2.6. (We more generally define liftings to (M, Υ)-candidates).
2) Note that here probably it is more natural if in the definition of k-system $\mathscr{Y}$, we replace the relations "$f \subseteq g$", "$f = g \restriction A$", "$f \supseteq g \restriction A$" on $\mathscr{F}$ by abstract ones (so $\mathscr{F}$ will be an index set). Also in Definition 2.6 we could demand more properties which naturally holds (similarly in Definition 2.13, e.g. if you satisfy the properties of "a set B is $\mathfrak{Z}$-support of x" you are one).

2.6 Definition. 1) Let $\mathscr{Y} = (M, I, \mathscr{F})$ be an k-system, M a τ-model, Υ is an inductive scheme for $\mathscr{L}^*(\tau^+)$ and $\mathbf{m}_1 = \mathbf{m}^\Upsilon_1$.

We say that $\mathfrak{Z} = (N, \bar{P}, G, R) = (N^\mathfrak{Z}, \bar{P}^\mathfrak{Z}, G^\mathfrak{Z}, R^\mathfrak{Z})$ is a Υ-lifting or $\mathbf{m}_1$-lifting of $\mathscr{Y}$ if

(a) $(N, \bar{P})$ is an $(M, \mathbf{m}_1)$-candidate so N is a transitive submodel of set theory i.e. of $V_\infty[M]$ with M as its set of urelements and the relations of M (see Definition 1.4(1))

(b) G is a function with domain $\mathscr{F}$

(c) for $f \in \mathscr{F}$

  (α) $G(f)$ is a function with domain $\subseteq N, f \subseteq G(f)$, moreover $f = G(f) \restriction M$ and

  (β) $G(f)$ is a partial automorphism of N

(d) if $f \in \mathscr{F}, g \in \mathscr{F}, f \subseteq g$ then $G(f) \subseteq G(g)$

(e) R is a two-place relation written xRy such that
$xRy \Rightarrow x \in I \;\&\; y \in N$
[we say: x is a $\mathfrak{Z}$-support of y]

(f)

(α) if ARy and $f \in \mathscr{F}, A \subseteq \text{Dom}(f)$, then
$y \in \text{Dom}(G(f))$ and $f \restriction A = \text{id}_A \Rightarrow G(f)(y) = y$

(β) if $f \in \mathscr{F}$ and $y \in \text{Dom}(G(f))$ (hence $y \in N$) then some a $\mathfrak{Z}$-support of y is included in $\text{Dom}(f)$

(g) $(\forall y \in N)(\exists A \in I)ARy$
[i.e. every element of N has a $\mathfrak{Z}$-support]

(h) if $A \in I$ and $A \subseteq \text{Dom}(f), y \in \text{Dom}(G(f))$ then
$ARy \Leftrightarrow f"(A)R(G(f)(y))$

(i) for $f \in \mathscr{F}$ we have $G(f^{-1}) = (G(f))^{-1}$

(j) for $f_1, f_2 \in \mathscr{F}, f = f_2 \circ f_1$ we have[3] $G(f) \subseteq G(f_2) \circ G(f_1)$

(k) if $\bar{a} \in {}^{\mathbf{m}_1(\ell)}(\text{Dom}(f))$ and $f \in \mathscr{F}$ and $f(\bar{a})$ is well defined, then $\bar{a} \in P^{\mathfrak{Z}}_\ell \equiv f(\bar{a}) \in P^{\mathfrak{Z}}_\ell$; moreover $\emptyset Rc_\ell$ if $P^{\mathfrak{Z}}_\ell$ is the individual constant $c_\ell = c^{\mathfrak{Z}}_\ell$ when c_ℓ is well defined (see Definition 1.1(2)(F); this implies that $G(f)$ is a partial automorphism of $(N, \bar{P})$).

We may write $\mathbf{m}_1 = \mathbf{m}^{\mathfrak{Z}}_1$, recall that $\mathbf{m}_1(k)$ gives the arity of P_k and the information is it a relation or (possibly partial) function.

2.7 Fact: Let $\mathscr{Y} = (M, I, \mathscr{F})$ be a k-system, and $\mathfrak{Z}$ be an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.
1) The $0 - \Upsilon$-lifting (in Definition 2.12) exists and is a lifting (see Definition 2.6).
2) If $f_1, f_2 \in \mathscr{F}$ and A is a $\mathfrak{Z}$-support of $y \in N$ then

$$f_1 \restriction A = f_2 \restriction A \ \& \ A \subseteq \text{Dom}(f_1) \quad \Rightarrow \quad f_1(y) = f_2(y).$$

3) From $\mathfrak{Z}$ we can reconstruct $\mathscr{Y}, \mathscr{F}, \mathbf{m}_1$; and if $I = \{A :$ for some $B, A \subseteq B$ and B is a $\mathfrak{Z}$-support of some $y \in N\}$ then we can reconstruct also I (so the whole $\mathscr{Y}$).

Proof. 1) Easy [compare with 2.4, 2.8].
2) Let $y' = G(f_2)(y)$, let $A_1 = A, A_2 = f"_2(A_1)$ hence as A is a $\mathfrak{Z}$-support of y and $A \subseteq \text{Dom}(f_1)$ necessarily y' is well defined and $A_2 R^{\mathfrak{Z}} y'$ (see Definition 2.6(1) clause (h)). We know that f_2^{-1} and $f_2^{-1} \circ f_1$ belongs to $\mathscr{F}$ (see Definition 2.6(1) clauses (i) and (j)). We also know that $A_2 \subseteq \text{Dom}(f_2^{-1})$ so as $A_2 R^{\mathfrak{Z}} y'$ (see above) we have $y' \in \text{Dom}(G(f_2^{-1}))$ (see Definition 2.6(1), clause $(f)(\alpha)$) and $(G(f_2^{-1}))(y') = y$ (by Definition 2.6(1), clause (i) as $(G(f_2))(y) = y'$ by the choice of y').
Clearly $A = A_1 \subseteq \text{Dom}(f_2^{-1} \circ f_1)$ hence (see Definition 2.6(1), clause $(f)(\beta)$) we have $y \in \text{Dom}(G(f_2^{-1} \circ f))$.

[3] so maybe x even has support A_ℓ, $\text{id}_{A_\ell} \subseteq f_\ell$ for $\ell = 1, 2$ but x has no support $\subseteq \text{Dom}(f)$

But $\mathrm{id}_A \subseteq f_2^{-1} \circ f_1$, so as $AR^{\mathfrak{Z}} y$ we have $y = (G(f_2^{-1} \circ f))(y)$. By Definition 2.6(1), clause (j) we have, as the left side is well defined:

$$(G(f_2^{-1} \circ f_1))(y) = ((G(f_2^{-1})) \circ (G(f_1)))(y)$$

and trivially

$$(G(f_2^{-1})) \circ (G(f_1)))(y) = (G(f_2^{-1}))((G(f_1))(y)).$$

By the last three equations $y = (G(f_2^{-1} \circ f_1))(y) = (G(f_2^{-1}))((G(f_1))(y))$, but by the above we note $y = (G(f_2^{-1}))(y')$. So, as $G(f_2^{-1})$ is one-to-one (having an inverse), we have $(G(f_1))(y) = y'$, now as y' was defined as $(G(f_2))(y)$ we are done.

3) Straightforward. $\square_{2.7}$

Note that 2.8 is close to 2.4 and 2.12(2)-(4).

2.8 Definition/Claim. Let $\mathscr{Y} = (M, I, \mathscr{F})$ be a k-system and $\mathfrak{Z} = (N, \bar{P}, G, R)$ be an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.

1) For $B \subseteq N$ let $\mathbf{E}_B = \mathbf{E}_B^{\mathscr{Y},\mathfrak{Z}}$ be the following 2-place relation on N:

$x_1 \mathbf{E}_B x_2$ <u>iff</u> for some $f \in \mathscr{F}$ we have $\mathrm{id}_B \subseteq f$ and $(G(f))(x_1) = x_2$.

2) If $B \in I[k-2]$ so $k \geq 3$ <u>then</u>

(α) $\mathbf{E}_B$ is an equivalence relation on N

(β) if $f \in \mathscr{F}, B \subseteq \mathrm{Dom}(f)$ <u>then</u> f maps $\mathbf{E}_B$ to $\mathbf{E}_{f``(B)}$ which means:

$$f \restriction B \subseteq g \in \mathscr{F} \ \ \& \ \bigwedge_{\ell<2} (G(g))(x_\ell) = y_\ell \Rightarrow [x_1 \mathbf{E}_B x_2 \equiv y_1 \mathbf{E}_{f``(B)} y_2]$$

(γ) if $B \subseteq \mathrm{Dom}(f)$ <u>then</u> there is a one-to-one function $F = F_{f,B}$ from $N/\mathbf{E}_B$ onto $N/\mathbf{E}_{f``(B)}$ such that:

$(*)_1$ for $x_1, x_2 \in N$ we have: $(\exists g)(f \restriction B \subseteq g \in \mathscr{F} \ \& \ G(g)(x_1) = x_2) \Leftrightarrow F(x_1/\mathbf{E}_B) = x_2/\mathbf{E}_{f``(B)}$

(δ) if $x \in N$ and $AR^{\mathfrak{Z}} x$ and $\bar{a} \in \mathrm{Seq}_{I,A}$ <u>then</u> there is an equivalence relation $E \in \mathscr{E}_{\mathscr{Y},1}(B)$ with domain $\{f(\bar{a}) : f \text{ extend } \mathrm{id}_B \text{ and } A \subseteq \mathrm{Dom}(f)\}$ such that:

$(*)_2$ if $f_\ell \in \mathscr{F}, \mathrm{id}_B \subseteq f_\ell$ and $A \subseteq \mathrm{Dom}(f_\ell)$ for $\ell = 1, 2$ <u>then</u> $G^{\mathfrak{Z}}(f_1)(x) = G^{\mathfrak{Z}}(f_2)(x) \Leftrightarrow f_1(\bar{a}) E f_2(\bar{a})$

$(*)_3$ $|x/\mathbf{E}_B| = |\mathrm{Dom}(E)/E|$

(ε) if f, F are as in clause (γ) and $x_1, x_2 \in N$ <u>then</u>

$(*)_4$ if $F(x_1/\mathbf{E}_B) = x_2/\mathbf{E}_B$ and $\mathscr{Y}$ is a counting k-system, <u>then</u> $|x_1/\mathbf{E}_B| = |x_2/\mathbf{E}_B|$

$(*)_5$ if $F(x_1/\mathbf{E}_B) = x_2/\mathbf{E}_B$ and $\mathscr{Y}$ is a medium $(\mathbf{t}, k)$-system, <u>then</u> $|x_1/\mathbf{E}_B| = |x_2/\mathbf{E}_B|$ or both are $> \mathbf{t}(M)$.

Proof of (2).

Clause (α):

Reflexivity:

For $x \in N$ choose a $\mathfrak{Z}$-support $A \in I$, so we can find $f \in \mathscr{F}$ extending $\mathrm{id}_{B\cup A}$ hence $G(f)$ maps x to itself, hence $x\mathbf{E}_B x$.

Symmetry:

If $f \in \mathscr{F}$ witnesses $x\mathbf{E}_B y$ then $f^{-1} \in \mathscr{F}$ witnesses $y\mathbf{E}_B x$.

Transitivity:

If $x_0\mathbf{E}_B x_1, x_1\mathbf{E}_B x_2$ let f_1 witness $x_0\mathbf{E}_B x_2$ and let f_2 witness $x_1\mathbf{E}_B x_2$, now let $A_0 \subseteq \mathrm{Dom}(f_1)$ be a $\mathfrak{Z}$-support of x_0, so $A_1 = f"_1(A_0) \subseteq \mathrm{Rang}(f_1)$ is a $\mathfrak{Z}$-witness of x_1, now let $A_1^* \subseteq \mathrm{Dom}(f_2)$ be a $\mathfrak{Z}$-support of X_1, so $B \cup A_1^* \in I[k-1]$ hence without loss of generality $A_1 \subseteq \mathrm{Dom}(f_2)$ hence $A_2 = f"_2(A_1)$ is a $\mathfrak{Z}$-support of x_2, so $x_1 \in \mathrm{Dom}(G(f_2 \circ f_1))$ and $G(f_2 \circ f_1) \subseteq G(f_1) \circ G(f_2)$ hence $G(f_2 \circ f_1)(x_1) = x_2$ as required.

Clause (β):

So assume $f \restriction B \subseteq g \in \mathscr{F}$ and $(G(g))(x_\ell) = y_\ell$, for $\ell = 1,2$ and we should prove $x_1\mathbf{E}_B x_2 \equiv y_1\mathbf{E}_{f"(B)}y_2$. It suffices to prove $x_1\mathbf{E}_B x_2 \Rightarrow y_1\mathbf{E}_{f"(B)}y_2$ (as applying it to $B' = f"(B), f' = f^{-1}, g' = g^{-1}, y_1, y_2, x_1, x_2$ we get the other implication). As $x_1\mathbf{E}_B x_2$ we can find a witness h, i.e., $\mathrm{id}_B \subseteq h \in \mathscr{F}$ and $(G(h))(x_1) = x_2$. Let $A_1 \subseteq \mathrm{Dom}(h)$ be a $\mathfrak{Z}$-support of x_1 and let $A_2 = h"(A_1)$, so A_2 is a $\mathfrak{Z}$-support of x_2.

If $B \in I[k-4]$ without loss of generality $A_1, A_2 \subseteq \mathrm{Dom}(g)$, and let $A_1^* = f"(A_1)$, $A_2^* = f"(A_2)$, lastly let $g^* = g \circ h \circ g^{-1}$. Now $(G(g^*))(y_1) = y_2$, $\mathrm{id}_{f"(B)} \subseteq g^*$ so g^* witnesses $y_1\mathbf{E}_{f"(B)}y_2$ as required.

But maybe $B \notin I[k-4]$, still $B \in I[k-2]$; now for $\ell = 1,2$, as $(G(f))(x_\ell) = y_\ell$ there is a $\mathfrak{Z}$-support C_ℓ of x_ℓ such that $C_\ell \subseteq \mathrm{Dom}(g)$ and let $C'_\ell = g"(C_\ell)$. So we can find, for $\ell = 1,2$ a function $g_1 \in \mathscr{F}$ such that $g \restriction (B \cup C_1) \subseteq g_1$ and $A_1 \subseteq \mathrm{Dom}(g_2)$ and also there is $h_1 \in \mathscr{F}$ such that $h \restriction (B \cup A_1) \subseteq h_1$ and $C_1 \subseteq \mathrm{Dom}(h_1)$. So $h_1 \circ g_1^{-1}$ extends $(g \restriction B)^{-1}$ and $C'_1 \subseteq \mathrm{Dom}(g_1^{-1}), (g_1^{-1})"(C'_1) = C_1 \subseteq \mathrm{Dom}(h_1)$ but C'_1 is a $\mathfrak{Z}$-support of y_1. Hence $(G(h \circ g_1^{-1}))(y_1)$ is well defined and equal to $(G(h) \circ G(g_1^{-1}))(y_1) = (G(h))((G(g_1^{-1}))(y_1)) = (G(h_1))(x_1) = x_2$. Let $g_2 \in \mathscr{F}$ be such that $g \restriction (B \cup A_2) \subseteq g_2$ and $(h_1 \circ g_1^{-1})"(C'_1) \subseteq \mathrm{Dom}(g_2)$ and similarly we get $g_2^{-1} \circ h_1 \circ g_1^{-1}$ extends $\mathrm{id}_{g"(B)}$ and $(G(g_2^{-1} \circ h_1 \circ g_1^{-1}))(y_1) = y_2$, so we are done.

Clause $(\gamma), (\delta), (\varepsilon)$:

Should be clear. $\square_{2.8}$

Quite naturally for such a $\mathbf{m}_1$-lifting $\mathfrak{Z}$ of $\mathscr{Y}$ the family $\{G(f) : f \in \mathscr{F}^{\mathscr{Y}}\}$ helps us to understand first order logic on $(N^{\mathfrak{Z}}, \bar{P}^{\mathfrak{Z}})$.

2.9 Claim. *Assume $\mathscr{Y} = (M, I, \mathscr{F})$ is a k-system and $\mathfrak{Z} = (N, \bar{P}, G, R)$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.*
<u>*Then*</u>

(∗) *Assume $\varphi(\bar{x})$ is first order and $k \geq$ quantifier depth$(\varphi(\bar{x})) + \ell g(\bar{x})$, or just $\varphi(\bar{x}) \in \mathscr{L}_{\infty,k}$ which means:*
every subformula of $\varphi(\bar{x})$, (e.g. φ itself) has $\leq k$ free variables.

If $\bar{a} \in {}^{\ell g(\bar{x})}N, A_\ell R a_\ell$ and $A_\ell \subseteq \mathrm{Dom}(f)$ (hence $a_\ell \in$ Dom$(G(f))$) for $\ell < \ell g(\bar{x})$ and $f \in \mathscr{F}$ <u>*then*</u>

$$(N, \bar{P}) \models \text{“}\varphi[\ldots, a_\ell, \ldots]_{\ell < \ell g(\bar{x})}\text{”} \Leftrightarrow (N, \bar{P}) \models \text{“}\varphi[\ldots, G(f)(a_\ell), \ldots]_{\ell < \ell g(\bar{x})}\text{”}$$

Proof. We prove this by induction on the quantifier depth of φ. Let $m = \ell g(\bar{x})$ so $\bar{x} = \langle x_\ell : \ell < m \rangle$ and without loss of generality $m \leq k$.

<u>Case 1</u>: φ atomic.
As $G(f)$ is a partial automorphism of N and even $(N, \bar{P})$ this should be clear.

<u>Case 2</u>: $\varphi = \neg\psi$ or $\varphi = \psi_1 \wedge \psi_2$ or $\varphi = \varphi_1 \vee \varphi_2$.
Straight.

<u>Case 3</u>: $\varphi = \varphi(\bar{x}) = (\exists y)\psi(y, \bar{x})$.
Without loss of generality y is not a dummy variable in ψ, that is has a free occurrence. Let $\bar{x} = \langle x_0, \ldots, x_{m-1} \rangle$ so $m < k$.
As $G(f^{-1}) = G(f)^{-1}$ it is enough to prove $N \models$ “$\varphi[a_0, \ldots, a_{m-1}]$” $\Rightarrow N \models$ “$\varphi[G(f)(a_0), \ldots, G(f)(a_{m-1})]$”.
So we assume the left side, i.e.

$(*)_1$ $N \models \varphi[a_0, \ldots, a_{m-1}]$,
hence for some $a^* \in N$ we have

$(*)_2$ $N \models \psi[a^*, a_0, \ldots, a_{m-1}]$.

Necessarily a^* has a $\mathfrak{Z}$-support $A^* \in I$.
Now $k \geq m + 1$ and $\mathscr{Y}$ is a k-system hence there is $f^* \in \mathscr{F}$ such that $f \restriction (\bigcup\limits_{\ell < m} A_\ell) \subseteq f^*$ and $A^* \subseteq \mathrm{Dom}(f^*)$. So each of $a^*, a_0, \ldots, a_{m-1}$ has a $\mathfrak{Z}$-support included in $\mathrm{Dom}(f^*)$ hence by the induction hypothesis applied to $\psi[a^*, a_0, \ldots, a_{m_1}]$ and $(*)_2$ we have

$(*)_3$ $N \models \psi[G(f^*)(a^*), G(f^*)(a_0), \ldots, G(f^*)(a_{m-1})]$.

So by the definition of $\models$ we get

$(*)_4$ $N \models (\exists y)\psi(y, G(f^*)(a_0), \ldots, G(f^*)(a_{m-1})$.

But for $\ell < m$, the set A_ℓ is a $\mathfrak{Z}$-support of a_ℓ and $f^* \restriction A_\ell = f \restriction A_\ell$ hence $(G(f^*))(a_\ell) = G(f)(a_\ell)$ so

$(*)_5$ $N \models (\exists y)\psi(y, G(f)(a_0), \ldots, G(f)(a_{m-1})]$.

But $\varphi(\bar{x}) = (\exists y)\psi(y, \bar{x})$ so we are done. $\square_{2.9}$

Having dealt with first order logic we should deal with cardinality logic (actually any of the variants we mention). Here we use the counting version, really naturally the medium version suffices but for it we have to use more "bookkeeping" of the various things used, and the reader can use only this smoother case. Note that if we like to add cardinality quantifiers on pairs we need $s \geq 2$, etc., but we may create the set of pairs in N_t so not so necessary.

2.10 Claim. *Assume that $\mathscr{Y}$ is a counting k-system (see Def 2.3(1)) and $\mathfrak{Z} = (N, \bar{P}, G, R)$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.*
Then

(∗) *assume $\varphi(\bar{x}) \in \mathscr{L}_{\text{card},\mathbf{T}}$ or $\varphi(\bar{x}) \in \mathscr{L}_{\text{card}}$ (can have both kinds quantifiers; recall that $\mathscr{L}_{\text{f.o. + na}}$ is included in a special case of $\mathscr{L}_{\text{card},\mathbf{T}}$) and every subformula of $\varphi(\bar{x})$, including $\varphi(\bar{x})$ itself, has $< k$ free variables and $m = \ell g(\bar{x}) < k$*

$(\alpha)_{\varphi(\bar{x})}$ *if $f \in \mathscr{F}$ and A_ℓ is a $\mathfrak{Z}$-support of a_ℓ and $A_\ell \subseteq$ Dom(f) for $\ell < m$ then:*

$$(N, \bar{P}) \models \text{“}\varphi[a_0, \ldots, a_{m-1}]\text{”} \quad \textit{iff} \quad (N, \bar{P}) \models \text{“}\varphi[G(f)(a_0), \ldots, G(f)(a_{m-1})]\text{”}$$

$(\beta)_{\varphi(\bar{x})}$ *if $f \in \mathscr{F}$ and A_ℓ is a $\mathfrak{Z}$-support of a_ℓ for $\ell = 1, \ldots, m-1$ and $A_\ell \subseteq$ Dom(f) then the sets $\{b \in N : (N, \bar{P}) \models \text{“}\varphi[b, a_1, \ldots, a_{m-1}]\text{”}\}$, and $\{b \in N : (N, \bar{P}) \models \text{“}\varphi[b, G(f)(a_1), \ldots, G(f)(a_{m-1})]\text{”}\}$ have the same number of elements.*

Proof. We prove by induction on the quantifier depth of φ.
We first show that

⊠ $(\alpha)_{\varphi(\bar{x})} \Rightarrow (\beta)_{\varphi(\bar{x})}$

Why? So assume $(\alpha)_{\varphi(\bar{x})}$, and $a_1, \ldots, a_{m-1} \in N$ be given (where $\ell g(\bar{x}) = m$) and also $f \in \mathscr{F}$ and $A_1, \ldots, A_{m-1} \in N, A_\ell$ is a $\mathfrak{Z}$-support of a_ℓ for $\ell = 1, \ldots, m-1$ such that $A_\ell \subseteq$ Dom(f), and we should prove the equality in $(\beta)_{\varphi(\bar{x})}$. Let a^i_ℓ be a_ℓ if $i = 1$ and $(G(f))(a_\ell)$ if $i = 2$. Let A^i_ℓ be A_ℓ if $i = 1$ and $(G(f))$"(A_ℓ) if $\ell = 2$.
Let $B_i = \bigcup\limits_{\ell=1}^{m-1} A^i_\ell$ so $B_i \in I[m-1]$ and f maps B_1 onto B_2.

By Definition 2.8(1) we know that $\mathbf{E}_{B_\ell}$ is an equivalence relation on N and, see Definition 2.8(2), clause $(\gamma), (\delta)$, the function $F = F_{f,B_1}$ satisfies

(i) F is a one-to-one function from $N/\mathbf{E}_{B_1}$ onto $N/\mathbf{E}_{B_2}$
(ii) $f \restriction B_1 \subseteq g \in \mathscr{F}$ & $(G(g))(x_1) = x_2 \Rightarrow (F(x_1)/\mathbf{E}_{B_1}) = x/\mathbf{E}_{B_2}$

(*iii*) for every $x_1 \in N, A_1 \in I$ such that $A_1 R^3 x_1$, and $\bar{a}_1 \in \text{Seq}_{I,A_1}$ <u>for some</u> $E = E_{x_1} \in \mathscr{E}_{\mathscr{Y},1}(B_1)$ we have $\text{Dom}(E) = x_1/\mathbf{E}_{B_1}$ and

(∗) if $[\text{id}_{B_1} \subseteq f_\ell \;\&\; A_1 \subseteq \text{Dom}(f_\ell)]$ for $\ell = 1, 2$ then $f_1(\bar{a}_1) E_x f_2(\bar{a}_2) \Leftrightarrow (G(f_1))(x_1) = (G(f_2))(x_1)$

(*iv*) if $x_1 \in N, x_2 \in N, (G(g))(x_1) = x_2$ for some $g \in \mathscr{F}$ such that $f \restriction B_1 \subseteq g$, then $|(x_1/\mathbf{E}_{B_1})/E_{x_1}| = |(x_2/\mathbf{E}_{B_2}/E_{x_2}|$.

Hence it suffices to prove, assuming $F(x_1/\mathbf{E}_{A_1}) = x_2/\mathbf{E}_{A_2}$ that

$$N \models \text{“}\varphi[x_1, a_1, \dots, a_{m-1}]\text{”} \Leftrightarrow N \models \text{“}\varphi[x_2, G(f)(a_1), \dots, G(f)(a_{m-1})]\text{”}.$$

As we can replace f by a suitable extension of $f \restriction B_1$, without loss of generality there are $y_1 \in x_1/\mathbf{E}_{B_1}$ and $y_2 \in x_2/\mathbf{E}_{B_2}$ such that $(G(f))(y_1) = y_2$. We can find $C_1 \subseteq \text{Dom}(f)$ which is a $\mathfrak{Z}$-support of y_1.

As $x_1 \mathbf{E}_{B_1} y_1$ we can find $f_1 \in \mathscr{F}$ such that $\text{id}_{B_1} \subseteq f_1$ and $(G(f_1))(x_1) = y_1$; as $m < k$, without lose of generality $C_1 \subseteq \text{Rang}(f_1)$ and let $C_0 = (f_1^{-1})\text{”}(C_1)$, let $C_2 = f\text{”}(C_1)$. As $x_2 \mathbf{E}_{B_1} y_2$ we can find $f_2 \in \mathscr{F}$ such that $\text{id}_{B_2} \subseteq f_2$ and $(G(f_2))(y_2) = x_2$; as $m < k$ without lose of generality $C_2 \subseteq \text{Dom}(f_2)$ and let $C_3 = f\text{”}_2(C_2)$. So $f' = f_2 \circ f \circ f_1$ belongs to $\mathscr{F}$ and extends $f \restriction B_1$ and it maps C_0 onto C_3 hence $y_1 \in \text{Dom}(G(f'))$, and clearly $(G(f_1))(x_1) = y_1, (G(f))(y_1) = y_2, (G(f_2))(y_2) = x_2$ hence $(G(f'))(x_1) = x_2$ and, of course, $B_\ell \subseteq \text{Dom}(f')$ hence applying $(\alpha)_{\varphi(\bar{x})}$ to $f', x_1, a_1, \dots, a_{m-1}$ we get

$$(N, \bar{P}) \models \text{“}\varphi[x_1, a_1, \dots, a_{m-1}]\text{”} \Leftrightarrow (N, \bar{P}) \models \text{“}\varphi[x_2, G(f)(a_1), \dots, G(f)(a_{m-1})]\text{”}$$

recalling $x_2 = (G(f'))(x_1)$. So ⊠ holds.

Now the inductive proof of $(\alpha)_{\bar{\varphi}}$ is separated to cases. The case φ atomic, $\varphi = \neg\varphi, \varphi = \psi_1 \wedge \psi_2, \varphi = (\exists y)(\psi(y, \bar{x}))$ works as in the proof of 2.9. The new cases hold because $(\beta)_\psi$ hold by the induction hypothesis + ⊠. $\square_{2.10}$

2.11 Claim. *We can weaken in 2.10 the assumption on $\mathscr{Y}$ to “medium $(\mathbf{t}, k)$-dichotomical” provided that:*

⊠ *if $B \in I[m], 1 \leq m \leq r$, <u>then</u> every equivalence class of $\mathbf{E}_B^{\mathscr{Y},3}$ (so a subset of N^3, see Definition 2.8(1)) has $\leq \mathbf{t}(M^{\mathscr{Y}})$ members.*

Proof. Straightforward.

2.12 Definition. For $\mathscr{Y} = (M, I, \mathscr{F})$ a k-system, the $0-\Upsilon$-lifting or the $0-\mathbf{m}_1$-lifting if $\mathbf{m}_1 = \mathbf{m}_1^\Upsilon$ <u>is</u> $(M, \bar{P}, G, R)$ where

(*a*) G is the identity on $\mathscr{F}$

(*b*) $ARy \Leftrightarrow A \in I \;\&\; y \in A$

(*c*) each P_ℓ is the empty relation.

Clearly our intention requires us for a k-system $\mathscr{Y}$, to move from an Υ-lifting $\mathfrak{z}_t = (N_t[M,\Upsilon,\mathbf{t}], \bar P_t[M,\Upsilon,t], G_t, R_t)$ to a Υ-lifting

$$\mathfrak{z}_{t+1} = (N_{t+1}[M,\Upsilon], \bar P_{t+1}[M,\Upsilon], G_{t+1}, R_{t+1}).$$

Toward this aim naturally in Definition 2.13 below we define for Υ-lifting $\mathfrak{z}$ some successors, and in Claim 2.16 we prove what they satisfy.
In Definition 2.13(6) we can define "$\mathfrak{z}'$ is <u>the</u> full $\mathbf{t}$-successor of $\mathfrak{z}$" (both $\mathbf{m}_1$-liftings of a k-system $\mathscr{Y}$).

2.13 Definition. Let $\mathscr{Y} = (M, I, \mathscr{F})$ be a k-system and $\mathfrak{z} = (N, \bar P, G, R)$ be an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.
1) We say X is good or $(\mathscr{Y},\mathfrak{z})$-good if

- (a) X a subset of N
- (b) for some $A \in I$ we have "A supports X" (for our $\mathscr{Y}$ and $\mathfrak{z}$) which means: <u>if</u> $f \in \mathscr{F}, BRy$ (so $B \in I, y \in N$) and $A \cup B \subseteq \operatorname{Dom}(f)$, and $f \restriction A = \operatorname{id}_A$ <u>then</u> $y \in X \Leftrightarrow (G(f))(y) \in X$
  (note: $(G(f))(y)$ is well defined by clause (f) of 2.6)
- (c) $X \notin N$.

2) Let $\mathscr{P} = \mathscr{P}_{\mathscr{Y},\mathfrak{z}}$ be the family of good subsets of N, let $\mathscr{R} = \mathscr{R}_{\mathscr{Y},\mathfrak{z}}$ be the two-place relation defined by: $A\mathscr{R}X$ iff A supports X, i.e. (b) of part (1) holds.
3) For $f \in \mathscr{F}$ we define a function $G^+(f) = G^+_{\mathscr{Y},\mathfrak{z}}(f)$ with domain $\subseteq \mathscr{P}_{\mathscr{Y},\mathfrak{z}} \cup N$ as follows, (well, now $(G^+(f))(X_1) = X_2$ is just a relation, but by 2.14(1) clause (ii) below it is a function)

- (α) For good X such that $A\mathscr{R}X, A \in I$ when $A \subseteq \operatorname{Dom}(f)$ we let $(G^+(f))(X)$ $= \{y \in N :$ for some $g \in \mathscr{F}$ and $y' \in X$ we have $f \restriction A \subseteq g$ and $G(g)(y') = y\}$
- (β) $G^+(f) \restriction N = G(f)$.
  Note that no contradition arises between clauses (α) and (β) because of clause (c) in part (1).

4) We define $E = E_{\mathscr{Y},\mathfrak{z}}$ as the following two place relation: $X_1 E X_2$ <u>iff</u> X_1, X_2 are good subsets of $N^{\mathfrak{z}}$ and for some $f \in \mathscr{F}$ we have $(G^+(f))(X_1) = X_2$; this is an equivalence relation (see 2.15(2) below).
5) $\mathfrak{z}' = (N', \bar P', G', R')$ is a successor of $\mathfrak{z}$ <u>if</u>:

- (a) $N \subseteq N' \subseteq N \cup \mathscr{P}_{\mathscr{Y},\mathscr{Z}}$
- (b) $X_1 E_{\mathscr{Y},\mathfrak{z}} X_2 \;\;\&\;\; X_1 \in \mathscr{P}_{\mathscr{Y},\mathfrak{z}} \;\;\&\;\; X_2 \in \mathscr{P}_{\mathscr{Y},\mathfrak{z}} \Rightarrow [X_1 \in N' \leftrightarrow X_2 \in N']$
- (c) G' is a function with domain $\mathscr{F}$ and for $f \in \mathscr{F}$ the function $G'(f)$ is defined as $G^+(f)$ from part (3) restricted to N'
- (d) R' is $R \cup [\mathscr{R} \restriction (I \times N')]$
- (e) the pair $(N', \bar P')$ is an $(M, \mathbf{m}_1)$-candidate; so P'_ℓ is an $\mathbf{m}_1(\ell)$-ary relation or function as dictated by $\mathbf{m}_1$

6) We say $\mathfrak{Z}'$ is a $\bar{\varphi}$-reasonable successor of $\mathfrak{Z}$ if it is a successor and

$(e)'$ $P'_\ell = \{\bar{b} \in N : (N, \bar{P}) \models \varphi_\ell(\bar{b})\}$ but when Υ is i.c. (see Definition 1.1(2), clause (F)) this is so only if $P'_\ell \in N'$ and $P'_\ell = \emptyset$ otherwise (for each $\ell < \mathbf{m}_1$ (so we demand $P'_\ell \in N'$).

We may omit $\bar{\varphi}$ if clear from the content.
7) We say that $\mathfrak{Z}' = (N', \bar{P}', G', R')$ is a full successor of $\mathfrak{Z}$ if it is a successor of $\mathfrak{Z}$ and $N' = N \cup \mathscr{P}_{\mathscr{Y},\mathfrak{Z}}$. We say it is the full successor if in addition $P'_\ell = \emptyset$ and it is the full reasonable successor (or reasonable full successor) if it is a full successor which is a reasonable successor.
8) $\mathfrak{Z}' = (N', \bar{P}', G', R')$ is a full $\mathbf{t}$-successor of $\mathfrak{Z}$ if it is a successor of $\mathfrak{Z}$ and

$(a)^*_8$ $N' = N \cup \{X \in \mathscr{P}_{\mathscr{Y},\mathfrak{Z}} : |X/E_{\mathscr{Y},\mathfrak{Z}}| \le \mathbf{t}(\|M\|)\}$.

So if we omit $\mathbf{t}$ we mean $\mathbf{t}(\|N\|) = \infty$.
9) $\mathfrak{Z}' = (N', \bar{P}', G', R')$ is the true $(\Upsilon, \mathbf{t})$-successor of $\mathfrak{Z}$ if $\mathbf{m}_1 = \mathbf{m}_1^\Upsilon$ and $\mathfrak{Z}'$ is a reasonable successor of $\mathfrak{Z}$ and:

$(a)^*_9$ $(N', \bar{P}')$ is the $(\Upsilon, \mathbf{t})$-successor of $(N, \bar{P})$, (see Definition 1.8(2)).

10) $\mathfrak{Z}' = (N', \bar{P}', G', R')$ is the true Υ-successor of $\mathfrak{Z}$ if it is a $\bar{\varphi}^\Upsilon$-reasonable successor of $\mathfrak{Z}$ and:

$(a)^*_{10}$ $(N', \bar{P}')$ is the Υ-successor of $(N, \bar{P})$, (see Definition 1.8(2A)) [this just means the true (Υ, ∞)-successor of $\mathfrak{Z}$].

Note that the names above indicate our intentions, but we have to prove that "$\mathfrak{Z}'$ is a successor of $\mathfrak{Z}$" implies that "$\mathfrak{Z}$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$" (done in 2.16), the true $(\Upsilon, \mathbf{t})$-successor of $\mathfrak{Z}$ is a $\mathbf{t}$-successor of $\mathfrak{Z}$ (done in 2.17, 2.18, 2.20) and similarly without the $\mathbf{t}$.

2.14 Claim. *Assume $\mathscr{Y}$ is a k-system, Υ an inductive scheme (so τ is common) and $\mathfrak{Z}$ is an Υ-lifting of $\mathscr{Y}$.*
1) In Definition 2.13(3), if $k \ge 3$ then for $f \in \mathscr{F}$ and $(\mathscr{Y}, \mathfrak{Z})$-good X we have:
(i) if the relation $X_2 = (G^+(f))(X_1)$ holds and $(G(f))(x_1) = x_2$ then $x_1 \in X_1 \equiv x_2 \in X_2$
(ii) the value $(G^+(f))(X)$ does not depend on A, so $G^+(f)$ is well defined.
(iii) if the relation $X_2 = (G^+(f))(X_1)$ holds then X_2 is a $(\mathscr{Y}, \mathfrak{Z})$-good
2) There is a unique object $\mathfrak{Z}'$ which is the full successor of $\mathfrak{Z}$. ; there is a unique object $\mathfrak{Z}'$ which is the reasonable full successor of $\mathfrak{Z}$ and there is a unique object which is the reasonable $\mathbf{t}$-full successor of $\mathfrak{Z}$.
3) If the $\mathfrak{Z}'$ is the true $(\Upsilon, \mathbf{t})$-successor of $\mathfrak{Z}$, then $\mathfrak{Z}'$ is a reasonable successor of $\mathfrak{Z}$ which implies $\mathfrak{Z}'$ is a successor of $\mathfrak{Z}$.
4) There is at most one true successor $\mathfrak{Z}'$ of $\mathfrak{Z}$.

Proof. Easy, using 2.16(2) below for part (2); e.g.
1) Clause (i) So assume that X_1 is $(\mathscr{Y}, \mathfrak{Z})$-good, A_1 is a $\mathfrak{Z}$-support of X_1,

$A_1 \subseteq \mathrm{Dom}(f), f \in \mathscr{F}$ and $X_2 = \{(G(g))(x) : x \in \mathrm{Dom}(g), f \restriction A \subseteq g\} \subseteq N$ and $(G(f))(x_1) = x_2$. First $x_1 \in X_1 \Rightarrow x_2 \in X_2$ by the definition of X_2. Second assume that $x_2 \in X_2$, so for some $y \in X_1$ and $g \in \mathscr{F}$ we have $f \restriction A_1 \subseteq g$ and $(G(g))(y) = x_2$. Let $B_2 \subseteq \mathrm{Dom}(g)$ be a $\mathfrak{Z}$-support of y. Let $B_1 \subseteq \mathrm{Dom}(f)$ be a $\mathfrak{Z}$-support of x_1; as $k \geq 3$ there is $f_1 \in \mathscr{F}$ such that $f \restriction (A_1 \cup B_1) \subseteq f_1$ and $g"(B_2) \subseteq \mathrm{Rang}(f_1)$ so $(G(f_1))(x_1) = x_2$. Now $(G(g^{-1} \circ f_1))(x_1)$ is well defined as $B_2 \subseteq \mathrm{Dom}(g^{-1} \circ f_1)$ hence is equal to $((G(g^{-1}) \circ (G(f_1)))(x_1) = y$ and $(g^{-1} \circ f_1) \restriction A_1 = \mathrm{id}_{A_1}$ hence $x_1 \in X_1 \equiv y \in X_1$ but $y \in X_1$ hence $x_1 \in X_1$ as required.

Clause (ii) So assume that $f \in \mathscr{F}, X$ is good and for $\ell = 1, 2$ the set $A_\ell \in I$ is a support of X and $A_\ell \subseteq \mathrm{Dom}(f)$. For $\ell = 1, 2$ let $X_\ell =: \{y \in N :$ for some $g \in \mathscr{F}$ and $y' \in X$ we have $f \restriction A_\ell \subseteq g$ and $(G(g))(y') = y\}$.
By the symmetry it is enough to show that $y \in X_1 \Rightarrow y \in X_2$. So assume $y \in X_1$ hence there are $g \in \mathscr{F}$ and $y' \in X$ such that $f \restriction A_1 \subseteq g$ and $(G(g))(y') = y$. As $y' \in X \subseteq N$, by Definition 2.6(1), clause (g) there is $B \in I$ which is a $\mathfrak{Z}$-support of y'. As $(G(g))(y') = y$ without loss of generality B is such that $B \subseteq \mathrm{Dom}(g)$ (see Definition 2.6(1), clause (f)). As $\mathscr{Y}$ is a k-system and $k \geq 3$ there is $f^* \in \mathscr{F}$ such that $f \restriction (A_1 \cup A_2) \subseteq f^*$ and $g"(B) \subseteq \mathrm{Rang}(f^*)$ hence for some $B^* \subseteq \mathrm{Dom}(f^*)$ we have $(f^*)"(B^*) = g"(B)$ so for some $y^* \in N$ we have $(G(f^*))(y^*) = y$. By clause (i) applied to A_2, f^*, y^*, y we have $y^* \in X \equiv y \in X_2$, so it is enough to prove that $y^* \in X$. Now easily $g^{-1} \circ f^* \in \mathscr{F}$, $B^* \subseteq \mathrm{Dom}(g^{-1} \circ f^*)$, $\mathrm{id}_{A_1} \subseteq g^{-1} \circ f^*$ and so $y^* \in \mathrm{Dom}(G(g^{-1} \circ f^*))$ and $(G(g^{-1} \circ f^*))(y^*) = y'$. Hence, as X is good (see Definition 2.13(1) clause (b)), we have $y^* \in X \equiv y' \in X$, but $y' \in X$ by its choice, so we are done.

Clause (iii) Easy, or see the proof of $\boxtimes(*)_1$ inside the proof of 2.16 below. $\square_{2.14}$

Now for the definition of successor for liftings of $\mathscr{Y}$, we naturally ask whether there is any.

2.15 Claim. *Assume $\mathscr{Y}$ is a k-system $k \geq 3$, Υ an inductive scheme and $\mathfrak{Z}$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$ and $\mathbf{t} \in \mathbf{T}$.*
1) There is a $\bar{\varphi}^\Upsilon$-reasonable full $\mathbf{t}$-successor of $\mathfrak{Z}$ (and it is unique), similarly without $\mathbf{t}$.
2) $E_{\mathscr{Y},\mathfrak{Z}}$, defined in 2.13(4) is an equivalence relation on $\mathscr{P}_{Y,\mathfrak{Z}}$ (see Definition 2.13) and for every $f \in \mathscr{F}$, the function $G^+(f) : \mathscr{P}_{\mathscr{Y},\mathfrak{Z}} \to \mathscr{P}_{\mathscr{Y},\mathfrak{Z}}$ preverse the $E_{\mathscr{Y},\mathfrak{Z}}$-equivalence class.

Proof. 1) All is straight modulo part (2) (recalling 2.14).
2) Let $\mathfrak{Z}^*$ be the $\bar{\varphi}^\Upsilon$-reasonable full successor of $\mathfrak{Z}$ which exists by 2.14(2), and is a successor of $\mathfrak{Z}$ by 2.14, and is an $\mathbf{m}_1$-lifting by 2.16 below.

Why is $E_{\mathscr{Y},\mathfrak{Z}}$ an equivalence relation on $P_{\mathscr{Y},\mathfrak{Z}} = N^{\mathfrak{Z}^*} \backslash N^{\mathfrak{Z}}$? In short, by the properties of $\mathfrak{Z}$; in details:

$E_{\mathscr{Y},\mathfrak{Z}}$ is reflexive:

Let $X \in P_{\mathscr{Y},\mathfrak{Z}}$, so for some $A \in I$ we have $A \mathscr{R}^{\mathfrak{Z}^*} X$ (or equivalently $A \mathscr{R} X$) (see Definition 2.13(2)) and there is $f \in \mathscr{F}$ which is the identity on A, hence (see Definition 2.13), $X \in$ Dom$(G^{\mathfrak{z}^*}(f))$, and $(G^{\mathfrak{z}^*}(f))(X) = X$ (as clause (f)) of Definition 2.6(1) holds as $\mathfrak{z}^*$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.

<u>$E_{\mathscr{Y},\mathfrak{z}}$ is symmetric</u>:
Use $G^{\mathfrak{z}^*}(f^{-1}) = (G^{\mathfrak{z}^*}(f))^{-1}$.

<u>$E_{\mathscr{Y},\mathfrak{z}}$ is transitive</u>:
Use $G^{\mathfrak{z}^*}(f_1 \circ f_2) \subseteq G^{\mathfrak{z}^*}(f_1) \circ G^{\mathfrak{z}^*}(f_2)$.
[This is similar to 2.4.]

By the definition of $E_{\mathscr{Y},\mathfrak{z}}$ (see Definition 2.13(4)), clearly for $f \in \mathscr{F}$ the mapping $G^+(f) = G^{\mathfrak{z}^*} \restriction \mathscr{P}_{\mathscr{Y},\mathfrak{z}}$ preserves the $E_{\mathscr{Y},\mathfrak{z}}$-equivalence class or use 2.16. $\square_{2.15}$

Clearly for "reasonable" cases, everything can be interpreted in $N^{\mathfrak{z}^{\text{full}}_{\mathscr{Y},\mathfrak{t}}}$, see later. We now prove that Definition 2.13(1)-(5) works as intended, i.e. any successor of $\mathfrak{z}$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$. In particular, we have to show that the functions defined are functions with the right domain and range and the E's are equivalence relations. This is included in the proof of 2.16.

2.16 Claim. *Assume $\mathscr{Y}$ is a k-system and $\mathfrak{z}$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$ (see Definition 2.1(2),Definition 2.6) and $k \geq 3$.*
Any successor $\mathfrak{z}'$ of $\mathfrak{z}$ is an $\mathbf{m}_1$-lifting of $\mathscr{Y}$.

Proof. We check the clauses in Definition 2.6. Let $G^+, G', R', N', \bar{P}'$ be as in Definition 2.13.

<u>Clause (a)</u>: As N is transitive with M its set of urelements, and $X \in N' \backslash N \Rightarrow X \in \mathscr{P}_{\mathscr{Y},\mathfrak{z}} \Rightarrow X \subseteq N$ also N' is transitive with M its set of urelements. Clearly N has the right vocabulary $\tau^+ = \tau_M \cup \{\in\}$ and $Q \in \tau_M \Rightarrow Q^{N'} = Q^M$. So N' is as required. Also $\bar{P}' = \langle P'_\ell : \ell < m_1 \rangle$, each P'_ℓ as required by $\mathbf{m}_1$.

<u>Clause (b)</u>: By Definition 2.13(5)(c) we have that G' is $G^+ \restriction N'$ where the function G^+ is defined in part (3) of Definition 2.13 and $f \in \mathscr{F}$ implies $G^+(f)$ is a partial function with domain $\subseteq N \cup \mathscr{P}_{\mathscr{Y},\mathfrak{z}}$ (see 2.14(1)). So G' is a function with domain $\mathscr{F}$ and $G'(f)$ by its definition is a partial function with domain $\subseteq N'$.

<u>Clause (c)</u>:

<u>Subclause (α)</u>:
For $f \in \mathscr{F}$ we know that $G(f)$ is a function, $G(f) \restriction M = f$ (see Definition 2.6(1)), clause (c)(i)) and $G(f) = (G^+(f)) \restriction N$ (see Definition 2.13(3) particularly subclause (β), remembering that "X good $\Rightarrow X \notin N$" by Definition 2.13(1), clause (c)). As $M \subseteq N \subseteq N'$ and $G'(f) = G^+(f) \restriction N'$, together we get $f = (G^+(f)) \restriction M = (G'(f)) \restriction M$.

<u>Subclause (β)</u>: Let $f \in \mathscr{F}$, $G(f) = f^1$, $G'(f) = f^2$ and let $x, y \in N'$ belongs to the domain of f' and we should prove

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(α) $f^2(x) \in N'$
(β) $x \in N' \backslash N \Rightarrow f^2(x) \in N' \backslash N$
(γ) if $x \neq y$ are from N' then $f^2(x) \neq f^2(y)$
(δ) for every predicate $Q \in \tau_M$, f^2 preserve Q and $\neg Q$
(ε) $N' \models$ "$y \in x$" $\Leftrightarrow N' \models$ "$f^2(y) \in f^2(x)$"
(we shall do more toward proving clause (g) of Definition 2.6(1) below).

Note that for clause (α), as $f^2 \restriction N = G(f') \restriction N = G(f) = f^1$, it is enough to check it for $x \in N' \backslash N$, which is done in $\boxtimes, (*)_4 + (*)_1 + (*)_6$ below (as a good subset of N does not belong to N). Clause (β) also follows from $\boxtimes, (*)_4 + (*)_1 + (*)_6$ below. As for clause (γ), if $x, y \in N$ use $G(f') \restriction N = G(f)$; if $x \in N$ & $y \in N' \backslash N$ note that $G(f')(x) = (G(f))(x) \in N$ and $(G(f'))(y) \notin N$ by clause (β); similarly if $x \in N' \backslash N$ & $y \in N$; lastly if $x, y \in N' \backslash N$ we use clause (ε) proved below and N' being transitive (as $f'(x), f'(y)$ are subsets of N so $\notin M$). Now clause (δ) is easy as $G'(f) \restriction M = f$ and f being a partial automorphism of M being from $\mathscr{F}$.

Lastly, we consider clause (ε), so we let $x, y \in N'$. If $x \in N$, then $f^2(x)$ is necessarily in N too, but N is transitive, hence $N' \models$ "$y \in x$" $\Rightarrow y \in N$ and $N' \models$ "$z \in f^2(x)$" $\Rightarrow z \in N$, so as $f^2 \restriction N = f^1$ we are done. So we can assume $x \in N' \backslash N$, so x is a good subset of N, so for some $A_0 \in I, A_0 R^{\mathfrak{Z}'} x$. We define

$$\otimes_1 \qquad z =: \big\{b \in N : \text{for some } g \in \mathscr{F} \text{ and } b' \in x \text{ we have } f \restriction A_0 \subseteq g \text{ and } G(g)(b') = b\big\}.$$

We need the following, and it suffices

$\boxtimes$ assume $x \in N' \backslash N$ and z is defined as in $\otimes_1$.
<u>Then</u>
$(*)_1$ z is a good subset of N with $A_1 =: f"(A_0)$ a support of z
$(*)_2$ $x = \{b' \in N :$ for some $g \in \mathscr{F}$ and $b \in z$ we have $f^{-1} \restriction (f"(A_0)) \subseteq g$ and $G(g)(b) = b'\}$
$(*)_3$ z does not belong to N
$(*)_4$ $z = f^2(x)$
$(*)_5$ if B is another $\mathfrak{Z}$-support of x, then $z' = z$ when $z' = \{b \in N :$ for some $g \in \mathscr{F}$ and $a \in x$ we have $f \restriction A \subseteq g$ and $G(g)(a) = b\}$.
$(*)_6$ $z \in N'$

Proof of $(*)_1$. We should check clauses (a),(b),(c) of Definition 2.13(1). Now clause (a) is trivial and clause (c) is dealt with in $(*)_3$ which we prove below (and we do not use it till then, so no vicious circle). So we concentrate on proving clause (b). So suppose:

(i) $a, b \in N$ and
(ii) $g_1 \in \mathscr{F}$ satisfies $A_1 \subseteq \text{Dom}(g_1)$ and $g_1 \restriction A_1$ is the identity and
(iii) $a \in \text{Dom}[G(g_1)]$ and $b = G(g_1)(a)$.

Now we should prove that $a \in z \Leftrightarrow b \in z$. It is enough to prove $\Rightarrow$ as applying it to g_1^{-1} we get the other implication. As $b = G(g_1)(a)$ necessarily by clause (i) of Definition 2.6 for some $\mathfrak{Z}$-support B_1 of a we have $B_1 \subseteq \mathrm{Dom}(g_1)$.

Assume $a \in z$ then by the definition of z we can find $g \in \mathscr{F}$ and $a' \in X$ such that $f \restriction A_0 \subseteq g$ and $G(g)(a') = a$. By Definition 2.6(1), clause $(f)(\beta)$ there is $B_2 \in I$ such that B_2 is a $\mathfrak{Z}$-support of a' and $B_2 \subseteq \mathrm{Dom}(g)$. As $k \geq 3$ and as we can replace g by any g^* such that $g \restriction (A_0 \cup B_2) \subseteq g^* \in \mathscr{F}$,without loss of generality $B_1 \subseteq \mathrm{Rang}(g)$. So, possibly changing B_2 without loss of generality $B_1 = g"(B_2)$ (see clause (b) of Definition 2.6(1)).

Let $g' = g_1 \circ g$, so $A_0 \cup B_2 \subseteq \mathrm{Dom}(g'), g' \restriction A_0 = g \restriction A_0 = f \restriction A_0$.
[Why? As $g \restriction A_0 = f \restriction A_0, f"(A_0) = A_1$ and $g_1 \restriction A_1 = \mathrm{id}_{A_1}$; also $B_2 \subseteq \mathrm{Dom}(g)$ and $g"(B)$ is equal to B_1 which is $\subseteq \mathrm{Dom}(g_1)$]. Hence $a' \in \mathrm{Dom}(G(g'))$ and so $G(g')(a') = (G(g_1))\,(G(g)(a')) = (G(g_1))\,(a) = b$. (See Definition 2.6(1), clause (j).)

So g', a' witness $b \in z$; so $b \in z$ has been proved under the assumption $a \in z$. So by symmetry we have proved $a \in z \Leftrightarrow b \in z$.

Proof of $(*)_2$.

Call the set in the right side x'.

First assume that $a \in x$, so a has a $\mathfrak{Z}$-support B_1 hence for some $g_1 \in \mathscr{F}$ we have $A_0 \cup B_1 \subseteq \mathrm{Dom}(g_1)$ and $f \restriction A_0 \subseteq g_1$, hence $a \in \mathrm{Dom}(G(g_1))$ and let $b =: (G(g))(a)$, so $b \in z$ by the definition of z, also b has $\mathfrak{Z}$-support $B_2 =: g"(B_1)$. Let $g_2 = g_1^{-1}$ so $g_2 \in \mathscr{F}$ and $G(g_2) = G(g_1)^{-1}$ hence $(G(g_2))(b) = a$. Lastly, as $f \restriction A_0 \subseteq g_1$ clearly $f^{-1} \restriction (f"(A_0)) \subseteq g_2$. Together g_2, b witness that $a \in x'$. So we have proved $a \in x \Rightarrow a \in x'$.

Second, assume that $a \in x'$, so we have witnesses g, b for this, i.e. $g \in \mathscr{F}, b \in z, f^{-1} \restriction (f"(A_0)) \subseteq g$ and $(G(g))(b) = a$. So we can find $B_1 \subseteq \mathrm{Dom}(g)$ a $\mathfrak{Z}$-support of b, so $B_0 = g"(B_1)$ is a $\mathfrak{Z}$-support of a. As $b \in z$ there are witnesses for it, that is, there are $g_1 \in \mathscr{F}$ and $b' \in x$ such that $f \restriction A_0 \subseteq g_1$ and $(G(g_1))(b') = b$, hence $g_1^{-1} \in \mathscr{F}, G(g_1^{-1}) = (G(g_1))^{-1}$ so without loss of generality $B_1 \subseteq \mathrm{Rang}(g_1)$ and let $B_2 = (g_1^{-1})"(B_1)$, but B_1 is a $\mathfrak{Z}$-support of b hence B_2 is a $\mathfrak{Z}$-support of b'. Let $g' = g \circ g_1 \in \mathscr{F}$. Now $A_0 \subseteq \mathrm{Dom}(g_1)$ and $g"_1(A_0) = f"(A_0) \subseteq \mathrm{Dom}(g)$ hence $A_0 \subseteq \mathrm{Dom}(g')$, and as $g_1 \restriction A_0 = f \restriction A_0$, and $g \restriction f"(A_0) = f^{-1} \restriction f"(A_0)$ clearly $g' \restriction A_0 = \mathrm{id}_{A_0}$. Also $B_2 \subseteq \mathrm{Dom}(g_1)$, $B_1 = g"(B_2) \subseteq \mathrm{Dom}(g)$, hence $B_2 \subseteq \mathrm{Dom}(g')$ and $(G(g'))(b') = (G(g \circ g_1))(b') = G(g)((G(g_1))(b')) = (G(g))(b) = a$, but as $g' \restriction A_0 = \mathrm{id}_{A_0}$ and $(*)_1$ we have $b' \in x \Leftrightarrow (G(g'))(b') \in x$ which means $b' \in x \Leftrightarrow a \in x$. But we have chosen $b' \in x$ hence $a \in x$. So we have proved that $a \in x' \Rightarrow a \in x$. Thus finishing the proof of $(*)_2$.

Proof of $(*)_3$. If $z \in N$ there is $A^* \in I$ such that A^* is a $\mathfrak{Z}$-support of z.

Now there is $f_1 \in \mathscr{F}, f \restriction A_0 \subseteq f_1$ such that $A^* \subseteq \mathrm{Rang}(f_1)$. So $z_1 = G(f_1^{-1})(z)$ is well defined and by $(*)_2$ we can check that $\{b \in N : b \in z_1\} = x$; contradiction to "$x \notin N$".

Proof of $(*)_4$.

Should be clear.

Proof of $(*)_5$.
By 2.14(1).

Proof of $(*)_6$.
By clause (b) of 2.13(5).

We continue checking the clauses in Definition 2.6.

Clause (d):
Easy as for $f, g \in \mathscr{F}$ we have $f \subseteq g \Rightarrow G(f) \subseteq G(g)$.

Clause (i):
By the symmetry it is enough to show that $G'(f^{-1}) \subseteq G'(f)^{-1}$.
So let $(G'(f^{-1}))(x) = z$.
Now we know that both $G(f)$ and $G(f^{-1})$ maps N to N and $N'\backslash N$ to $N'\backslash N$ (that is when defined), so if $x \in N$ we have $z \in N$ and we use "$\mathfrak{Z}$ satisfies Definition 2.6 (1), clause (i)" to get $(G(f)^{-1})(z) = x$ as required. So assume $x \in N'\backslash N$ hence $z \in N'\backslash N$. By $\boxtimes(*)_4 + \otimes_1$ and $\boxtimes(*)_2$ we are done.

Clause (j):
Assume $x_0 \in$ Dom$(G'(f))$, hence x_0 has a $\mathfrak{Z}'$-support $A_0 \subseteq$ Dom(f), so by the definition of $f_2 \circ f_1 = f$ we have $A_0 \subseteq$ Dom(f_1) and $f"_1(A_0) \subseteq$ Dom(f_2). So we have $x_0 \in$ Dom$(G'(f_1))$ and $x_1 =: (G'(f_1))(x_0)$ has $\mathfrak{Z}'$-support $A_1 =: f"_2(A_0)$. Similarly $x_2 =: (G'(f_2))(x_1)$ is well defined and has $\mathfrak{Z}'$-support $A_2 =: f"_1(A_1)$ which is $\subseteq$ Rang$(f_2 \circ f_1) =$ Rang(f). Now we would like to show that $x_2 = (G'(f))(x_0)$; if $x_0 \in N$ this should be clear so assume that $x_0 \in N'\backslash N$ hence $x_1, x_2 \in N'\backslash N$. Let $x'_2 = (G'(f))(x_0)$, it is well defined as x_0 has $\mathfrak{Z}'$-support $A_0, A_0 \subseteq$ Dom(f) and it suffices to prove that $x_2 = x'_2$. So let $y \in N$ and we shall prove that $(y \in x_2) \equiv (y \in x'_2)$.
Let B_2 be a $\mathfrak{Z}'$-support, equivalently $\mathfrak{Z}$-support of y and let $y_2 = y$. We can find $f'_2 \in \mathscr{F}$ such that $f'_2 \restriction A_1 = f_2 \restriction A_1$ and $B_2 \subseteq$ Rang(f'_2) so as A_1 is a $\mathfrak{Z}'$-support of x_1, clearly $(G'(f'_2))(x_1) = (G'(f_2))(x_1) = x_2$. Let $B_1 = ((f'_2)^{-1})"(B_2)$. Also we can find $f'_1 \in \mathscr{F}$ such that $f'_1 \restriction A_0 = f_1 \restriction A_0$ and $B_1 \subseteq$ Rang(f'_1) so as A_0 is a $\mathfrak{Z}'$-support of x_0 clearly $(G'(f'_2))(x_0) = x_1$ and let $B_0 = ((f'_1)^{-1})"(B_1)$. Let $y_1 =: (G'((f'_2)^{-1}))(y_2)$, so $y_1 \in N$ has $\mathfrak{Z}$-support B_1, and let $y_0 = (G'((f'_1)^{-1}))(y_1)$, so $y_0 \in N$ has $\mathfrak{Z}$-support B_0. As $G'(f'_2)$ maps x_1 to x_2 and y_1 to y_2 we have by clause $(c)(\beta)$ on $\mathfrak{Z}'$ which we have already proved that $(y_2 \in x_2) = (y_1 \in x_1)$. Similarly as $G'(f'_1)$ maps x_0 to x_1 and y_0 to y_1 we have by clause $(c)(\beta)$ that $(y_1 \in x_1) \equiv (y_0 \in x_0)$ so together $(y_2 \in x_2) \equiv (y_0 \in x_0)$. Now $f' = f'_2 \circ f'_1 \in \mathscr{F}$ and its domain include $A_0 \cup B_0$ and $G'(f')$ maps y_0 to y_2 (by clause (j) for $\mathfrak{Z}$!); also as x_0 is in its domain (as $A_0 \subseteq$ Dom(f') is a $\mathfrak{Z}'$-support of x_0) and as $f \restriction A_0 = f' \restriction A_0$ we have $(G'(f'))(x_0) = (G'(f))(x_0)$ but the later is x'_2. So $(G'(f'))(x_0) = x'_2$, so as $y_0 \in$ Dom$(G'(f'))$ by clause (c) we have

$y_0 \in x_0 \equiv (G'(f'))(y_0)) \in x'_2$ but $(G'(f'))(y_0) = y_2$ so $(y_0 \in x_0) \equiv (y_2 \in x'_2)$. As earlier we have gotten $(y_2 \in x_2) \equiv (y_0 \in x_0)$ together $(y_2 \in x_2) \equiv (y_2 \in x'_2)$ but $y_2 = y$ so we are done.

Clause (e): See Definition of $R^{\mathfrak{Z}'}$ in Definition 2.13(5), clause (d).

Clause (f):

Subclause (α):

So assume $AR^{\mathfrak{Z}'}y, f \in \mathscr{F}$ and $A \subseteq \mathrm{Dom}(f)$. First, if $y \in N$ then we use $G'(f) \restriction N = G(f)$ and $\mathfrak{Z}$ satisfying Definition 2.6(1), clause $(f)(\alpha)$. Second, if $y \in N' \backslash N$ then $AR^{\mathfrak{Z}'}y$ means $A\mathscr{R}y$ and clearly y is a good subset of N and by the definition of $G'(f)(= G^+(f))$, necessarily $y \in \mathrm{Dom}(G'(f))$. If in addition $f \restriction A = \mathrm{id}_A$, we should prove that $(G'(f))(y) = y$. Now by $\boxtimes(*)_4$ apply to y, A instead x, A_0 we have

$$(G'(f))(y) = \{b \in N : \text{for some } g \in \mathscr{F} \text{ and } b' \in y \text{ we have } f \restriction A \subseteq g \text{ (i.e. } \mathrm{id}_A \subseteq g) \text{ and } G(g)(b') = b\}.$$

But as $A\mathscr{R}y$ we have:

$$\mathrm{id}_A \subseteq g \ \& \ b \in \mathrm{Dom}(G(g)) \Rightarrow [b \in y \equiv G(g)(b) \in y]$$

which means that $(G'(f))(y) = y$, as required.

Subclause (β)(of (f)):

So assume $f \in \mathscr{F}$ and $y \in \mathrm{Dom}(G'(f))$ (hence $y \in N'$). First, if $y \in N$, recall that $G'(f) \restriction N = G(f)$ and use $\mathfrak{Z}$ satisfying clause $(f)(\beta)$ of Definition 2.6(1) and

$\boxtimes_2$ $R' \restriction (I \times N) = R$.

Second, if $y \in N' \backslash N$ see the definition of $G'(f) = G^+(f)$ and R'.

Clause (g):

See the choice of $R', \mathscr{R}$.

Clause (h):

The new case is: assume $A \subseteq \mathrm{Dom}(f), A \in I, X \in \mathrm{Dom}(G'(f)), X \in N' \backslash N$. We have to show $AR^{\mathfrak{Z}'}X \Leftrightarrow f"(A)R^{\mathfrak{Z}'}(G'(f))(X)$; now by by clause (i) it is enough to prove the implication $\Leftarrow$.

Let $A^* =: f"(A)$ and $X^* =: (G'(f))(X)$, so we know that $A, A^* \in I$ and $X, X^* \in N' \backslash N$ and $A^*\mathscr{R}X^*$. We have to show that $A\mathscr{R}X$. If $\neg A\mathscr{R}X$, then we can find $g \in \mathscr{F}, g \restriction A = \mathrm{id}_A$, and $z_0 \in \mathrm{Dom}(G(g)), z_1 = G(g)(z_0)$, such that $z_0 \in X \equiv z_1 \notin X$. We can find $B_0 \in I$ such that $B_0 R z_0$ and $B_0 \subseteq \mathrm{Dom}(g)$ and let $B'_1 =: g"(B_0)$. We can find $f_1, f \restriction A \subseteq f_1, B_0 \cup B_1 \subseteq \mathrm{Dom}(f_1), f_1 \in \mathscr{F}$ and without loss of generality $f_1 = f$ and $\mathrm{Dom}(g) = A \cup B_0$. Let $g^* = f \circ g \circ f^{-1}, B^*_0 = f"(B_0)$ and $B^*_1 = f"(B_1)$. Clearly $B^*_0, B^*_1 \in I$ and $f^{-1}, g \circ f^{-1}, g^* = f \circ g \circ$

$f^{-1} \in \mathscr{F}$. Also $B_0^* \subseteq \text{Dom}(f^{-1}), (f^{-1})"(B_0^*) = B_0 \subseteq \text{Dom}(g), g"(B_0) = B_1$ and $f"(B_1) = B_1^*$ hence together $g^*(B_0^*) = B_1^*$. Let $z_0^* =: (G(f))(z_0), z_1^* = (G(f))(z_1)$, so $(G(f^{-1}))(z_0^*) = z_0, (G(g))(z_0) = z_1, (G(f))(z_1) = z_1^*$, and as B_0^* is $\mathfrak{Z}$-support of $z_0^*, B_0^* \subseteq \text{Dom}(g^*)$ necessarily $(G(g^*))(z_0^*) = z_1^*$.

We can also show that $(z_0 \in X) \equiv (z_0^* \in X^*)$ and $(z_1 \in X) \equiv (z_1^* \in X^*)$ by clause $(c)(\beta)$ which we already proved so remembering $(z_0 \in X) \equiv (z_1 \notin X)$ we get a contradiction to "$A \mathscr{R} X^*$" which we have assumed.

Clause (k):

Trivial. $\square_{2.16}$

Well we have Υ-successors of candidates (in Definition 1.8, implicitly in Definition 1.1) and we have successors of $\mathbf{m}_1$-liftings $\mathfrak{Z}$ of $\mathscr{Y} = (M, I, \mathscr{F})$ where $\mathfrak{Z}$ has in it a candidate $(N^{\mathfrak{Z}}, \bar{P}^{\mathfrak{Z}})$.

Of course, we like to connect then, specifically show that true $(\Upsilon, \mathbf{t})$-successor of $\mathfrak{Z}$ exists. This is not always true, as Definition 1.8 can lead us to elements of $N' \backslash N$ with no support in I. In Definition 2.13 we restrict ourselves to elements with support in I, and we can change the definition in 1.1, 1.8 to have it, but it seems to me not so convincing for a logic. Rather we show that the dichotomy assumptions (as in 2.1(3), 2.3) help.

When we use $\mathscr{L}^* = \mathscr{L}_{\text{f.o.}}$, then dealing with Υ-successor is easier, we have to look less carefully at cardinalities, still we need a dichotomy property (see Definition 2.6) in order to get a $\mathfrak{Z}$-support to every member.

2.17 Claim. *Assume that:*

- (a) *$\mathscr{Y} = (M, I, \mathscr{F})$ is a k-system, $k \geq 3$*
- (b) *Υ is an inductive scheme for $\mathscr{L}_{\text{f.o.}}(\tau_M^+)$*
- (c) *for every $\ell < m_0^\Upsilon$ any subformula of ψ_ℓ^Υ has $\leq k$ free variables; also for every $\ell < m_1^\Upsilon$ any subformula of φ_ℓ^Υ has $\leq k$ free variables*
- (d) $\mathbf{t} \in \mathbf{T}$
- (e) *$\mathscr{Y}$ is a $\mathbf{t}$-dichotomical k-system*
- (f) *for every $\ell < m_0^\Upsilon$ the formula ψ_ℓ^Υ has $\leq k/2$ free variables.*

Then

- (α) *if $\mathfrak{Z}$ is an $\mathbf{m}_1^\Upsilon$-lifting of Y then $\mathfrak{Z}$ has a true $(\Upsilon, \mathbf{t})$-successor (see Definition 2.13(9))*
- (β) *for every $t \leq \infty$ there is an $\mathbf{m}_1^\Upsilon$-lifting $\mathfrak{Z}^t$ of Y such that recalling Definition 1.1(3A), we have $(N^{\mathfrak{Z}^t}, \bar{P}^{\mathfrak{Z}^t}) = (N_t[M, \Upsilon, \mathbf{t}], \bar{P}_t[M, \Upsilon, \mathbf{t}])$*
- (γ) *if $t \leq t_\iota[M, \Upsilon, \mathbf{t}]$ and $\iota \in \{1, \ldots, 7\}$ or $t \leq t_\iota[M, \Upsilon, \mathbf{t}]$ & $\iota = 11, \ldots, 17$ then there is an $\mathbf{m}_1^\Upsilon$-lifting $\mathfrak{Z}^{\mathbf{t}}$ of $\mathscr{Y}$ such that $(N^{\mathfrak{Z}^t}, \bar{P}^{\mathfrak{Z}^t})$ is equal to*

$$(N_t[M, \Upsilon], \bar{P}_t[M, \Upsilon]) \quad \text{or} \quad (N_t[M, \Upsilon, \mathbf{t}], \bar{P}_t[M, \Upsilon, \mathbf{t}]),$$

respectively; similalry for $\iota = 21, \ldots, 27$.

Proof. Clearly clause (β) follows from clause (α), just prove by induction on t. Also clause (γ) follows from (β) and clause (α), so we deal with clause (α). Let $\mathfrak{Z} = (N, \bar{P}, G, R)$.

The main point is to prove

$\boxtimes$ assume $\ell < m_0^\Upsilon$ and for $\bar{a} \in {}^{\ell g(\bar{y})}N$ and $X = X_{\bar{a}} = \{b \in N : (N, \bar{P}) = \psi_\ell[b, \bar{a}]\}$. <u>Then</u> $A \mathscr{R} X$ for some $A \in I$ <u>or</u> $\{\{b \in N : (N, \bar{c}) \models \psi_\ell[b, \bar{a}']\} : \bar{a}' \in {}^{\ell g(\bar{y})}N\}| > \mathbf{t}(\|M\|)$.

Let $m = \ell g(\bar{y})$, so $2(m+1) \leq k$ and $\bar{a} = \langle a_\ell : \ell < m \rangle$, let $A_\ell \in I$ be a $\mathfrak{Z}$-support of a_ℓ and let $\bar{b}_\ell$ be a list of A_ℓ without repetitions, and define a function h^* with domain $[m]$ with $h^*(\ell) = \bar{b}_\ell$. We define a 2-place relation E on $h^*/E^0_{I,m}$, see Definition 2.1(3)(γ):

$\oplus_1$ if for $j = 1, 2$, $f_j \in \mathscr{F}$ and $\bigcup\limits_{\ell \in [m]} \text{Rang}(h^*(\ell)) \subseteq \text{Dom}(f_j)$ and $h_j = f_j * h^*$ then $h_1 E h_2 \Leftrightarrow X_{G(f_1)(\bar{a})} = X_{G(f_2)(\bar{a})}$ (where $G(f_j)(\langle a_\ell : \ell < m \rangle) = \langle G(f_j)(a_\ell) : \ell < m \rangle$.

Now

$\oplus_2$ E belongs to $\mathscr{E}_{I,m}(\emptyset)$,
see Def 2.1(3)(δ).

Why? by traslating this means that:

$(*)$ if $\bar{a}^1, \bar{a}^2, \bar{a}^3, \bar{a}^4 \in \{(G(f))(\bar{a}) : \bigcup\limits_{\ell < m} A_\ell \subseteq \text{Dom}(f)$ and $f \in \mathscr{F}\}$, and $(G(f))(\bar{a}^1 {}^\frown \bar{a}^2) = \bar{a}^3 {}^\frown \bar{a}^4$, then $N \models \psi(\bar{a}^1, \bar{a}^2) \equiv \psi(\bar{a}^3, \bar{a}^4)$

where $\psi(\bar{y}_1, \bar{y}^2) =: (\forall x)[\varphi(x, \bar{y}_1) \equiv \varphi(x, \bar{y}_2)]$.

Now every subformula of $\psi(\bar{y}_1, \bar{y}_2)$ has at most $2m + 1 < k$ free variables or is a subformula of $\varphi(x, \bar{y})$ hence has $\leq k$ free variables, hence by 2.9 we have $N \models \psi(\bar{a}^1, \bar{a}^2)$ iff $N \models \psi(\bar{a}^3, \bar{a}^4)$, so we are done showing $\oplus_2$.

Now $m \leq k/2$ and we are assuming that $\mathscr{Y}$ is a $\mathbf{t}$-dichotomical k-system, so $(\beta)_1$ or $(\beta)_2$ of Definition 2.1(4) holds. Now $(\beta)_2$ gives the desirable first possible conclusion in $\boxtimes$ and $(\beta)_1$ gives that $|\{b : N \models \psi_\ell(b, \bar{a}') : \bar{a}' \in {}^m N\}| > \mathbf{t}(\|M\|)$, hence second possible conclusion in $\boxtimes$. $\square_{2.17}$

2.18 Claim. *1) Assume that*

(a) *$\mathscr{Y}$ is a counting k-system with $k \geq 3$*

(b) *Υ is an inductive scheme, in $\mathscr{L}_{\text{f.o.}}$ or in $\mathscr{L}_{\text{f.o.} + \text{na}}$ or $\mathscr{L}_{\text{card}}$ or $\mathscr{L}_{\text{card},\mathbf{T}}$*

(c) *for $\ell < m_0^\Upsilon$, every subformula of $\psi_\ell(y, \bar{x})$ has at most $k-1$ free variables and for $\ell < m_1^\Upsilon$ every subformula of φ_ℓ has at most $k-1$ free variables*

(d) $\mathbf{t} \in \mathbf{T}$

(e) *$\mathscr{Y}$ is a $\mathbf{t}$-dichotomical k-system*

(*f*) *for* $\ell < m_0^\Upsilon$ *the formula* ψ_ℓ *has* $\le k/2$ *free variables.*
<u>*Then*</u>

(α) *if* $\mathfrak{z}$ *is an* $\mathbf{m}_1^\Upsilon$*-lifting,* <u>*then*</u> $\mathfrak{z}$ *has a true* $(\mathbf{t}, \Upsilon)$*-successor*

(β) *for every* t *there is an* $\mathbf{m}_1^\Upsilon$*-lifting* $\mathfrak{z}^t$ *such that* $(N^{\mathfrak{z}^t}, \bar P^{\mathfrak{z}^t}) = (N_t[M, \Upsilon, \mathbf{t}], \bar P_t[M, \Upsilon, \mathbf{t}])$.

Proof. The proof is as in 2.17 , but we know by 2.10 that the partial automorphism $G(f)$ preserves also $\psi_1(y, \bar x)$ and even $(\exists^k y)\psi_1(y, \bar x)$ when every subformula of ψ_1 has $< k$ free variables; (note that only now having $2m + 1 < k$ rather than $2m + 1 \le k$ seem helpful or repeat the proof of 2.10 as we can use the $< k$ just for subformulas of φ). $\square_{2.18}$

2.19 Remark. Why do we still need in 2.18 the "**t**-dichotomical"? Just to guarantee that the true $(\mathbf{t}, \Upsilon)$-successor is included in the full one.

2.20 Claim. *Assume*

(*a*) $\mathscr{Y}$ *is a* k*-system with* $k \ge 3$

(*b*) Υ *is an inductive scheme in* $\mathscr{L}_{\text{f.o.}}$

(*c*) *for* $\ell < m_0^\Upsilon$*, every subformula of* $\psi_\ell(y, \bar x)$ *has at most* $k - 1$ *free variables and for* $\ell < m_1^\Upsilon$ *every subformula of* φ_ℓ *has at most* $k - 1$ *free variables*

(*d*) $\mathbf{t} \in \mathbf{T}$

(*e*) $\mathscr{Y}$ *is medium* $\mathbf{t}$*-dichotomical*

(*f*) *for* $\ell < m_0^\Upsilon$ *the formula* ψ_ℓ *has* $\le k/2$ *free variables.*

<u>*Then*</u>*: the conclusion of 2.18 holds, so if* $t \le t_\iota(M, \Upsilon, \mathbf{t})$*,* <u>*then*</u> *for some* $\mathbf{m}_1^\Upsilon$*-lifting* $\mathfrak{z}^t$ *we have* $(N^{\mathfrak{z}^t}, \bar P^{\mathfrak{z}^t}) = (N_t[M, \Upsilon, \mathbf{t}], \bar P_t[M, \Upsilon, \mathbf{t}])$.

Proof. Like the proof of 2.17, 2.18.

2.21 Definition. 1) We say that $\mathscr{H}$ is a witness to the k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$ if

(*a*) for $\ell = 1, 2$ we have $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is a k-system

(*b*) $\mathscr{H}$ is a family of partial isomorphisms from M_1 into M_2

(*c*) for every $g \in \mathscr{H}$, we have $\text{Dom}(g) \in I_1$, $\text{Rang}(g) \in I_2$

(*d*) if $g \in \mathscr{H}$ and $f_1 \in \mathscr{F}_1$ then $g \circ f_1 \in \mathscr{H}$

(*e*) if $g \in \mathscr{H}$ and $f_2 \in \mathscr{F}_2$ then $f_2 \circ g \in \mathscr{H}$

(*f*) if $g \in \mathscr{H}$ and $A \in I_1[k-1]$ and $B \in I_1$, <u>then</u> for some $g_1 \in \mathscr{H}$ we have $g \restriction A \subseteq g_1$ and $B \subseteq \text{Dom}(g_1)$

(*g*) if $g \in \mathscr{H}$ and $A \in I_2[k-1]$ and $B \in I_2$, <u>then</u> for some $g_1 \in \mathscr{H}$ we have $g^{-1} \restriction A \subseteq g_1^{-1}$ and $B \subseteq \text{Rang}(g_1)$.

2) We say that $\mathscr{H}$ is a witness to the dichotomical (k,r)-equivalence of $(\mathscr{Y}_1, \mathbf{t}_1)$ and $(\mathscr{Y}_2, \mathbf{t}_2)$ if

- (i) $\mathscr{Y}_\ell$ is a $(\mathbf{t}_\ell, k, r)$-dichotomical k-system for $\ell = 1, 2$
- (ii) $\mathscr{H}$ is a witness to the k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$
- (iii) each $g \in \mathscr{H}$ preserved the possibility chosen in the definition of $(\mathbf{t}, k, r)$-dichotomical.

If we omit r, we mean $s = [k/2]$.
3) We say that $\mathscr{H}$ is a witness to the counting k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$ if

- (i) $\mathscr{Y}_\ell$ is a counting $(\mathbf{t}, k, r)$-system for $\ell = 1, 2$
- (ii) $\mathscr{H}$ is a witness to the k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$
- (iii) each $g \in \mathscr{H}$ preserve the cardinalities involved in the definition of "counting $(\mathbf{t}, k, r)$-system".

4) Similarly with "medium dichotomy".

<u>2.22 Main Conclusion</u>: Assume

- (a) $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is a $\mathbf{t}_\ell$-dichotomical k-system, and $\tau(M_\ell) = \tau$ for $\ell = 1, 2$
- (b) $\mathscr{H}$ is a witness to the k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$
- (c) $\chi \in \mathscr{L}_{\text{f.o.}}(\tau^+)$, i.e. a first order sentence in the vocabulary $\tau^+ = \tau \cup \{\in\}$,
- (d) Υ is an inductive scheme for $\mathscr{L}_{\text{f.o.}}$
- (e) every subformula of χ and of ψ_ℓ^Υ and of φ_ℓ^Υ has at most $< k$ free variables
- (f) $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{T}$
- (g) every formula ψ_ℓ^Υ has $\le k/2$ free variables and $k \ge 3$ of course.

<u>Then</u>

- (α) let[4] $\iota \in \{2,3,4,5\}$; the truth value of $\theta_{\Upsilon,\chi,\mathbf{t}_1}$ in M_1 and $\theta_{\Upsilon,\chi,\mathbf{t}_2}$ in M_2 under $\models_\iota$ are equal except possibly when: for some $\ell \in \{1,2\}$ we have the truth value of $\theta_{\Upsilon,\chi,\mathbf{t}_\ell}$ in M_ℓ is undefined whereas that of $\theta_{\Upsilon,\chi,\mathbf{t}_{3-\ell}}$ in $M_{3-\ell}$ is well defined and $t_\iota[M_\ell, \Upsilon, \mathbf{t}_\ell] < t_\iota[M_{3-\ell}, \Upsilon, \mathbf{t}_{3-\ell}]$.
- (β) For any t, if $N^\ell = N_t[M_\ell, \Upsilon, \mathbf{t}_\ell]$ is well defined for $\ell = 1, 2$, <u>then</u> for every sentence $\theta \in \mathscr{L}_{\text{f.o.}}(\tau^+)$ such that every subformula has at most k free variables, we have $N^1 \models \theta \Leftrightarrow N^2 \models \theta$
- (γ) if $\iota \in \{2,3,4,5\}$ and $\theta^\ell = \theta_{\Upsilon,\chi,\mathbf{t}^\ell} \in \mathscr{L}_\iota^{\mathbf{T}}(\mathscr{L}_{\text{f.o.}}(\tau))$ is ι-good for $\ell = 1, 2$: <u>then</u> $M_1 \models_\iota$ "θ^1" iff $M_2 \models_\iota$ "θ^2"
- (δ) for any t, if $N^\ell = N_t[M_\ell, \Upsilon, \mathbf{t}_\ell]$ is well defined for $\ell = 1, 2$ and $\theta = \theta(x_1, \dots, x_n) \in \mathscr{L}_{\text{f.o.}}(\tau^+)$ is a formula such that every subformula has at most k free variables <u>then</u>:
  - $\oplus$ let $a_1, \dots, a_m \in N^1$ and $g \in \mathscr{H}$, <u>then</u> $N^1 \models_\iota \theta[a_1, \dots, a_m]$ iff $N^2 \models_\iota \theta[(G(f))(a_1), \dots, (G(f))(a_m)]$.

[4]here and below, for $\iota \in \{6,7\}$ the conclusion is similar but expressed more cumbersomely

Proof. First we can prove clause (δ) by induction on the quantifier depth of θ, as in 2.9.

Second, note that clause (α) follows from clause (β).

Third, note that clause (β) follows from clause (δ)

and the definition of satisfaction 1.1.

Lastly, concerning clause (γ) follows the definition of $\mathscr{L}$-good (and clause (β)).

$\square_{2.22}$

2.23 Conclusion. Assume

(a) $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is a counting k-system, $\tau(M_\ell) = \tau$ for $\ell = 1, 2$

(b) $\mathscr{H}$ is a witness for the counting k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$

(c) $\chi \in L_{\mathrm{card},\mathbf{T}}$

(d) Υ is an inductive scheme for $\mathscr{L}^* = \mathscr{L}_{\mathrm{card},T}(\tau^+)$ or $\mathscr{L}_{\mathrm{card}}(\tau^+)$

(e) every subformula of χ and of ψ^Υ_ℓ (for $\ell < m^\Upsilon_0$) and of φ^Υ_ℓ (for $\ell < m^\Upsilon_1$) has at most $k-1$ free variables

(f) $\mathbf{t} \in \mathbf{T}$

(g) if $\ell < m^\Upsilon_0$ then ψ^Υ_ℓ has $\leq k/2$ free variables and $k \geq 3$ of course.

Then

(α) if $\theta = \theta_{\Upsilon,\chi,\mathbf{t}}$ and $\iota \in \{2,3,4,5,6,7,11,22\}$ then $M_1 \models_\iota \theta \Leftrightarrow M_2 \models_\iota \theta$ and $M_1 \models_\iota \neg\theta \Leftrightarrow M_2 \models_\iota \neg\theta$

(β) For any t if $N^\ell = N_t[M_\ell, \Upsilon, \mathbf{t}]$ is well defined for $\ell = 1, 2$, then for every sentence $\theta \in \mathscr{L}_{\mathrm{f.o.}}(\tau^+)$ such that every subformula has at most $k-1$ free variables, we have $N^1 \models_\iota \theta \Leftrightarrow N^2 \models_\iota \theta$

(γ) for any t, if $N^\ell = N_t[M_\ell, \Upsilon, \mathbf{t}]$ are well defined (for $\ell = 1, 2$), and $\theta = \theta(x_1, \ldots, x_n) \in \mathscr{L}_{\mathrm{f.o.}}(\tau^+)$ is a formula such that every subformula has at most $k-1$ free variables we have:

- $\oplus$ if $a_1, \ldots, a_m \in N^1$ and $g \in \mathscr{H}$, each $(G(f))(a_\ell)$ is well defined then $N^1 \models_\iota \theta[a_1, \ldots, a_m]$ iff $N^2 \models_\iota \theta[(G(f))(a_1), \ldots, (G(f))(a_m)]$.

Proof. Straight.

Now

2.24 Conclusion. Assume

(a) $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is a medium $\mathbf{t}$-dichotomical k-system, $\tau(M_\ell) = \tau$ for $\ell = 1, 2$

(b) $\mathscr{H}$ is a witness for the medium $\mathbf{t}$-dichotomical k-equivalence of $\mathscr{Y}_1$ and $\mathscr{Y}_2$

(c) $\chi \in \mathscr{L}_{\text{f.o.}}(\tau^+)$

(d) Υ is an inductive scheme for $\mathscr{L}^* = \mathscr{L}_{\text{f.o.}}(\tau^+)$

(e) every subformula of χ and of ψ^Υ_ℓ (for $\ell < m^\Upsilon_0$) and of φ^Υ_ℓ (for $\ell < m^\Upsilon_1$) has at most $k-1$ free variables

(f) $\mathbf{t}_\ell \in \mathbf{T}$

(g) if $\ell < m^\Upsilon_0$ then ψ^Υ_ℓ has $\le k/2$ free variables and $k \ge 3$ of course.

Then

(α) if $\theta = \theta_{\Upsilon,\chi,\mathbf{t}}$ and $\iota \in \{1,\dots,5,6,7,11,22\}$ then $M_1 \models_\iota \theta \Rightarrow M_2 \models_\iota \theta$ and $M_1 \models_\iota \neg\theta \Leftrightarrow M_2 \models_\iota \neg\theta$

(β) For any t if $N^\ell = N_t[M_\ell,\Upsilon,\mathbf{t}_\ell]$ is well defined for $\ell = 1,2$, then for every sentence $\theta \in \mathscr{L}_{\text{f.o.}}(\tau^+)$ such that every subformula has at most $(k-1)$-free variables, we have $N^1 \models \theta \Leftrightarrow N^2 \models \theta$.

Proof. Straightforward.

2.25 Discussion We consider now some variants.
1) We have to consider the stopping times. If $\mathscr{L}^* = \mathscr{L}_{\text{car},\mathbf{T}}$ or $\mathscr{L}_{\text{card},\mathbf{T}}$ this is natural, (and they are stronger logics than the earlier variants). If we still would like to analyze in particular for the others, we should be careful how much information can be gotten by the time.
2) We can modify Υ such that in N_{t+1} we can reconstruct the sequence $\langle (N_\ell,\bar P_\ell) : \ell \le s\rangle$ (see §4).
3) We can change our presentation: first proving the equivalences for $N^{3[\mathscr{Y}_\ell,\mathbf{t}_\ell]}$ for $\ell = 1,2$, (see Definition 5.5) and then proving that $(N_t[M^{Y_\ell},\Upsilon,\mathbf{t}],\bar P_t[M,\Upsilon,\mathbf{t}])$ is interpretable in $N^{3^{\text{full}}_{\mathscr{Y}_\ell,\mathbf{t}}}$ uniformly.

§3 The canonical example

We apply §2 to the canonical example: random enough graph.

3.1 Definition. Let τ be a fixed (finite) vocabulary consisting of predicates only. We say M is a $(\mathbf{s},k)$-random τ-model if every quantifier free 1-type over $A \subseteq M, |A| < k$ (not explicitly inconsistent) is realized in M by at least $\mathbf{s}(\|M\|)$ elements. If $\mathbf{s}$ is constantly ∞ we may write k-random.

Remark. We can restrict the set of allowable quantifier free types if it is nice enough e.g. R two-place symmetric irreflexive. More generally see e.g. [BlSh 528].

3.2 Definition. $\mathbf{T}_{\rm pol}$ is $\mathbf{T}_{\mathbb{Q}}$, where for a set $Q \subseteq \mathbb{R}$ containing an unbounded set of reals > 0 let $\mathbf{T}_Q$ be $\{f_q : q \in Q, q > 0\}$ where $f_q : \omega \to \omega$ is $f_q(n) = n^q$, or more exactly, $\lceil n^q \rceil$ the least integer $\geq n^q$.

3.3 Claim. *Assume*

(a) q^*, k *are integers* > 1 *and* $k^* = q^* k$

(b) $s \leq k, s > 0$ *integer*

(c) M_ℓ *is* $(\mathbf{s}_\ell, 3k^*)$*-random* τ*-model for* $\ell = 1, 2$

(d) $\mathbf{t}_\ell(\|M_\ell\|) < (\mathbf{s}_\ell(\|M_\ell\|))^{q^*+1}/(q^*+1)!$ *and* $\mathbf{s}_\ell(\|M_\ell\|) > q^*$

(e) Υ *is an inductive scheme for* $\mathscr{L}_{\rm f.o.}(\tau_{[2]})$*,* χ *a sentence in* $\mathscr{L}_{\rm f.o.}(\tau_{[2]})$ *and each subformula of any formula among* ψ^Υ_ℓ $(\ell < m^\Upsilon_0), \varphi^\Upsilon_m(m < m^\Upsilon_1)$ *and* χ *has at most* s *free variables*

<u>Then</u>

$\oplus$ *if* $\iota \in \{2, 3, 4, 5, 6\}$ *and* $\theta_{\Upsilon,\chi,\mathbf{t}_\ell}$ *is* ι*-good (at least for* M_ℓ *that is, the truth values below are defined) for* $\ell = 1, 2$*,* <u>then</u> $M_1 \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}_1} \Leftrightarrow M_2 \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}_2}$.

3.4 Remark. 1) Compare clause (β) with 2.21(2).
2) Why in 3.3 do we use clause (d)? As there we use $N_t[M_\ell, \Upsilon, \mathbf{t}_\ell]$ so for some t we may add the sets in $\mathscr{P}_t[M_1, \Upsilon, \mathbf{t}_1]$ to $N_t[M_1, \Upsilon, \mathbf{t}_1]$ (in defining $N_{t+1}[M_1, \Upsilon, \mathbf{t}_1]$ but do not add $\mathscr{P}_\ell[M_2, \Upsilon, \mathbf{t}_2]$ to $N_t[M_2, \Upsilon, \mathbf{t}_2]$ in defining $N_{t+1}[M_2, \Upsilon, \mathbf{t}_2]$.
3) We concentrate on ι-good sentences (or local versions) in order to have neat results. Otherwise we have really to be more careful, e.g. about the cardinalities of the $N_t[M, \Upsilon, \mathbf{t}]$'s. This is very reasonable for counting logic.
4) We have ignored in this claim, and others in this section, the cases $10 < \iota$. We can deal with them, if we note the following required changes. We have to note that the function $\mathbf{t}$ is split to two functions; one, $\mathbf{t}^{sp}$, for telling us how to increase N_t to N_{t+1}, that is which additional families of subsets of N_t are allowed to be added, and for this function the parallel of clause (d) should be demanded. Secondly we have to consider what families are added in each stage (so for the counting and the medium analog our situation may be better).

Proof. We let $I_\ell = \{A \subseteq M_\ell : |A| \leq q^*\}$ and

$$\mathscr{F}_\ell = \{f : f \text{ is a partial automorphism of } M_\ell \text{ and } \mathrm{Dom}(f) \text{ has } \leq q^* k \text{ elements}\}$$

$(*)_1$ $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is a k-system
[why? the least obvious clause in Definition 2.1(1) is clause (D) which holds by Definition 3.1 above.]

$(*)_2$ $\mathscr{Y}_\ell = (M_\ell, I_\ell, \mathscr{F}_\ell)$ is $(\mathbf{t}_\ell, s)$-dichotomical, (see Def 2.1(4)).

Why? The proof of $(*)_2$ takes most of the proof. Let $m \in \mathbb{N}$ be $1 \le m \le s$ and let E be an equivalence relation from $\mathscr{E}^0_{I,m}(\emptyset)$ so it is an equivalence relation on $h^*/E^0_{\mathscr{Y}_\ell,m}$ where $h^* : [m] \to \mathrm{Seq}_{I_\ell}, E$ satisfying $(*)$ of clause (δ) of Definition 2.1(3). For $h \in h^*/E^0_{\mathscr{Y}_\ell,m}$ let $\bar{b}_h$ be the concatenation of $h(1), h(2), \ldots, h(m)$. Without loss of generality h^* is one-to-one and even $\bar{b}_{h^*}$ is with no repetitions. Let $t^* = \ell g(\bar{b}_{h^*})$ so $t^* \le q^* k \le k^*$. By clause (c) and the definition of $\mathscr{F}$ there is a quantifier free formula $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}(\tau)$ with $\ell g(\bar{x}) = \ell g(\bar{y}) = t^*$ such that $h_1 E h_2$ iff $M_\ell \models \varphi[\bar{b}_{h_1}, \bar{b}_{h_2}]$.

Clearly without loss of generality $\varphi(\bar{x}, \bar{y})$ tells us that the quantifier free type of $\bar{x}$ and of $\bar{y}$ (same as that of $\bar{b}_{h^*} = h^*(1)\hat{\ }h^*(2)\hat{\ }\ldots\hat{\ }h^*(m)$), call it $p(\bar{x})$ and let $p[M] = \{\bar{a} \in {}^{t^*}M : \bar{a} \text{ realizes } p(\bar{x})\}$, so we can look at E restricted to this set.
Can there be $\bar{a}_0, \bar{a}_1 \in {}^{t^*}(M_\ell)$ realizing the same quantifier free type $p(\bar{x})$ (over the empty set) which are not E-equivalent? If not, then $|p[M_\ell]/E| = 1$ and we are done so assume there are. We can find $\bar{a}_2 \in {}^{t^*}(M_\ell)$ realizing the same quantifier free type $p(\bar{x})$ and disjoint to $\bar{a}_0\hat{\ }\bar{a}_1$ (use "M_ℓ is $(\mathbf{s}_\ell, 3k^*)$-random"), so $\bar{a}_2, \bar{a}_j$ are not E-equivalent for some $j \in \{0,1\}$; so without loss of generality $\bar{a}_0, \bar{a}_1$ are disjoint. Now we ask "are there disjoint $\bar{b}_0, \bar{b}_1 \in {}^{t^*}(M_\ell)$ realizing $p(\bar{x})$ which are E-equivalent"? If yes, we easily get a contradiction to "E an equivalence relation" (by finding $\bar{b}'$, a sequence from M_ℓ realizing $p(\bar{x})$ such that both $\bar{b}'\hat{\ }\bar{a}_0, \bar{b}'\hat{\ }\bar{a}_1$ realize the same quantifier free type as $\bar{b}_0\hat{\ }\bar{b}_1$; contradiction). So: no disjoint $\bar{b}_0, \bar{b}_1 \in p[M_\ell]$ are E-equivalent. Next we claim that

$\bigotimes$ for some $u \subseteq [0, t^*)$ and subgroup $\mathbf{g}$ of the group of permutations of u, moreover of $\mathbf{g}^* = \mathbf{g}^*_u = \{\sigma \in \mathrm{Per}(u)$: if $\bar{a} \in {}^{t^*}(M_\ell)$ realizes $p(\bar{x})$ then $\langle a_{\sigma(i)} : i \in u\rangle$ realizes the same quantifier type as $\bar{a} \restriction u\}$, we have:
$E \restriction \{\bar{a} \in {}^{t^*}(M_\ell) : \bar{a} \text{ realizes } p(\bar{x})\} = E_{\mathbf{g}} \restriction \{\bar{a} \in {}^{t^*}(M_\ell) : \bar{a} \text{ realizes } p(\bar{x})\}$
where for any subgroup $\mathbf{g}'$ of $\mathbf{g}^*, E_{\mathbf{g}'} = E^{t^*, M_\ell}_{\mathbf{g}'}$ is defined by: for $\bar{a}, \bar{b} \in {}^{t^*}(M_\ell), \bar{a} E_{\mathbf{g}'} \bar{b} \Leftrightarrow (\exists \sigma \in \mathbf{g})(\langle a_{\sigma(t)} : t \in u\rangle = \langle b_r : t \in u\rangle)$ (this is an equivalence relation).

[Why? It is enough to show: assume $\bar{a}, \bar{b}, \bar{c} \in {}^{t^*}(M_\ell)$ realize $p(\bar{x})$ and $\bar{a}, \bar{b}$ are E-equivalent, $r^* < t^*$ and $a_{r^*} \notin \{b_t : t < t^*\}$ then $\bar{c}/E$ does not depend on c_{r^*} (i.e. if $\bar{c}' \in {}^{t^*}(M_\ell)$ realizes $p(\bar{x})$ and $r' < t^*$ & $r' \ne r^* \Rightarrow c_{r'} = c'_{r'}$ then $\bar{c} E \bar{c}'$.)
Toward this end, let σ be the partial function from $[0, t^*)$ to $[0, t^*)$ such that $\sigma(r_1) = r_2 \Leftrightarrow a_{r_1} = b_{r_2}$. Clearly σ is one to one and $r^* \notin \mathrm{Dom}(\sigma)$.
We choose by induction on $j^* \le t!$ a sequence $\bar{a}^j \in p[M_\ell]$ such that $\bar{a}^0 = \bar{a}, \bar{a}^1 = \bar{b}$, the quantifier free type of $\bar{a}^j\hat{\ }\bar{a}^{j+1}$ in M_ℓ is the same as the quantifier free type of $\bar{a}^0\hat{\ }\bar{a}^1 = \bar{a}\hat{\ }\bar{b}$ in M_ℓ and $(\forall r)[a^{j+1}_r \notin \{a^j_r : r < t^*\} \Rightarrow a^j_r \notin \{a_r : r < t^*\}]$. Clearly $j < t^*! \Rightarrow \bar{a}^j E \bar{a}^{j+1}$, hence $j \le t^*! \Rightarrow \bar{a} E \bar{a}^j$. Let σ^j be the partial function from $[0, t^*)$ to $[0, t^*)$ defined by $\sigma^j(r_1) = r_2 \Leftrightarrow a_{r_1} = a^j_{r_2}$. Clearly $\sigma^0 = \mathrm{id}_{[0,t^*)}$ and $\sigma^{j+1} = \sigma \circ \sigma^j$. Clearly $\sigma^{t^*!}$ is the identity function on some subset of $[0, t^*)$ and $a^{t^*!}_{r_2} = a^0_{r_1} (= a_{r_1}) \Leftrightarrow r_1 = r_2$ & $\sigma^{t^*!}(r_1) = r_1$. Now given $\bar{c}'$ as above we can find $\bar{b} \in p[M_1]$ such that $\bar{c}\hat{\ }\bar{b}$ and $\bar{c}'\hat{\ }\bar{b}$ realizes the same quantifier free type as $\bar{a}^0\hat{\ }\bar{a}^{t!}$, hence $\bar{c} E \bar{b}$ & $\bar{c}' E \bar{b}$ hence $\bar{c} E \bar{c}'$. Easily we are done proving

$\boxtimes$ $p[M_\ell]/E$ has cardinality $\ge (\mathbf{s}_\ell(\|M_\ell\|))^{|u|} \cdot (|\mathbf{g}^*|/|\mathbf{g}|)$.

[Why? Clear by ⊠ and Definition 3.1.]
This number is $\geq (\mathbf{s}_\ell(\|M_\ell\|)^{|u|}/|u|!$. Hence if $|u| > q^*$ as $\mathbf{s}_\ell(\|M\|) > k^* = q^*k \geq |u| > q^*$ (see assumption (d)) the number is $\geq (\mathbf{s}_\ell(\|M_\ell\|))^{q^*+1}/(q^*+1)!$ which by assumption (d) is $> \mathbf{t}_\ell(\|M_\ell\|)$ and we get one of the allowable answers in Definition 2.1(4). So we can assume that $|u| \leq q^*$ and this gives the second possibility so we have finished proving $(*)_2$.
Let

$$\mathscr{H} = \big\{f : f \text{ is a partial embedding of } M_1 \text{ into } M_2 \text{ with } \mathrm{Dom}(f) \text{ having } \leq q^*k \text{ members}\big\}.$$

$(*)_3$ $\mathscr{H}$ is a (k,s)-witness to the equivalence of $(\mathscr{Y}_1, \mathbf{t}_1)$ and $(\mathscr{Y}_2, \mathbf{t}_2)$
[why? straight.]

So we can apply 2.22 and get the desired result. $\square_{3.3}$

Lastly, we can conclude the answer for the question in [BGSh 533]:

3.5 Theorem. *1) Assume $\iota \in \{2,3,4,5\}, \tau = \{R\}, R$ binary symmetric irreflexive, $\mathbf{p} \in (0,1)$ and $\mathbf{T}$ are given and each $\mathbf{t} \in \mathbf{T}$ is bounded by a polynomial. The logic $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.}})(\tau)$ satisfies the undecided 0-1 law for finite random enough model, that is graph with a fix probability $\mathbf{p} \in (0,1)$ which means; if $\theta_1 = \theta_{\Upsilon,\chi,\mathbf{t}} \in \mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.}})(\tau)$ and $\theta_0 = \theta_{\Upsilon,\neg\chi,\mathbf{t}}$ then $\langle Min\{Prob(\mathscr{M}_n \models \theta_0), Prob(\mathscr{M}_n \models \theta_1)\} : n < \omega\rangle$ converge to zero[5], where $\mathscr{M}_n$ is $G(n,\mathbf{p})$, the random graph on n with edge probability $\mathbf{p}$.*
2) Moreover also the undecided$^+$ $0-1$-law hold; which means:
if $\theta_1 = \theta_{\Upsilon,\chi,\mathbf{t}} \in \mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.}})(\tau)$ and $\theta_0 = \theta_{\Upsilon,\neg\chi,\mathbf{t}}$ then for some $\ell \in \{0,1\}$ the sequence $\langle\{Prob(\mathscr{M}_n \models \theta_\ell) : n < \omega\rangle$ converges to zero,
3) Similarly for any fixed (finite) vocabulary τ consisting of predicates only $\bar{\mathbf{p}} = \langle \mathbf{p}_R : R \in \tau\rangle, \mathbf{p}_R \in (0,1)_{\mathbb{R}}$.

Proof. 1) Let $\theta_0, \theta_1, \Upsilon, \chi, \mathbf{t}$ be as above and $\varepsilon > 0$. Let s be large enough such that assumption (e) of 3.3 holds, choose $k = 3s$ so assumption (b) there holds. We choose $\mathbf{s}_\ell$ as $\mathbf{s}(n) = (n-k) \times (\mathrm{Min}\{\mathbf{p}^k/2, (1-\mathbf{p})^k/2\})$ and let q^* be integer > 0 such that for n large enough $\mathbf{s}(n)^{q^*+1} \geq \mathbf{t}(n)(q^*+1)!$ and let $k^* = q^*k$.
Let n be large enough and $\mathscr{M}^1_n, \mathscr{M}^2_n$ be random enough (for $G(n,\mathbf{p})$). We would like to apply Claim 3.3 with $M_1 = \mathscr{M}^1_n, M_2 = \mathscr{M}^2_n$ and Υ, χ as in the definition of θ_0, θ_1 and $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}$ and $\mathbf{s}_1 = \mathbf{s}_2$ large enough. This is straight, noting that the case of the truth value of θ_1 in $\mathscr{M}_n$ is undefined, i.e. we run out of resources, just help us.
2) Similarly, this time for n is large enough, $n_1 \geq n, n_2 \geq n$ and $\mathscr{M}^1_{n_1}, \mathscr{M}^2_{n_2}$ are random enough (for $G(n_1,\mathbf{p}), G(n_2,\mathbf{p})$ respectively).
3) Similarly $\square_{3.5}$

[5] we do not ask that for some $\ell < 2$ the probability for the satisfaction of θ_ℓ converges to 1, as the decision when to stop may be complicated. If we e.g. use an inside "clock" to tell us when to stop, this disappears

3.6 Discussion: 1) It is reasonable to consider the undecided law if we know that the $(N_t[M_\ell, \Upsilon, \mathbf{t}_\ell], \bar{P}_t[M_\ell, \Upsilon, \mathbf{t}_\ell])$ for $\ell = 1, 2$ are quite equivalent for every t, when $\|M_1\| = \|M_2\|$, but we do not have information otherwise.
2) We might prefer to have the usual zero-one law. There are some avenues to get (at some price), see also 3.7,3.8 below; we may consider all sentences, $\iota \in \{2, \dots, 5\}$ and the usual 0-1 law.

We have to try to use $\mathbf{t}$ which tries to diagonalize the right sets. That is, using 3.3 for $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}$, we can get strong enough equivalence of $N_t^+[M_1, \Upsilon], N_t^+[M_2, \Upsilon]$, which is fine if $\|M_1\| = \|M_2\| < \|M_2\|$, so it is enough if $N_{t_1}^+[M_\ell, \Upsilon], N_{t_2}^+[M_\ell, \Upsilon]$ with $t_1 = t_\iota[M_1, \Upsilon, \mathbf{t}_1], t_2 = t_\iota[M_2, \Upsilon, t_1]$ and choose $\mathbf{t}$ such that they are quite equivalent. As in $N_t^+[M_\ell, \Upsilon]$ we can define $\mathbb{N} \restriction \{0, \dots, t-1\}$, this requires $\mathbf{t}(\|M_\ell\|)$ to be quite large compare to $\|M_\ell\|$. So we can get our desired 0-1 laws and all possible ι's, but for a logic remote from our intention.

On the other hand, we may restrict our family of sentences (here)

3.7 Theorem. *If in Theorem 3.5 we restrict ourselves to the good sentences, i.e. the logic is $\mathscr{L}_\iota^{\mathbf{T}}(\mathscr{L}_{\text{f.o.}})^{\text{good}}$ and $\iota \in \{2, 3, 4, 5\}$, then the usual 0-1 law holds.*

Proof. Similar. $\square_{3.7}$

3.8 Theorem. *1) Assume $\iota \in \{2, 3, 4, 5\}, \tau = \{R\}, R$ binary symmetric irreflexive predicate, $\mathbf{p} \in (0, 1)_{\mathbb{R}}$ and $\mathbf{T}$ are given and for each $\mathbf{t} \in \mathbf{T}$ for some integer r and $\varepsilon \in (0, 1/2)_{\mathbb{R}}$ we have $0 = lim\langle \mathbf{t}(n)/n^{r+1-\varepsilon} : n \in \mathbb{N}\rangle$ and $\infty = lim\langle \mathbf{t}(n)/n^{r+\varepsilon} : n \in \mathbb{N}\rangle$. Then the logic $\mathscr{L}_\iota^{\mathbf{T}}(\mathscr{L}_{\text{f.o.}})(\tau)$ satisfies the results in 3.3 - 3.7.*
2) Similarly for any fixed (finite vocabulary τ consisting of predicates only, $\bar{p} = \langle \mathbf{p}_R : R \in \tau\rangle, \mathbf{p}_R \in (0, 1)_{\mathbb{R}}$.

Proof. 1) Suppose that $M^* \models \theta_{\Upsilon, \chi, \mathbf{t}}$ and this because in stage t^* the run stop, i.e., in a good way; and assume further that M is random enough graph (for our given Υ and χ). We can find E_ℓ for $\ell < \ell^*$ such that E_ℓ is a quantifier free formula with $2m_\ell$ variable defining an equivalence relation on $p_\ell(M)$ for every random enough graph M, $p_\ell(\bar{x})$ a complete quantifier free type with m_ℓ variables said to be pairwise distinct. We can find non negative integers r_ℓ^t for $\ell < \ell^*, t \leq t^*$ such that: if M is random enough graph and $N_t = N_t[M, \Upsilon]$ then $\|N_t\| = \Sigma_{\ell < \ell^*} r_\ell^t |({}^{m_\ell}M)/E_\ell|$. Now the expected value of $|({}^{m_\ell}M)/E_\ell|$ is of the form $\mathbf{p}' \times \text{binomial}(n, m_\ell)$ for some constant $\mathbf{p}'$. The distribution is similar enough to normal (see [Sh 550]) to ensure that the run on $\mathscr{M}_n$ will not stop for $t \leq t^*$ for over using resources.
2) Similarly $\square_{3.8}$

What we have done for random graphs we can do to unary predicate. The point is to replace claim 3.3 by a parallel one (the rest will follow).

3.9 Claim. *1) Assume that the vocabulary τ is $\{P\}$. P a unary predicate and*

(a) *q^+, q^-, k are integers > 1 and h is a decreasing (not necessarily strictly) function from $\{0, 1, \ldots, q^+\}$ to $\{0, 1, \ldots, q^-\}$,*
(b) *s is an integer $\leq k$ but > 0*
(c) *$M_\ell = (|M_\ell|, P^{M_\ell})$ for $\ell = 1, 2$*
(d) *if $q \leq q^+$ then*
   (i) *$\mathbf{t}_\ell(\|M_\ell\|)$ is at least $|P^{M_\ell}|^q \times |(M_\ell \setminus P^{M_\ell})|^{h(q)}$*
   (ii) *$\mathbf{t}_\ell(\|M_\ell\|)$ is strictly smaller than*
$$\text{binomial}(|P^{M_\ell}|, q+1) \times \text{binomial}(|M_\ell \setminus P^{M_\ell}|, h(q)),$$
   (iii) *$\mathbf{t}_\ell(\|M_\ell\|)$ is strictly smaller than*
$$\text{binomial}(|P^{M_\ell}|, q) \times \text{binomial}(|M_\ell \setminus P^{M_\ell}|, h(q)+1).$$

2) The parallel of 3.5- 3.8 holds

Proof. Similar but letting $I_\ell = \{A \subseteq M_\ell$: for some $q \leq q^+$ we have $|A \cap P^{M_\ell}| \leq q$ and $|A \setminus P^{M_\ell}| \leq h(q)\}$. $\square_{3.9}$

3.10 Definition. 1) We say M is a τ-model with k-elimination of quantifiers <u>if</u> for every subsets A_0, A_1 of M, $|A_0| = |A_1| < k$ and an isomorphism f from $M \restriction A_0$ onto $M \restriction A_1$ and $a_0 \in M$ there is $a_1 \in M$ such that $f = f \cup \{\langle a_0, a_1\rangle\}$ is an isomorphism from $M \restriction (A_0 \cup \{a_0\})$ onto $M \restriction (A_1 \cup \{a_1\})$.
2) We replace "quantifiers" by "quantifier and counting" if we add: and the two sets $\{a'_0 \in M : a'_0, a_0$ realize the same quantifier free type over $A_0\}$ and $\{a'_1 \in M : a'_1, a_1$ realize the same quantifier free type over $A_1\}$ has the same number of elements (we can then get it to equivalence relations on m-tuples).

3.11 Claim. *1) Assume (a), (b), (e) in 3.3 replacing first order by counting logic and*

$(c)^-$ (α) *M_ℓ are τ-models which has k-elimination of quantifier for $\ell = 1, 2$*
(β) *if $\varphi(\bar{x}, \bar{y})$ is a quantifier free formula defining an equivalence relation and $\ell g(\bar{x}) = \ell g(\bar{y}) \leq k^*$ <u>then</u> the number of classes is $> \mathbf{t}_\ell(M_\ell)$ for $\ell = 1, 2$ <u>or</u> for some $u \subseteq [0, \ell g(\bar{x}))$ with $\leq q^*$ elements, some $\varphi'(\bar{x} \restriction u, \bar{y} \restriction u)$ defines the equivalence relation in M_1 and in M_2*
(γ) *if $2r + s \leq k$ and for $\ell = 1, 2$, $\bar{a}_\ell \in {}^s(M_\ell), \bar{x}^j = \langle x^j_i : i < s\rangle$, for $j = 1, 2$,*
$\varphi_\ell = \varphi_\ell(\bar{x}^1, \bar{x}^2, , \bar{a}_\ell)$ is first order and defines in M_0 an equivalence relation E_ℓ and $\varphi_1 = \varphi_2$ and the quantifier free types of $\bar{a}_1$ in M_1 and $\bar{a}_2$ in M_2 are equal,
<u>then</u> $|^r(M_1)/E_\ell|, |^r(M_2)/E_2|$ are equal or
$|^r(M_1)/E_1| > \mathbf{t}_1(\|M_1\|)$ & $|^r(M_2)/E_2| > \mathbf{t}_2(\|M_2\|)$.

<u>*Then*</u>

$\oplus$ *if* $\iota \in \{1-7, 11-17, 21-27\}$ *then* $M_1 \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}_1} \Leftrightarrow M_2 \models_\iota \theta_{\Upsilon,\chi,\mathbf{t}_2}$.

2) We have a theorem like 3.3, 3.8 (for ι *as above) using 3.11(1) instead of 3.3.*
3) We can in parts (1), (2) and in 3.3, 3.5, 3.7, 3.8 replace $\mathscr{L}_{\text{f.o.}}$ *by* $\mathscr{L}_{\text{f.o. + na}}$.

Proof. Straight.

3.12 Claim. *1) Choiceless polynomial time does not capture counting logic.*
2) Similarly for the pair $(\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.}}), \mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.+na}}))$*, the pair* $(\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.+na}}), \mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{card}}))$ *and the pair* $(\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{f.o.+na}}), \mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_{\text{card},\mathbf{T}}))$.
3) We can apply 3.11 to show that the pairs (M^1_n, M^2_n) *of models from [GuSh 526] are not distinguished by our logics (for a sentence* θ *for* n *large enough.*

Proof. 1) Use 2.22, 3.9 on the question: $|P^M| \geq \|M\|/2$ with $\tau = \{P\}$.
2),3) Also easy $\square_{3.12}$

3.13 Remark. : Y. Gurevich asks for 0-1 laws, as in 3.3 - 3.8, for the general framework of §2. The answer is quite straight by 3.14, 3.15, when we use constant I.

3.14 Definition. Let τ be a fixed vocabulary consisting of predicates only. We say M is $(\mathbf{s}, k)$-random model <u>if</u>: every quantifier free 1-type over $A \subseteq M, |A| < k$ (not explicitly inconsistent) is realized in M by at least $\mathbf{s}(\|M\|)$ elements.

We are, of course, using

3.15 Claim. *Let* $k, s > 0$ *be integers, let* $\tau = \{R\}, R$ *symmetric irreflexive and* $\mathbf{p} \in (0, \frac{1}{2}]_{\mathbb{R}}$*. The probability that: for* $\mathscr{M}_n$ *a* $G(n;\mathbf{p})$ *random graph (so set of vertices in* $[n]$*),* $(\mathscr{M}_n, I)$ *is not* (s,k)*-random is* $\leq \Sigma_{\ell<k}\text{binomial}(n,\ell)\times$ Prob*(flipping* $n-\ell$ *times a coin with probability* $\mathbf{p}/2^\ell$ *for a head we get* $< s$ *heads).*

§4 Relating the definitions in [BGSh 533] to the one here

<u>4.1 Discussion</u> If we just like to replace the creation of $N_t[M, \Upsilon, \mathbf{t}]$ by ASM, we can note that we can straightforwardly code the actions of the ASM by a monotone Υ; the waste is small <u>except</u> that we are not allowed to omit old elements so for fine measurements this make a difference. But we can just replace $N_t[M, \Upsilon, \mathbf{t}]$ by a situation of the ASM with no lose and no real difference in the proofs. Still, the reader may instead of just accepting or understanding this observation choose to read the formal translation below. Though this seem trivial, writing the details of a translation is tedious.

<u>4.2 Discussion</u>: How do we relate between the definitions above and [BGSh 533]?

(i) an infinite structure I there corresponds to a τ-model here

(ii) a state A there corresponds to a model of the form $N = N_t[M, \Upsilon]$ in 1.3 and $(N, \bar{P})$ in 1.8

(iii) dynamic function there corresponds to $P_{t,\ell}$ here

(iv) an object x is active at A in 5.1 there, corresponds to $x \in N$

(v) a program 5.7 there corresponds to an Υ in 1.1(2) here (mainly the first order formulas used);

(vi) the counting function in 5.5 there corresponds to the cardinality quantifier (1.1(6)) here

(vii) the polynomial functions p, q in 5.1 there corresponds to $\mathbf{t}^{\rm sp}, \mathbf{t}^{\rm tm} \in \mathbf{T}$ here

($viii$) the logic there corresponds to $\mathscr{L}^{\mathbf{T}}_\iota(\mathscr{L}_*), \mathscr{L}_* \in \{\mathscr{L}_{\rm f.o.}, \mathscr{L}_{\rm r.o.+\ na}, \mathscr{L}_{\rm card}\}$ here.

If we insist on P_ℓ being individual constants this still can be done with a price. The $P_{t+1,\ell}$ can in the usual set theory manner be actually 7-place function from N_t to N_t or 7-place relation on N_t, or be the universe of N_t. Understanding this to interpret the successor step there to here we need that all parts of the program are expressible in $\mathscr{L}_{\rm f.o.}$ (or $\mathscr{L}_{\rm card}$). For the other direction we need to show f.o. operations can be expressed by the programs of ASM there (see 6.1 there), no problem (and not needed to show our results solved problems there).

4.3 Lemma. *1) Let π be a program concerned with τ-models (in [BGSh 533, 4.7]'s sense). We can find a natural number $r^* \geq 1$ and Υ such that:*

$\boxtimes_1(a)$ *Υ is an inductive scheme in $\mathscr{L}_{\rm f.o.}$ with $\tau^\Upsilon = \tau^\pi$*

(b) *for every integer polynomials $p(n), q(n)$ and τ-model M such that $p(n) \geq 2, q(n) > n+2$ we have $M \models \theta_{\Upsilon, p^*(n), q^*(n)}$ (in the sense of Definition 1.3) iff $M \models \bar{\pi}$ (in the sense of [BGSh 533]) when*

($*$) $p^*(n) = 3 + r^* p(n), q^*(n) = 2 + 2q(n)$.

2) If $\bar{\pi}$ is as in [BGSh 533, §11], that is with being able to compute the cardinality of M (= set of atoms) then a similar result holds but Υ is in $\mathscr{L}_{\rm f.o.\ +\ na}$.
3) If $\bar{\pi}$ is as in [BGSh 533] for cardinality logic (see §4 there), then a similar result holds but Υ is in $\mathscr{L}_{\rm card}$ and spaces do not use $\theta_{\Upsilon, 3+n+r^*(p), 2+n+2q(n)}$.
4) We can replace p, q by arbitrary function from $\mathscr{T}$ and get similar results.

Before proving 4.3:

4.4 Observation. 1) If we identify truth, false with the sets $\emptyset, |M|$ and $\mathbf{m}_1$ is as in 1.1, then the state (for π, i.e. Υ there, etc.) of [BGSh 533] are the same $(M, \mathbf{m}_1)^+$-candidate here when we identify a state there with its set of active elements (O.K., as they carry the same information). Sometimes we use any transition set $\subseteq V_\infty(M)$ containing the active member.

The main point is

4.5 Claim. *Let π be a program concerned with τ-models and the states correspond to $(M, \mathbf{m}_1)$-candidates.*
1) For every term σ for some natural number $r(\sigma)$ (actually at most its depth) and pure inductive scheme Υ which is monotonic and $\psi_0 = [y = x_0]$ we have:

$\boxtimes_1$ *if $(N_i, \bar{P}_i)$ is a $(M, \mathbf{m}_1)$-candidate for $i = 0, \dots, r(\sigma)$ and $(N_{i+1}, \bar{P}_{i+1})$ is the (M, Υ)-successor of $(N_i, \bar{P}_i)$ and $\bar{\zeta}$ is a variable assignment of $\bar{x}$, a sequence listing the free variables of σ into N_σ, then the interpretation from σ under $\bar{\zeta}$ in N_i satisfies $d =: val_{N_i,\zeta}(\sigma) \in N_{r(\sigma)}$.*

2) Moreover in (1) we can find also formulas $\langle \psi_{\ell,r}(x, \bar{y}_\ell, \bar{z}) : \ell < m_1^\Upsilon, r < r(\sigma) \rangle$ such that

$\boxtimes_2$ *in $\boxtimes_1$ above we can add:*
if $r \le r(\sigma)$ and let $N'_r = N_0 \cup (TC(d) \cap N_r)$ where TC is transitive closure, then $N'_{r+1} = N_r \cup \{a :$ for some $\bar{b} \in {}^{\ell g(\bar{y})}N_r, (N'_r, P) \models \psi_{\ell,2}(a, \bar{b}, \bar{\zeta})\}$ (identifying $\bar{\zeta}$ with a sequence of members of N_0).

3) For every rule $\mathbf{R}$ (see [BGSh 533, 4.5]) there are $r(\mathbf{R}) \in \mathbb{N}$ and an inductive scheme Υ in $\mathscr{L}_{\text{f.o.}}$ such that (P_0 is zero place relation):

$\boxtimes_3$ *if $(N_i, \bar{P}_i)$ is a (M, Υ)-candidate for $i \le r(\mathbf{R})$ and $(N_{i+1}, \bar{P}_{i+1})$ is the Υ-successor of $(N_i, \bar{P}_i)$ for $i < r(\mathbf{R})$ and $P_{i,0} =$ truth, then $i < r(\mathbf{R}) \Rightarrow N_i \subseteq N_{i+1}$, the stationary $(N_{i(\mathbf{R})}, \bar{P}^-_{i(\mathbf{R})})$ is the $\mathbf{R}$-udpate of $(N_0, \bar{P}^-_0)$, $P_{i,r(\mathbf{R})} =$ truth where $\bar{P}^- = \bar{P} \restriction [1, m_1^\Upsilon)$.*

4) In (3) we can even have

$\boxtimes_4$ *$N_r = N_0 \cup \{x : x$ active in $N_{r^*}\}$.*

Proof. 1) By induction on the term.
2) As N_{r+2} can be (uniformly) interpreted in N_r.
3) By induction on the rule $\mathbf{R}$. $\square_{4.5}$

Proof of 4.3.
We describe what is an Υ-successor rather than let $r^* = r(\mathbf{R}^\pi) + 1, \mathbf{R}^\pi$ is the rule which π is. Then say formally what is Υ.

Now the predicates (and function symbols) $\{P_k^\Upsilon : k < m_0^\Upsilon\}$ serve some purposes:

kind 1: The dynamic predicate and function symbols of π, say P_k for $k \in w(1)$, say $P_{k(1,0)}$ will denote $\emptyset, P_{k(1,a)}$ will denote the set of atoms.
For notation simplicity

kind 2: $P_{k(2,0)}$ unary predicate will serve to denote the set of active elements; and

kind 3: $P_{k(3,1)}, \dots, P_{k(3,r^*)}$ will be zero place relations, they will denote the time modulo r^*, say for $t = 0$ they are all false; for $t = 1$ we get true, true false ..., for $t = 2$ we get true, true, true, false... (without loss of generality $r^* \geq 3$ for $t = 3 + r^*s + r, r < r^*$ we have $P_{k(3,r')} \equiv r' = r$). The reason is that in our translation one step for $\bar{\pi}$ will be translated to r^* steps in the construction of the N_t's and the translation begins only with $t = 1$. Now we can describe almost a translation.

Now Υ is such that:

(a) $N_1[M, \Upsilon]$ is $M \cup \{M, \emptyset\}$,
$N_2[M, \Upsilon]$ is $M_1[M, \Upsilon] \cup \{1\}$, recall $1 = \{\emptyset\}$
$N_3[N, \Upsilon]$ is $N_1[M, \Upsilon] \cup \{1, 2\}$,

(b) if $t^* = 3 + r^*s$, then $P_{t^*,k(3,0)} =$ true, $P_{t^*,k(3,1)}, \dots, c_{t^*,k(3,r^*)}$ are false and we take care that $P_{t^*+r,k(3,r')} = r' = 0 \bmod r^*$ for $r' \leq r^*$ and $\langle (N_{t^*+2}, \bar{P}_{t^*+r}) : i \leq r(\mathbf{R}^*) \rangle$ is as in 4.4(3)+(4). Moreover $\bar{P}_{t^*+r^*} = \bar{P}_{t^*+r(\mathbf{R}^*)}$ and $N_{t^*+r^*}$ is the set of active members of $(N_{t^*+r(\mathbf{R}^*)}, \bar{P}_{t^*+r(\mathbf{R}^*)}$.

Well, as in $\theta = \theta_{\Upsilon,1+r^*p,q}$ and $\iota = 1$, the stopping decision for time will be the same, but we still have to deal with the space (up to now using $r + r^*p$ would be O.K.). However, between t^* and $t^* + r^*$ we have to preserve N_{t^*} till creating $N_{t^*+r(\mathbf{R}^*)}$ and only then can we omit the elements of N_{t^*} no longer necessary.

So we will have

<u>kind 4</u>: individual constants $P_{k(4,1)}, P_{k(4,2)}$

(c) if $t^* = 3 + r^*s$ then: $P_{t,k(4,0)} < P_{t,k(4,1)} < P_{t,k(4.2)}$ are the three last ordinals, $P_{t,k(4,0)}$ is an active ordinal but not $P_{t,k(4,1)}, P_{t,k(4,2)}$ and $x \in N_{t^*}$ is active <u>iff</u> $P_{t,k(4,2)} \cup x \in N_{t^*}$.

So $\|N_{t^*}\| = 2 + 2\|\{x \in N_{t^*} : x \text{ active}\}\|$.

Now starting with N_{t^*} in deciding N_{t^*} we omit and non-active elements except the two last ordinals and then do as earlier by 4.4(3)+(4) for $r = 1, \dots, r(\mathbf{R}^\pi)$ taking care to have the right natural numbers.

So defining $N_{i^*+2^*}$, we take care of the "doubling".

2), 3), 4) Similarly. $\square_{4.3}$

Less critical is the inverse relation

4.6 Lemma. *1) Let Υ be an inductive scheme for $\mathscr{L}_{\text{f.o.}}, \iota = 7, \tau = \tau^\Upsilon, \chi \in \mathscr{L}_{\text{f.o.}}(\tau)$.*
<u>Then</u> we can find $r^* \geq 1$ and a program π for the same vocabulary as in [BGSh 533, §4] such that for every integer polynomials $p(n), q(n)$ and τ-model M such that:

> *$M \models_\iota \theta_{\Upsilon,\chi,p,q}$ (see Definition 1.3) <u>iff</u> $M \models \bar{\pi}$ where $\bar{\pi} = (\pi, p^*, q^*)$ and $\models$ is as in [BGSh 533, §4] where: $p^*(n) = r^*, p^*(n) + r^*, q^*(n) = q(n) + 2$*

4.7 Lemma. *1) If $\bar{\pi}$ is as in [BGSh 533, §11], that is with being able to compute the cardinality of M (= set of atoms) then a similar result holds but Υ is in $\mathscr{L}_{\text{f.o. + na}}$.*
2) If $\bar{\pi}$ is as in [BGSh 533] for cardinality logic (see §4 there), then a similar result holds but Υ is in $\mathscr{L}_{\text{card}}$ and spaces do not use $\theta_{\Upsilon,3+n+r^*(p),2+n+2q(n)}$.
3) We can replace p,q by arbitrary function from T and get similar results.

Proof. 1) We ignore too short runs for simplicity. The "$q(n) = q(n)+2$" comes from [BGSh 533] starting with two extra elements so during the computation we preserve having two entry elements (except when we notice we are to stop-see below).

Now every step of the computation for $\bar{\pi}$ is translated to r^* step during the computation of the $N_t[M,\Upsilon]$'s.
What do we do in those r^* steps? First we compute the relations on N_t definable first order subformulas of the ψ_ℓ. We also translate χ to be equivalent to what should be in the next state and then add the new elements (so $N_{t+1} \models \chi$ was computed in N_t, as in 4.4).
2), 3), 4) Similar to part (1) + 4.3.

[How do we "compute" the first order formulas? Where $P_{\varphi(\bar{x})}$ code $\varphi(\bar{x})$, we of course represent all subformulas and do it inductively.]

Atomic are given

negation is by "if $P_{\varphi(\bar{x})}(\bar{a})$ = truth then $P_{\varphi(\bar{x})}(\bar{a})$ = false, else $P_\varphi(\bar{a})$ is truth (and appropriate "for all" also adding dummy variables is possible by "for all".

For conjunctions $\varphi(\bar{x}) = \varphi_1(\bar{x})$ use "if $P_{\varphi(\bar{x})}(\bar{a})$ = truth then $P_{\varphi(\bar{x})}(\bar{a}) = P_{\varphi_2(\bar{x})}(\bar{a})$ else $P_{\varphi(\bar{x})}(\bar{a})$ = false.

For existential quantifier, $\varphi(\bar{x}) = (\exists y)\psi(y,\bar{x})$ use
"if $P_{\psi(\bar{x},bary)}(\bar{a})$ = truth then $P_{\varphi(\bar{x})}(\bar{a})$ = truth else do nothing". $\square_{4.3}$

§5 Closing comments

We may consider a context is $(K, \mathscr{I})$ such that logics related to our proof in §2. The first version in 5.1(3) changes the satisfaction the second (in 5.1(4),(4A)) changes also the syntax.

5.1 Definition. 1) A context is a pair $(K, \mathscr{I})$ such that

- (a) K be a class of models with vocabulary (= the set of predicates) τ
- (b) $\mathscr{I}$ is a function
- (c) $\mathrm{Dom}(\mathscr{I}) = K$
- (d) $\mathscr{I}(M)$ is a family of subsets of M, whose union is $|M|$, and closed under subsets.

1A) We call $(K, \mathscr{I})$ invariant if

- (e) $\mathscr{I}$ is preserved by isomorphisms, i.e. if f is an isomorphism from $M_1 \in K$ onto $M_2 \in K$ and $A \in \mathscr{I}(M_1)$ then $f"(A) \in \mathscr{I}(M_2)$.

1B) We say "f is an $\mathscr{I}$-isomorphism from $M_1 \in K$ onto $M_2 \in K$" <u>if</u> f is an isomorphism from M_1 onto M_2 such that $\mathscr{I}(M_2) = \{f"(A) : A \in \mathscr{I}(M_1)\}$. Recall that $\mathscr{I}(M_2)[k] = \{\bigcup\limits_{\ell<k} A_\ell : A_\ell \in \mathscr{I}[M]\}$.

2) In 1) let

$$\mathrm{Seq}^\alpha_{\mathscr{I}}(M) = \{\bar{a} : \bar{a} \text{ a sequence of members of } M \text{ of length } \alpha,\ \mathrm{Rang}(\bar{a}) \in \mathscr{I}(M)\}.$$

3) Let $\mathscr{L} = \mathscr{L}_{\text{f.o.}}$ or $\mathscr{L} = \mathscr{L}_{\lambda,\kappa}$; recalling that the logic $\mathscr{L}_{\lambda,\kappa,\alpha}$ for λ, κ cardinals, is defined like first order logic but we allow conjunctions of $\bigwedge\limits_{i<\alpha}$, for $\alpha < \lambda$ and existential quantifier $(\exists \bar{x})$ with $\bar{x}$ a sequence of variables of length $< \kappa$, and the depth of the formulas is $< \alpha$. We define $\mathscr{L}^{[k]} = \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$, logics with the same syntax but with a difference in the definition of the satisfaction relation, $M \models^{[k]}_{\mathscr{I}} \varphi[\bar{a}]$ or $(M, \mathscr{I}) \models^{[k]} \varphi(\bar{a})$ is defined inductively on α as usual, except that

- $(*)$ we demand $\mathrm{Rang}(\bar{a}) \in \mathscr{I}[M][k]$ (otherwise not defined) , that is
  $M \models^{[k]}_{\mathscr{I}} (\exists \bar{x})\varphi(\bar{x}, \bar{b})$ <u>iff</u>
  $\mathrm{Rang}(\bar{b}) \in \mathscr{I}[M][k-1]$ and for some $\bar{a} \in {}^{\ell g(\bar{x})}M$ with
  $\mathrm{Rang}(\bar{a}) \in \mathscr{I}(M)$ we have $M \models^k_{\mathscr{I}} \varphi[\bar{a}, \bar{b}]$.

Let $\mathscr{L}_{\lambda,\kappa,\alpha;\text{card}}, \mathscr{L}_{\lambda,\kappa;+\text{na}}, \mathscr{L}_{\lambda,\kappa;\text{card},\mathbf{T}}$ be defined similarly (so

$$M \models^{[k]}_I \exists^{!\mu} \bar{x}\varphi(\bar{x}, \bar{b}) \text{ iff } (\bar{b} \in {}^{\ell g(\bar{b}))}M),\ \mathrm{Rang}(\bar{b}) \in \mathscr{I}(M)[k-1]$$
$$\text{and } \mu = |\{a : \{a\} \in I \text{ and } M \models^{[k]}_I \varphi[a, \bar{b}]\}|.$$

Omitting α means some α.
If λ, κ, α are omitted they are $\aleph_0$ (so $L_{\lambda,\kappa,\alpha}$ is first order).
4) We define a logic $\mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$. Let us define the formulas in $\mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$ by induction on α, each formula φ has the form $\varphi(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{k_1-1}), k_1 \le k$, where the $\bar{x}_\ell$'s are pairwise disjoint sequences of variables of length $< \kappa$ (so if $\kappa = \aleph_0$, finite sequences) and every variable appearing freely in φ appear in one of those sequences (so any formula is coupled with such $\langle \bar{x}_0, \ldots, \bar{x}_{k_1-1} \rangle$, probably some not actually appearing).

$\underline{\alpha = 0}$: quantifier free formula; i.e. any Boolean combination of atomic ones (with the right variables, of course).

$\underline{\alpha + 1}$: α non-limit $\varphi(\bar{x}_0, \ldots, \bar{x}_{k_1-1})$ is from $\mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$ or is a Boolean combination of formulas of the form $(\exists \bar{y})\psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k_2-2}}, \bar{y})$ where $k_2 \le k, \psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k_2-2}}, \bar{y}) \in \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$.

$\underline{\alpha \text{ limit}}$: $\mathscr{L}^{[k]}_{\lambda,\kappa,\alpha} = \bigcup\limits_{\beta<\alpha} \mathscr{L}^{[k]}_{\lambda,\kappa,\beta}$.

$\underline{\alpha + 1, \alpha \text{ limit}}$: $\mathscr{L}^{[k]}_{\lambda,\kappa,\alpha+1}$ is the set of $\varphi \in \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$ or φ a Boolean combinations of members of $\mathscr{L}^{*}_{\lambda,\kappa,\alpha}$ of the right variables.

Let $\mathscr{L}^{k}_{\lambda,\kappa} = \bigcup\limits_{\alpha} \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$ and $\mathscr{L}^{[*]}_{\lambda,\kappa,\alpha} = \bigcup\limits_{k<\omega} \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}$ and $\mathscr{L}^{[k]}_{\alpha} = L^{[k]}_{\aleph_0,\aleph_0,\alpha}$ and $\mathscr{L}^{[k]}_{<\alpha} = \bigcup\limits_{\beta<\alpha} \mathscr{L}^{[k]}_{\beta}$ and $\mathscr{L}^{[k]} = \mathscr{L}^{[k]}_{\aleph_0,\aleph_0}$ and $L^{[*]} = \bigcup\limits_{k<\omega} L_k$.
4A) We now define a satisfaction relation $M \models \varphi(\bar{a}_0, \ldots, \bar{a}_{k_1-1})$ where $k_1 \le k$ (depending on $\mathscr{I}$).
I.e. we define by induction on α, for $\varphi(\bar{x}_0, \ldots, \bar{x}_{k_1-1}) \in \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}, \bar{a}_\ell \in \text{Seq}^{\ell g(\bar{x}_\ell)}_{\mathscr{I}}(M)$, when does $M \models \varphi[\bar{a}_0, \ldots, \bar{a}_{k_1-1}]$ and when $M \models \neg\varphi[\bar{a}_0, \ldots, \bar{a}_{k_1-1}]$. This is done naturally, in particular $M \models (\exists \bar{y})\varphi(\bar{a}_0, \ldots, \bar{a}_{k_2-2}, \bar{y})$ iff for some $\bar{b} \in \text{Seq}^{\ell g(\bar{y})}_{\mathscr{I}}(M)$, (so $\text{Rang}(\bar{b}) \in \mathscr{I}(M)$) we have $M \models \varphi[\bar{a}_0, \ldots, \bar{a}_{k_2-2}, \bar{b}]$.
5) We can define for $\mathscr{L}$ one of the above, $(\mathscr{L})^{\text{card}}$ similarly adding the quantifiers $[\varphi'(\bar{x}; \bar{z})/\varphi''(\bar{x}, \bar{y}, \bar{z})]$ saying: $\varphi''(\ldots, \ldots; \bar{z})$ define an equivalence relation on $\{\bar{x} : \varphi'(\bar{x})\}$ with exactly s equivalence classes.
6) We can above replace models M by pairs $(M, I), I \subseteq \mathscr{P}(M)$ closed under subsets.

<u>5.2 Discussion</u>: We may replace M by M^+, adding elements coding each $A \in \mathscr{I}(M)$, with decoding by functions, but

(a) this does not capture $\models^{[k]}$ and

(b) for $\models^{[k]}$ this requires infinitely many functions, we need to actually code any sequence listing each $A \in \mathscr{I}(M)$.

Still this framework seems to work quite smoothly for its purposes. We could have made it more central (use 5.5 below).

Note that

5.3 Observation. For any k-system $\mathscr{Y}$ and $\mathfrak{z}$ and $\mathbf{m}_1^{\Upsilon}$-lifting of $\mathscr{Y}$, letting $M^{\mathscr{Y}} = M$ we have

$$\mathscr{Y}_{\mathfrak{z}} = (N^{\mathfrak{z}}, \{A : (\exists f \in \mathscr{F})[\mathrm{Dom}(f) \in I^{\mathscr{F}} \ \& \ A \subseteq \mathrm{Dom}(G(f))]\}, \{G(f) : f \in \mathscr{F}\})$$

is a k-system.

Now easily (and it makes a connection with §1, §2):
5.4 Observation: 1) Let $(K, \mathscr{I})$ be a context and $M_1, M_2 \in K$ be (finite as always), τ-models, τ a vocabulary, and $k < \omega$.
The following are equivalent:

- (A) there are k-systems
$\mathscr{Y}_\ell = (M_\ell, \mathscr{I}(M_\ell), \mathscr{F}_\ell)$ for $\ell = 1, 2$ and $\mathscr{H}$ as in Definition 2.21(1),(2)
- (B) for every sentence $\psi \in \mathscr{L}^{[k]}_{\text{f.o.}}(\tau)$ we have
$M_1 \models^{[k]}_{\mathscr{I}} \psi \Leftrightarrow M_2 \models^{[k]}_{\mathscr{I}} \psi$
- (C) for infinite λ, κ, α, for every sentence $\psi \in \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}(\tau)$ we have
$M_1 \models^{[k]}_{\mathscr{I}} \psi \Leftrightarrow M_2 \models^{[k]}_{\mathscr{I}} \psi$
- (D) for every t and infinite λ, κ, α for every sentence $\psi \in \mathscr{L}^{[k]}_{\lambda,\kappa,\alpha}(\tau)$ we have
$N^{\mathfrak{z}^{\text{full}}_{\mathscr{Y}_1,t}} \models^{[k]}_{I^{\mathscr{Y}_1}_t} \psi \Leftrightarrow N^{\mathfrak{z}^{\text{full}}_{\mathscr{Y}_2,t}} \models^{[k]}_{I^{Y_2}_t} \psi$ where $N^{\mathfrak{z}^{\text{full}}_{\mathscr{Y}_\ell,t}}$ is defined in 2.13(7) and
$I^{\mathscr{Y}}_t = \{\mathrm{Dom}(f) : f \in G^{\mathfrak{z}^{\text{full}}_{\mathscr{Y},t}}\}$.

$\square_{5.4}$

Proof. Straight.

5.5 Definition. 1) We say Υ (from Definition 1.1) is pure if $m_1[\Upsilon] = 0$ so no P_ℓ.
2) Let $\mathscr{Y} = (M, I, \mathscr{F})$ be a k-system; let "$\mathfrak{z}^*$ is the full $\mathbf{t}$-successor of $\mathfrak{z}$" be as defined in 2.13. We define by induction on $t; \mathfrak{z}_t = \mathfrak{z}^t[\mathscr{Y}, \mathbf{t}]$ as follows: for $t = 0, N^{\mathfrak{z}^t} = M, P^{\mathfrak{z}^t}_\ell$ is the empty set, $G^{\mathfrak{z}^t}$ is the identity on $\mathscr{F}$ and $R^{\mathfrak{z}} = \{(A, x) : A \in I, x \in A\}$; for $t = s + 1$ let $\mathfrak{z}^t$ be the full $\mathbf{t}$-successor of $\mathfrak{z}^s$. Let $\mathscr{I}^{\mathfrak{z}} = I_{\mathfrak{z},\mathscr{Y}} = \{B : B \subseteq S_{\mathfrak{z}}(A)$ for some $A \in I\}$ where $S_{\mathfrak{z}}(A) = S_{\mathfrak{z},A} = \{x \in N^{\mathfrak{z}} : A$ is a $\mathfrak{z}$-support of $x\}$.

We can also see:

5.6 Claim. *1) Assume*

(a) $\mathscr{Y}_\ell = (M_\ell, \mathscr{I}_\ell, \mathscr{F}_\ell)$ *is a* $\mathfrak{t}$*-dichotomical* k*-system for* $\ell = 1, 2$

(b) $\mathfrak{Z}^\ell_t = \mathfrak{Z}_t[M_\ell, I_\ell]$*, so* $\mathfrak{Z}^\ell_{t+1}$ *is the full successor of* $\mathfrak{Z}^\ell_t$*.*

Then the following are equivalent:

(α) $(M_1, I_1), (M_2, I_2)$ *are* $\mathscr{L}^{[k]}$*-equivalent*

(β) *for every* $t < \infty$ *the pairs* $(M_1^{\mathfrak{Z}_t}, I^{\mathfrak{Z}_t}), (M_2^{\mathfrak{Z}_t}, I^{\mathfrak{Z}_2})$ *are* $\mathscr{L}^{[k]}$*-equivalent.*

2) For any given (M, I) *there in a sentence* $\psi \in \mathscr{L}^{[k]}_{\text{f.o.}}(\tau_M)$ *satisfied by* (M, I) *and implying any other sentence* $\psi' \in L^{[k]}_{\infty,\infty}(\tau_\mu)$ *satisfied by* (M, I)*.*

5.7 Remark. 1) So in part (2) we can apply 2.22, 2.23, 2.24.
2) Note that by 5.3, $\mathscr{L}^{[k]}$ satisfies addition theorems.

5.8 Fact: For any Υ we can find Υ' which is equivalent if we use in Definition 1.3 the case $\iota = 4$ (well when $\mathbf{t}(\|M_\ell\|)$ always is ≥ 2). In fact, we can reconstruct (i.e. define by a formula in $\mathscr{L}^*$) the sequence of $\langle P_{t,\ell} : t' < t\rangle$ in N_t.

5.9 Conclusion 1) Assume $\mathscr{Y}$ is counting k-system (see 2.3). Then we can define R_t, G_t for every t (N_t the "computation" in time t) such that

$$(M, \bar{P}_0, G_0, R_0) \text{ is 0-lifting}$$

$$(N_{t+1}, \bar{P}_{t+1}, G_{t+1}, R_{t+1}) \text{ is a lifting, successor of } (N_t, \bar{c}_t, G_t, R_t).$$

2) So the formula the $\bar{\varphi}$ defines is preserved by $f \in \mathscr{F}_0$.

Proof. Straight.

* * *

5.10 Discussion: 1) In §2 and in 5.5 we can allow infinite models M and define $N_t[M] = N_t[M, \Upsilon, \mathbf{t}]$ for every ordinal t, for this better assume Υ is standard, monotonic and pure (or strongly monotonic, i.e. demand $P_{\ell,t}[M, \Upsilon, \mathbf{t}]$ is increasing with t; for t limit we take union and so $V[M, \Upsilon, \mathbf{t}] = \cup\{N_\alpha[M, \Upsilon, \mathbf{t}] : \alpha \text{ an ordinal}\}$, see below. Now as in the case $\iota = 4$, the analysis in §2 works for this but it is not clear if we can get any interesting things.
Note that those definitions remind us of Gödel's construction of L, particularly of $L_{\alpha+1}$ from L_α, and the Frankel-Mostowski models (which use automorphisms).

May can this give interesting proofs of consistency for set theory with no choice but with urelements? It seems they can be reduced to the classical case.
2) We can prove various equivalence and 0-1 laws by 5.5, by proving that the relevant model N_t can be interpreted in $\mathfrak{Z}^t[M, I]$ from 5.5, using f.o. logic which suffices.

Proof. Straight.

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