MORE ON WEAK DIAMOND

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ABSTRACT. We deal with the combinatorial principle Weak Diamond, showing that we always either a local version is not saturated or we can increase the number of colours. Then we point out a model theoretic consequence of Weak Diamond.

0. Basic definitions

In this section we present basic notations, definitions and results.

Notation 0.1. (1) $\kappa, \lambda, \theta, \mu$ will denote cardinal numbers and α, β, δ, ε, ξ, ζ, γ will be used to denote ordinals.

(2) Sequences of ordinals are denoted by ν, η, ρ (with possible indexes).

(3) The length of a sequence η is $\ell g(\eta)$.

(4) For a sequence η and $\ell \leq \ell g(\eta)$, $\eta{\restriction}\ell$ is the restriction of the sequence η to ℓ (so $\ell g(\eta{\restriction}\ell) = \ell$). If a sequence ν is a proper initial segment of a sequence η then we write $\nu \vartriangleleft \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).

(5) For a set A and an ordinal α, α_A stands for the function on A which is constantly equal to α.

(6) For a model M, $|M|$ stands for the universe of the model.

(7) The cardinality of a set X is denoted by $\|X\|$. The cardinality of the universe of a model M is denoted by $\|M\|$.

Definition 0.2. Let λ be a regular uncountable cardinal and θ be a cardinal number.

(1) *A* (λ, θ)*–colouring* is a function $F : \mathrm{DOM} \longrightarrow \theta$, where DOM is either ${}^{<\lambda}2 = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}2$ or $\bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$. In the first case we will write $\mathrm{DOM}_\alpha = {}^{1+\alpha}2$, in the second case we let $\mathrm{DOM}_\alpha = {}^{1+\alpha}(\mathcal{H}(\lambda))$ (for $\alpha \leq \lambda$).

If λ is understood we may omit it; if $\theta = 2$ then we may omit it too (thus *a colouring* is a $(\lambda, 2)$–colouring).

(2) For a (λ, θ)–colouring F and a set $S \subseteq \lambda$, we say that a function $\eta \in {}^{S}\theta$ is *an* F*–weak diamond sequence for* S if for every $f \in \mathrm{DOM}_\lambda$

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the set

$$\{\delta \in S : \eta(\delta) = F(f{\restriction}\delta)\}$$

is stationary.

(3) WDmId_λ is the collection of all sets $S \subseteq \lambda$ such that for some colouring F there is no F–weak diamond sequence for S.

Remark 0.3. In the definition of WDmId_λ (0.2(3)), the choice of DOM (see 0.2(1)) does not matter; see [Sh:f, AP, §1], remember that $\|\mathcal{H}(\lambda)\| = 2^{<\lambda}$.

Theorem 0.4 (Devlin Shelah [DvSh 65]; see [Sh:f, AP, §1] too)**.** *Assume that $2^\theta = 2^{<\lambda} < 2^\lambda$ (e.g. $\lambda = \mu^+$, $2^\mu < 2^\lambda$). Then for every λ–colouring F there exists an F–weak diamond sequence for λ. Moreover,* WDmId_λ *is a normal ideal on λ (and $\lambda \notin \mathrm{WDmId}_\lambda$).*

Remark 0.5. One could wonder why the weak diamond (and WDmId_λ) is interesting. Below we list some of the applications, limitations and related problems.

(1) Weak diamond is really weaker than diamond, but it holds true for some cardinals λ in ZFC. Note that under GCH, $\diamondsuit_{\mu^+}$ holds true for each $\mu > \aleph_0$, so the only interesting case then is $\lambda = \aleph_1$.
(2) Original interest in this combinatorial principle comes from Whitehead groups:
    if G is a strongly λ–free Abelian group and $\Gamma(G) \notin$ WDmId_λ
    then G is Whitehead.
(3) A related question was: can we have stationary subsets $S_1, S_2 \subseteq \omega_1$ such that $\diamondsuit_{S_1}$ but $\neg\diamondsuit_{S_2}$? (See [Sh 64].)
(4) Weak diamond has been helpful particularly in problems where we have some uniformity, e.g.:
    $(*)_1$ *Assume $2^\lambda < 2^{\lambda^+}$. Let $\psi \in L_{\lambda^+,\omega}$ be categorical in λ, λ^+. Then $(\mathrm{MOD}_\psi, \prec_{\mathrm{Frag}(\psi)})$ has the amalgamation property in λ.*
    $(*)_2$ *If G is an uncountable group then we can find subgroups G_i of G (for $i < \lambda$) non-conjugate in pairs (see* [Sh 192]*).*
(5) One may wonder if assuming $\lambda = \mu^+$, $2^\lambda > 2^\mu$ (and e.g. μ regular) we may find a regular $\sigma < \mu$ such that

$$\{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\} \notin \mathrm{WDmId}_\lambda(\lambda).$$

Unfortunately, this is not the case (see [Sh 208]).
(6) We would like to prove
    (a) WDmId_λ is not λ^+–saturated or
    (b) a strengthening, e.g. weak diamond for more colours.
    We will get (a variant of) a local version of the disjunction, where we essentially fix F. There are two reasons for interest in **(a)**: understanding λ^+–saturated normal ideals (e.g. we get more information on the case CH + "$\mathcal{D}_{\omega_1}$ is $\aleph_2$–saturated"; see also Zapletal Shelah

[ShZa 610]), and non λ^+–saturation helps in "non-structure theorems" (see [Sh 87b], [Sh 576]). That is, having $2^\mu < 2^{\mu^+} < 2^{\mu^{++}}$ and some "bad" (i.e. "nonstructure") properties for models in μ we get $2^{\mu^{++}}$ models in μ^{++} when $\mathrm{WDmId}_{\lambda^+}$ is not λ^{++}–saturated (and using the local version does not hurt).

(7) Note that for $S \notin \mathrm{WDmId}_\lambda$ we have a weak diamond sequence $f \in {}^S 2$ such that the set of "successes"(=equalities) is stationary, but it does not have to be in $(\mathrm{WDmId}_\lambda)^+$. We would like to start and end in the same place: being positive for the same ideal. Also, in **(b)** above the set of places we guess was stationary, when we start with $S \in (\mathrm{WDmId}_\lambda)^+$.

Note that it may well be that $\lambda \in \mathrm{WDmId}_\lambda$ (if $(\exists \theta < \lambda)(2^\theta = 2^\lambda)$ this holds), but some "local" versions may still hold. E.g. in the Easton model, we have F–weak diamond sequences for all F which are reasonably definable (see [Sh:f, AP, §1]; define

$$F(f) = 1 \ \Leftrightarrow\ \mathrm{L}[X, f] \models \varphi(X, f)$$

for a fixed first order formula φ, where $X \subseteq \lambda$ depends on F only). So the case $\mathrm{WDmId}_\lambda = \mathcal{P}(\lambda)$ has some interest.

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1. When colourings are almost constant

Definition 1.1. Let λ be a regular uncountable cardinal.

(1) Let $S \subseteq \lambda$ and let F be a (λ, θ)–colouring. We say that *a sequence* $\eta \in {}^S\theta$ *is coded by* F if there exists $f \in \mathrm{DOM}_\lambda$ such that

$$\alpha \in S \quad \Leftrightarrow \quad \eta(\alpha) = F(f{\restriction}(1+\alpha)).$$

We let

$$\mathfrak{B}(F) \stackrel{\mathrm{def}}{=} \{\eta \in {}^\lambda\theta : \eta \text{ is coded by } F\}.$$

(2) For a family $\mathcal{A}$ of subsets of λ let $\mathrm{ideal}_\lambda(\mathcal{A})$ be the λ–complete normal ideal on λ generated by $\mathcal{A}$ (i.e. it is the closure of $\mathcal{A}$ under unions of $< \lambda$ elements, diagonal unions, containing singletons, and subsets).

[Note that $\mathrm{ideal}_\lambda(\mathcal{A})$ does not have to be a proper ideal.]

(3) For a λ–colouring F (so $\theta = 2$) we define by induction on α:

$$\mathrm{ID}^-_0(F) = \emptyset, \qquad \mathrm{ID}_0(F) = \{S \subseteq \lambda : S \text{ is not stationary }\},$$

for a limit α

$$\mathrm{ID}^-_\alpha(F) = \bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F), \qquad \mathrm{ID}_\alpha(F) = \mathrm{ideal}_\lambda(\bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F)),$$

and for $\alpha=\beta+1$

$$\begin{aligned}\mathrm{ID}^-_\alpha(F)= &\ \big\{S\subseteq\lambda: \text{for each } S^*\subseteq S \text{ there is } f\in \mathrm{DOM}_\lambda \text{ such that}\\ &\qquad \{\delta<\lambda:\delta\in S^*\Leftrightarrow F(f\restriction\delta)=0\}\in \mathrm{ID}_\beta(F)\big\};\\ \mathrm{ID}_\alpha(F)= &\ \mathrm{ideal}_\lambda\big(\mathrm{ID}^-_\alpha(F)\big).\end{aligned}$$

Finally we let $\mathrm{ID}(F)=\bigcup\limits_\alpha \mathrm{ID}_\alpha(F)$.

(4) We say that F is *rich* if $\mathrm{DOM}(F)=\bigcup\limits_{\alpha<\lambda}{}^\alpha\mathcal{H}(\lambda)$, and for every function $f\in\mathrm{DOM}_\lambda$ and $\alpha<\lambda$ and a set $A\subseteq\alpha$ there is $f'\in\mathrm{DOM}_\lambda$ such that

$$(\forall i<\lambda)(f(1+i)=f'(1+i)\ \&\ F(f\restriction(\alpha+i))=F(f'\restriction(\alpha+i)))$$

and $(\forall j<\alpha)(F(f'\restriction j)=1\ \Leftrightarrow\ j\in A)$.

Definition 1.2. Let λ be a regular uncountable cardinal and let F be a λ–colouring.

(1) $\mathrm{WDmId}_\lambda(F)$ is the family of all sets $S\subseteq\lambda$ with the property that for every $S^*\subseteq S$ there is $f\in\mathrm{DOM}_\lambda$ such that the set

$$\{\delta\in S:\delta\in S^*\ \Leftrightarrow\ F(f\restriction\delta)=1\}$$

is not stationary.

(2) $\mathfrak{B}^+(F)$ is the closure of

$$\mathfrak{B}(F)\cup\{S\subseteq\lambda: S \text{ is not stationary }\}$$

under unions of $<\lambda$ sets, complement and diagonal unions (here, in $\mathfrak{B}(F)$, we identify a subset of λ with its characteristic function).

(3) $\mathrm{ID}^1(F)\overset{\mathrm{def}}{=}\{S\subseteq\lambda:(\exists X\in\mathfrak{B}^+(F))(S\subseteq X\ \&\ \mathcal{P}(X)\subseteq\mathfrak{B}^+(F))\}$.

(4) $\mathrm{ID}^2(F)$ is the collection of all $S\subseteq\lambda$ such that for some $X\in\mathfrak{B}^+(F)$ we have: $S\subseteq X$ and there is a partition X_0,X_1 of X such that

(α) $\mathcal{P}(X_\ell)=\{Y\cap X_\ell: Y\in\mathfrak{B}^+(F)\}$ for $\ell=0,1$, and

(β) there is no $Y\in\mathfrak{B}^+(F)$, $\ell<2$ satisfying

$$Y\setminus X_\ell\in\mathrm{ID}^1(F)\quad\&\quad Y\notin\mathrm{ID}^1(F).$$

Proposition 1.3. *Assume λ is a regular uncountable cardinal and F is a λ–colouring.*

(1) *If $\mathcal{A}$ is a family of subsets of λ such that*

($\circledast_{\mathcal{A}}$) *if $S_0\subseteq S_1$ and $S_1\in\mathcal{A}$ and $A\in[\lambda]^{<\lambda}$ then $S_0\cup A\in\mathcal{A}$,*

then $\mathrm{ideal}_\lambda(\mathcal{A})$ is the collection of all diagonal unions $\mathop{\nabla}\limits_{\xi<\lambda}A_\xi$ such that $A_\xi\in\mathcal{A}$ for $\xi<\lambda$.

(2) *The condition ($\circledast_{\mathrm{ID}^-_\alpha(F)}$) (see above) holds true for each α. Consequently, if $\alpha=\beta+1$ then $\mathrm{ID}_\alpha(F)=\{\mathop{\nabla}\limits_{i<\lambda}A_i:\langle A_i:i<\lambda\rangle\subseteq \mathrm{ID}^-_\alpha(F)\}$, and if α is limit then $\mathrm{ID}_\alpha(F)=\{\mathop{\nabla}\limits_{i<\lambda}A_i:\langle A_i:i<\lambda\rangle\subseteq \bigcup\limits_{\beta<\alpha}\mathrm{ID}_\beta(F)\}$.*

(3) $\mathrm{ID}(F)$ *and* $\mathrm{ID}_\alpha(F)$ *are* λ*–complete normal ideals on* λ *extending the ideal of non-stationary subsets of* λ *(but they do not have to be proper). For* $\alpha < \gamma$ *we have* $\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}_\gamma(F)$ *and hence* $\mathrm{ID}(F) = \mathrm{ID}_\alpha(F)$ *for every large enough* $\alpha < (2^\lambda)^+$.

(4) *Suppose* $\bar{B} = \langle B_\ell : \ell \leq m \rangle$, *where* $B_\ell \subseteq B_{\ell+1}$ *(for* $\ell < m$*) and* $B_m \in \mathrm{ID}(F)$. *Then* $\bar{B}$ *has an* F*–representation, which means that there are a well founded tree* $T \subseteq {}^{\omega>}\lambda$, *sequences* $\langle B^\ell_\eta : \eta \in T,\ \ell \leq \ell_\eta \rangle$, *and* $\langle f^k_\eta : \eta \in T,\ k \leq k_\eta \rangle$ *such that* $k_\eta \leq \ell_\eta + 1$ *and*

  (a) $B^\ell_{\langle\rangle} = B$, $\ell_{\langle\rangle} = m$, $B^\ell_\eta \subseteq B^{\ell+1}_\eta \subseteq \lambda$, $f^\ell_\eta \in {}^\lambda 2$,

  (b) $(\forall \eta \in T \setminus \max(T))(\forall i < \lambda)(\eta^\frown\langle i \rangle \in T)$,

  (c) *for each* $\eta \in T \setminus \max(T)$ *there is* $\alpha_\eta < \lambda$ *such that for all* $\delta \in \lambda \setminus \alpha_\eta$

    $(\oplus)$ $\delta \in B^\ell_\eta$ *iff*

    $(\exists i < \delta)(\delta \in B^\ell_{\eta^\frown\langle i\rangle})$ *or*

    $F(f^\ell_\eta \restriction \delta) = 1 \ \&\ \neg(\exists i < \delta)(\exists k)(\delta \in B^k_{\eta^\frown\langle i\rangle})$,

  (d) *for each* $\eta \in \max(T)$, B_η *is a bounded subset of* λ *with* $\min(B_\eta) > \max(\{\eta(n) : n < \ell g(\eta)\})$.

(5) *If for some* $f^* \in {}^\lambda 2$ *we have* $(\forall \alpha < \lambda)(F(f^* \restriction \alpha) = 0)$ *then in part (4) above we can demand that* $k_\eta = \ell_\eta + 1$.

(6) *If* F *is rich then in part (4) above we can add*

  (e) α_η*=0 for* $\eta \in T \setminus \max(T)$ *and* $B_\eta = \emptyset$ *for* $\eta \in \max(T)$.

(7) $\mathrm{ID}(F)$ *is the minimal normal filter on* λ *such that there is no* $S \in (\mathrm{ID}(F))^+$ *satisfying*

$$(\forall S^* \subseteq S)(\exists A \in \mathfrak{B}(F))(S^* \vartriangle A \in \mathrm{ID}(F)).$$

Proof. (1)–(2) Should be clear.

(3) By induction on $\gamma < \lambda$ and then by induction on $\alpha < \gamma$ we show that $(\forall \gamma < \lambda)(\forall \alpha < \gamma)(\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}_\gamma(F))$. If $\gamma = 1$ then this follows immediately from definitions; similarly if γ is limit. So suppose now that $\gamma = \gamma_0 + 1$ and we proceed by induction on $\alpha \leq \gamma_0$. There are no problems when $\alpha = 0$ nor when α is limit. So suppose that $\alpha = \beta + 1 < \gamma$ (so $\beta < \gamma_0$). By the inductive hypothesis we know that $\mathrm{ID}_\beta(F) \subseteq \mathrm{ID}_{\gamma_0}(F)$. Let $A \in \mathrm{ID}_{\beta+1}(F)$. By (2) there are $A_\xi \in \mathrm{ID}^-_{\beta+1}$ (for $\xi < \lambda$) such that $A = \mathop{\nabla}\limits_{\xi<\lambda} A_\xi$. Now look at the definition of $\mathrm{ID}^-_{\beta+1}(F)$: since $\mathrm{ID}_\beta(F) \subseteq \mathrm{ID}_{\gamma_0}(F)$ we see that $A_\xi \in \mathrm{ID}^-_{\gamma_0+1}(F)$. Hence $A \in \mathrm{ID}_\gamma$.

(4) By induction on α we show that if $\bar{B} = \langle B_\ell : \ell \leq m \rangle$, where $B_\ell \subseteq B_{\ell+1}$ (for $\ell < m$) and $B_m \in \mathrm{ID}_\alpha(F)$ then $\bar{B}$ has an F–representation.

Case 1: $\alpha = 0$.

Thus the set B_m is not stationary and we may pick up a club E of λ disjoint from B_m. Let $E = \{\alpha_\zeta : \zeta < \lambda\}$ be the increasing enumeration. Put $T = \{\langle\rangle\} \cup \{\langle i \rangle : i < \lambda\}$, $\alpha_{\langle\rangle} = 1$, $\ell_{\langle\rangle} = \ell_{\langle i\rangle} = m$, $B^\ell_{\langle\rangle} = B_\ell$ and $B^\ell_{\langle i\rangle} = B_\ell \cap \alpha_{i+1}$. Now check.

Case 2: α is limit.
It follows from (2) that $B_\ell = \mathop{\nabla}\limits_{i<\lambda} B_{\ell,i}$ for some $B_{\ell,i} \in \bigcup\limits_{\beta<\alpha} \mathrm{ID}_\beta(F)$. Let $B'_{\ell,i}$ be defined as follows:

if $i = (m+1)j + t$, $\ell < t \le m$ then $B'_{\ell,i} = \emptyset$,
if $i = (m+1)j + t$, $t \le m$, $t \le \ell$ then $B'_{\ell,i} = B_{\ell,i}$.

Then for each i, ℓ we may find $\langle B^{i,\ell}_\eta, f^{i,\ell'}_\eta, \alpha^i_\eta : \eta \in T_i,\ \ell < \ell^{i,1}_\eta,\ \ell' < \ell^{i,2}_\eta \rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,i,k}_{\langle\rangle} : k \le k^1_\eta \rangle = \langle B'_{\ell,i} : \ell \le m \rangle$ (by the inductive hypothesis). Put

$$\begin{aligned} & T = \{\langle\rangle\} \cup \{\langle i \rangle^\frown \eta : \eta \in T_i\}, \\ & \ell_{\langle\rangle} = m, \quad \ell'_{\langle\rangle} = 0, \quad \ell_{\langle i \rangle^\frown \eta} = \ell^{i,1}_\eta, \quad \ell_{\langle i \rangle^\frown \eta} = \ell^{i,2}_\eta \\ & B^\ell_{\langle\rangle} = B_\ell, \quad B^\ell_{\langle i \rangle^\frown \eta} = B^{i,\ell}_\eta, \quad f^{\ell'}_{\langle i \rangle^\frown \eta} = f^{i,\ell'}_\eta, \\ & \alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle i \rangle^\frown \eta} = \alpha^i_\eta. \end{aligned}$$

Checking that $\langle B^\ell_\eta, f^{\ell'}_\eta, \alpha_\eta : \eta \in T,\ \ell \le \ell_\eta,\ \ell' \le \ell'_\eta \rangle$ is as required is straightforward.
Case 3: $\alpha = \beta + 1$.
By (2) above and the proof of Case 2 we may assume that $B_m \in \mathrm{ID}^-_\alpha(F)$. It follows from the definition of $\mathrm{ID}^-_\alpha(F)$ that there are $f_\ell \in {}^\lambda 2$ (for $\ell \le m$) such that

$$B^\oplus_\ell \stackrel{\mathrm{def}}{=} \{\delta < \lambda : \delta \text{ is limit and } F(\eta{\restriction}\delta) = 0 \Leftrightarrow \delta \in B_\ell\} \in \mathrm{ID}_\beta(F),$$

and hence $B^\oplus \stackrel{\mathrm{def}}{=} \bigcup\limits_{\ell \le m} B^\oplus_\ell \in \mathrm{ID}_\beta(F)$. Therefore $B^*_\ell \stackrel{\mathrm{def}}{=} B_\ell \cap B^\oplus \in \mathrm{ID}_\beta(F)$. Now apply the inductive hypothesis for β and $\bar{B}^* = \langle B^*_\ell : \ell \le m \rangle$ to get the sequences $\langle B^{\ell,*}_\eta, f^{k,*}_\eta : \eta \in T^*,\ \ell \le \ell^*_\eta,\ k \le k^*_\eta \rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,*}_{\langle\rangle} : \ell \le \ell^*_\eta \rangle = \langle B^*_\ell : \ell \le m \rangle$. Put

$$\begin{aligned} & T = \{\langle\rangle\} \cup \{\langle i \rangle : i < \lambda\} \cup \{\langle 0 \rangle^\frown \eta : \eta \in T^*\}, \\ & \ell_{\langle 0 \rangle^\frown \eta} = \ell^*_\eta, \quad k_{\langle\rangle} = m+1, \quad k_{\langle 0 \rangle^\frown \eta} = k_\eta, \\ & B^\ell_{\langle 0 \rangle^\frown \eta} = B^{\ell,*}_\eta, \quad B^\ell_{\langle 0 \rangle^\frown \langle i \rangle} = B_\ell \cap (i + \omega), \\ & f^k_{\langle\rangle} = f_k, \quad f^k_{\langle 0 \rangle^\frown \eta} = f^{k,*}_\eta, \\ & \alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle 0 \rangle^\frown \eta} = \alpha^*_\eta. \end{aligned}$$

(5) If f^ℓ_η is not defined then choose f^* as it. □

Remark 1.4. Note that it may happen that $\lambda \in \mathrm{ID}(F)$. However, if $\eta \in {}^\lambda 2$ is a weak diamond sequence for F then the set $\{\gamma < \lambda : \eta(\gamma) = 0\}$ witnesses $\lambda \notin \mathrm{ID}^-_1(F)$. And conversely, if $\lambda \notin \mathrm{ID}^-_1(F)$ and $S^* \subseteq \lambda$ witnesses it, then the function $0_{S^*} \cup 1_{\lambda \setminus S^*}$ is a weak diamond sequence for F.

Definition 1.5. For a λ–colouring F we define λ–colourings $F^\oplus$ and $F^\otimes$ as follows.

(1) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\oplus}$–standard if there is a tuple $(T, \bar{f}, \bar{\alpha}, \bar{A})$ (called a witness) such that
   (i) $T \subseteq {}^{\omega>}\gamma$ is a well founded tree (so $\langle\rangle \in T$, $\nu \lhd \eta \in T \Rightarrow \nu \in T$ and T has no ω–branch);
   (ii) $\bar{f} = \langle f^\ell_\eta : \eta \in T,\ \ell \le k_\eta\rangle$, where $f^\ell_\eta \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;
   (iii) $\bar{\alpha} = \langle \alpha_\eta : \eta \in T\rangle$, where $\alpha_\eta < \lambda$;
   (iv) $\bar{A} = \langle A^\ell_\eta : \eta \in T,\ \ell \le \ell_\eta\rangle$, where $A^\ell_\eta \subseteq \alpha_\eta$;
   (v) $g(\beta) = (T \cap {}^{\omega>}\beta, \langle f^\ell_\eta \restriction \beta : \eta \in T \cap {}^{\omega>}\beta,\ \ell < k_\eta\rangle, \langle \alpha_\eta : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle A^\ell_\eta : \eta \in T \cap {}^{\omega>}\beta,\ \ell \le \ell_\eta\rangle)$ for each $\beta < \gamma$.

(2) $\mathrm{DOM}(F^{\oplus}) = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:
   $(\oplus)_\alpha$ if $\gamma = 0$ then $F^{\oplus}(g) = 0$,
   $(\oplus)_\beta$ if $\gamma > 0$ and g is not standard then $F^{\oplus}(g) = 0$,
   $(\oplus)_\gamma$ if $\gamma > 0$ and g is standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\oplus}(g) = \mathbf{t}^0_{F,g}(\langle\rangle)$, where $\mathbf{t}^\ell_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$, $\ell = 0, 1$) are defined by downward induction as follows.
      If $\eta \in \max(T)$ then $\mathbf{t}^\ell_{F,g}(\eta) = 1$ iff $\gamma \in A_\eta$,
      if $\eta \in T \setminus \max(T)$, $\gamma < \alpha_\eta$ then $\mathbf{t}^\ell_{F,g}(\eta) = 1$ iff $\gamma \in A_\eta$,
      if $\eta \in T \setminus \max(T)$, $\gamma \ge \alpha_\eta$ then

$$\begin{array}{lll} \mathbf{t}^1_{F,g}(\eta) = 1 & \text{iff} & F(f_\eta) = 1 \text{ or } (\exists i < \gamma)(\mathbf{t}^1_{F,g}(\eta^\frown\langle i\rangle) = 1), \\ \mathbf{t}^0_{F,g}(\eta) = 1 & \text{iff} & (\exists i < \gamma)(\mathbf{t}^0_{F,g}(\eta^\frown\langle i\rangle) = 1) \text{ or} \\ & & F(f'_\eta) = 1 \ \&\ (\forall i < \gamma)(\mathbf{t}^1_{F,g}(\eta^\frown\langle i\rangle) = 0). \end{array}$$

(3) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\otimes}$–standard if there is a tuple $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ (called a witness) such that
   (i) $T \subseteq {}^{\omega>}\gamma$ is a well founded tree;
   (ii) $\bar{f} = \langle f_\eta : \eta \in T\rangle$, where $f_\eta \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;
   (iii) $\bar{\ell} = \langle \ell_\eta : \eta \in T\rangle$, where $\ell_\eta : {}^{3}\{0,1\} \longrightarrow \{0,1\}$;
   (iv) $\bar{\alpha} = \langle \alpha_\eta : \eta \in T\rangle$, where $\alpha_\eta < \lambda$;
   (v) $\bar{A} = \langle A_\eta : \eta \in T\rangle$, where $A_\eta \subseteq \alpha_\eta$;
   (vi) $g(\beta) = (T \cap {}^{\omega>}\beta, \langle f_\eta \restriction \beta : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle \ell_\eta : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle \alpha_\eta : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle A_\eta : \eta \in T \cap {}^{\omega>}\beta\rangle)$ for each $\beta < \gamma$.

(4) $\mathrm{DOM}(F^{\otimes}) = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:
   $(\otimes)_\alpha$ if $\gamma = 0$ then $F^{\otimes}(g) = 0$,
   $(\otimes)_\beta$ if $\gamma > 0$ and g is not $F^{\otimes}$–standard then $F^{\otimes}(g) = 0$,
   $(\otimes)_\gamma$ if $\gamma > 0$ and g is $F^{\otimes}$–standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\otimes}(g) = \mathbf{t}_{F,g}(\langle\rangle)$, where $\mathbf{t}_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$) are defined by downward induction as follows.
      If $\eta \in \max(T)$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_\eta$,
      if $\eta \in T \setminus \max(T)$, $1 + \gamma < \alpha_\eta$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_\eta$,
      if $\eta \in T \setminus \max(T)$, $1 + \gamma \ge \alpha_\eta$ then

$$\mathbf{t}_{F,g}(\eta) = \ell_\eta(F(f_\eta), \max\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}, \min\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}).$$

Proposition 1.6. *Let F be a λ–colouring. Then $F^{\oplus}$ is a λ–colouring and*

(a) *if $S \in \mathrm{ID}(F)$ then $0_S \cup 1_{\lambda\setminus S} \in \mathfrak{B}(F^{\oplus})$ and $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\oplus})$,*
(b) $\mathrm{ID}(F) \subseteq \mathrm{ID}_1(F^{\oplus}) = \mathrm{ID}_1^-(F^{\oplus}) = \mathrm{ID}(F^{\oplus})$,

Proof. (a) Check.
(b) $\mathrm{ID}(F) \subseteq \mathrm{ID}_1(F^{\oplus})$.
Suppose that $B \in \mathrm{ID}(F)$. We are going to show that then $B \in \mathrm{ID}_1^-(F^{\oplus})$. So suppose that $B' \subseteq B$. We want to find $g \in \mathrm{DOM}_\lambda(F^{\oplus})$ such that the set

$$\{\delta < \lambda : \delta \text{ is limit and } F(g{\restriction}\delta) = 0 \Leftrightarrow \delta \in B'\}$$

is in $\mathrm{ID}_0(F^{\oplus})$ (what just means that it is non-stationary). Since $B \in \mathrm{ID}(F)$ we have $B' \in \mathrm{ID}(F)$, so by 1.3(4) we may find $\langle B^\ell_\eta, f^k_\eta, \alpha_\eta : \eta \in T,\ \ell \leq \ell_\eta,\ k < k_\eta\rangle$ such that the clauses (a)–(d) of 1.3(4) are satisfied with $\ell_{\langle\rangle} = 0$, $B' = B^0_{\langle\rangle}$. Define g as follows. For $\beta < \lambda$ let $T_\beta = T \cap {}^{\omega>}\beta$ and

$$g(\beta) = \big(T_\beta,\ \langle f^k_\eta : \eta \in T_\beta, k \leq k_\eta\rangle,\ \langle \alpha_\eta : \eta \in T_\beta\rangle,\ \langle B^\ell_\eta \cap \alpha_\eta : \ell \leq \ell_\eta, \eta \in T_\beta\rangle\big).$$

Now look at the demands in 1.5(2) – they are exactly what 1.3(4) guarantees us. $\square$

Definition 1.7. Let F_1, F_2 be λ–colourings (with $\mathrm{DOM}(F_\ell)$ being either ${}^{\lambda>}2$ or $\bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$, see 0.2(1)).

(1) We say that $F_1 \leq F_2$ if there is $h : \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that
  (a) $\eta \trianglelefteq \nu \quad\Rightarrow\quad h(\eta) \trianglelefteq h(\nu)$,
  (b) $h(\eta) = \lim\limits_{\alpha<\delta} h(\eta \restriction \alpha)$, for every $\eta \in {}^{\delta}2$, δ a limit,
  (c) $(\forall \eta \in \mathrm{DOM}(F_1))(0 < \ell g(\eta) = \ell g(h(\eta)) \ \Rightarrow\ F_1(\eta) = F_2(h(\eta)))$.
(2) We say that $F_1 \leq^* F_2$ if there is $h : \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that the clauses (a)–(c) above hold but
  (d) if $\eta \in \mathrm{DOM}_\lambda(F_1)$ and $\lim\limits_{\alpha<\lambda} h(\eta{\restriction}\alpha)$ has length $< \lambda$ then $F_1(\eta{\restriction}\alpha) = 0$ for every large enough α.

Proposition 1.8. (1) *$\leq^*$ and $\leq$ are transitive relations on λ–colourings, $\leq^* \subseteq \leq$.*
(2) *$\leq$ is λ^+ directed.*

Proposition 1.9. (1) *For every colouring $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$ there is a colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ such that $F_1 \leq F_2 \leq^* F_1$.*
(2) *For every λ–colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ there is a λ–colouring $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ such that $F_2 \leq F_1 \leq^* F_2$.*

Proof. 1) Let $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$. Let h_0 be a one-to-one function from $\mathcal{H}(\lambda)$ to ${}^{\lambda>}2$, say $h_0(\eta) = \langle \ell_{\eta,i} : i < \ell g(h_0(\eta))\rangle$. Define a function

$h_1 : \mathcal{H}(\lambda) \longrightarrow {}^{\lambda>}2$ by:

$\ell g(h_1(\eta)) = \ell g(h_0(\eta)) + 2,$
$h_1(\eta)(2i) = h_0(\eta)(i), \quad h_1(\eta)(2i+1) = 0 \quad$ for $i < \ell g(h_0(\eta)),$ and
$h_1(\eta)(2\ell g(h_0(\eta))) = h_1(\eta)(2\ell g(h_0(\eta)+1)) = 1.$

Next, by induction on $\ell g(\eta)$, we define a function $h^+ : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow {}^{\lambda>}2$ as follows:

$$h^+(\langle\rangle) = \langle\rangle, \qquad h^+(\eta^\frown\langle x\rangle) = h^+(\eta)^\frown h_1(x).$$

Finally we define a colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ by

$$F_2(\nu) = \begin{cases} F_1(\eta) & \text{if } \nu = h^+(\eta), \\ 0 & \text{if } \nu \notin \operatorname{rng}(h^+). \end{cases}$$

□

Proposition 1.10. *Assume that F_1, F_2 are λ–colourings such that $F_1 \le F_2$, or just $F_1 \le^* F_2$. Then:*

(1) *For every $\eta \in {}^{\lambda}2$ there are $\nu \in {}^{\lambda}2$ and a club E of λ such that*

$$(\forall\delta \in E)(F_1(\eta{\restriction}\delta) = F_2(\nu{\restriction}\delta)).$$

(2) $\mathrm{ID}_\alpha(F_1) \subseteq \mathrm{ID}_\alpha(F_2)$, $\mathrm{ID}^-_\alpha(F_1) \subseteq \mathrm{ID}^-_\alpha(F_2)$; *hence* $\mathrm{ID}(F_1) \subseteq \mathrm{ID}(F_2)$ *and* $\mathfrak{B}^+(F_1) \subseteq \mathfrak{B}^+(F_2)$.

(3) *For every colouring F there is a colouring F' such that $F \le F'$ and* $\mathrm{ID}^2(F) \subseteq \mathrm{ID}(F')$.

Proof. Straightforward. □

Conclusion 1.11. Assume that λ is a regular uncountable cardinal and $F : {}^{\lambda>}2 \longrightarrow 2$ is a λ–colouring. Let

$$F^{\otimes} : \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$$

be the colouring defined for F in Definition 1.5(4). Then:

(a) $F \le F^{\otimes}$.

(b) $\mathrm{ID}(F^{\otimes})$ is a normal ideal on λ.

(c) $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\otimes})$ and $\mathrm{ID}(F) \subseteq \mathrm{ID}(F^{\otimes}) = \mathrm{WDmId}_\lambda(F^{\otimes})$.

(d) $F^{\otimes}$ relates to itself as it relates to F, i.e. *if* $\alpha^* < \lambda^+$, $\langle S_\alpha : \alpha < \alpha^*\rangle$ is increasing continuous modulo $\mathrm{ID}(F^{\otimes})$, $S_{\alpha+1} = S_\alpha \cup A_\alpha$ mod $\mathrm{ID}(F^{\otimes})$, $A_\alpha \in \mathfrak{B}(F^{\otimes})$, $\ell_\alpha \in 2$,

then for some $f \in {}^{\lambda}(\mathcal{H}(\lambda))$

$$\{\alpha < \lambda : F(f{\restriction}\alpha) = 1\}/\mathcal{D}_\lambda$$

is, in $\mathcal{P}(\lambda)/\mathcal{D}_\lambda$, the least upper bound of the family $\{(A_\alpha \setminus S_\alpha)/\mathcal{D}_\lambda : \ell_\alpha = 1\}$ (where $\mathcal{D}_\lambda$ stands for the club filter).

(e) The family $\mathfrak{B}(F^{\otimes})$ is closed under complements, unions and intersections of less than λ sets, diagonal unions and diagonal intersections and it includes bounded subsets of λ. Moreover $\mathfrak{B}^+(F^{\otimes}) = \mathfrak{B}(F^{\otimes})$.

(f) If $\mathcal{P}(\lambda)/\mathrm{ID}(F^\otimes)$ is λ^+–saturated then
for every set $X \subseteq \lambda$ there are sets $A, B \in B(F^\otimes)$ such that
(α) $A \subseteq X \subseteq B$,
(β) for every $Y \in \mathfrak{B}(F^\otimes)$ one of the following occurs:
(i) the sets $(X \setminus A) \cap Y$, $(X \setminus A) \setminus Y$, $(B \setminus X) \cap Y$, $(B \setminus X) \setminus Y$ are[1] not in $\mathrm{ID}(F^\otimes)$,
(ii) $Y \cap (B \setminus A) \in \mathrm{ID}(F^\otimes)$,
(iii) $(B \setminus A) \setminus Y \in \mathrm{ID}(F^\otimes)$.
In the situation as above we denote $A = \max_{F^\otimes}(X)$, $B = \min_{F^\otimes}(X)$ (note that these sets are unique modulo $\mathrm{ID}(F^\otimes)$). Moreover
(g) if $A \subseteq \min_{F^\otimes}(B)$ then $\min_{F^\otimes}(A) \subseteq \min_{F^\otimes}(B) \mod \mathrm{ID}(F^\otimes)$.
(h) If $X \subseteq \lambda$, $X \notin \mathrm{ID}(F^\otimes)$ then for some $Y_1, Y_2 \subseteq X$ which are not in $\mathrm{ID}(F^\otimes)$ we have

$$\max_{F^\otimes}(Y_1) = \max_{F^\otimes}(Y_2) = \emptyset \quad \text{and} \quad \min_{F^\otimes}(Y_1) = \min_{F^\otimes}(Y_2) \notin \mathrm{ID}(F^\otimes).$$

Proof. CLAUSES (A) AND (B): Should be clear.

CLAUSE (E): Note that as $\theta = 2$ we identify a sequence $\eta \in {}^{\lambda}2$ with $\{i < \lambda : \eta(i) = 1\}$.

$\mathfrak{B}(F^\otimes)$ **is closed under complementation.**
Suppose that $A \in \mathfrak{B}(F^\otimes)$. If A is bounded then let g, $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ be as in 1.5(3) with $T = \{\langle\rangle\} \cup \{\langle i \rangle : i < \lambda\}$, $A_{\langle\rangle} = \alpha_{\langle\rangle} \setminus A$, $\alpha_{\langle\rangle} > \sup(A)$, $\ell_{\langle\rangle}$ constantly 1. Then $(\forall \alpha < \lambda)(F^\otimes(g{\restriction}(1+\alpha)) = 1 \Leftrightarrow \alpha \in A)$, so F codes $\lambda \setminus A$. So suppose that $\sup(A) = \lambda$. Pick g such that

$$(\forall \alpha < \lambda)(F^\otimes(g{\restriction}(1+\alpha)) = 1 \Leftrightarrow \alpha \in A).$$

By our assumption, for arbitrarily large $\beta < \lambda$ we have $F^\otimes(g{\restriction}\beta) = 1$, so $g(\beta)$ is

$$\big(T_\beta, \langle f^\beta_\eta : \eta \in T_\beta\rangle, \langle \alpha^\beta_\eta : \eta \in T_\beta\rangle, \langle \ell^\beta_\eta : \eta \in T_\beta\rangle, \langle \alpha^\beta_\eta : \eta \in T_\beta\rangle, \langle A^\beta_\eta : \eta \in T_\beta\rangle\big)$$

and it is as in 1.5(3). If $\beta_1 < \beta_2$ then the two values necessarily cohere, in particular $T_{\beta_1} = T_{\beta_2} \cap {}^{\omega>}(\beta_1)$. Consequently there is $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ such that $T = \bigcup\limits_{\beta<\lambda} T_\beta \subseteq {}^{\omega>}\lambda$ is closed under initial segments and is well founded (as T_β increase with β and $\mathrm{cf}(\lambda) > \aleph_0$). Thus we have proved

($\boxtimes$) if $A \subseteq \lambda$ is unbounded and $F^\otimes$ coded by g then there is $\mathbf{p} = (T, \bar{f}, \bar{\bar{\ell}}, \bar{\alpha}, \bar{A})$ such that the clauses (i)–(vi) of 1.5(3) hold for $\gamma = \lambda$ and $g(\beta) = \mathbf{p}{\restriction}\beta$.

Now define $\mathbf{p}'$ like $\mathbf{p}$ (with the same T etc) except that $\ell^{\mathbf{p}'}_{\langle\rangle} = 1 - \ell^{\mathbf{p}}_{\langle\rangle}$ and $A^{\mathbf{p}'}_{\langle\rangle} = A^{\mathbf{p}}_{\langle\rangle}$.

$\mathfrak{B}(F^\otimes)$ **contains all bounded subsets of λ.**
By the first part of the arguments above all co-bounded subsets of λ are in $\mathfrak{B}(F^\otimes)$, so (by the above) their complements are there too.

[1] hence none of $X \setminus A$, $B \setminus A$ includes (modulo $\mathrm{ID}(F^\otimes)$) a member of $\mathfrak{B}(F^\otimes) \setminus \mathrm{ID}(F^\otimes)$

$\mathfrak{B}(F^{\otimes})$ **is closed under unions of length** $< \lambda$**.**
Let $B = \bigcup\limits_{i<\alpha} B_i$ where $\alpha < \lambda$ and $B_i \in \mathfrak{B}(F^{\otimes})$. Let $w = \{i < \alpha : \sup(B_i) = \lambda\}$ and for $i \in w$ let B_i be represented by $g_i \in {}^{\lambda}(\mathcal{H}(\lambda))$ which, by ($\boxtimes$), comes from $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$. We may assume that $w = \beta \le \alpha$. Let

$$T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\} \cup \{\langle i\rangle^\frown\eta : \eta \in T^i,\ i < \beta\},$$
$$f_{\langle i\rangle^\frown\eta} = f^i_\eta, \text{ etc}$$
$$\alpha_{\langle\rangle} \text{ is the first } \gamma \ge \omega \text{ such that } \gamma \ge \alpha \ \&\ (\forall i \in [\beta, \alpha))(B_i \subseteq \gamma),$$
$$B_{\langle i\rangle} = \emptyset \quad \text{if } i \ge \beta,$$
$$A_{\langle\rangle} = \bigcup_{i<\alpha} B_i \cap \alpha_{\langle\rangle},$$
$$\ell_{\langle\rangle}(i_0, i_1, i_2) = i_1.$$

Checking is straightforward.

$\mathfrak{B}(F^{\otimes})$ **is closed under diagonal unions.**
Let $B = \mathop{\nabla}\limits_{i<\lambda} B_i$, where each $B_i \in \mathfrak{B}(F^{\otimes})$ is represented by $g_i \in {}^{\lambda}(\mathcal{H}(\lambda))$ which, by ($\boxtimes$), comes from $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$. Let $T = \{\langle\rangle\} \cup \{\langle i\rangle^\frown\eta : \eta \in T_i,\ i < \lambda\}$, $f_{\langle i\rangle^\frown\eta} = f^i_\eta$, etc, $\alpha_{\langle\rangle} = \omega$, $B_{\langle\rangle} = B \cap \omega$ and $\ell_{\langle\rangle}(i_0, i_1, i_2) = i_1$.

Clause (c): First note that $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\otimes})$ as $\mathfrak{B}(F) \subseteq \mathfrak{B}^+(F) \subseteq \mathfrak{B}^+(F^{\otimes}) = \mathfrak{B}(F^{\otimes})$ (the second inclusion by (a) and 1.10, the last equality by (e)). Next note that

$$\mathrm{WDmId}_\lambda(F^{\otimes}) \subseteq \mathrm{ID}^-_1(F^{\otimes}) \subseteq \mathrm{ID}_1(F^{\otimes}) \subseteq \mathrm{ID}(F^{\otimes}).$$

Now by induction on α we are proving that $\mathrm{ID}_\alpha(F^{\otimes}) \subseteq \mathrm{WDmId}_\lambda(F^{\otimes})$. So suppose that we have arrived to a stage α.
If $\alpha = 0$ then we use the fact that every non-stationary subset of λ is in $\mathfrak{B}(F^{\otimes})$ (by (e)).
If α is limit then, by the induction hypothesis, $\mathrm{ID}^-_\alpha(F^{\otimes}) \subseteq \mathfrak{B}(F^{\otimes})$ and hence $\mathrm{ID}_\alpha \subseteq \mathfrak{B}(F^{\otimes})$ (as $gB(F^{\otimes})$ is closed under diagonal unions by (e); remember 1.3(3)).
So suppose that $\alpha = \beta + 1$ and $B \in \mathrm{ID}_\alpha(F^{\otimes})$. Suppose $B' \subseteq B$ (so $B' \in \mathrm{ID}^-_\alpha(F^{\otimes})$). There is $B'' \in \mathfrak{B}(F)$ such that $B'' \triangle B' \in \mathrm{ID}_\beta(F)$. By the first part we know that $B'' \in \mathfrak{B}(F^{\otimes})$ and by the induction hypothesis $B' \triangle B'' \in \mathfrak{B}(F^{\otimes})$. Consequently $B' \in \mathfrak{B}(F^{\otimes})$.

Together we have proved that $\mathrm{ID}(F^{\otimes}) = \mathrm{WDmId}_\lambda(F^{\otimes})$. The inclusion $\mathrm{ID}(F) \subseteq \mathrm{ID}(F^{\otimes})$ is easy. □

Proposition 1.12. *Let λ be a regular uncountable cardinal and F be a λ–colouring.*

1. *If $\mathrm{ID}_\alpha(F)$ is λ^+–saturated then for some $\beta < \lambda^+$ we have $\mathrm{ID}_{\alpha+\beta}(F) = \mathrm{ID}(F)$.*
2. $\mathrm{ID}_\alpha(F) \subseteq \mathrm{WDmId}_\lambda$.
3. *If $\mathrm{ID}_\alpha(F)$ is λ^+–saturated and $\lambda \notin \mathrm{WDmId}_\lambda$ then $\mathrm{WDmId}_\lambda = \mathrm{ID}_1(F')$ for some λ–colouring F'.*

(4) $\mathrm{ID}^2(F)$ *is a normal ideal, and* $\mathrm{ID}^1(F) \subseteq \mathrm{ID}^2(F) \subseteq \mathrm{WDmId}_\lambda$.
(5) $\mathrm{ID}^1(F^\otimes) = \mathrm{WDmId}_\lambda(F^\otimes)$.

Proof. 1) It follows from 1.3(3) that $\mathrm{ID}_\gamma(F)$ increases with γ, so the assertion should be clear.

2) By 1.11(c).

3) Assume that $\mathrm{ID}_\alpha(F)$ is λ^+–saturated and $\lambda \notin \mathrm{WDmId}_\lambda$. By induction on $\beta < \lambda^+$ we try to define colourings F_β such that

(a) $\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}(F_0)$,
(b) if $\beta < \gamma$ then $\mathrm{ID}(F_\beta) \subseteq \mathrm{ID}(F_\gamma)$,
(c) $\mathrm{ID}(F_\beta) \neq \mathrm{ID}(F_{\beta+1})$.

So we let $F_0 = F$. If β is limit then we use 1.9(2) to choose F_β so that $(\forall\gamma < \beta)(F_\gamma \leq F_\beta)$. Finally, if $\beta = \gamma + 1$ then we let $F'_\beta = (F_\gamma)^\otimes$ (so $\mathrm{ID}(F_\gamma) \subseteq \mathrm{ID}_1(F'_\beta) = \mathrm{ID}(F'_\beta) \subseteq \mathrm{WDmId}_\lambda$). If $\mathrm{ID}(F'_\beta) \neq \mathrm{WDmId}_\lambda$ then we choose a set $A \in \mathrm{WDmId}_\lambda \setminus \mathrm{ID}(F'_\beta)$ and F^*_β witnessing $A \in \mathrm{WDmId}_\lambda$. We may assume that $(\forall\alpha \in \lambda \setminus A)(\forall\eta \in {}^{\alpha}2)(F^*_\beta(\eta) = 0)$. Now take a colouring F_β such that $F'_\beta, F^*_\beta \leq F_\beta$.

After carrying out the construction choose $S^0_\beta \in \mathrm{ID}(F_{\beta+1}) \setminus \mathrm{ID}(F_\beta)$ (for $\beta < \lambda^+$) and let $S_\beta = S^0_\beta \setminus \mathop{\nabla}\limits_{\gamma<\beta} S^0_\gamma$. Then $\langle S_\beta : \beta < \lambda^+\rangle$ is a sequence of pairwise disjoint members of $\mathcal{P}(\lambda) \setminus \mathrm{ID}(F_0) \subseteq \mathcal{P}(\lambda) \setminus \mathrm{ID}_\alpha(F)$, contradicting our assumptions. □

For the rest of this section we will assume the following

Hypothesis 1.13. Assume that

(a) λ is a regular uncountable cardinal,
(b) F is a λ–colouring,
(c) $\lambda \notin \mathrm{ID}(F^\otimes)$, and
(d) $\mathrm{ID}(F^\otimes)$ is λ^+–saturated, that is there is no sequence $\langle A_\alpha : \alpha < \lambda^+\rangle$ such that for each $\alpha < \beta < \lambda^+$

$$A_\alpha \notin \mathrm{ID}(F^\otimes) \quad \text{and} \quad \|A_\alpha \cap A_\beta\| < \lambda.$$

For each limit ordinal $\alpha \in [\lambda, \lambda^+)$ fix an enumeration $\langle \varepsilon^\alpha_i : i < \lambda\rangle$ of α.

Construction 1.14. Fix a sequence $\eta \in {}^{\lambda}2$ for a moment. We define a sequence

$$\langle S_\alpha[\eta], A_\alpha[\eta], B_\alpha[\eta], \ell_\alpha[\eta], m_\alpha[\eta], f_\alpha[\eta] : \alpha < \alpha^*[\eta]\rangle$$

as follows. By induction on $\alpha < \lambda^+$ we try to choose $S_\alpha[\eta] = S_\alpha$, $A_\alpha[\eta] = A_\alpha$, $B_\alpha[\eta] = B_\alpha$, $\ell_\alpha[\eta] = \ell_\alpha$, $m_\alpha[\eta] = m_\alpha$, $f_\alpha[\eta] = f_\alpha$ such that

(a) $S_\alpha, A_\alpha, B_\alpha \subseteq \lambda$, $\ell_\alpha, m_\alpha \in \{0, 1\}$, $f_\alpha \in {}^{\lambda}2$,
(b) $A_\alpha \notin \mathrm{ID}(F^\otimes)$, $A_\alpha \cap S_\alpha = \emptyset$,
(c) $S_{\alpha+1} = S_\alpha \cup A_\alpha$; if $\alpha < \lambda$ is limit then $S_\alpha = \bigcup\limits_{\beta<\alpha} S_\alpha$; if $\alpha \in [\lambda, \lambda^+)$ is limit then $S_\alpha = \{\gamma < \lambda : (\exists i < \gamma)(\gamma \in S_{\varepsilon^\alpha_i})\}$, $S_0 = \emptyset$,

(d) $B_\alpha \in \mathrm{ID}(F^\otimes)$,

(e) for every $\delta \in \lambda \setminus (S_\alpha \cup B_\alpha)$

$$\eta(\delta) = m_\alpha \qquad \Rightarrow \qquad F(f_\alpha \restriction \delta) = \ell_\alpha,$$

(f) $A_\alpha = \{\delta \in \lambda \setminus S_\alpha : F(f_\alpha \restriction \delta) = 1 - \ell_\alpha\}$.

It follows from 1.13 that at some stage $\alpha^* = \alpha^*[\eta] < \lambda^+$ we get stuck (remember clause (b) above). Still, we may define then S_{α^*} as in the clause (c).

Proposition 1.15. *Assume 1.13. Then:*

(1) *There exists $\eta \in {}^\lambda 2$ such that*

$$\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}(F^\otimes).$$

(2) *If $S \in \mathfrak{B}(F^\otimes) \setminus \mathrm{ID}(F^\otimes)$ then we can demand $S \subseteq S_{\alpha^*[\eta]}[\eta]$.*

Proof. Assume not. Then for each $\eta \in {}^\lambda 2$ the set $B_{\alpha^*}[\eta] \stackrel{\text{def}}{=} \lambda \setminus S_{\alpha^*[\eta]}$ is in $\mathrm{ID}(F^\otimes)$. Now,

$$\{\alpha \in B_{\alpha^*}[\eta] : \eta(\alpha) = 1\} \in \mathrm{ID}(F^\otimes) \subseteq \mathfrak{B}(F^\otimes)$$

(see 1.6).

Claim 1.15.1. *For each α, $S_\alpha \in \mathfrak{B}(F^\otimes)$.*

Proof of the claim. We show it by induction on α. If $\alpha = 0$ then $S_\alpha = \emptyset \in \mathfrak{B}(F^\otimes)$ (see 1.11(c)). If $\alpha < \lambda$ is a limit ordinal then $S_\alpha = \bigcup\limits_{\beta<\alpha} S_\beta$ and by the inductive hypothesis $S_\beta \in \mathfrak{B}(F^\otimes)$, so by 1.11(e) we are done (as $\mathfrak{B}(F^\otimes)$ is closed under unions of $< \lambda$ elements). If $\alpha \in [\lambda, \lambda^+)$ is limit then we use the fact that $\mathfrak{B}(F^\otimes)$ is closed under diagonal unions. If $\alpha = \beta + 1$ then $A_\beta \in \mathfrak{B}(F)$ or $\lambda \setminus A_\beta \in \mathfrak{B}(F)$ and hence we may conclude that $A_\beta \in \mathfrak{B}(F^\otimes)$ (remember 1.11(e)). Since $\mathfrak{B}(F^\otimes)$ is closed under unions of length $< \lambda$ we are done. $\square$

Claim 1.15.2. *For each α, $Y_\alpha \stackrel{\text{def}}{=} \{\beta < \lambda : \eta(\beta) = 1\} \cap S_\alpha \in \mathfrak{B}(F^\otimes)$.*

Proof of the claim. We prove it by induction on α. If $\alpha = 0$ then $Y_\alpha = \emptyset$ and there is nothing to do. The case of limit α is handled like that in the proof of 1.15.1. So suppose that $\alpha = \beta + 1$. It suffices to show that the set $Y_\alpha \cap (S_\alpha \setminus S_\beta)$ is in $\mathfrak{B}(F)$, what means that $Y_\alpha \cap A_\alpha$ is there (remember clauses (e) and (f)). Note that if $\delta \in A_\alpha \setminus B_\alpha$ then $F(f_\alpha \restriction \delta) = 1 - \ell_\alpha \neq \ell_\alpha$ and hence $\eta(\delta) \neq m_\alpha$ so $\eta(\delta) = 1 - m_\alpha$. Consequently $Y_\alpha \cap (A_\alpha \setminus B_\alpha) \in \{A_\alpha \setminus B_\alpha, \emptyset\}$. But $\mathcal{P}(B_\alpha) \subseteq \mathfrak{B}(F^\otimes)$ so together we are done. $\square$

It follows from 1.15.1, 1.15.2 that

$$\{\beta : \eta(\beta) = 1\} \cap S_{\alpha^*[\eta]}[\eta] \in \mathfrak{B}(F^\otimes).$$

But $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}(F^\otimes)$, so $\mathcal{P}(\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \subseteq \mathfrak{B}(F^\otimes)$ so we get a contradiction. $\square$

Conclusion 1.16. Assume 1.13. Let $\eta \in {}^{\lambda}2$, $X_\ell[\eta] = (\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \cap \eta^{-1}(\{\ell\})$ (for $\ell = 0, 1$). Then one of the following occurs:

(A) $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}(F^{\otimes})$,

(B) $X_0[\eta], X_1[\eta] \notin \mathrm{ID}(F^{\otimes})$, and $X_0[\eta] \cup X_1[\eta] \in \mathfrak{B}(F^{\otimes})$, $X_0[\eta] \cap X_1[\eta] = \emptyset$, and for every $f \in {}^{\lambda}2$,

either the sequence $\langle F(f \restriction \delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \rangle$ is $\mathrm{ID}(F^{\otimes})$–almost constant
or both sequences $\langle F(f \restriction \delta) : \delta \in X_0[\eta] \rangle$ and $\langle F(f \restriction \delta) : \delta \in X_1[\eta] \rangle$ are not $\mathrm{ID}(F^{\otimes})$–almost constant.

Proof. Assume that the first possibility fails, so $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}(F^{\otimes})$.

Assume $X_0[\eta] \in \mathrm{ID}(F^{\otimes})$. Take any $f_{\alpha^*[\eta]} \in {}^{\lambda}2$ and choose $\ell_{\alpha^*[\eta]} \in \{0, 1\}$ so that

$$\{\delta \in \lambda \setminus S_{\alpha^*[\eta]}[\eta] : F(f_{\alpha^*[\eta]} \restriction \delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \mathrm{ID}(F^{\otimes}).$$

Putting $m_{\alpha^*[\eta]} = 0$ and $B_{\alpha^*[\eta]} = X_0[\eta]$ we get a contradiction with the definition of $\alpha^*[\eta]$. Similarly one shows that $X_1[\eta] \notin \mathrm{ID}(F^{\otimes})$.

Suppose now that $f \in {}^{\lambda}2$ and the sequence $\langle F(f \restriction \delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \rangle$ is not $\mathrm{ID}(F^{\otimes})$–almost constant but, say, the sequence $\langle F(f \restriction \delta) : \delta \in X_0[\eta] \rangle$ is $\mathrm{ID}(F^{\otimes})$–almost constant (and let the constant value be $\ell_{\alpha^*[\eta]}$). Let $m_{\alpha^*[\eta]} = 0$, $B_{\alpha^*[\eta]} = \{\delta \in X_0[\eta] : F(f \restriction \delta) = 1 - \ell_{\alpha^*[\eta]}\}$. Then $B_{\alpha^*[\eta]} \in \mathrm{ID}(F^{\otimes})$ and since necessarily

$$\{\delta \in X_0[\eta] \cup X_1[\eta] : F(f \restriction \delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \mathrm{ID}(F^{\otimes}),$$

we immediately get a contradiction. Similarly in the symmetric case. $\square$

Remark 1.17. Note that if $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}(F^{\otimes})$ then there is $\eta \in {}^{\lambda}2$ such that $\eta^{-1}[\{0\}] \supseteq \lambda \setminus S$ and above $X_0, X_1 \subseteq S$ and possibility (A) fails.

Proposition 1.18. *Assume 1.13.*

(1) *We can find $S^* = S^*_F$, S^*_0 and S^*_1 such that:*

(a) $S^* \in \mathfrak{B}(F^{\otimes})$,

(b) $S^* = S^*_0 \cup S^*_1$, $S^*_0 \cap S^*_1 = \emptyset$,

(c) *if* $S^* \neq \lambda$ *then* $\mathrm{ID}^2(F^{\otimes}) \restriction \mathcal{P}(\lambda \setminus S^*) = \mathrm{WDmId}_\lambda(F^{\otimes}) \restriction \mathcal{P}(\lambda \setminus S^*)$, $\lambda \setminus S^* \notin \mathrm{ID}^2(F^{\otimes})$.

(d) *if* $S^* \neq \emptyset$ *then* $S^* \notin \mathrm{ID}(F^{\otimes})$ *and*

$$\{(S^*_0 \cap F^{\otimes}(f)/\mathrm{ID}(F^{\otimes}), S^*_1 \cap F^{\otimes}(f)/\mathrm{ID}(F^{\otimes})) : f \in \mathrm{DOM}_\lambda\}$$

is an isomorphism from $\mathcal{P}(S^*_0)/\mathrm{ID}(F^{\otimes})$ onto $\mathcal{P}(S^*_1)/\mathrm{ID}(F^{\otimes})$.

(2) *If in 1.16, $S_F \subseteq S_{\alpha^*[\eta]}[\eta] \mod \mathrm{ID}(F)$ then we can add*

($\circledast$) *for some $\rho \in {}^{X_1}2$ for every $f \in {}^{\lambda}2$ we have*

$$\{\delta \in X_1 : F(f \restriction \delta) = \rho(\delta)\} \neq \emptyset \mod \mathrm{ID}(F^{\otimes}).$$

Proof. 1) We try to choose by induction on $\alpha < \lambda^+$ sets $S_\alpha, S_{\alpha,0}, S_{\alpha,1}$ such that

(a) $S_\alpha \subseteq \lambda$,

(b) $S_\alpha = S_{\alpha,0} \cup S_{\alpha,1}$, $S_{\alpha,0} \cap S_{\alpha,1} = \emptyset$,

(c) if $\beta < \alpha$ and $\ell < 2$ then

$$S_\beta \subseteq S_\alpha \mod \mathrm{ID}(F^\otimes) \quad \text{and} \quad S_{\beta,\ell} \subseteq S_{\alpha,\ell} \mod \mathrm{ID}(F^\otimes),$$

(d) the sets $S_{\alpha,0}, S_{\alpha_1}$ witness that $S \in \mathrm{ID}^2(F^\otimes)$ (see 1.2(4)).

At some stage $\alpha < \lambda^+$ we have to be stuck (as $\mathrm{ID}(F^\otimes)$ is λ^+–saturated) and then $(S_\alpha, S_{\alpha,0}, S_{\alpha,1})$ can serve as (S^*_F, S^*_0, S^*_1).

2) By the choice of S_F, for some $\ell < 2$ we have

$$\mathcal{P}(X_\ell) \neq \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda\},$$

so let $Y \subseteq X_\ell$ be such that $Y \notin \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda\}$. Let $\rho = 0_Y \cup 1_{X_\ell \setminus Y}$. Since without loss of generality $\ell = 1$, we are done. □

Remark 1.19. (1) If $\lambda \notin \mathrm{WDmId}_\lambda$ ten $S^* \neq \lambda$.

(2) Recall: $\mathrm{ID}^1(F^\otimes) = \mathrm{ID}(F^\otimes) = \mathrm{WDmId}_\lambda(F^\otimes)$ is a normal ideal and $\mathrm{ID}^2(F^\otimes)$ is a normal ideal extending it.

2. Weak diamond for more colours

In this section we deduce a weak diamond for, say, three colours, assuming the weak diamond for two colours and assuming that a certain ideal is saturated.

Proposition 2.1. *Assume that λ is a regular uncountable cardinal and $\mu \leq 2^{<\lambda}$. Let $F_i : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be λ–colourings for $i < \mu$. Then there is a colouring $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $F_i \leq F$ for every $i < \mu$.*

Proof. Case 1. $\mu \leq 2^{\|\alpha\|}$ for some $\alpha < \lambda$.

Let $\rho_i \in {}^\alpha 2$ for $i < \mu$ be distinct. For $\eta \in {}^{\lambda>}2$ let $h_i(\eta) = \rho_i{}^\frown\eta$. Define F by:

$$F(\nu) = \begin{cases} 0 & \text{if } \ell g(\nu) < \alpha, \text{ or } \ell g(\nu) \geq \alpha \\ & \text{but } \nu \restriction \alpha \notin \{\rho_i : i < \nu\}, \\ F_i(\langle \nu(\alpha+\varepsilon) : \varepsilon < \ell g(\nu) - \alpha\rangle) & \text{if } \ell g(\nu) \geq \alpha \text{ and } \nu \restriction \alpha = \rho_i. \end{cases}$$

It is easy to see that $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ and h_i exemplifies that $F_i \leq F$.

Case 2. $\mu = \lambda$.

For $\eta \in {}^{\lambda>}2$, $i < \mu$ and $\gamma < \lambda$ let

$$h_i(\eta)(\gamma) = \begin{cases} 0 & \text{if } \gamma < i, \\ 1 & \text{if } \gamma = i, \\ \eta(\gamma - (i+1)) & \text{otherwise.} \end{cases}$$

Next, for $\nu \in {}^{\lambda>}2$ define:

$$F(\nu) = \begin{cases} F_i(\langle \nu(i+1+\gamma) : \gamma < \ell g(\nu) - (i+1)\rangle) & \text{if } i = \min\{j : \nu(j) = 1\} \\ 0 & \text{if there is no such } i. \end{cases}$$

Now check.

Case 3. Otherwise, for each $\alpha<\lambda$ choose $F^\alpha : {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $(\forall i < 2^{\|\alpha\|})(F_i \leq F^\alpha)$ (exists by Case 1). Let $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that $(\forall\alpha<\lambda)(F^\alpha \leq F)$ (exists by Case 2).

The proposition follows. □

Theorem 2.2. *Assume that λ is a regular uncountable cardinal. Let $F^{\rm tr} : {}^{\lambda>}2 \longrightarrow 3$. For $i<3$ let $F_i : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that*

$$F_i(\eta) = \begin{cases} 1 & \textit{if } F^{\rm tr}(\eta) = i, \\ 0 & \textit{otherwise,} \end{cases}$$

and let $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that $(\forall i<3)(F_i \leq F)$. Assume that $\lambda \notin {\rm ID}^2(F^\otimes)$ (remember 1.10(3)), and ${\rm ID}(F^\otimes)$ is λ^+–saturated, i.e. there is no sequence $\langle A_\alpha : \alpha < \lambda^+\rangle$ such that

$$(\forall\alpha<\beta<\lambda^+)(A_\alpha \notin {\rm ID}(F) \quad \& \quad \|A_\alpha \cap A_\beta\| < \lambda).$$

Then there is a weak diamond sequence for $F^{\rm tr}$, even for every $S \in \mathfrak{B}(F^\otimes)\setminus {\rm ID}^2(F^\otimes)$.

Proof. Let S^*_F be as in 1.18. Since $\lambda\notin{\rm ID}^2(F^\otimes)$ necessarily $\lambda\setminus S^*_F \notin {\rm ID}(F^\otimes)$. Recall that ${\rm ID}^2(F^\otimes) = {\rm ID}(F) + S_F$.

It follows from 1.15 and 1.16 that there are disjoint sets $X_0, X_1 \subseteq \lambda$ (even disjoint from S^*_F from 1.18) such that $X_0, X_1 \notin {\rm ID}(F^\otimes)$, $X_0\cup X_1 \in \mathfrak{B}(F^\otimes)$ and for every $f \in {}^\lambda 2$ we have one of the following:

(a) the sequence $\langle F(f{\restriction}\delta) : \delta \in X_0\cup X_1\rangle$ is ${\rm ID}(F^\otimes)$–almost constant, or

(b) both sequences $\langle F(f{\restriction}\delta) : \delta\in X_0\rangle$ and $\langle F(f{\restriction}\delta) : \delta \in X_1\rangle$ are not ${\rm ID}(F^\otimes)$–almost constant.

It follows from 1.18(2) that we may assume that there is $\eta \in {}^{X_1}2$ such that for every $f\in{}^\lambda 2$ the set

$$\{\delta\in X_1 : F(f{\restriction}\delta) = \eta(\delta)\}$$

is stationary. Define a function $\rho \in {}^\lambda 2$ as follows:

$$\rho(\alpha) = \begin{cases} 1+\eta(\alpha) & \text{if } \alpha\in X_1, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 2.2.1. *ρ is a weak diamond sequence for $F^{\rm tr}$ even on $X_0\cup X_1$.*

Proof of the claim. Let $f\in{}^\lambda 2$. If $\{\alpha\in X_0 : F^{\rm tr}(f{\restriction}\alpha) = 0\} \notin {\rm ID}(F)$ then we are done (remember 1.3(3)). Otherwise, we have

$$\{\alpha \in X_0 : F_0(f\restriction\alpha) = 1\} \in {\rm ID}(F).$$

For $\ell<3$ let $f_\ell \in {}^\lambda 2$ be such that the set $\{\alpha<\lambda : F_\ell(f{\restriction}\alpha) = F(f_\ell{\restriction}\alpha)\}$ contains a club of λ (exists by 1.10); we first use f_0. Then

$$\{\alpha\in X_0 : F(f_0{\restriction}\alpha) = 1\} \in {\rm ID}(F^\otimes),$$

and hence, by the choice of the sets X_0, X_1,

$$\{\alpha\in X_1 : F(f_0{\restriction}\alpha) = 1\}\in{\rm ID}(F^\otimes).$$

Consequently,

$$\{\alpha \in X_1 : F^{\rm tr}(f{\restriction}\alpha) = 0\} = \{\alpha \in X_1 : F_0(f{\restriction}\alpha) = 1\} \in {\rm ID}(F^{\otimes}).$$

Now we use the choice of η. We know that the set

$$Y = \{\delta \in X_1 : F(f_1{\restriction}\delta) = \eta(\delta)\}$$

is stationary. Hence for some $k \in \{0,1\}$ the set

$$Y_k = \{\delta \in X_1 : F(f_1{\restriction}\delta) = k = \eta(\delta)\}$$

is stationary, but $\{\delta \in X_1 : F(f_1{\restriction}\delta) = F_1(f{\restriction}\delta)\}$ contains a club. Hence

$$Y^*_k = \{\delta \in X_1 : F(f_1{\restriction}\delta) = k = \eta(\delta) \text{ and } F(f_1{\restriction}\delta) = F_1(f{\restriction}\delta)\}$$

is stationary. Finally note that if $k = 1$ then

$$\delta \in Y_k \quad\Rightarrow\quad F(f_1{\restriction}\delta) = \eta(\delta) = F_1(f{\restriction}\delta) = 1 \quad\Rightarrow\quad F^{\rm tr}(f{\restriction}\delta) = 1.$$

The claim and the theorem are proved. □

□

Theorem 2.3. *Suppose $F^{\rm tr}$ is a (λ,θ)–colouring, $\theta \le \lambda$ and F_i (for $i < \theta$) are given by*

$$F_i(f) = \begin{cases} 1 & \text{if } F(f) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F : {}^{\lambda>}2 \longrightarrow 2$ be such that $(\forall i < \theta)(F_i \le F)$ and let $F^{\otimes}$ be as in 1.5 for F. Suppose that ${\rm ID}(F^{\otimes})$ is λ^+–saturated, and $S^*_{F^{\otimes}} \ne \lambda$ (i.e. $\lambda \notin {\rm ID}^2(F^{\otimes})$). Furthermore, assume that

($\otimes$) *there are sets $Y_i \subseteq \lambda \setminus S^*_{F^{\otimes}}$ for $i < \theta$ such that*
- (a) $(\forall i < \theta)(Y_i \notin {\rm ID}(F^{\otimes}))$,
- (b) *the sets Y_i are pairwise disjoint or at least*

$$(\forall i < j < \theta)(Y_i \cap Y_j \in {\rm ID}(F^{\otimes})),$$

- (c) $\bigcap\limits_{i<\theta} \min_{F^{\otimes}}(Y_i) \notin {\rm ID}(F^{\otimes})$, *see 1.11(h).*

Then

($\bigstar$) *there is a weak diamond sequence $\eta \in {}^{\lambda}\theta$ for $F^{\rm tr}$, i.e.*

$$(\forall f \in {}^{\lambda}2)(\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = \eta(\delta)\} \text{ is stationary });$$

moreover

$$(\forall f \in {}^{\lambda}2)(\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = \eta(\delta)\} \notin {\rm ID}(F^{\otimes})).$$

Proof. We may assume that the sets $\langle Y_i : i < \theta\rangle$ are pairwise disjoint (otherwise we use $Y_i' = Y_i \setminus \bigcup\limits_{j<i} Y_j$). Let $\eta \in {}^{\lambda}\theta$ be such that $(\forall i < \theta)(\eta{\restriction}Y_i = i)$. Note that if

$$\{\delta \in Y_i : F^{\rm tr}(f{\restriction}\delta) = i\} \in {\rm ID}(F^{\otimes})$$

then we also have

$$\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = i\} \in \mathfrak{B}(F^{\otimes})$$

(use $F_i \le F \le F^{\otimes}$). Consequently, in this case, we have

$$\{\delta \in \min_{F^{\otimes}}(Y_i) : F^{\rm tr}(f\restriction\delta) = i\} \in {\rm ID}(F^{\otimes}).$$

If this occurs for every $i < \theta$ then

$$\{\delta \in \bigcap_{i<\theta} \min_{F^{\otimes}}(Y_i) : (\exists i < \theta)(F(f\restriction\delta) = i)\} \in {\rm ID}(F^{\otimes}),$$

but for each δ, for some $i < \theta$ we have $F(f\restriction\delta) = i$, a contradiction. $\square$

Proposition 2.4. *Under the assumptions of 2.2 (so the ideal* ${\rm ID}(F^{\otimes})$ *is* λ^+*–saturated), if* $X \subseteq \lambda \setminus S^*_{F^{\otimes}}$*,* $X \notin {\rm ID}(F^{\otimes})$ *then there is a partition* (X_0, X_1) *of* X *(so* $X_0 \cup X_1 = X$*,* $X_0 \cap X_1 = \emptyset$*) such that*

$$X_0, X_1 \notin {\rm ID}(F^{\otimes}), \quad \textit{and} \quad \min_{F^{\otimes}}(X_0) = \min_{F^{\otimes}}(X_1) = \min_{F^{\otimes}}(X).$$

Proof. Let

$$\mathcal{A}_{F^{\otimes}} \stackrel{\rm def}{=} \{Z \subseteq \lambda : \ Z \notin {\rm ID}(F^{\otimes}) \text{ and there is a partition } (Z_0, Z_1) \text{ of } Z \text{ such that } \min_{F^{\otimes}}(Z_1) = \min_{F^{\otimes}}(Z_2) \mod {\rm ID}(F^{\otimes})\}.$$

Note that, by 1.11(h),

(∗) $\quad (\forall Y \in {\rm ID}(F^{\otimes})^+)(\exists Z \in \mathcal{A}_{F^{\otimes}})(Z \subseteq Y)$.

Let $X \subseteq \lambda$, $X \notin {\rm ID}(F^{\otimes})$ and let $\langle Z_\alpha : \alpha < \alpha^* \rangle$ be a maximal sequence such that for each $\alpha < \alpha^*$:

$$Z_\alpha \in \mathcal{A}_{F^{\otimes}}, \quad Z_\alpha \subseteq X, \quad \text{and} \quad (\forall \beta < \alpha)(Z_\alpha \cap Z_\beta \in {\rm ID}(F^{\otimes})).$$

Necessarily $\alpha^* < \lambda^+$, so without loss of generality $\alpha^* \le \lambda$, $\min(Z_\alpha) > \alpha$ and $Z_\alpha \cap Z_\beta = \emptyset$ for $\alpha < \beta < \alpha^*$. Let $\langle Z^0_\alpha, Z^1_\alpha \rangle$ be a partition of Z_α witnessing $Z_\alpha \in \mathcal{A}_{F^{\otimes}}$. Put

$$Z_0 \stackrel{\rm def}{=} \bigcup_{\alpha<\alpha^*} Z^0_\alpha \quad \text{and} \quad Z_1 \stackrel{\rm def}{=} \bigcup_{\alpha<\alpha^*} Z^1_\alpha.$$

Then $Z_0 \cap Z_1 = \emptyset$, $Z_0 \cup Z_1 \subseteq X$. Note that $\bigcup\limits_{\alpha<\alpha^*} Z_\alpha$ is equal to the diagonal union and, by (∗) above, $X \setminus \bigcup\limits_{\alpha<\alpha^*} Z_\alpha \in {\rm ID}(F^{\otimes})$. Consequently we may assume $Z_0 \cup Z_1 = \bigcup\limits_{\alpha<\alpha^*} Z_\alpha = X$. Next, since

$$\min_{F^{\otimes}}(Z_0) \supseteq \min_{F^{\otimes}}(Z^0_\alpha) \supseteq Z^0_\alpha \cup Z^1_\alpha = Z_\alpha,$$

we get

$$\min_{F^{\otimes}}(Z_0) \supseteq \bigcup_{\alpha<\alpha^*} Z_\alpha = X = Z_0 \cup Z_1,$$

and similarly one shows that $\min_{F^{\otimes}}(Z_1) \supseteq X$. Now we use 1.11(h) to finish the proof. $\square$

Proposition 2.5. *Under the assumptions of 2.3:*

1. *If* $2^\theta < \lambda$ *then there is a sequence* $\langle Y_i : i < \theta \rangle$ *as required in 2.3(*$\oplus$*).*
2. *Similarly if* $\theta \le \aleph_0$*.*
3. *In both cases, if* $S \notin {\rm ID}(F^{\otimes})$ *then we can demand* $(\forall i < \theta)(Y_i \subseteq S)$*.*

Proof. 1) By induction on $\alpha \le \theta$ we choose sets $X_\eta \subseteq \lambda$ for $\eta \in {}^{\alpha}2$ such that:

(i) $X_{\langle\rangle} \notin \mathrm{ID}(F^\otimes)$,

(ii) if α is limit then $X_\eta = \bigcap\limits_{i<\alpha} X_{\eta\restriction i}$,

(iii) if $\alpha = \beta+1$, $\eta \in {}^{\beta}2$ and $X_\eta \in \mathrm{ID}(F^\otimes)$ then $X_{\eta\frown\langle 0\rangle} = X_\eta$, $X_{\eta\frown\langle 1\rangle} = \emptyset$; if $\alpha = \beta + 1$, $\eta \in {}^{\beta}2$ and $X_\eta \notin \mathrm{ID}(F^\otimes)$ then $(X_{\eta\frown\langle 0\rangle}, X_{\eta\frown\langle 1\rangle})$ is a partition of X_η such that $\min_{F^\otimes}(X_{\eta\frown\langle 0\rangle}) = \min_{F^\otimes}(X_{\eta\frown\langle 1\rangle}) = \min_{F^\otimes}(X_\eta)$.

It follows from 2.4 that we can carry out the construction.

Clearly $\langle X_\eta : \eta \in {}^{\theta}2\rangle$ is a partition of $X_{\langle\rangle}$, so (as $2^\theta < \lambda$ and $\mathrm{ID}(F^\otimes)$ is λ–complete) we can find a sequence $\eta \in {}^{\theta}2$ such that $X_\eta \notin \mathrm{ID}(F^\otimes)$. Then

$$(\forall \alpha < \theta)(X_{\eta\restriction\alpha} \notin \mathrm{ID}(F^\otimes))$$

(as each of these sets includes X_η). Moreover, for each $\alpha < \theta$ and for $\ell = 0, 1$ we have

$$\min_{F^\otimes}(X_{\eta\restriction\alpha\frown\langle\ell\rangle}) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Put $Y_\alpha = X_{\eta\restriction\alpha\frown\langle 1-\eta(\alpha)\rangle}$. Then $\langle Y_\alpha : \alpha < \theta\rangle$ is a sequence of pairwise disjoint sets (as $X_{\eta\restriction\alpha\frown\langle 0\rangle} \cap X_{\eta\restriction\alpha\frown\langle 1\rangle} = \emptyset$) and for every $\alpha < \theta$

$$Y_\alpha \notin \mathrm{ID}(F^\otimes) \quad \text{and} \quad \min_{F^\otimes}(Y_\alpha) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Hence $\bigcap\limits_{\alpha<\theta} \min_{F^\otimes}(Y_\alpha) \notin \mathrm{ID}(F^\otimes)$. Let $Z_\alpha = Y_\alpha \cap \min_{F^\otimes}(X_\eta)$. Note that $\min_{F^\otimes}(Z_\alpha) = \min_{F^\otimes}(X_\eta)$ (the "$\le$" is clear; if $\min_{F^\otimes}(Z_\alpha) < \min_{F^\otimes}(X_\eta)$ then $\min_{F^\otimes}(X_\eta) \setminus \min_{F^\otimes}(Z_\alpha)$ contradicts the definition of $\min_{F^\otimes}(Y_\alpha)$). Thus the sequence $\langle Z_\alpha : \alpha < \theta\rangle$ is as required. Moreover

$$\min_{F^\otimes}(Z_\alpha) = \bigcup_\beta \min_{F^\otimes}(Z_\beta).$$

2) Let $X \subseteq \lambda$, $X \notin \mathrm{ID}(F^\otimes)$. By induction on n we choose sets X'_n, X''_n such that $X'_n \cap X''_n = \emptyset$, $X'_n \cup X''_n \supseteq X$, and

$$\min_{F^\otimes}(X'_n) = \min_{F^\otimes}(X''_n) = \min_{F^\otimes}(X).$$

For $n = 0$ we use 2.4 for X to get X'_0, X''_0. For $n + 1$ we use 2.4 for X''_n to get X'_{n+1}, X''_{n+1}.

Finally we let $Y_n = X''_n$ (note that $\min_{F^\otimes}(Y_n) = \min_{F^\otimes}(X)$). □

Conclusion 2.6. Assume that

(A) λ is a regular uncountable cardinal,

(B) F is a (λ, θ)–colouring such that $\lambda \notin \mathrm{ID}(F)$ and $\mathrm{ID}(F)$ is λ^+–saturated,

(C) $2^\theta < \lambda$ or $\theta = \aleph_0$,

(D) $(\exists \mu < \lambda)(2^\mu = 2^{<\lambda} < 2^\lambda)$ or at least $\lambda \notin \mathrm{WDmId}_\lambda$ or at least $\lambda \notin \mathrm{ID}^2(F)$.

Then there is a weak diamond sequence for F. Moreover, there is $\eta \in {}^{\lambda}\theta$ such that for each $f \in \mathrm{DOM}_\lambda(F)$ we have

$$\{\delta < \lambda : F(f{\restriction}\delta) = \eta(\delta)\} \notin \mathrm{ID}(F).$$

3. An application of Weak Diamond

In this section we present an application of Weak Diamond in model theory. For more on model–theoretic investigations of this kind we refer the reader to [Sh 576] and earlier work [Sh 88], and to an excellent survey my Makowsky, [Mw85].

Definition 3.1. Let $\mathfrak{K}$ be a collection of models.

(1) For a cardinal λ, $\mathfrak{K}_\lambda$ stands for the collection of all members of $\mathfrak{K}$ of size λ.

(2) We say that a partial order $\leq_{\mathfrak{K}}$ on $\mathfrak{K}_\lambda$ is λ–nice if
 - (α) $\leq_{\mathfrak{K}}$ is a suborder of $\subseteq$ and it is closed under isomorphisms of models (i.e. if $M, N \in \mathfrak{K}_\lambda$, $M \leq_{\mathfrak{K}} N$ and $f : N \longrightarrow N' \in \mathfrak{K}_\lambda$ is an isomorphism then $f[M] \leq_{\mathfrak{K}} N'$),
 - (β) $(\mathfrak{K}_\lambda, \leq_{\mathfrak{K}})$ is λ–closed (i.e. any $\leq_{\mathfrak{K}}$–increasing sequence of length $\leq \lambda$ of elements of $\mathfrak{K}_\lambda$ has a $\leq_{\mathfrak{K}}$–upper bound in $\mathfrak{K}_\lambda$) and
 - (γ) if $\bar{M} = \langle M_\alpha : \alpha < \lambda\rangle$ is an $\leq_{\mathfrak{K}}$–increasing sequence of elements of $\mathfrak{K}_\lambda$ then $\bigcup\limits_{\alpha<\lambda} M_\alpha$ is the $\leq_{\mathfrak{K}}$–upper bound to $\bar{M}$ (so $\bigcup\limits_{\alpha<\lambda} M_\alpha \in \mathfrak{K}_\lambda$).

(3) Let $N \in \mathfrak{K}_\lambda$, $A \subseteq |N|$. We say that *the pair (A, N) has the amalgamation property in* $\mathfrak{K}_\lambda$ if for every $N_1, N_2 \in \mathfrak{K}_\lambda$ such that $N \leq_{\mathfrak{K}} N_1$, $N \leq_{\mathfrak{K}} N_2$ there are $N^* \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$–embeddings f_1, f_2 of N_1, N_2 into N^*, respectively, such that $f_1{\restriction}A = f_2{\restriction}A$. (In words: N_1, N_2 can be amalgamated over (A, N).)

(4) We say that $(\mathfrak{K}, \leq_{\mathfrak{K}})$ has *the amalgamation property for* λ if for every $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$ such that $M_0 \leq_{\mathfrak{K}} M_1$, $M_0 \leq_{\mathfrak{K}} M_2$ there are $M \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$–embeddings f_1, f_2 of M_1, M_2 into M, respectively, such that

$$M_0 \leq_{\mathfrak{K}} M \quad \text{and} \quad f_1{\restriction}M_0 = f_2{\restriction}M_0 = \mathrm{id}_{M_0}.$$

Theorem 3.2. *Assume that λ is a regular uncountable cardinal for which the weak diamond holds (i.e. $\lambda \notin \mathrm{WDmId}_\lambda$). Suppose that $\mathfrak{K}$ is a class of models, $\mathfrak{K}$ is categorical in λ (i.e. all models from $\mathfrak{K}_\lambda$ are isomorphic), it is closed under isomorphisms of models, and $\leq_{\mathfrak{K}}$ is a λ–nice partial order on $\mathfrak{K}_\lambda$ and $M \in \mathfrak{K}_\lambda$. Let $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence of subsets of $|M|$ such that*

$$(\forall \alpha < \lambda)(\|A_\alpha\| < \lambda) \quad \textit{and} \quad \bigcup_{\alpha<\lambda} A_\alpha = M.$$

Then the set

$$S^{\bar{A}}_M \stackrel{\text{def}}{=} \{\alpha < \lambda\colon (A_\alpha, M) \text{ does not have the amalgamation property}\}$$

is in WDmId_λ.

Proof. Assume that $S^{\bar A}_M \notin \mathrm{WDmId}_\lambda$.

We may assume that $|M| = \lambda$. By induction on $i < \lambda$ we choose pairs (B_η, N_η) and sequences $\langle C^\eta_j : j < \lambda\rangle$ for $\eta \in {}^i 2$ such that

(a) $\|B_\eta\| < \lambda$, $N_\eta \in \mathfrak{K}_\lambda$, $B_\eta \subseteq |N_\eta| \subseteq \lambda$,
(b) $\langle C^\eta_j : j < \lambda\rangle$ is increasing continuous, $\bigcup\limits_{j<\lambda} C^\eta_j = |N_\eta|$, $\|C^\eta_j\| < \lambda$,
(c) if $\nu \lhd \eta$ then $N_\nu \leq_{\mathfrak{K}} N_\eta$ and $B_\nu \subseteq B_\eta$,
(d) if $j_1, j_2 < i$ then $C^{\eta\restriction j_1}_{j_2} \subseteq B_\eta$,
(e) if the pair (B_η, N_η) does not have the amalgamation property in $\mathfrak{K}_\lambda$ then $N_{\eta\frown\langle 0\rangle}$, $N_{\eta\frown\langle 1\rangle}$ witness it (i.e. they cannot be amalgamated over B_η),
(f) if i is limit and $\eta \in {}^i 2$ then $B_\eta = \bigcup\limits_{j<i} B_{\eta\restriction j}$, $\bigcup\limits_{j<i} N_{\eta\restriction j} \subseteq N_\eta$.

There are no problems with carrying out the construction (remember that $\leq_{\mathfrak{K}}$ is a nice partial order), we can fix a partition $\langle D_i : i < \lambda\rangle$ of λ into λ sets each of cardinality λ, and demand that the universe of N_η is included in $\bigcup\{D_j : j < 1 + \ell g(\eta)\rangle$. Finally, for $\eta \in {}^\lambda 2$ we let $B_\eta = \bigcup\limits_{i<\lambda} B_{\eta\restriction i}$ and $N_\eta = \bigcup\limits_{i<\lambda} N_{\eta\restriction i}$. Clearly, by 3.1(2$\gamma$), we have $N_\eta \in \mathfrak{K}$ and $B_\eta \subseteq |N_\eta|$ for each $\eta \in {}^\lambda 2$. Moreover,

$$|N_\eta| = \bigcup_{j<\lambda} |N_{\eta\restriction j}| = \bigcup_{j<\lambda}\bigcup_{i<\lambda} C^{\eta\restriction j}_i = \bigcup_{j^*<\lambda}\ \bigcup_{j_1,j_2<j^*} C^{\eta\restriction j_1}_{j_2} \subseteq \bigcup_{j^*<\lambda} B_{\eta\restriction j^*} = B_\eta,$$

and thus $B_\eta = |N_\eta|$. Since $\mathfrak{K}$ is categorical in λ, for each $\eta \in {}^\lambda 2$ there is an isomorphism $f_\eta : N_\eta \xrightarrow{\text{onto}} M$.

Fix $\eta \in {}^\lambda 2$ for a moment.

Let $E_\eta = \{\delta < \lambda : f_\eta[B_{\eta\restriction\delta}] = A_\delta = \delta\}$. Clearly, E_η is a club of λ. Note that if $\delta \in E_\eta$ then:

$$(\boxtimes)\quad \delta \in S^{\bar A}_M \begin{array}[t]{ll} \Rightarrow & (A_\delta, M) \text{ does not have the amalgamation property}\\ \Rightarrow & (B_{\eta\restriction\delta}, N_\eta) \text{ fails the amalgamation property}\\ \Rightarrow & (B_{\eta\restriction\delta}, N_{\eta\restriction\delta}) \text{ fails the amalgamation property}\\ \Rightarrow & N_{\eta\restriction\delta\frown\langle 0\rangle},\ N_{\eta\restriction\delta\frown\langle 1\rangle} \text{ cannot be amalgamated}\\ & \qquad\qquad \text{over } (B_{\eta\restriction\delta}, N_{\eta\restriction\delta})\\ \Rightarrow & \text{for each } \nu \in {}^\lambda 2 \text{ such that } \eta\restriction\delta\frown\langle 1-\eta(\delta)\rangle \lhd \nu\\ & \text{we have } f_\nu\restriction B_{\eta\restriction\delta} \neq f_\eta\restriction B_{\eta\restriction\delta}. \end{array}$$

We define a colouring

$$F : \bigcup_{\alpha<\lambda} {}^\alpha(\mathcal{H}(\lambda)) \longrightarrow \{0,1\}$$

by letting, for $f \in \mathrm{DOM}_\alpha$, $\alpha < \lambda$,

$$F(f) = 1 \quad\text{iff}\quad (\exists \eta \in {}^\lambda 2)\big(\eta(\alpha) = 0 \ \&\ (\forall i < \alpha)(f(i) = (\eta(i), f^{-1}_\eta(i)))\big).$$

We have assumed $S^{\bar{A}}_M \notin \mathrm{WDmId}_\lambda$, so there is $\rho \in {}^\lambda 2$ such that for each $f \in \mathrm{DOM}_\lambda$ the set

$$S_f = \{\delta \in S^{\bar{A}}_M : \rho(\delta) = F(f{\restriction}\delta)\}$$

is stationary. Let $f \in \mathrm{DOM}_\lambda$ be defined by $f(i) = (\rho(i), f^{-1}_\rho(i))$ (for $i < \lambda$). Note that

if $\alpha \in E_\rho$, $\rho(\alpha) = 0$

then ρ is a witness to $F(f{\restriction}\alpha) = 1$ and hence $\alpha \notin S_f$.

Since S_f is stationary and E_ρ is a club of λ we may pick $\delta \in S_f \cap E_\rho$. Then $\rho(\delta) = 1$ and hence $F(f{\restriction}\delta) = 1$, so let $\eta_\delta \in {}^\lambda 2$ be a witness for it. It follows from the definition of F that then $\eta_\delta(\delta) = 0$, and $\eta_\delta{\restriction}\delta = \rho{\restriction}\delta$, and $f^{-1}_{\eta_\delta}{\restriction}\delta = f^{-1}_\rho{\restriction}\delta$. Hence $f_{\eta_\alpha}{\restriction}B_{\eta_\delta{\restriction}\delta} = f_\rho{\restriction}B_{\rho{\restriction}\delta}$, so both have range $A_\delta = \delta$ (and $\delta \in E_{\eta_\delta} \cap E_\rho \cap S^{\bar{A}}_M$). But now we get a contradiction with (⊠). □

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